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THE H-FUNCTION AND PROBABILITY DENSITY FUNCTIONS OF CERTAIN ALGEBRAIC COMBINATIONS OF INDEPENDENT RANDOM VARIABLES WITH H-FUNCTION PROBABILITY DISTRIBUTIONS

~~Publication No.~~ ______

Ivy Dewey Cook, Jr., Ph.D., Major, USAF,
The University of Texas at Austin, 1981, 239 pages.

Supervising Professor: J. Wesley Barnes

A practical technique is presented for determining the exact probability density function and cumulative distribution function of a sum of any number of terms involving any combination of products, quotients, and powers of independent random variables with H-function distributions. The H-function is the most general named function, encompassing as special cases most of the other special functions of mathematics and many of the classical statistical distributions. Its unique properties make it a powerful tool for statistical analysis. In particular, the product, quotient, and powers of independent H-function variates are also H-function variates, and the Laplace and Fourier transforms and the derivatives of an H-function are readily-determined H-functions.

This dissertation [redacted] provides background material, including

history on H- functions and the algebra of random variables

~~transformations of random variables~~ and definition, properties and special cases of the H- function. For determining whether convergence of a general Mellin- Barnes integral or an H- function occurs with left-half-plane versus right-half-plane summation of residues, evaluation guidelines are formally established and applied to the known special cases, the Laplace transform, and the derivatives of the H- function. Then, a new, improved formulation for evaluation of an H- function by summing residues is derived.

The definition, special cases, and transformation theorems for the H- function distribution are presented. A new formula for finding the constant of an H- function distribution is derived. Also, the cumulative distribution function of an H- function distribution is shown to be a convergent H- function, and a more efficient way to compute it is found.

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THE H-FUNCTION AND PROBABILITY DENSITY FUNCTIONS OF CERTAIN ALGEBRAIC COMBINATIONS OF INDEPENDENT RANDOM VARIABLES WITH H-FUNCTION PROBABILITY DISTRIBUTIONS

by

IVY DEWEY COOK, JR., B.S., M.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

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THE UNIVERSITY OF TEXAS AT AUSTIN

May 1981

THE H- FUNCTION AND PROBABILITY DENSITY FUNCTIONS OF CERTAIN ALGEBRAIC COMBINATIONS OF INDEPENDENT RANDOM VARIABLES WITH H- FUNCTION PROBABILITY DISTRIBUTIONS

APPROVED BY SUPERVISORY COMMITTEE:

J. Wesley Barnes

Paul A. Jensen

Charles Beightler

W. T. Guy Jr.

Leon Lasdon

LIFE, KNOWLEDGE, OPPORTUNITIES, AND DREAMS:

To God for giving me each of these to pursue,

To my parents for teaching me the value of each
and for guiding my early pursuits,

To my wife Patricia for her love and support
during my continued pursuits,

And to Barry Eldred and Wes Barnes
for motivating and befriending me
in this most recent pursuit.

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I. D. C.

ABSTRACT

THE H- FUNCTION AND PROBABILITY DENSITY FUNCTIONS OF CERTAIN ALGEBRAIC COMBINATIONS OF INDEPENDENT RANDOM VARIABLES WITH H- FUNCTION PROBABILITY DISTRIBUTIONS

Ivy Dewey Cook, Jr., Ph.D.
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This dissertation first provides background material, including

history on H- functions and the algebra of random variables, definitions and properties of integral transforms, theorems on transformations of random variables, and definition, properties and special cases of the H- function. For determining whether convergence of a general Mellin- Barnes integral or an H- function occurs with left-half-plane versus right-half-plane summation of residues, evaluation guidelines are formally established and applied to the known special cases, the Laplace transform, and the derivatives of the H- function. Then, a new, improved formulation for evaluation of an H- function by summing residues is derived. This formulation is combined with a Laplace transform numerical inversion method to give a second new formulation.

The definition, special cases, and transformation theorems for the H- function distribution are presented. A new formula for finding the constant of an H- function distribution is derived. Also, the cumulative distribution function of an H- function distribution is shown to be a convergent H- function, and a more efficient way to compute it is found. Demonstration of the practical technique for handling sums is accompanied by an implementing computer program. Some examples of areas of application are discussed.

Throughout this dissertation, a number of new H- function formulas are found, including relations between given H- functions and other named functions or lower order H- functions, special-case derivative rules, and improved transform and derivative formulas.

TABLE OF CONTENTS

* Indicates chapter or section contains significant new material.

CHAPTER 1

INTRODUCTION AND REVIEW

1.1. PURPOSE AND SCOPE

Suppose one wishes to determine the exact probability density function of the sum of N independent random variables, $X_1,\ldots,X_N$, with known probability density functions $f_i(x_i)$, $i=1,\ldots,N$, respectively, such that $f_i(x_i)=0$ for $x_i \leq 0$, $i=1,\ldots,N$. That the desired answer is the inverse Laplace transform of the product of the N Laplace transforms of the f_i is well-established. This transform technique has been used for many special cases of f_i, particularly when the f_i are identical. To date, due to the integrations needed to find both the Laplace transforms and the inverse Laplace transform, each special case has been handled individually.

Now, suppose one has a general function which has as special cases all of the probability density functions f_i in some group of interest. If the transform technique is applied to the problem with f_i each taking the form of the general function, then the resulting solution covers all those problems involving any combination of the special cases. This is the motivation for using a general function.

Further, suppose an added bonus. Suppose that the product $\prod_{j=1}^{M} Y_j^{P_j}$, where each random variable Y_j has a probability density function expressible in the general function form and each P_j is a positive or negative rational constant, $j=1,\ldots,M$, is known to be a random variable with a probability density function that is easily and immediately expressible in the general function form. Then, the problem of finding the probability density function of

$$\sum_{i=1}^{N} \left(\prod_{j=1}^{M_i} X_{ij}^{P_{ij}} \right),$$

where the independent random variables X_{ij} all have general function forms for their probability density functions and P_{ij} are rational constants, reduces to finding the probability density function of

$$\sum_{i=1}^{N} Y_i ,$$

where the independent random variables Y_i all have general function forms for their probability density functions, a problem already covered by the general function transform solution.

The primary purpose of this dissertation is to develop a general technique, presented in Chapter 4, for determining the probability density function and the cumulative distribution function of the random variable

$$Z = \sum_{i=1}^{N} \left(\prod_{j=1}^{M_i} X_{ij}^{P_{ij}} \right),$$

where the X_{ij} are independent random variables with probability density functions expressible as H-functions and the P_{ij} are rational constants. The general function known as the H-function is chosen for several reasons. First, the H-function is the most general of the special functions and includes nearly every named function as a special case. Second, the H-function distribution, presented in

Chapter 4, has the added bonus that the products, quotients and rational powers of independent H-function variates are also H-function variates and are easily characterized. Third, the Laplace transform of an H-function is a related, known H-function, which means an integration is not needed for finding Laplace transforms for H-functions.

In the course of developing the above general technique, some secondary purposes became evident. One is the attempt, in Chapter 2, to relate H-functions to known elementary or special functions and to other simpler H-functions. The general form for H-functions is a contour integral containing gamma functions and is not readily identified by this form. The H-function contour integral can be evaluated using residues under certain convergence conditions. The aim of Chapter 3 is to develop practical guidelines for when left half plane residues versus right half plane residues should be summed in order to evaluate a given H-function. These guidelines are then applied to known formulas for the H-function that represent special cases, the Laplace transform, or the derivative.

Chapter 5 presents an improved formulation for using residues to numerically evaluate the H-function. This numerical evaluation is needed both to implement the general technique and to just evaluate a single given H-function.

Determination of distributions of algebraic combinations of independent random variables has application in virtually every aspect of probability and statistics. The few applications given in

Chapter 6 are intended to prod the imagination as to the vast number of potential areas of application and not to limit the extent of possible usage.

Some important limitations to the scope of this dissertation must be stated. For instance, only independent random variables are considered, though distributions of functions of dependent variates may be expressed as H-functions. Also, no exact method has been found to deal, in general terms, with certain linear combinations of independent random variables, particularly differences and also the product, quotient, or powers of sums and differences. Although many attempts were made, there was no success; general evaluation of such combinations will require some theoretical breakthroughs.

After much effort, no closed-form solution for the general technique has been found. The accompanying computer program, however, can be used to find values of the desired probability density function to any desired accuracy.

The only alternatives at present to the exact determination of a probability density function are various approximating methods based upon either the moments of a distribution or simulation. Such methods have many disadvantages and have been addressed extensively by others (7,21). Obviously, an exact, complete determination of a probability density function is preferable to any approximation, therefore these approximating methods will not be treated here.

Only real-valued variates and functions are considered. When inverting the Laplace transform, the computer program can handle complex values of the transform argument, but it is not designed to handle random variables or probability density functions that assume complex values. Also, the general H-function definition permits some parameters to be complex numbers, but the computer program can only handle real parameters.

The H-function is not defined for a zero or negative real value of its argument. Therefore, only probability density functions that are defined to be zero for nonpositive arguments are treated. Probability density functions defined non-zero for both positive and negative argument values can be handled by dividing such functions into two components, a technique presented by Epstein (8) for the case of two variables and extended by Springer and Thompson (320,321) to n variables. The computer program can then be used to evaluate pieces of the component derivation, for H-function components.

Since the general technique requires a Laplace transform inversion method, a review of such methods was made, and one was chosen that seemed suitable to H-function evaluation. However, the numerical inversion of the Laplace transform is a considerably large area to study by itself. The scope of this dissertation is not meant to include a comparison of the various methods or to find the best inversion method. Instead, the intent is to demonstrate feasibility of the general technique with at least one inversion method.

When background on the H-function and the H-function distribution is presented, the mathematical proofs have been omitted. Full understanding of the H-function requires a high level of mathematical knowledge and maturity. Not burying main ideas in non-contributory mathematical details should help bring out the power and simplicity inherent in usage of the H-function. Of course, the mathematical details are presented for all new material. New material is indicated by asterisks throughout this work.

The algebra of random variables is a vast field of study, so that a complete coverage is not reasonably within the scope of this dissertation. However, combining the advantages of a general function and of certain properties of the H-function with a practical technique for finding the exact probability density function of any member of a large class of algebraic combinations of independent random variables is, hopefully, a meaningful contribution to this field of study.

1.2. LITERATURE SURVEY

The development of probability and statistics has focused primarily upon the analysis of probability distributions of random variables and of algebraic combinations of random variables. Since the 1920's, many mathematicians and statisticians have directed their attention to the algebra of random variables, that is, to the problem of determining the probability distributions of sums, differences, products, quotients, and rational powers of random variables. In a recent book, Springer provides an excellent discussion and complete bibliography on this subject (21).

Considerable attention has been given to deriving the distributions of sums and differences of random variables, so that systematic, well-defined procedures now exist. Many early authors, including Aroian (243), Baten (244,245), Church (249), Craig (250), Cramer (212, 213,251), Dodd (252), Irwin (258), Levy (259), and Wintner (229,267), have presented detailed discussions concerning sums and differences of independent random variables. The usage of Fourier and Laplace transforms as powerful tools for dealing with sums and differences of independent random variables is well-established, and a large number of fine references are available, including Lukacs (217 - 219), Kawata (215), and Newcomb and Oliveira (220). Section G of the bibliography lists some of the basic references on integral transforms (224 - 228).

Papers on particular cases of sums of random variables are given in section I of the bibliography. Many of these treat the distribution of quadratic forms, such as the sums of squares of normally distributed variates (247,253,255 - 257,263 - 265).

The problem of deriving the distribution of products and quotients of random variables has not received the same extensive treatment as sums and differences (21:1). From 1929 to 1942, concentrating on normal variates, Craig made some of the early investigations into the distribution of the product and quotient of two random variables (270 - 272). He used an approximation method involving moments or semi-invariants. In 1930, Geary (275) developed an approximation for the quotient of two normal variables that became widely-used. Other approximations were developed: Tukey and Wilks (288) in 1946 for the product of beta variables, Aroian (268) in 1947 for the product of two normal variables, and Shellard (287) in 1952 for the product of several random variables.

In 1939, Huntington (278) presented the proofs of four theorems resulting in a mathematical formulation for determining distributions of the sum, difference, product, and quotient of two random variables. Other early contributors to the theory of products and quotients of random variables, including Camp (269), Curtiss (273), Gurland (276), Haldane (277), Levy (281), Rietz (285), and Sakamoto (286), dealt with specific probability density functions, usually normal.

In finding the distribution of a product or quotient of two random variables, the Fourier integral transform, or characteristic function, was useful for a number of special cases, beginning with studies by Kullback (279,280) in the 1930's and continuing through the 1960's with the references listed in section K of the bibliography.

A few general results for certain distributions were found using characteristic functions. For example, Kullback (279) in 1934 determined the distribution of the geometric mean for n uniform or gamma variates, and Jambunathan (293) in 1954 derived the distribution of products of special cases of beta and gamma independent random variables. Using a logarithmic transformation, Schulz-Arenstorff and Morelock (299) found the probability density function of the product of n uniform independent random variables.

The first practical, systematic, general approach for dealing with products and quotients of independent random variables was presented in 1948 by Epstein (8). His approach was the first usage of the Mellin integral transform to analyze the distribution of the product or quotient of two variates. Epstein demonstrated that the Mellin integral transform is a natural and powerful tool for finding the probability density function of products and quotients of independent random variables, by deriving directly and easily the probability density functions of the Student t and Fisher F statistics and of the product of two standardized, normal variates. His work was limited to two random variables. Surprisingly, additional application of the Mellin transform did not arise until the 1960's.

In 1959, Levy (282,283) derived some results for products of two independent random variables and posed the question of constructing a general theory for multiplication of random variables. Zolotarev (289) began this construction in 1962, focusing on a sequence of theorems, without proofs, that showed the similarities and differences between the results for addition of independent random variables and the results for multiplication. Then, in 1964, Springer and Thompson (320, 321) presented a general method for determining the probability density function of the product of n independent random variables that are not necessarily non-negative nor identically distributed. They applied the Mellin transform to analyses of products, quotients, and geometric means of rectangular, monomial, Cauchy, Gaussian, and gamma variates. Mellin transforms were then employed by Lomnicki (310) in 1967 to products of beta, gamma, Weibull, and normal variates.

A number of authors in the 1960's, as shown in section L of the bibliography, used the Mellin transform to treat the product and quotient of independent random variables. Most of the work was for two variables: Wells, Anderson and Cell (323) for central and for non-central chi-square variates, Srodka (322) for generalized gamma, Maxwell, and Weibull variates, Kotz and Srinivasan (309) for Bessel variates, Malik (311 - 314) for generalized gamma, non-central beta, and Pareto variates, and Pruett (317) for some nonstandardized variates, including the nonstandardized normal.

A significant contribution to analyzing algebraic combinations of independent random variables was made in 1970 by Prasad (223). He provided formulas for finding the Mellin transform of a function directly from its Laplace or Fourier transform, and vice versa, without having to determine the function itself. For example, if one wants to find the probability density function h(y) for the random variable Y given by

$$Y = X_1 + \frac{X_2 + X_3}{X_4},$$

where the independent random variables X_i have known probability density functions $f_i(x_i)$, $i = 1,2,3,4$. The analysis is considerably simplified if one can convert the Laplace (or Fourier) transform of the density function $g_1(u)$ for $U = X_2 + X_3$ into a Mellin transform, and then convert the Mellin transform of the density function $g_2(v)$ for $V = U/X_4$ into a Laplace (or Fourier) transform. Prasad's formulas can be used for these conversions, so that the probability density function $h(y)$ can be determined directly without first determining the probability density functions $g_1(u)$ and $g_2(v)$ (21:4-5).

Another important development in the analysis of the algebra of random variables has been the use of the G- and H-functions. These functions are general forms of many of the common and special functions of mathematics, including most of the common probability density functions. As early as 1958, Kabe (333) expressed some multivariate test statistics' density functions as G-functions, after recognizing that the moments of these statistics could be expressed as products of gamma functions. Similarly, Consul (328) in 1967 expressed the distributions of likelihood ratio criteria for testing independence as G-functions. In the early 1970's, Mathai (16,335-344) indicated many statistical applications for the G-function, including finding the distributions of various multivariate test statistics, the distribution of the product of independent beta variates, and examples to counter some proposed characterizations of probability laws.

Using Mellin transforms, a few authors expressed the probability density functions of products and quotients of selected independent random variables in terms of G- or H- functions. Dwivedi (303,304) in 1966 and 1970 introduced a confluent hypergeometric density function and demonstrated that the distribution of a product or quotient of variates each with such a density function was expressible as an H- function. Also, in 1970, Springer and Thompson (352) expressed the distributions of the products of beta, gamma, and Gaussian variates as G- functions. And, in 1974, Shah and Rathie (319) showed that distributions for products of generalized F- variates could be expressed as G- and H- functions.

Gupta and Jain (12) in 1966 proved that the Mellin convolution of two H- functions is another H- function. This led to the most significant advances in the usage of H- functions in statistical analysis, by Bradley D. Carter in 1972 (4,5). He tied together the physical science work on H- functions and the probability work on Mellin transforms into a meaningful general theory.

Carter introduced a new probability distribution, the H- function distribution, which is simply an H- function multiplied by a constant that makes the integral over the relevant range equal to unity. He showed that the H- function distribution includes, as special cases, ten common classical distributions - gamma, exponential, chi-square, Weibull, Rayleigh, Maxwell, half-normal, beta, half-Cauchy, and general hypergeometric. Most important, Carter proved that the probability density functions of products, quotients, and rational powers of independent H- function variates are also H- functions. This closure property does not hold for the classical distributions and thus makes the H- function a powerful general form.

In 1979, Eldred (7) implemented Carter's results by developing an operational computer program to calculate H- function values to any desired accuracy and to calculate values for the probability density function of combinations of products, quotients, and powers of H- function variates. He also expressed the half-Student and F distributions as H- function distributions. Springer (21) has reproduced the results of Carter and Eldred.

Additional background is in order with respect to the H- function history. The H- function is a Mellin-Barnes integral first introduced in 1961 by Charles Fox (10) as a symmetric Fourier kernel to the Meijer G- function, which is also a Mellin-Barnes integral and a special case of the H- function. Mellin-Barnes integrals have been used extensively in physics and engineering and are considered the most important of all integrals containing gamma functions in their integrands (9:49). Such contour integrals, introduced in 1888 by Pincherle (9:49), have long been used in solving differential equations, starting with Barnes (2) in 1908 for complete integration of the hypergeometric differential equation and Mellin (19) in 1910. In the 1940's, Meijer (360) introduced the G- function, in terms of which all significant particular solutions of a hypergeometric differential equation can be expressed.

Much work has been done on the G-function, notably by Luke (14) and by Mathai and Saxena (16), who provide an extensive bibliography. Nearly every special function of applied mathematics is a special case of the G-function and of the H-function. The bibliography for this dissertation (1,9,section N) lists the basic references for special functions. Mathai and Saxena (18:10-11,151-159) provide H-function formulas for the following special functions: Gauss' hypergeometric function, the confluent hypergeometric function, the generalized hypergeometric function, the generalized hypergeometric functions of Wright and Maitland, MacRobert's E-function, Meijer's G-function, the functions of Mittag-Leffler and Boersma (357), the Bessel and associated functions, and Wright's generalized Bessel function. Of course, all special cases of the above functions are also special cases of the H-function, including the elementary power, exponential, trigonometric, inverse trigonometric, and logarithmic functions.

In the same way that the work of Meijer (360-362) formed the basis for much of the later work on the G-function, Braaksma (3) presented properties, identities, asymptotic expansions and analytic continuations which became the foundation for H-function work. The decade following Braaksma's 1964 paper brought great numbers of works on differentiation, integration, identities, recurrence relations, expansions and series involving H-functions. For the H-function of one variable, these works are listed in bibliography sections A to F. The most prolific contributors were Anandani, Bajpai, K. C. Gupta, Kalla, V. C. Nair, R. K. Saxena, Shah, and Taxak. Some of the more significant contributions were the Laplace transform of an H-function by K. C. Gupta (11), identities and recurrence relations by K. C. Gupta (11) and Anandani (34-37), and derivative formulas (section B, 35), especially those by A. N. Goyal and G. K. Goyal (25) and by K. C. Gupta and U. C. Jain (26). Most of the H-function work of recent years has been for H-functions of more than one variable and for the H-function transform. The bibliography does not list these, but many are given by Mathai and Saxena (18).

The majority of H-function work has been highly theoretical, unwieldy, and usually directed to special cases instead of development of general theory. Almost no applications are given in the literature and the few given are for physics and engineering, particularly for heat production in a cylinder and differential equation solution. Most of the articles are by authors from India and are published in foreign or little known journals, often not easily accessible to the U. S. researcher. Comparison of the Mathai and Saxena bibliographies (16,18) shows that much of the H-function work simply extends earlier G-function results to the H-function by directly paralleling the earlier developments. Often articles will duplicate or involve only minor changes to previous articles by the same or another author. Due to the lack of general theory development and instead the treatment of many special cases, there is continual repetition of the same techniques. Most formulas have been derived by switching the order of the H-function contour integral and another operation, such as a differentiation, a summation, or another integration. Recurrence relations are often found by equating different derivative formulas.

One must be cautious when dealing with H-function literature because of the frequent errors. While some errors are probably just misprints, many are due to the failure to verify the existence or convergence conditions that enable evaluation of an H-function or permit switching of the order of operations. For example, a check against H-function convergence conditions easily shows that six of the H-function special case formulas tabulated by Mathai and Saxena (18:146,154,156) diverge for all values of the arguments.

Other errors result from the failure to check theoretical results by presenting at least one special case with known results. For example, in one of the few H-function papers concerning the algebra of random variables, Mathai and Saxena (17) committed the following error. Their equation (16) has a term based upon Braaksma's series expansion for the H-function, which has the form

$$\sum_{k=0}^{\infty} \left(f(i,h,k) \cdot (a+t)^{g(i,h,k)} \right) \text{, where, in their notation,}$$

$\nu_i = k$ and g and f are functions of the integer k and the i-th and h-th parameters of the H-function. They wrongly factor the second term out of the summation, so that their derivation for the density function of a linear combination of n variates, each with a density function of a form involving an H-function, has a term

$$(a+t)^{g(i,h,k)} \cdot \sum_{k=0}^{\infty} f(i,h,k) \text{, which is meaningless with}$$

the part depending on k that is outside the summation. Any attempt to use the derivation on even the simplest example would have shown this error to the authors.

Once Carter showed that the distribution of the products, quotients, and rational powers of H-function variates could be easily expressed as another H-function distribution, then practical application of H-function required only the ability to evaluate the H-function inversion integral. For an H-function where no denominator singularity in the integrand coincides with any pole, Mathai and Saxena (16:177-185; 18:70-75) in 1973 and 1978 presented a somewhat complicated computable representation. In 1977, Lovett (13) attempted a numerical evaluation of the general H-function inversion integral, but fell far short of success. In 1979, Barry S. Eldred (7) performed the first successful H-function evaluation by developing a simpler model and a computer program to calculate to any desired accuracy the values for a general H-function inversion integral.

When numerical inversion of a Laplace transform involving a product of H-functions is desired, two methods seem well-suited: one by Crump (231) which is an improvement of that by Dubner and Abate (232) and another by Jagerman (234) based on the well-known Widder inversion formula. Piessens (237,238) provides a bibliography on this subject and other good references are listed in section H of the bibliography of this dissertation; however, the appropriateness of these other techniques with H-functions has not been investigated.

1.3. INTEGRAL TRANSFORMS

1.3.1. Definitions (7:68 - 72; 21:27 -31).

Laplace Transform: A real function $f(x)$, defined and single-valued almost everywhere for $x \geq 0$, with x a real variable, is said to be Laplace transformable if the integral

$$\int_0^{\infty} |f(x)| \, e^{-kx} \, dx$$

converges for some real value k. Then,

$$L_r\{f(x)\} = \int_0^{\infty} e^{-rx} f(x) \, dx \tag{1.1}$$

is the Laplace transform of $f(x)$, where r is a complex variable. The inverse Laplace transform or inversion integral is given by

$$f(x) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} e^{rx} L_r\{f(x)\} \, dr. \tag{1.2}$$

Equations (1.1) and (1.2) constitute a transform pair. The function $f(x)$ is determined uniquely by (1.2) if $L_r\{f(x)\}$ is analytic in a strip consisting of that portion of the plane to the right of and including the Bromwich path $(c - i\infty, c + i\infty)$. This strip may or may not include the entire right half plane.

Fourier Transform: A real function $f(x)$, defined and single-valued almost everywhere for $-\infty < x < \infty$, with x a real variable, is said to be Fourier transformable if the integral

$$\int_{-\infty}^{\infty} |f(x)| \, e^{ikx} \, dx$$

converges for some real value k. Then,

$$F_t\{f(x)\} = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \tag{1.3}$$

is the Fourier transform of $f(x)$ and is called the characteristic function of $f(x)$, while e^{itx} is called the kernel. The inverse Fourier transform or inversion integral is given by

$$f(x) = (1/2\pi)\int_{-\infty}^{\infty} e^{-itx} F_t\{f(x)\}\, dt. \tag{1.4}$$

Many authors, including Eldred (7), Springer (21), and Tranter (227), use the transform pair defined above by equations (1.3) and (1.4). However, others, including Erdelyi (9), Titchmarsh (226), and Whittaker and Watson (368), use the transform pair with kernal e^{-itx}:

$$F_t\{f(x)\} = \int_{-\infty}^{\infty} e^{-itx} f(x)\, dx$$

$$f(x) = (1/2\pi)\int_{-\infty}^{\infty} e^{itx} F_t\{f(x)\}\, dt.$$

Which transform pair is used is not important as long as consistency is maintained. Changing from one pair to the other moves the poles of the transform from a strip in the right half plane to a strip in the left half plane, or vice versa.

Mellin Transform: A real function $f(x)$, defined and single-valued almost everywhere for $x \geq 0$, with x a real variable, is said to be Mellin transformable if the integral

$$\int_{0}^{\infty} |f(x)|\, x^{k-1}\, dx$$

converges for some real value k. Then,

$$M_s\{f(x)\} = \int_0^{\infty} x^{s-1} f(x)\, dx \qquad (1.5)$$

is the Mellin transform of $f(x)$, where s is a complex number. The Mellin transform inversion integral, or inverse Mellin transform, is given by

$$f(x) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} x^{-s} M_s\{f(x)\}\, ds. \qquad (1.6)$$

1.3.2. Properties (7:76-78; 21:34-36).

1.3.2.1. Linearity:

$$L_r\{c_1f_1(x) + c_2f_2(x)\} = c_1L_r\{f_1(x)\} + c_2L_r\{f_2(x)\}$$

$$F_t\{c_1f_1(x) + c_2f_2(x)\} = c_1F_t\{f_1(x)\} + c_2F_t\{f_2(x)\}$$

$$M_s\{c_1f_1(x) + c_2f_2(x)\} = c_1M_s\{f_1(x)\} + c_2M_s\{f_2(x)\}$$

1.3.2.2. First translation or shifting:

$$L_r\{e^{ax} f(x)\} = L_{r-a}\{f(x)\}$$

$$F_t\{e^{ax} f(x)\} = F_{t-a}\{f(x)\}$$

$$M_s\{x^{-a} f(x)\} = M_{s-a}\{f(x)\}$$

1.3.2.3. Second translation or shifting:

$$L_r\{f(x-a)\} = e^{-ar} L_r\{f(x)\},\ x > a$$

$$F_t\{f(x-a)\} = e^{-iat} F_t\{f(x)\}$$

1.3.2.4. Scaling with $a > 0$:

$$L_r\{f(ax)\} = a^{-1} L_{r/a}\{f(x)\}$$

$$F_t\{f(ax)\} = a^{-1} F_{t/a}\{f(x)\}$$

$$M_s\{f(ax)\} = a^{-s} M_s\{f(x)\}$$

1.3.2.5. Multiplication by x^n:

$$L_r\{x^n f(x)\} = (-1)^n L_r^{(n)}\{f(x)\}$$

$$F_t\{x^n f(x)\} = (-i)^n F_t^{(n)}\{f(x)\}$$

$$M_s\{x^n f(x)\} = M_{s+n}\{f(x)\}$$

1.3.2.6. Division by x, provided $\lim_{x \to 0} (f(x)/x)$ exists:

$$L_r\{f(x)/x\} = \int_r^\infty L_r\{f(x)\}\,dr$$

$$F_t\{f(x)/x\} = \int_t^\infty F_t\{f(x)\}\,dt$$

$$M_s\{f(x)/x\} = M_{s-1}\{f(x)\}$$

1.3.2.7. Transform of an integral:

$$L_r\{\int_0^x f(u)\,du\} = L_r\{f(x)\}/r$$

$$F_t\{\int_0^x f(u)\,du\} = F_t\{f(x)\}/(it)$$

1.3.2.8. Argument to a power:

$$M_s\{f(x^a)\} = a^{-1} M_{s/a}\{f(x)\}$$

1.3.3. Relations between transforms.

To obtain the Laplace or Fourier transform of a function directly from its Mellin transform, and vice versa, one can use the following relations, derived by Prasad (223) and presented formally by Lew (216) and Springer (21:412-417).

If the Laplace transform of f(x), $x \geq 0$, is analytic and of order $O(r^{-k})$, $k > 1$, for all r such that $\mathrm{Re}(r) > b$, $b < 0$, then the Mellin transform of f(x) is given by

$$M_s\{f(x)\} = M_{1-s}\{L_r\{f(x)\}\} / \Gamma(1-s)$$
$$= (\Gamma(s)/2\pi i)\int_{c-i\infty}^{c+i\infty} L_r\{f(x)\}(-r)^{-s}\,dr,$$

$\mathrm{Re}(s) > 0$, $b < c < 0$.

If the Fourier transform of f(x) is analytic and of order $O((it)^{-k})$, $k > 1$ and $\mathrm{Im}(t) \neq 0$, then the Mellin transform of f(x) is given by

$$M_s\{f(x)\} = (\Gamma(s)/2\pi)\int_{-\infty}^{\infty}\Big[F_t\{f(x)\}(it)^{-s}\Big|_{\mathrm{Im}(t)>0}$$
$$+ (-1)^{s-1} F_t\{f(x)\}(-it)^{-s}\Big|_{\mathrm{Im}(t)<0}\Big]dt.$$

If (1) the Mellin transform of f(x) is absolutely convergent on $a < \mathrm{Re}(s) < b$, $a < 1$ (216:582), or (2) f(x) is of bounded variation and measurable on (0,1) and on $(1,\infty)$ and, for $a < b$,

$$\int_0^1 \left|x^{a-\frac{1}{2}} f(x)\right|^2 dx < \infty \text{ and } \int_1^{\infty} \left|x^{b-\frac{1}{2}} f(x)\right|^2 dx < \infty$$

(21:175), then,

$$L_r\{f(x)\} = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} M_s\{f(x)\}\Gamma(1-s)\, r^{s-1}\,ds,$$

$a < c < \min(1,b)$, and

$$F_t\{f(x)\} = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} M_s\{f(x)\}\Gamma(1-s)\,(it)^{s-1}\,ds.$$

1.4. TRANSFORMATIONS OF RANDOM VARIABLES

Primary emphasis in this work is on the use of integral transforms to obtain probability density functions and cumulative distribution functions for certain transformations of independent random variables, that is, for algebraic combinations. First, a review of related basic probability concepts should be made.

A one-to-one transformation $h(x)$ from a set S into a set T means that for each y, element of T, there exists one and only one x, element of S, such that $h(x) = y$. When a function $h(x)$ is a one-to-one transformation from S into T, then the inverse transformation $h^{-1}(y)$, from T onto S, exists and $h^{-1}(h(x)) = x$. The set of positivity for a transformation $h(x)$ is the set of values x for which $h(x)$ is positive.

Two random variables X and Y are independent if and only if their joint probability density function $f_{X,Y}(x,y)$ equals the product of the individual densities $f_X(x)$ and $f_Y(y)$, associated with X and Y, respectively. That is,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \text{ , for all } (x,y).$$

This means that any variation in the outcome of X will in no way affect the outcome of Y, and vice versa. Or, knowledge of the value taken by X yields no information about nor affects the probability distribution of Y, and vice versa.

THEOREM 1.1: Let X be a random variable with continuous probability density function $f_X(x)$ and $y = h(x)$ be a one-to-one transformation from S, the set of positivity of $f_X(x)$, onto T, the image of S under

$h(x)$. If $h^{-1}(y)$ is differentiable and its derivative is continuous on T, then the probability density function of Y is given by

$$f_Y(y) = \begin{cases} f_X(h^{-1}(y))\,|d(h^{-1}(y))/dy| , & y \in T \\ 0 , & \text{otherwise.} \end{cases}$$

THEOREM 1.2: Let $\underset{\sim}{X} = (X_1,\ldots,X_k)$ be a set of k random variables having the joint continuous probability density function $f_{\underset{\sim}{X}}(\underset{\sim}{x})$. Let $\underset{\sim}{Y} = h(\underset{\sim}{X}) = (h_1(\underset{\sim}{X}), h_2(\underset{\sim}{X}),\ldots,h_k(\underset{\sim}{X}))$ be a set of relations forming a one-to-one transformation from S, the k-dimensional set of positivity of $f_{\underset{\sim}{X}}$, onto T, the k-dimensional image of S under $h(\underset{\sim}{X})$. The inverse transformation, $\underset{\sim}{X} = h^{-1}(\underset{\sim}{Y}) = (g_1(\underset{\sim}{Y}), g_2(\underset{\sim}{Y}),\ldots,g_k(\underset{\sim}{Y}))$. If the partial derivatives of $h^{-1}(\underset{\sim}{Y})$ exist and are continuous,

$$g_{ij} = \partial(g_i(\underset{\sim}{Y}))/\partial Y_j ,$$

then the joint probability density function of $\underset{\sim}{Y}$ is given by

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \begin{cases} f_{\underset{\sim}{X}}(g_1(\underset{\sim}{y}), g_2(\underset{\sim}{y}),\ldots,g_k(\underset{\sim}{y}))\cdot|J| , & \underset{\sim}{y} \in T \\ 0 , & \text{otherwise,} \end{cases}$$

where J is the Jacobian, the determinant of first partial derivatives,

$$J = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ g_{21} & g_{22} & \cdots & g_{2k} \\ \vdots & & & \\ g_{k1} & g_{k2} & \cdots & g_{kk} \end{vmatrix} .$$

<u>Example</u>: Suppose the probability density function of $Z = X + Y$ is desired. Let $W = Y$, so that $X = Z - W$ and $Y = W$ and

$$J = \begin{vmatrix} \partial X/\partial Z & \partial X/\partial W \\ \partial Y/\partial Z & \partial Y/\partial W \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 .$$

By Theorem 1.2, for the appropriate ranges of z and w,

$$f_{Z,W}(z,w) = f_{X,Y}(z-w,w)\,|J|\ .$$

If X and Y are independent, then

$$f_{Z,W}(z,w) = f_X(z-w)\cdot f_Y(w)\ .$$

The marginal distribution of $Z=X+Y$ is found by integrating the above joint distribution $f_{Z,W}(z,w)$ over the proper range of w:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\cdot f_Y(y)\,dy\ .$$

Using Theorem 1.2 similarly to find the distributions for the difference, product, and quotient of two independent random variables gives the following theorem (278).

<u>THEOREM 1.3</u>: If X and Y are continuous independent random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then

(1) the probability density function of the random variable $Z=X+Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\cdot f_Y(y)dy = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)dx,$$

(2) the probability density function of the random variable $Z=X-Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y)\cdot f_Y(y)dy = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z+x)dx,$$

(3) the probability density function of the random variable $W=X\cdot Y$ is given by

$$f_W(w) = \int_{-\infty}^{\infty} |x^{-1}| f_X(x)\cdot f_Y(\tfrac{w}{x})dx = \int_{-\infty}^{\infty} |y^{-1}| f_X(\tfrac{w}{y}) f_Y(y)dy,$$

(4) and the probability density function of the random variable $W = X/Y$ is given by

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) \cdot f_Y(wx) dx = \int_{-\infty}^{\infty} |y| f_X(wy) \cdot f_Y(y) dy .$$

Theorems 1.2 and 1.3 have been applied to many distribution problems. However, each case must be treated separately and special care must be taken to determine the proper integration limits and ranges for the variables. Integral transforms can help simplify the process. Consider the following formulas:

$$F_t\{f(x)\} \cdot F_t\{g(y)\} = F_t\left\{\int_{-\infty}^{\infty} f(x)\ g(y-x)\ dx\right\}$$

$$L_r\{f(x)\} \cdot L_r\{g(y)\} = L_r\left\{\int_{0}^{\infty} f(x)\ g(y-x)\ dx\right\}$$

$$M_s\{f(x)\} \cdot M_s\{g(y)\} = M_s\left\{\int_{0}^{\infty} x^{-1}\ f(x)\ g(y/x)\ dx\right\} .$$

Combining these formulas with Theorem 1.3 gives Theorems 1.4 to 1.8 below. The following integral transform theorems are straightforward, powerful tools for determining probability density functions of sums, differences, products, quotients, and powers of independent random variables. A distinct advantage to transform use is the convenient extension of Theorem 1.3 to more than two variables. However, even though integral transforms have assisted considerably in analyzing probability density functions and have been used a great deal, each case must still be handled separately because of the requirement of finding the transforms.

THEOREM 1.4: Distribution of Linear Combination. If $X_1, X_2, \ldots, X_N$ are continuous independent random variables with probability density functions $f_1, \ldots, f_N$, respectively, then the probability density function of the random variable

$$Y = \sum_{i=1}^{N} a_i X_i \ , \ a_i > 0, \ i = 1, \ldots, N,$$

is given by

$$f_Y(y) = F_y^{-1}\left\{\prod_{i=1}^{N} F_{a_i t}\{f_i(x_i)\}\right\},$$

where F_y^{-1} is the inverse Fourier transform operation.

THEOREM 1.5: Given the same conditions as in Theorem 1.4 and $P\{X_i < 0\} = 0$ for $i = 1, \ldots, N$, then the probability density function of Y is given by

$$L_y^{-1}\left\{\prod_{i=1}^{N} L_{a_i r}\{f_i(x_i)\}\right\}, \ y \geq 0,$$

where L_y^{-1} is the inverse Laplace transform operation.

THEOREM 1.6: Distribution of a Difference. If X_1 and X_2 are continuous independent random variables with probability density functions $f_1(x_1)$ and $f_2(x_2)$, respectively, then the probability density function of the random variable $Y = X_1 - X_2$ is given by

$$f_Y(y) = F_y^{-1}\left\{F_t\{f_1(x_1)\} \cdot F_t\{f_2(-x_2)\}\right\},$$

where F_y^{-1} is the inverse Fourier transform operation.

THEOREM 1.7: Distribution of Product. If $X_1,\ldots,X_N$ are continuous independent random variables with probability density functions $f_1,\ldots,f_N$, respectively, where $P\{X_i<0\}=0$ for $i=1,\ldots,N$, then the probability density function of the random variable

$$Y=\prod_{i=1}^{N} a_i X_i \ , \ a_i>0,\ i=1,\ldots,N,$$

is given by

$$f_Y(y)=M_y^{-1}\left\{\prod_{i=1}^{N} a_i^{\,s-1}\, M_s\{f_i(x_i)\}\right\},\ y\geq 0,$$

where M_y^{-1} is the inverse Mellin transform operation.

THEOREM 1.8: Distribution of Quotient. If X_1 and X_2 are continuous independent random variables with probability density functions $f_1(x_1)$ and $f_2(x_2)$, respectively, where $P\{X_1<0\}=P\{X_2<0\}=0$, then the probability density function of the random variable $Y=X_1/X_2$ is given by

$$f_Y(y)=M_y^{-1}\left\{M_s\{f_1(x_1)\}\, M_{-s+2}\{f_2(x_2)\}\right\},\ y\geq 0,$$

where M_y^{-1} is the inverse Mellin transform operation.

THEOREM 1.9: Distribution of Rational Power. If X is a continuous random variable with probability density function f_X, where $P\{X<0\}=0$, then the probability density function of the random variable $Y=X^a$, a rational, is given by

$$f_Y(y)=M_y^{-1}\left\{M_{as-a+1}\{f_X(x)\}\right\},\ y\geq 0,$$

where M_y^{-1} is the inverse Mellin transform operation.

THEOREM 1.10: Moments of a Distribution. If X is a continuous random variable with probability density function $f_X(x)$, then

(1) if $f_X(x)$ is defined on the whole real line,

$$E(X^k) = (1/i^k)\,\frac{d^k}{dt^k}\,F_t\{f_X(x)\}\Big|_{t=0}\,, \text{ and}$$

(2) if $P\{X<0\} = 0$,

$$E(X^k) = (-1)^k\,\frac{d^k}{dr^k}\,L_r\{f_X(x)\}\Big|_{r=0} = L_r\{x^k\,f_X(x)\}\Big|_{r=0}$$

$$= M_s\{f_X(x)\}\Big|_{s=k+1}\,.$$

CHAPTER 2

THE H-FUNCTION

2.1. GENERAL REMARKS

Once the initiate to H-functions proceeds beyond the difficult mathematical hurdles of understanding the definition, convergence, and evaluation of an H-function (section 2.2. and chapters 3 and 5), he can then begin to appreciate the unique advantages of using H-functions. Foremost, the H-function is the most general special function, encompassing as special cases most of the other special functions and elementary functions of mathematics. Thus, anything accomplished with the *general form* for the H-function is valid for all special cases and has been accomplished therefore for every member of a large class of functions. When the accomplishment is a procedure involving differentiation or integration, the general nature of that procedure's applicability is particularly meaningful.

The properties of the H-function which are presented in this chapter are readily seen to be no more than simple adjustments of given parameters. The simple parameter changes needed to find the Laplace, Fourier, and Mellin transforms or the derivatives of an H-function are trivial compared to performing these same operations for the various special cases. Treating the many different types of special cases separately requires remembering a large number of differentiation formulas and integration methods or compiling long tables of the results. Another advantage is that the derivatives, Laplace transform, and Fourier transform of an H-function are themselves H-functions. Formulas, procedures, and computer programs used to handle an H-function can also be used to handle its derivatives and integral transforms.

Chapter 4 presents the H-function distribution, a probability density function, expressed in terms of an H-function times an appropriate constant. Many of the classical statistical distributions are special cases of the H-function distribution. Moreover, use of this general H-function distribution has a singular advantage: the probability density function of the products, quotients, and rational powers of independent H-function distributed variates is another H-function distribution. This new H-function is easily determined by combining and adjusting the parameters of the given H-function distributions for the variates. This closure property is not common to the classical distributions; for example, the product of normally-distributed independent random variables is not distributed normally.

On the other hand, the probability density function of the sum or difference of two H-function variates is not in general an H-function distribution. By making use of the simple relation for finding Laplace transforms of H-functions, a straight-forward technique can still be used to determine the distribution of the sum of H-function variates. This technique, shown in Chapter 4, provides a numerical evaluation to any desired accuracy.

2.2. DEFINITION

The H-function is defined by either of two forms (10; 4:35-37; 7:98-102; 18:2-3; 21:195-198):

$$H(z) = H^{m\,n}_{p\,q}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{\prod_{i=1}^{m} \Gamma(b_i + B_i s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod_{i=m+1}^{q} \Gamma(1 - b_i - B_i s)} z^{-s}\, ds \tag{2.1}$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{\prod_{i=1}^{m} \Gamma(b_i - B_i s) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i - A_i s) \prod_{i=m+1}^{q} \Gamma(1 - b_i + B_i s)} z^{+s}\, ds, \tag{2.2}$$

where z and all a_i and b_i are real or complex numbers, all A_i and B_i are positive real numbers, and m, n, p and q are integers such that $0 \leq m \leq q$ and $0 \leq n \leq p$. Empty products are defined to be equal to unity (1). C_1 is a contour in the complex s-plane running from $w - i\infty$ to $w + i\infty$, such that all poles of $\prod \Gamma(b_i + B_i s)$ lie to the left of C_1 and all poles of $\prod \Gamma(1 - a_i - A_i s)$ lie to the right. Similarly, C_2 is a contour running from $v - i\infty$ to $v + i\infty$, such that all poles of $\prod \Gamma(b_i - B_i s)$ lie to the right of C_2 and all poles of $\prod \Gamma(1 - a_i + A_i s)$ lie to the left.

Form (2.1) above is that of a Mellin transform inversion integral (refer to section 1.3.1). Form (2.2) is that of a type of the general Mellin-Barnes integral (refer to section 3.2).

Form (2.2) above is easily found from form (2.1) by substituting -s for s everywhere in (2.1), letting v equal -w and recognizing that

$$\int_a^b f(-s)\, d(-s) \;=\; \int_b^a f(-s)\, ds \,.$$

Form (2.1) of the H-function definition is used hereafter, because of the direct relation for the Mellin transform, as shown in the next section.

2.3. PROPERTIES

2.3.1. Reciprocal argument (4:36; 7:101; 18:4; 21:196):

$$H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= H_{q\,p}^{n\,m}\left[\tfrac{1}{z} : \{(1-b_i,B_i)\} ; \{(1-a_i,A_i)\}\right]. \qquad (2.3)$$

2.3.2. Argument to a real power:

$$H_{p\,q}^{m\,n}\left[z^k : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= k^{-1}\, H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i/k)\} ; \{(b_i,B_i/k)\}\right],$$

for $k>0$ (4:36; 7:101; 21:196); (2.4)

or equivalently, for $k>0$ (18:4),

$$H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= k\, H_{p\,q}^{m\,n}\left[z^k : \{(a_i,kA_i)\} ; \{(b_i,kB_i)\}\right]. \qquad (2.5)$$

Combining properties (2.3) and (2.4), when $k<0$,

$$* \quad H_{p\,q}^{m\,n}\left[z^k : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= (-k)^{-1} H_{q\,p}^{n\,m}\left[z : \{(1-b_i,-B_i/k)\} ; \{(1-a_i,-A_i/k)\}\right] \tag{2.6}$$

and, combining properties (2.3) and (2.5), when $k<0$,

$$* \quad H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= -k\, H_{q\,p}^{n\,m}\left[z^k : \{(1-b_i,-kB_i)\} ; \{(1-a_i,-kA_i)\}\right]. \tag{2.7}$$

2.3.3. Multiply by the argument to a power k (4:36; 7:102; 18:4; 21:196):

$$z^k H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= H_{p\,q}^{m\,n}\left[z : \{(a_i+kA_i,A_i)\} ; \{(b_i+kB_i,B_i)\}\right]. \tag{2.8}$$

2.3.4. If one of the (a_i,A_i), $i\leq n$, is equal to one of the (b_j,B_j), $j>m$, or one of the (a_i,A_i), $i>n$, is equal to one of the (b_j,B_j), $j\leq m$, then the H-function reduces to one of lower order:

$$H_{p\,q}^{m\,n}\left[z : \{(a_i,A_i)\} ; (b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)\right]$$

$$= H_{p-1,q-1}^{m,n-1}\left[z : (a_2,A_2),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right],$$

provided $n>0$ and $q>m$ (18:4); (2.9)

$$* \quad H^{m\,n}_{p\,q}\left[z : (a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(b_1,B_1) \; ; \; \{(b_i,B_i)\}\right]$$

$$= H^{m-1,n}_{p-1,q-1}\left[z : \{(a_i,A_i)\} \; ; \; (b_2,B_2),\ldots,(b_q,B_q)\right] ,$$

provided $m > 0$ and $p > n$. (2.10)

2.3.5. Mellin transform.

Form (2.1) of the H- function definition is exactly that of a Mellin transform inversion integral (refer to section 1.3.1), so that the Mellin transform of the H- function is directly given as (4:37; 7:102; 21:199):

$$M_s\{H(cz)\} = c^{-s} \frac{\prod_{i=1}^{m} \Gamma(b_i + B_i s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod_{i=m+1}^{q} \Gamma(1 - b_i - B_i s)} . \tag{2.11}$$

2.3.6. The Laplace and Fourier transforms of an H- function are themselves H- functions (4:38-39; 7:102; 21:199-201):

$$L_r\{H(cz)\} = c^{-1} H^{n+1,m}_{q,p+1}\left[r/c : \{(1 - b_i - B_i, B_i)\} \; ; \; (0,1), \{(1 - a_i - A_i, A_i)\}\right] \tag{2.12}$$

$$F_t\{H(cz)\} = c^{-1} H^{n+1,m}_{q,p+1}\left[-it/c : \{(1 - b_i - B_i, B_i)\} \; ; \; (0,1), \{(1 - a_i - A_i, A_i)\}\right] \tag{2.13}$$

2.3.7. Derivatives.

The r-th derivative of an H- function can be shown to be an H- function by using Skibinski's derivative formula (20; 18:5-7, 12-14):

$$z^r \frac{d^r}{dz^r} H_{p\ q}^{m\ n}\left[z^k : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= H_{p+1,q+1}^{m,n+1}\left[z^k : (0,k), \{(a_i,A_i)\} ; \{(b_i,B_i)\}, (r,k)\right]$$

for $k > 0$ and r a non-negative integer. (2.14)

Multiplying both sides of (2.14) by z^{-r} and then applying property (2.8) to the right side gives, for $k>0$:

$$\frac{d^r}{dz^r} H(z^k) = H^{(r)}(z^k)$$

$$= H_{p+1,q+1}^{m,n+1}\left[z^k : (-r,k), \{(a_i - \tfrac{r}{k}A_i, A_i)\} ; \{(b_i - \tfrac{r}{k}B_i, B_i)\}, (0,k)\right]. \quad (2.15)$$

Similarly, for $k < 0$:

$$H^{(r)}(z^k) = (-1)^r H_{p+1,q+1}^{m,n+1}\left[z^k : (1,-k), \{(a_i + \tfrac{r}{-k}A_i, A_i)\} ; \{(b_i + \tfrac{r}{-k}B_i, B_i)\}, (1+r,-k)\right]. \quad (2.16)$$

When r is zero, (2.14) through (2.16) reduce to the trivial $H(z^k) = H(z^k)$, using property (2.9) or (2.10). Skibinski's rule, formula (2.14), will be used in section 2.5. to derive new relations between H- functions and between H- functions and well-known elementary functions.

2.3.8. Parameters differing by an integer r (18:4-5):

$$H^{m,n+1}_{p+1,q+1}\left[z : (c,C), \{(a_i,A_i)\} ; \{(b_i,B_i)\}, (c+r,C)\right]$$

$$= (-1)^r H^{m+1,n}_{p+1,q+1}\left[z : \{(a_i,A_i)\}, (c,C) ; (c+r,C), \{(b_i,B_i)\}\right] \tag{2.17}$$

$$H^{m+1,n}_{p+1,q+1}\left[z : \{(a_i,A_i)\}, (c-r,C) ; (c,C), \{(b_i,B_i)\}\right]$$

$$= (-1)^r H^{m,n+1}_{p+1,q+1}\left[z : (c-r,C), \{(a_i,A_i)\} ; \{(b_i,B_i)\}, (c,C)\right] \tag{2.18}$$

2.3.9. Recurrence relations.

Throughout the literature, a great number of recurrence or contiguity formulas relating H- functions of the same order (m,n,p,q) can be found (11; 33 to 53; 18:7-8,17-19). A few typical examples are given below (18:7-8):

$$(a_1-a_2)\, H^{m\,n}_{p\,q}\left[z : (a_1,A_1),(a_2,A_1),(a_3,A_3),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right]$$

$$= H^{m\,n}_{p\,q}\left[z : (a_1,A_1),(a_2-1,A_1),(a_3,A_3),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right]$$

$$- H^{m\,n}_{p\,q}\left[z : (a_1-1,A_1),(a_2,A_1),(a_3,A_3),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right]$$

where $n \geq 2$ (note that $A_1 = A_2$).

$$(b_1A_1 - a_1B_1 + B_1)\, H^{m\,n}_{p\,q}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= B_1\, H^{m\,n}_{p\,q}\left[z : (a_1 - 1,A_1),(a_2,A_2),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right]$$

$$+ A_1\, H^{m\,n}_{p\,q}\left[z : \{(a_i,A_i)\} ; (b_1 + 1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\right]$$

where $m \geq 1$ and $n \geq 1$.

$$(b_qA_1 - a_1B_q + B_q)\, H^{m\,n}_{p\,q}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$$

$$= B_q\, H^{m\,n}_{p\,q}\left[z : (a_1 - 1,A_1),(a_2,A_2),\ldots,(a_p,A_p) ; \{(b_i,B_i)\}\right]$$

$$- A_1\, H^{m\,n}_{p\,q}\left[z : \{(a_i,A_i)\} ; (b_1,B_1),\ldots,(b_{q-1},B_{q-1}), (b_q + 1,B_q)\right]$$

where $n \geq 1$ and $q > m$.

$$(a_p - ka_1)\, H^{m\,n}_{p\,q}\left[z : (1 + a_1,A_1),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}), (1 + a_p,kA_1) ; \{(b_i,B_i)\}\right]$$

$$= H^{m\,n}_{p\,q}\left[z : (1 + a_1,A_1),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,kA_1) ; \{(b_i,B_i)\}\right]$$

$$+ k\, H^{m\,n}_{p\,q}\left[z : (a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(1 + a_p,kA_1) ; \{(b_i,B_i)\}\right]$$

where $k > 0$ and $1 \leq n < p$.

2.4. KNOWN SPECIAL CASES

An extensive list of elementary special functions expressed as special cases of the H-function is given by Mathai and Saxena (18:10-12,145-159). The most important and most familiar of these cases are given below:

2.4.1. Exponential and Power Functions.

$$e^{-z} = H_{0\,1}^{1\,0}\left[z : (0,1)\right] \qquad (4:40)$$

$$z^b e^{-z} = H_{0\,1}^{1\,0}\left[z : (b,1)\right] \qquad (18:151)$$

$$B^{-1} z^{b/B} e^{-z^{1/B}} = H_{0\,1}^{1\,0}\left[z : (b,B)\right] \qquad (18:10)$$

$$z^b = H_{1\,1}^{1\,0}\left[z : (b+1,1) \; ; \; (b,1)\right] \qquad (18:152)$$

$$z^b (1-z)^{+a} = \Gamma(a+1)\, H_{1\,1}^{1\,0}\left[z : (a+b+1,1) \; ; \; (b,1)\right] \qquad (18:152)$$

$$z^b (1+z)^{-a} = H_{1\,1}^{1\,1}\left[z : (b-a+1,1) \; ; \; (b,1)\right]/\Gamma(a) \qquad (18:10)$$

2.4.2. Trigonometric and Hyperbolic Functions and Their Inverses.

$$\sin(z) = \sqrt{\pi}\, H_{0\,2}^{1\,0}\left[z^2/4 : (\tfrac{1}{2},1),(0,1)\right] \qquad (18:151)$$

$$\cos(z) = \sqrt{\pi}\, H_{0\,2}^{1\,0}\left[z^2/4 : (0,1),(\tfrac{1}{2},1)\right] \qquad (18:151)$$

$$\sinh(z) = -i\sqrt{\pi}\, H_{0\,2}^{1\,0}\left[-z^2/4 : (\tfrac{1}{2},1),(0;1)\right] \qquad (18:151)$$

$$\cosh(z) = \sqrt{\pi}\, H^{1\,0}_{0\,2}\left[-z^2/4 : (0,1),(\tfrac{1}{2},1)\right] \qquad (18{:}152)$$

$$\arcsin(z) = \tfrac{1}{2}\, H^{1\,2}_{2\,2}\left[-z^2 : (\tfrac{1}{2},1),(\tfrac{1}{2},1)\ ;\ (0,1),(-\tfrac{1}{2},1)\right] \qquad (18{:}152)$$

$$\arctan(z) = \tfrac{1}{2}\, H^{1\,2}_{2\,2}\left[z^2 : (1,1),(\tfrac{1}{2},1)\ ;\ (\tfrac{1}{2},1),(0,1)\right] \qquad (18{:}152)$$

$$\operatorname{arcsinh}(z) = (1/2\sqrt{\pi})\, H^{1\,2}_{2\,2}\left[z^2 : (1,1),(1,1);(\tfrac{1}{2},1),(0,1)\right] \qquad (18{:}152)$$

$$\operatorname{arctanh}(z) = \tfrac{1}{2}z\, H^{1\,2}_{2\,2}\left[-z^2 : (\tfrac{1}{2},1),(0,1)\ ;\ (0,1),(\tfrac{1}{2},1)\right] \qquad (18{:}152)$$

Applying property (2.4) with $k=2$ to the above eight formulas:

*

$$\sin(z) = \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right]$$

*

$$\cos(z) = \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right]$$

*

$$\sinh(z) = -\tfrac{1}{2}i\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}iz : (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right]$$

*

$$\cosh(z) = \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}iz : (0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right]$$

*

$$\arcsin(z) = \tfrac{1}{4}\, H^{1\,2}_{2\,2}\left[iz : (\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\ ;\ (0,\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2})\right]$$

*

$$\arctan(z) = \tfrac{1}{4}\, H^{1\,2}_{2\,2}\left[z : (1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\ ;\ (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right]$$

* $$\operatorname{arcsinh}(z) = \log(z+\sqrt{1+z^2})$$
$$= (1/4\sqrt{\pi})\, H_{2\,2}^{1\,2}\left[z : (1,\tfrac{1}{2}),(1,\tfrac{1}{2}) \,;\, (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right]$$

* $$\operatorname{arctanh}(z) = \tfrac{1}{2}\log((1+z)/(1-z))$$
$$= -\tfrac{1}{4}i\, H_{2\,2}^{1\,2}\left[iz : (1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2}) \,;\, (\tfrac{1}{2},\tfrac{1}{2}),(1,\tfrac{1}{2})\right]$$
$$= -\tfrac{1}{4}i\, H_{1\,1}^{1\,1}\left[iz : (\tfrac{1}{2},\tfrac{1}{2}) \,;\, (\tfrac{1}{2},\tfrac{1}{2})\right],$$

after also applying (2.9).

2.4.3. Logarithmic Function.

$$\log(1\pm z) = \pm H_{2\,2}^{1\,2}\left[\pm z : (1,1),(1,1) \,;\, (1,1),(0,1)\right]$$

(18:152)

2.4.4. Bessel Functions (18:10 - 11,152 - 153).

Starting with Mathai and Saxena's formulas with $\alpha = 0$ and applying property (2.4) with $k = 2$:

* $$J_v(z) = \tfrac{1}{2} H_{0\,2}^{1\,0}\left[\tfrac{1}{2}z : (\tfrac{1}{2}v,\tfrac{1}{2}),(-\tfrac{1}{2}v,\tfrac{1}{2})\right]$$

* $$K_v(z) = \tfrac{1}{4} H_{0\,2}^{2\,0}\left[\tfrac{1}{2}z : (\tfrac{1}{2}v,\tfrac{1}{2}),(-\tfrac{1}{2}v,\tfrac{1}{2})\right]$$

* $$Y_v(z) = \tfrac{1}{2} H_{1\,3}^{2\,0}\left[\tfrac{1}{2}z : (-\tfrac{1}{2}(v+1),\tfrac{1}{2}) \,;\, (\tfrac{1}{2}v,\tfrac{1}{2}),(-\tfrac{1}{2}v,\tfrac{1}{2}), (-\tfrac{1}{2}(v+1),\tfrac{1}{2})\right]$$

$$J_v^u(z) = H_{0\,2}^{1\,0}\left[z : (0,1),(-v,u)\right]$$

(Maitland's generalized Bessel function)

2.4.5. Confluent Hypergeometric Function (1:506).

$$M(a,b,-z) = {}_1F_1(a;b;-z)$$

$$= \frac{\Gamma(b)}{\Gamma(a)} H^{1\,1}_{1\,2}\left[z : (1-a,1) ; (0,1),(1-b,1)\right]$$

2.4.6. Hypergeometric Function (18:158).

$$\Gamma(a)\Gamma(b)\cdot{}_2F_1(a,b;c;-z)/\Gamma(c)$$

$$= H^{1\,2}_{2\,2}\left[z : (1-a,1),(1-b,1) ; (0,1),(1-c,1)\right]$$

2.4.7. Generalized Hypergeometric Functions (4:40; 7:101; 18:11, 159; 21:197-198).

$${}_pF_q(\{a_i\};\{b_i\};-z) = \left(\prod_{i=1}^{q}\Gamma(b_i) / \prod_{i=1}^{p}\Gamma(a_i)\right)\cdot$$

$$\cdot H^{1\,p}_{p\,q+1}\left[z : \{(1-a_i,1)\} ; (0,1), \{(1-b_i,1)\}\right],$$

for $p \leq q$ or for $p = q+1$ and $|z| < 1$.

$${}_p\Psi_q\left[\begin{matrix}\{(a_i,A_i)\} \\ \{(b_i,B_i)\}\end{matrix} ; -z\right]$$

$$= H^{1\,p}_{p\,q+1}\left[z : \{(1-a_i,A_i)\} ; (0,1), \{(1-b_i,B_i)\}\right]$$

(Maitland's or Wright's generalized hypergeometric function)

$$E(p;\{a_i\};q;\{b_i\};z)$$

$$= H^{p,1}_{q+1,p}\left[z : (1,1), \{(b_i,B_i)\} ; \{(a_i,A_i)\}\right]$$

(MacRobert's E-function)

2.4.8. Meijer's G-function (4:41; 7:101; 18:11,159; 21:197-198).

$$G^{m\,n}_{p\,q}\left[z : \begin{matrix}\{a_i\}\\ \{b_i\}\end{matrix}\right] = H^{m\,n}_{p\,q}\left[z : \{(a_i,1)\} ; \{(b_i,1)\}\right]$$

Extensive lists of elementary special functions expressed as G-functions are given by Luke (14:225-234) and Mathai and Saxena (16:53-68).

2.5. NEW USE OF SKIBINSKI'S DERIVATIVE RULE (Equation 2.14)

One of the more difficult aspects of dealing with H-functions is to relate the higher order H-functions both to elementary functions and to lower order H-functions. One tool for doing this is the derivative rule that has been presented in section 2.3.7.

2.5.1. Consider, for example, from section 2.4.1.,

$$H^{1\,0}_{0\,1}\left[z : (b,B)\right] = B^{-1}\, z^{b/B}\, e^{-z^{1/B}}. \qquad (2.19)$$

The derivative rule (2.14) can be applied to (2.19) to find some new formulas:

$$H^{1\,1}_{1\,2}\left[z : (0,1) ; (b,B),(1,1)\right] = z\frac{d}{dz} H^{1\,0}_{0\,1}\left[z : (b,B)\right]$$

$$= z\frac{d}{dz}\left(B^{-1}\, z^{b/B}\, e^{-z^{1/B}}\right)$$

$$= bB^{-2}\, z^{b/B}\, e^{-z^{1/B}} - B^{-2}\, z^{(b+1)/B}\, e^{-z^{1/B}}.$$

The above result combined with (2.17) gives the following new relations:

$$* \quad H^{1\,1}_{1\,2}\left[z : (0,1) ; (b,B),(1,1)\right]$$

$$= B^{-2}\, z^{b/B}\, e^{-z^{1/B}} (b - z^{1/B})$$

$$= bB^{-1}\, H^{1\,0}_{0\,1}\left[z : (b,B)\right] - B^{-1}\, H^{1\,0}_{0\,1}\left[z : (b+1,B)\right]$$

$$= -\, H^{2\,0}_{1\,2}\left[z : (0,1) ; (1,1),(b,B)\right] . \qquad (2.20)$$

Multiplying (2.20) by z^a and using property (2.8):

$$H^{1\,1}_{1\,2}\left[z : (a,1) ; (b+aB,B),(a+1,1)\right]$$

$$= B^{-2}\, z^{(b+aB)/B}\, e^{-z^{1/B}} (b - z^{1/B}) .$$

Substituting $x^{1/A} = z$, $b = b + aB$, and then $B = AB$, and using properties (2.4) and (2.17) gives the following new relations:

$$* \quad H^{1\,1}_{1\,2}\left[x : (a,A) ; (b,B),(a+1,A)\right]$$

$$= B^{-2}\, x^{b/B}\, e^{-x^{1/B}} (bA - aB - Ax^{1/B})$$

$$= B^{-1}(bA - aB)\, H^{1\,0}_{0\,1}\left[x : (b,B)\right] - AB^{-1}\, H^{1\,0}_{0\,1}\left[x : (b+1,B)\right]$$

$$= -\, H^{2\,0}_{1\,2}\left[x : (a,A) ; (a+1,A),(b,B)\right] . \qquad (2.21)$$

Next, consider (2.19) with $B = 1$; then, applying the derivative rule (2.14):

$$H^{1\,1}_{1\,2}\left[z : (0,1) ; (b,1),(r,1)\right] = z^r \frac{d^r}{dz^r} (z^b\, e^{-z}) =$$

$$= z^r \sum_{w=0}^{r} \binom{r}{w} \frac{d^{r-w}}{dz^{r-w}} (z^b) \frac{d^w}{dz^w} (e^{-z})$$

$$= z^r \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b+1)}{\Gamma(b+1-r+w)} z^{b-r+w} (-1)^w e^{-z} .$$

The above result combined with (2.17) gives the following new relations:

$$* \quad H_{1\,2}^{1\,1}[z : (0,1) ; (b,1),(r,1)]$$

$$= \Gamma(b+1)\, z^b e^{-z} \sum_{w=0}^{r} \binom{r}{w} (-z)^w / \Gamma(b+1-r+w)$$

$$= \Gamma(b+1) \sum_{w=0}^{r} \frac{\binom{r}{w} (-1)^w H_{0\,1}^{1\,0}[z: (b+w,1)]}{\Gamma(b+1-r+w)}$$

$$= (-1)^r H_{1\,2}^{2\,0}[z : (0,1) ; (r,1),(b,1)] ,$$

for non-negative integer r. (2.22)

Multiplying (2.22) by z^a and applying property (2.8):

$$H_{1\,2}^{1\,1}[z : (a,1) ; (b+a,1),(a+r,1)]$$

$$= \Gamma(b+1)\, z^{b+a} e^{-z} \sum_{w=0}^{r} \binom{r}{w} (-z)^w / \Gamma(b+1-r+w).$$

Substituting $b = b + a$ and $x^{1/B} = z$, and using properties (2.4) and (2.17) gives the following new relations:

$$* \quad H_{1\,2}^{1\,1}[x : (a,B) ; (b,B),(a+r,B)] =$$

(next page)

$$* \quad = B^{-1}\Gamma(b-a+1)x^{b/B}\, e^{-x^{1/B}} \sum_{w=0}^{r} \binom{r}{w}(-x^{1/B})^{w} / \,\Gamma(b-a+1-r+w)$$

$$= \sum_{w=0}^{r} \binom{r}{w}(-1)^{w} \frac{\Gamma(b-a+1)}{\Gamma(b-a+1-r+w)}\, H_{0\,1}^{1\,0}[x : (b+w,B)]$$

$$= (-1)^{r}\, H_{1\,2}^{2\,0}[x : (a,B) ; (a+r,B),(b,B)],$$

for non-negative integer r. (2.23)

Note that, when $r=0$, (2.23) reduces to (2.19) by property (2.9). Also, referring to section 2.4.5., if $b=0$, $B=1$ and $c=1-a$, then (2.23) becomes

$$* \quad {}_1F_1(c;c-r;-x) = e^{-x} \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(c-r)}{\Gamma(c-r+w)} (-x)^{w},$$

for non-negative integer r. (2.24)

2.5.2. Following the same procedure as in section 2.5.1., but starting with the following known H- function of section 2.4.1.,

$$H_{1\,1}^{1\,0}[z : (d,1) ; (b,1)] = z^{b}(1-z)^{d-b-1} / \,\Gamma(d-b),$$

then the following new relations are derived:

$$* \quad H_{2\,2}^{1\,1}[x : (a,B),(d,B) ; (b,B),(a+r,B)]$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a+1)\,(-1)^{w}\, x^{(b+w)/B}(1-x^{1/B})^{d-b-1-w}}{\Gamma(b-a+1-r+w)\,\Gamma(d-b-w)}$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a+1)\,(-1)^{w}}{\Gamma(b-a+1-r+w)}\, H_{1\,1}^{1\,0}[x : (d,B) ; (b+w,B)]$$

(next page)

$$= (-1)^r H^{2\,0}_{2\,2}\left[x : (a,B),(d,B) \;;\; (b,B),(a+r,B)\right],$$

for non-negative integer r. (2.25)

And, starting with the following known H- function,

$$H^{1\,1}_{1\,1}\left[z : (a,1) \;;\; (b,1)\right] = \Gamma(b-a+1)\, z^b\, (1+z)^{-b+a-1},$$

then the following new relations are found:

$$H^{1\,2}_{2\,2}\left[x : (a_1,B),(a_2,B) \;;\; (b,B),(a_1+r,B)\right]$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a_1+1)\,\Gamma(b-a_2+1+w)\, x^{(b+w)/B}}{\Gamma(b-a_1+1-r+w)\,(-1)^w\,(1+x^{1/B})^{b-a_2+1+w}}$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a_1+1)\,(-1)^w}{\Gamma(b-a_1+1-r+w} H^{1\,1}_{1\,1}\left[x : (a_2,B);(b+w,B)\right]$$

$$= (-1)^r H^{2\,1}_{2\,2}\left[x : (a_2,B),(a_1,B) \;;\; (b,B),(a_1+r,B)\right],$$

for non-negative integer r. (2.26)

2.5.3. Because the section 2.5.1. and section 2.5.2. results are summations, the second application of Skibinski's derivative rule produces somewhat more complicated results. Consider the p-th derivative of (2.23) where $B=1$:

$$H^{1\,2}_{2\,3}\left[z : (0,1),(a,1) \;;\; (b,1),(a+r,1),(p,1)\right]$$

$$= z^p \frac{d^p}{dz^p} H^{1\,1}_{1\,2}\left[z : (a,1) \;;\; (b,1),(a+r,1)\right] =$$

$$= z^p \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a+1)\ (-1)^w}{\Gamma(b-a+1-r+w)} \frac{d^p}{dz^p} (z^{b+w} e^{-z})$$

$$= z^p \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a+1)\ (-1)^w}{\Gamma(b-a+1-r+w)} z^{-p} \cdot$$

$$\cdot\ H_{1\ 2}^{1\ 1} [z : (0,1)\ ;\ (b+w,1),(p,1)] .$$

The above result provides the following new relations:

$$* \quad H_{2\ 3}^{1\ 2} [z : (0,1),(a,1)\ ;\ (b,1),(a+r,1),(p,1)]$$

$$= \sum_{w=0}^{r} \sum_{v=0}^{p} \binom{r}{w}\binom{p}{v} \frac{(-1)^{w+v}\ \Gamma(b-a+1)\ \Gamma(b+w+1)\ z^{b+w+v}\ e^{-z}}{\Gamma(b-a+1-r+w)\ \Gamma(b+w+1-p+v)}$$

$$= \sum_{w=0}^{r} \sum_{v=0}^{p} \binom{r}{w}\binom{p}{v} \frac{(-1)^{w+v}\ \Gamma(b-a+1)\ \Gamma(b+w+1)}{\Gamma(b-a+1-r+w)\ \Gamma(b+w+1-p+v)} \cdot$$

$$\cdot\ H_{0\ 1}^{1\ 0} [z : (b+w+v,1)]$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{\Gamma(b-a+1)\ (-1)^w}{\Gamma(b-a+1-r+w)} H_{1\ 2}^{1\ 1} [z : (0,1)\ ;\ (b+w,1),(p,1)]$$

$$= (-1)^p\ H_{2\ 3}^{2\ 1} [z : (a,1),(0,1)\ ;\ (p,1),(b,1),(a+r,1)]$$

$$= (-1)^{p+r}\ H_{2\ 3}^{3\ 0} [z : (0,1),(a,1)\ ;\ (a+r,1),(p,1),(b,1)]$$

$$= (-1)^r\ H_{2\ 3}^{2\ 1} [z : (0,1),(a,1)\ ;\ (a+r,1),(b,1),(p,1)] ,$$

for non-negative integers p and r, where the last three H-functions are differing applications of (2.17). (2.27)

Continuing with the section 2.5.1. procedure, the following new relations are also found:

$$H^{1\,2}_{2\,3}\left[x : (a_1,B),(a_2,B) ; (b,B),(a_1+r,B),(a_2+p,B)\right]$$

$$= \sum_{w=0}^{r}\sum_{v=0}^{p} \binom{r}{w}\binom{p}{v} \frac{(-1)^{w+v}\,\Gamma(b-a+1)\,\Gamma(b-a_2+w+1)}{\Gamma(b-a_1+1-r+w)\,\Gamma(b-a_2+w+1-p+v)} \cdot$$

$$\cdot\, x^{(b+w+v)/B}\, e^{-x^{1/B}}$$

$$= \sum_{w=0}^{r}\sum_{v=0}^{p} \binom{r}{w}\binom{p}{v} \frac{(-1)^{w+v}\,\Gamma(b-a_1+1)\,\Gamma(b-a_2+w+1)}{\Gamma(b-a_1+1-r+w)\,\Gamma(b-a_2+w+1-p+v)} \cdot$$

$$\cdot\, H^{1\,0}_{0\,1}\left[x : (b+w+v,B)\right]$$

$$= \sum_{w=0}^{r} \binom{r}{w} \frac{(-1)^{w}\,\Gamma(b-a_1+1)}{\Gamma(b-a_1+1-r+w)}\, H^{1\,1}_{1\,2}\left[x:(a_2,B);(b+w,B),(a_2+p,B)\right]$$

$$= (-1)^r\, H^{2\,1}_{2\,3}\left[x : (a_2,B),(a_1,B) ; (a_1+r,B),(b,B),(a_2+p,B)\right]$$

$$= (-1)^p\, H^{2\,1}_{2\,3}\left[x : (a_1,B),(a_2,B) ; (a_2+p,B),(b,B),(a_1+r,B)\right]$$

$$= (-1)^{p+r}\, H^{3\,0}_{2\,3}\left[x: (a_1,B),(a_2,B) ; (b,B),(a_1+r,B),(a_2+p,B)\right]$$

for non-negative integers p and r. (2.28)

2.6. NEW REDUCTION FORMULAS

The well-known relation $\Gamma(z+1) = z\,\Gamma(z)$ leads to a number of simple reduction formulas for special cases of the H-function. Using this relation,

$$\frac{\Gamma(1+b+Bs)\,\Gamma(kb+kBs)}{\Gamma(b+Bs)} = (b+Bs)\,\Gamma(kb+kBs)$$

$$= k^{-1}(kb+kBs)\,\Gamma(kb+kBs) = k^{-1}\,\Gamma(1+kb+kBs). \tag{2.29}$$

Applying (2.29) to the H-function definition (2.1), for $k > 0$:

$$* \quad H^{m+2,n}_{p+1,q+2}\left[z : \{(a_i,A_i)\},(b,B) \;;\; (b+1,B),(kb,kB), \{(b_i,B_i)\}\right]$$

$$= k^{-1}\,H^{m+1,n}_{p,q+1}\left[z : \{(a_i,A_i)\} \;;\; (1+kb,kB), \{(b_i,B_i)\}\right] \tag{2.30}$$

$$* \quad H^{m+1,n+1}_{p+2,q+1}\left[z : (1+ka,kA), \{(a_i,A_i)\},(a,A) \;;\; (a+1,A), \{(b_i,B_i)\}\right]$$

$$= -k^{-1}\,H^{m,n+1}_{p+1,q}\left[z : (ka,kA), \{(a_i,A_i)\} \;;\; \{(b_i,B_i)\}\right] \tag{2.31}$$

$$* \quad H^{m+1,n}_{p+2,q+1}\left[z : \{(a_i,A_i)\},(a+1,A),(ka,kA) \;;\; (a,A), \{(b_i,B_i)\}\right]$$

$$= k\,H^{m,n}_{p+1,q}\left[z : \{(a_i,A_i)\},(1+ka,kA) \;;\; \{(b_i,B_i)\}\right] \tag{2.32}$$

$$* \quad H^{m+1,n}_{p+1,q+2}\left[z : \{(a_i,A_i)\},(b+1,B) \;;\; (b,B), \{(b_i,B_i)\},(1+kb,kB)\right]$$

$$= -k\,H^{m,n}_{p,q+1}\left[z : \{(a_i,A_i)\} \;;\; \{(b_i,B_i)\},(kb,kB)\right] \tag{2.33}$$

Similarly,

$$\frac{\Gamma(1-a-As)\,\Gamma(-ka-kAs)}{\Gamma(-a-As)} = k^{-1}\,\Gamma(1-ka-kAs)$$

leads to the following reduction formulas for $k>0$:

$$* \quad H^{m,n+2}_{p+2,q+1}\left[z : (a,A),(1+ka,kA), \{(a_i,A_i)\} ; \{(b_i,B_i)\},(a+1,A)\right]$$

$$= k^{-1}\, H^{m,n+1}_{p+1,q}\left[z : (ka,kA), \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right] \tag{2.34}$$

$$* \quad H^{m+1,n+1}_{p+1,q+2}\left[z : (b,B), \{(a_i,A_i)\} ; (kb,kB), \{(b_i,B_i)\},(b+1,B)\right]$$

$$= -k^{-1}\, H^{m+1,n}_{p,q+1}\left[z : \{(a_i,A_i)\} ; (1+kb,kB), \{(b_i,B_i)\}\right] \tag{2.35}$$

$$* \quad H^{m,n+1}_{p+1,q+2}\left[z : (b+1,B), \{(a_i,A_i)\} ; \{(b_i,B_i)\},(b,B),(1+kb,kB)\right]$$

$$= k\, H^{m,n}_{p,q+1}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\},(kb,kB)\right] \tag{2.36}$$

$$* \quad H^{m,n+1}_{p+2,q+1}\left[z : (a+1,A), \{(a_i,A_i)\},(ka,kA) ; \{(b_i,B_i)\},(a,A)\right]$$

$$= -k\, H^{m,n}_{p+1,q}\left[z : \{(a_i,A_i)\},(1+ka,kA) ; \{(b_i,B_i)\}\right] \tag{2.37}$$

Using $\Gamma(z+1) = z\,\Gamma(z)$ also results in the relation

$$\frac{\Gamma(b+Bs)\,\Gamma(1+kb+kBs)}{\Gamma(1+b+Bs)} = \frac{\Gamma(1+kb+kBs)}{b+Bs}$$

$$= k\,\Gamma(kb+kBs). \tag{2.38}$$

Applying (2.38) to the H-function definition (2.1), we have the following reduction formulas for $k>0$:

$$* \quad H^{m+2,n}_{p+1,q+2}\left[z : \{(a_i,A_i)\},(b+1,B) \; ; \; (b,B),(1+kb,kB), \{(b_i,B_i)\}\right]$$

$$= k\, H^{m+1,n}_{p,q+1}\left[z : \{(a_i,A_i)\} \; ; \; (kb,kB), \{(b_i,B_i)\}\right] \tag{2.39}$$

$$* \quad H^{m+1,n+1}_{p+2,q+1}\left[z : (ka,kA), \{(a_i,A_i)\},(a+1,A) \; ; \; (a,A), \{(b_i,B_i)\}\right]$$

$$- \mathrm{Res}(-a/A) = -k\, H^{m,n+1}_{p+1,q}\left[z : (1+ka,kA), \{(a_i,A_i)\} \; ; \{(b_i,B_i)\}\right] \tag{2.40}$$

$$* \quad H^{m+1,n}_{p+2,q+1}\left[z : \{(a_i,A_i)\},(a,A),(1+ka,kA) \; ; \; (a+1,A), \{(b_i,B_i)\}\right]$$

$$= k^{-1}\, H^{m,n}_{p+1,q}\left[z : \{(a_i,A_i)\},(ka,kA) \; ; \{(b_i,B_i)\}\right] \tag{2.41}$$

$$* \quad H^{m+1,n}_{p+1,q+2}\left[z : \{(a_i,A_i)\},(b,B) \; ; \; (b+1,B), \{(b_i,B_i)\},(kb,kB)\right]$$

$$= -k^{-1}\, H^{m,n}_{p,q+1}\left[z : \{(a_i,A_i)\} \; ; \{(b_i,B_i)\},(1+kb,kB)\right] \tag{2.42}$$

Similarly,

$$\frac{\Gamma(-a-As)\,\Gamma(1-ka-kAs)}{\Gamma(1-a-As)} = k\,\Gamma(-ka-kAs)$$

leads to the following reduction formulas for $k>0$:

(next page)

$$* \quad H^{m,n+2}_{p+2,q+1}\left[z : (a+1,A),(ka,kA),\{(a_i,A_i)\} ; \{(b_i,B_i)\},(a,A)\right]$$

$$= k\, H^{m,n+1}_{p+1,q}\left[z : (1+ka,kA),\{(a_i,A_i)\};\{(b_i,B_i)\}\right] \tag{2.43}$$

$$* \quad H^{m+1,n+1}_{p+1,q+2}\left[z : (b+1,B),\{(a_i,A_i)\} ; (1+kb,kB),\{(b_i,B_i)\},(b,B)\right]$$

$$+ \mathrm{Res}(-b/B) = -k\, H^{m+1,n}_{p,q+1}\left[z : \{(a_i,A_i)\} ; (kb,kB),\{(b_i,B_i)\}\right] \tag{2.44}$$

$$* \quad H^{m,n+1}_{p+1,q+2}\left[z : (b,B),\{(a_i,A_i)\} ; \{(b_i,B_i)\},(b+1,B),(kb,kB)\right]$$

$$= k^{-1}\, H^{m,n}_{p,q+1}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\},(1+kb,kB)\right] \tag{2.45}$$

$$* \quad H^{m,n+1}_{p+2,q+1}\left[z : (a,A),\{(a_i,A_i)\},(1+ka,kA) ; \{(b_i,B_i)\},(a+1,A)\right]$$

$$= -k^{-1}\, H^{m,n}_{p+1,q}\left[z : \{(a_i,A_i)\},(ka,kA) ; \{(b_i,B_i)\}\right] \tag{2.46}$$

<u>Examples</u>: Consider the derivatives of sin(z) and cos(z) from section 2.4.2., using (2.15):

$$\frac{d}{dz}\sin(z) = \frac{d}{dz}\tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right]$$

$$= \tfrac{1}{2}\sqrt{\pi}\, H^{1\,1}_{1\,3}\left[\tfrac{1}{2}z : (-1,1) ; (0,\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2}),(0,1)\right]\cdot\tfrac{1}{2}. \tag{2.47}$$

Using (2.45) with $(b,B) = (-1,1)$ and $k = \frac{1}{2}$, (2.47) becomes

$$\frac{d}{dz} \sin(z) = (\tfrac{1}{2})^{-1} \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (0,\tfrac{1}{2}),(1-\tfrac{1}{2},\tfrac{1}{2})\right] \cdot \tfrac{1}{2}$$

$$= \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right],$$

which, by section 2.4.2., is cos(z) as expected.

Similarly,

$$\frac{d}{dz} \cos(z) = \frac{d}{dz} \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right]$$

$$= \tfrac{1}{2}\sqrt{\pi}\, H^{1\,1}_{1\,3}\left[\tfrac{1}{2}z : (-1,1) ; (-\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2}),(0,1)\right] \cdot \tfrac{1}{2} \tag{2.48}$$

Using (2.35) with $(b,B) = (-1,1)$ and $k = \frac{1}{2}$, then (2.48) becomes:

$$\frac{d}{dz} \cos(z) = -(\tfrac{1}{2})^{-1} \tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (1-\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right] \cdot \tfrac{1}{2}$$

$$= -\tfrac{1}{2}\sqrt{\pi}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (\tfrac{1}{2},\tfrac{1}{2}),(0,\tfrac{1}{2})\right],$$

which, by section 2.4.2., is $-\sin(z)$, as expected.

Consider the derivative of $\exp(-z^a)$ from section 2.4.1., using (2.15):

$$\frac{d}{dz} (e^{-z^a}) = \frac{d}{dz} a^{-1} H^{1\,0}_{0\,1}\left[z : (0,a^{-1})\right]$$

$$= a^{-1} H^{1\,1}_{1\,2}\left[z : (-1,1) ; (-a^{-1},a^{-1}),(0,1)\right]. \tag{2.49}$$

Using (2.35) with $(b,B) = (-1,1)$ and $k = 1/a$, (2.49) becomes

$$\frac{d}{dz}\left(e^{-z^a}\right) = (-a)\ a^{-1}\ H_{0\ 1}^{1\ 0}\left[z : (1-a^{-1}, a^{-1})\right]$$

$$= -\ H_{0\ 1}^{1\ 0}\left[z : \left(\frac{a-1}{a}, \frac{1}{a}\right)\right],$$

which, by section 2.4.1, is $(-a\ z^{a-1}\ e^{-z^a})$ as expected.

2.7. SPECIAL DERIVATIVE CASES

* The first derivative of an H-function H(z), by equation (2.15) with $k = 1$ and $r = 1$, has a numerator term $(a_i, A_i) = (-1,1)$ and a denominator term $(b_i, B_i) = (0,1)$. Combining this observation with equations (2.34), (2.35), (2.45), (2.46), where $a = -1$ and $b = -1$, leads to the following theorem.

* THEOREM 2.1: Given an H-function of order (m', n', p', q'), $H_{p'\ q'}^{m'\ n'}(z)$, where either $a_i = 1$ for any i, $i = 1, \ldots, p'$, or $b_i = 0$ for any i, $i = 1, \ldots, q'$, then the derivative of this H-function, $H'(z)$, is also an H-function of order (m', n', p', q') or less.

2.7.1. Let $H(z) = H_{p,q+1}^{m+1,n}\left[z : \{(a_i, A_i)\}\ ;\ (0,k), \{(b_i, B_i)\}\right]$;

then, by (2.15), $H'(z) =$

$$H_{p+1,q+2}^{m+1,n+1}\left[z : (-1,1), \{(a_i - A_i, A_i)\}\ ;\ (-k,k), \{(b_i - B_i, B_i)\}, (0,1)\right]. \tag{2.50}$$

Using (2.35) with $(b,B)=(-1,1)$, (2.50) becomes:

$$* \quad H'(z) = -k^{-1} H^{m+1,n}_{p,q+1}\left[z : \{(a_i-A_i,A_i)\} ; (1-k,k), \{(b_i-B_i,B_i)\}\right].$$

(2.51)

The cos(z) and $\exp(-z^a)$ examples of section 2.6. are in this class.

2.7.2. Let $H(z) = H^{m,n}_{p,q+1}\left[z : \{(a_i,A_i)\} ; \{(b_i,B_i)\}, (0,k)\right]$;

then, by (2.15), $H'(z)=$

$$H^{m,n+1}_{p+1,q+2}\left[z : (-1,1), \{(a_i-A_i,A_i)\} ; \{(b_i-B_i,B_i)\}, (-k,k), (0,1)\right].$$

Using (2.45) with $(b,B)=(-1,1)$,

$$* \quad H'(z) = k^{-1} H^{m,n}_{p,q+1}\left[z : \{(a_i-A_i,A_i)\} ; \{(b_i-B_i,B_i)\}, (1-k,k)\right].$$

(2.52)

The sin(z) example of section 2.6. is in this class.

2.7.3. Let $H(z) = H^{m,n+1}_{p+1,q}\left[z : (1,k), \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right]$;

then, by (2.15), $H'(z)=$

$$H^{m,n+2}_{p+2,q+1}\left[z : (-1,1), (1-k,k), \{(a_i-A_i,A_i)\} ; \{(b_i-B_i,B_i)\}, (0,1)\right].$$

Using (2.34) with $(a,A)=(-1,1)$,

$$* \quad H'(z) = k^{-1} H^{m,n+1}_{p+1,q}\left[z : (-k,k), \{(a_i-A_i,A_i)\} ; \{(b_i-B_i,B_i)\}\right].$$

(2.53)

2.7.4. Let $H(z) = H^{m,n}_{p+1,q}\left[z : \{(a_i,A_i)\}, (1,k) ; \{(b_i,B_i)\}\right]$;

then, by (2.15), $H'(z) =$

$$H^{m,n+1}_{p+2,q+1}\left[z : (-1,1), \{(a_i-A_i,A_i)\}, (1-k,k) ; \{(b_i-B_i,B_i)\}, (0,1)\right].$$

Using (2.46) with $(a,A) = (-1,1)$,

$$* \quad H'(z) = -k^{-1} H^{m,n}_{p+1,q}\left[z : \{(a_i-A_i,A_i)\}, (-k,k) ; \{(b_i-B_i,B_i)\}\right]. \qquad (2.54)$$

2.7.5. Examples.

Theorem 2.1 applies to most of the known special cases of the H-function that are given in section 2.4. Besides the sin(z), cos(z) and $\exp(-z^a)$ derivatives already treated in section 2.6., the following derivative formulas are a consequence of Theorem 2.1:

$$* \quad d(\arcsin(z))/dz = (1-z^2)^{-\frac{1}{2}}$$

$$= -\tfrac{1}{2}i\, H^{1\,2}_{2\,2}\left[iz : (0,\tfrac{1}{2}), (0,\tfrac{1}{2}) ; (\tfrac{1}{2},\tfrac{1}{2}), (-1,\tfrac{1}{2})\right].$$

$$* \quad d(\arctan(z))/dz = (1+z^2)^{-1}$$

$$= \tfrac{1}{2} H^{1\,2}_{2\,2}\left[z : (\tfrac{1}{2},\tfrac{1}{2}), (0,\tfrac{1}{2}) ; (0,\tfrac{1}{2}), (\tfrac{1}{2},\tfrac{1}{2})\right],$$

which reduces further, using equation (2.9), to

$$* \quad d(\arctan(z))/dz = (1+z^2)^{-1}$$

$$= \tfrac{1}{2} H^{1\,1}_{1\,1}\left[z : (0,\tfrac{1}{2}) ; (0,\tfrac{1}{2})\right].$$

* $d(\text{arcsinh}(z))/dz = (1+z^2)^{-\frac{1}{2}}$

$$= (1/2\sqrt{\pi})\cdot H^{1\,2}_{2\,2}\left[z : (\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2}) \;;\; (0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right],$$

which reduces further, using equation (2.9), to

* $d(\text{arcsinh}(z))/dz = (1+z^2)^{-\frac{1}{2}}$

$$= (1/2\sqrt{\pi})\cdot H^{1\,1}_{1\,1}\left[z : (\tfrac{1}{2},\tfrac{1}{2}) \;;\; (0,\tfrac{1}{2})\right].$$

The arctan(z) and arcsinh(z) results above can be verified by using the argument z^2 in the section 2.4.1. formula for $z^b(1+z)^{-a}$, with $b=0$ and $a=1$ or $\frac{1}{2}$, and then applying equation (2.4) with $k=2$.

Using Theorem 2.1, the derivatives for the Bessel functions of section 2.4. with $v=0$ are:

$$d(J_0(z))/dz = -\tfrac{1}{2}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2})\right] = -J_1(z)$$

$$= \tfrac{1}{2}\, H^{1\,0}_{0\,2}\left[\tfrac{1}{2}z : (-\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right] = J_{-1}(z),$$

using first derivative formula (2.51) and then (2.52);

$$d(K_0(z))/dz = -\tfrac{1}{4}\, H^{2\,0}_{0\,2}\left[\tfrac{1}{2}z : (\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2})\right] = -K_1(z),$$

and $d(Y_0(z))/dz = -\tfrac{1}{2}\, H^{2\,0}_{1\,3}\left[\tfrac{1}{2}z : (-1,\tfrac{1}{2});(\tfrac{1}{2},\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2}),(-1,\tfrac{1}{2})\right] = -Y_1(z)$

using (2.51). $J_0' = -J_1 = J_{-1}$, $K_0' = -K_1$ and $Y_0' = -Y_1$ are known results (1:376,361). Also, using (2.52),

* $$Y_{-1}'(z) = -Y_1'(z) = \tfrac{1}{2}\, H^{2\,0}_{1\,3}\left[\tfrac{1}{2}z : (-\tfrac{1}{2},\tfrac{1}{2}) \;;\; (-1,\tfrac{1}{2}),(0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right],$$

and, using (2.54),

$$* \quad Y_3'(z) = -Y_3'(z) = -\tfrac{1}{2}\, H^{2\,0}_{1\,3}\left[\tfrac{1}{2}z : (-\tfrac{1}{2},\tfrac{1}{2})\ ;\ (-2,\tfrac{1}{2}),(1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\right].$$

For Maitland's generalized Bessel function, using (2.51),

$$* \quad J^{u}_{v}{}'(z) = -H^{1\,0}_{0\,2}\left[z : (0,1),(-v-u,u)\right] = -J^{u}_{u+v}(z),$$

from which, for $v=0$,

$$J^{u}_{0}{}'(z) = -J^{u}_{u}(z) = -\,H^{1\,0}_{0\,2}\left[z : (0,1),(-u,u)\right],$$

which is also, by (2.52), equal to $u^{-1}\, H^{1\,0}_{0\,2}\left[z : (-1,1),(1-u,u)\right].$

Repeated application of (2.51) gives the r-th derivative of the hypergeometric functions of sections 2.4.:

$$\frac{d^r}{dz^r} M(a,b,-z) = \frac{d^r}{dz^r}\, {}_1F_1(a;b;-z)$$

$$= (-1)^r \frac{\Gamma(b)}{\Gamma(a)}\, H^{1\,1}_{1\,2}\left[z : (1-a-r,1)\ ;\ (0,1),(1-b-r,1)\right]$$

$$= (-1)^r \frac{\Gamma(b)\ \Gamma(a+r)}{\Gamma(a)\ \Gamma(b+r)}\, {}_1F_1(a+r;b+r;-z). \qquad (1{:}507)$$

$$\frac{d^r}{dz^r}\, {}_2F_1(a,b;c;-z) = (-1)^r \frac{\Gamma(c)\Gamma(a+r)\Gamma(b+r)}{\Gamma(a)\ \Gamma(b)\ \Gamma(c+r)}\, {}_2F_1(a+r,b+r;c+r;-z)$$

$$= (-1)^r \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, H^{1\,2}_{2\,2}\left[z : (1-a-r,1),(1-b-r,1);\ (0,1),(1-c-r,1)\right]. \qquad (1{:}557)$$

$$* \quad \frac{d^r}{dz^r}\, {}_pF_q(\{a_i\};\{b_i\};-z) = (-1)^r \frac{\prod_i \Gamma(b_i)}{\prod_i \Gamma(a_i)} \cdot$$

$$\cdot\, H^{1,p}_{p,q+1}[z : \{(1-a_i-r,1)\};\ (0,1),\ \{(1-b_i-r,1)\}]$$

$$= (-1)^r \frac{\prod_i \Gamma(b_i)/\Gamma(b_i+r)}{\prod_i \Gamma(a_i)/\Gamma(a_i+r)}\, {}_pF_q(\{a_i+r\};\{b_i+r\};-z)\ .$$

$$* \quad \frac{d^r}{dz^r}\, {}_p\Psi_q\left[\begin{matrix}\{(a_i,A_i)\}\\ \{(b_i,B_i)\}\end{matrix};-z\right] = (-1)^r\, {}_p\Psi_q\left[\begin{matrix}\{(a_i+rA_i,A_i)\}\\ \{(b_i+rB_i,B_i)\}\end{matrix};-z\right]$$

$$= (-1)^r H^{1,p}_{p,q+1}[z : \{(1-a_i-rA_i,A_i)\};\ (0,1),\ \{(1-b_i-rB_i,B_i)\}]$$

The hypergeometric function examples above demonstrate the following corollary to Theorem 2.1:

* <u>COROLLARY 2.1</u>: If the conditions of Theorem 2.1 are met so that one of the equations (2.51) or (2.52) is applicable, then Theorem 2.1 can be used repeatedly to find the r-th derivative when $k=1$.

For example, using (2.51) or (2.52) repeatedly:

$$* \quad \frac{d^r}{dz^r} H^{m+1,n}_{p,q+1}[z : \{(a_i,A_i)\};\ (0,1),\ \{(b_i,B_i)\}]$$

$$= (-1)^r H^{m+1,n}_{p,q+1}[z : \{(a_i-rA_i,A_i)\};\ (0,1),\ \{(b_i-rB_i,B_i)\}] \tag{2.55}$$

$$* \quad \frac{d^r}{dz^r} H^{m,n}_{p,q+1}[z : \{(a_i,A_i)\};\ \{(b_i,B_i)\},(0,1)] \tag{2.56}$$

$$= H^{m,n}_{p,q+1}[z : \{(a_i-rA_i,A_i)\};\ \{(b_i-rB_i,B_i)\},(0,1)]$$

CHAPTER 3

* CONVERGENCE OF MELLIN-BARNES INTEGRALS

3.1. GENERAL REMARKS

Convergence conditions for the general Mellin-Barnes integral were proven in 1936 by Dixon and Ferrar (6:81 - 96) and were later restated by Erdelyi (9:49 - 50). Luke (14:v.1) and Braaksma (3: 239 - 341) provide extensive theoretical treatment of convergence for the Mellin-Barnes subclasses G- functions and H- functions, respectively. However, none of the above references gives any straight-forward, practical, easily understood guidelines for when a given Mellin-Barnes integral should be evaluated as the sum of the left half plane (LHP) residues versus the negative of the sum of the right half plane (RHP) residues.

The derivation of evaluation guidelines which is presented below has been accomplished with the assistance of Dr. Barry S. Eldred and Dr. J. Wesley Barnes.

Lovett (13) stated that Jordan's Lemma is generally applicable to the H- function, which would allow the use of the residue theorem for all positive real values of the function variable. Lovett's attempted proof of this statement, reproduced by Springer (21:431 - 440), overlooks the oscillatory growth nature of $1/\Gamma(x)$ for negative values of real x. Thus, as given below, the correct development and results are somewhat more complicated than Lovett's.

As a particular example of the flaw in Lovett's proof, consider the form $g(s) = \Gamma(1-s)\,\Gamma(s)\,\Gamma(b+s)$. Then, $g(s)$ is a valid H-function kernel and

$$\lim_{s\to\infty} g(s) = \lim_{s\to\infty} \Gamma(b+s)\pi/\sin(s\pi).$$

This limit does not exist, since $\sin(s\pi)$ oscillates between -1 and +1 and $\Gamma(b+s)$ is unbounded as $s\to\infty$. However, Lovett's approach would indicate that $|\Gamma(1-b-s)|$ has a positive lower bound, so that he has

$$\lim_{s\to\infty} g(s) = \lim_{s\to\infty} \pi^2/(\sin(s\pi)\,\sin(b\pi + s\pi)\,\Gamma(1-b-s))$$

$$= 0 .$$

Lovett's approach is thus seen to be false by the oscillatory growth nature of $1/\Gamma(1-b-s)$ as $s\to\infty$; that is, $|\Gamma(1-b-s)|$ has no positive lower bound (1:255).

3.2. DERIVATION OF CONVERGENCE CONDITIONS

3.2.1. Definitions.

The general Mellin-Barnes integral is defined as (9:49):

$$f(z) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{\prod_{i=1}^{m}\Gamma(b_i - B_i s)\prod_{i=1}^{n}\Gamma(a_i + A_i s)}{\prod_{i=1}^{Q}\Gamma(c_i + C_i s)\prod_{i=1}^{P}\Gamma(d_i - D_i s)}\, z^s\, ds ,$$

where w is real and all A_i, B_i, C_i and D_i are positive real constants. The path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integral. When the poles of $\prod \Gamma(b_i - B_i s)$

lie entirely to the right of this path of integration and the poles of $\prod \Gamma(a_i + A_i s)$ lie entirely to the left, then this integral represents an H- function.

We wish to derive the conditions for which each of the following two relations are valid:

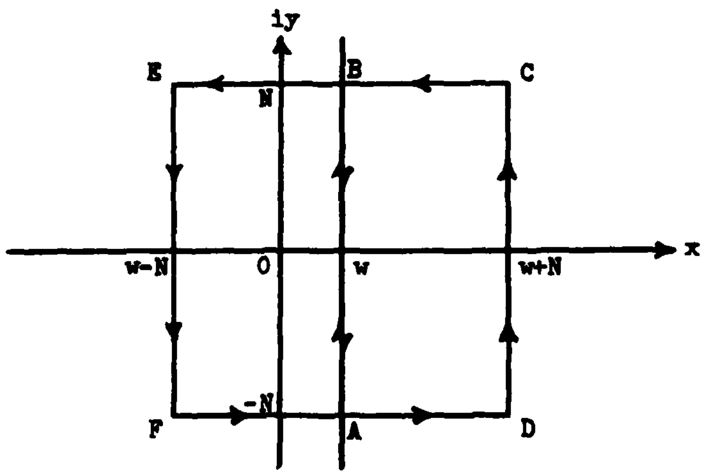
$$f(z) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty}(\bullet)\,ds = -\lim_{N\to\infty}\frac{1}{2\pi i}\int_{ADCBA}(\bullet)\,ds$$

$$= -\sum \text{RHP residues of } (\bullet); \qquad (3.1)$$

$$f(z) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty}(\bullet)\,ds = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{ABEFA}(\bullet)\,ds$$

$$= \sum \text{LHP residues of } (\bullet), \qquad (3.2)$$

where the integrand $(\bullet)$ is that of a Mellin-Barnes integral as defined above and the contours ADCBA and ABEFA are as shown below.



For simplification, the following parameters are defined:

$$\mathbb{D} = \sum_{i=1}^{n} A_i + \sum_{i=1}^{m} B_i - \sum_{i=1}^{Q} C_i - \sum_{i=1}^{P} D_i$$

$$\mathbb{E} = \sum_{i=1}^{n} A_i - \sum_{i=1}^{m} B_i - \sum_{i=1}^{Q} C_i + \sum_{i=1}^{P} D_i$$

$$L = \mathrm{Re}\Big(\sum_{i=1}^{n} a_i - \tfrac{1}{2}n + \sum_{i=1}^{m} b_i - \tfrac{1}{2}m - \sum_{i=1}^{Q} c_i + \tfrac{1}{2}Q - \sum_{i=1}^{P} d_i + \tfrac{1}{2}P\Big)$$

$$R = \prod_{i=1}^{n} A_i^{A_i} \prod_{i=1}^{P} D_i^{D_i} \Big/ \Big(\prod_{i=1}^{m} B_i^{B_i} \prod_{i=1}^{Q} C_i^{C_i}\Big) > 0$$

$$k = (2\pi)^{\frac{1}{2}(n+m-Q-P)} \cdot \frac{\prod_{i=1}^{n} A_i^{\mathrm{Re}(a_i)-\frac{1}{2}} \prod_{i=1}^{m} B_i^{\mathrm{Re}(b_i)-\frac{1}{2}}}{\prod_{i=1}^{Q} C_i^{\mathrm{Re}(c_i)-\frac{1}{2}} \prod_{i=1}^{P} D_i^{\mathrm{Re}(d_i)-\frac{1}{2}}}$$

$$K = R|z| > 0$$

$$\theta = \arg(z)$$

Let $s = u + vi$, u and v real, and note that

$$|z^s| = |z|^u e^{-v\theta} .$$

Using formula 6.145 in Abramowitz and Stegun (1:257), for b complex and B positive real, and noting $\lim_{|v|\to\infty} |\mathrm{Im}(b+Bs)| = \lim_{|v|\to\infty} B|v|$:

$$\lim_{|v|\to\infty} |\Gamma(b \pm Bs)| = \lim_{|v|\to\infty} \sqrt{2\pi}(B|v|)^{\mathrm{Re}(b) \pm Bu - \frac{1}{2}} \cdot \exp(-\tfrac{1}{2}\pi B|v|) .$$

Then,

$$\lim_{|v|\to\infty} (\bullet) = \lim_{|v|\to\infty} \left\{ \frac{\prod_{i=1}^{m} \sqrt{2\pi}(B_i|v|)^{\mathrm{Re}(b_i)-B_i u-\frac{1}{2}} \exp(-\frac{1}{2}\pi B_i|v|)}{\prod_{i=1}^{Q} \sqrt{2\pi}(C_i|v|)^{\mathrm{Re}(c_i)+C_i u-\frac{1}{2}} \exp(-\frac{1}{2}\pi C_i|v|)} \right.$$

(next page)

$$\cdot \frac{\prod_{i=1}^{n} \sqrt{2\pi}\,(A_i|v|)^{Re(a_i) + A_i - \frac{1}{2}} \exp(-\frac{1}{2}\pi A_i|v|)}{\prod_{i=1}^{p} \sqrt{2\pi}\,(D_i|v|)^{Re(d_i) - D_i - \frac{1}{2}} \exp(-\frac{1}{2}\pi D_i|v|)} |z|^u e^{-v\theta} \Bigg\}.$$

Collecting similar terms and using the parameters defined above:

$$\lim_{|v| \to \infty} |(\bullet)| = \lim_{|v| \to \infty} k|v|^{L + \mathbb{E}u} \exp(-\tfrac{1}{2}\pi \mathbb{D}|v| - v\theta)\, K^u \tag{3.3}$$

where $(\bullet)$ is the integrand of the Mellin-Barnes integral and $|\theta| < \pi$.

Since the Mellin-Barnes integral diverges for all z when $\mathbb{D} < 0$ (6:83; 9:50), hereafter we can restrict our attention to non-negative values of $\mathbb{D}$. Also, the branch point $z = 0$ is excluded.

3.2.2. Right Half Plane.

The equality

$$f(z) = \frac{1}{2\pi i} \int_{w - i\infty}^{w + i\infty} (\bullet)\, ds = - \lim_{N \to \infty} \frac{1}{2\pi i} \Big(\int_{ADCBA} (\bullet)\, ds - \int_A^D (\bullet)\, ds - \int_D^C (\bullet)\, ds - \int_C^B (\bullet)\, ds \Big)$$

will reduce to equation (3.1) if all of the last three integrals each approach zero as N tends to infinity.

Consider first the integral over the line AD, for which $s = x - Ni$ where $w \le x \le w + N$. Using equation (3.3),

$$\lim_{N \to \infty} \Big| \int_A^D (\bullet)\, ds \Big| \le \lim_{N \to \infty} \int_A^D |(\bullet)|\, ds =$$

(next page)

$$= \lim_{N \to \infty} \int_w^{w+N} k\, N^{L+\mathbb{E}x} \exp(-\tfrac{1}{2}\pi\mathbb{D}N + N\theta)\, K^x\, dx$$

$$= k \lim_{N \to \infty} \exp(-\tfrac{1}{2}\pi\mathbb{D}N + N\theta)\, N^{L} \int_w^{w+N} (N^{\mathbb{E}} K)^x\, dx$$

$$= k\, K^w \lim_{N \to \infty} \exp(-\tfrac{1}{2}\pi\mathbb{D}N + N\theta)\, N^{L+\mathbb{E}w}((N^{\mathbb{E}}K)^N - 1)/\log(N^{\mathbb{E}}K)) \tag{3.4}$$

Since we know that

$$\lim_{N \to \infty} \exp(-\tfrac{1}{2}\pi\mathbb{D}N + N\theta)\, N^{L+\mathbb{E}w} = \begin{cases} 0, & \text{if } \theta < \tfrac{1}{2}\pi\mathbb{D} \\ 0, & \text{if } \theta = \tfrac{1}{2}\pi\mathbb{D} \text{ and } L < -\mathbb{E}w \\ 1, & \text{if } \theta = \tfrac{1}{2}\pi\mathbb{D} \text{ and } L = -\mathbb{E}w \\ \infty, & \text{otherwise} \end{cases}$$

$$\lim_{N \to \infty} ((N^{\mathbb{E}}K)^N - 1)/\log(N^{\mathbb{E}}K) = \begin{cases} 0, & \text{if } \mathbb{E} < 0 \\ -1/\log(K), & \text{if } \mathbb{E} = 0 \text{ and } 0 < K < 1 \\ \infty, & \text{otherwise} \end{cases}$$

and $\lim_{K \to 1} (K^N - 1)/\log(K) = N$,

then (3.4) will equal zero under one of the following conditions:

(1) $\mathbb{D} > 0,\ \mathbb{E} < 0,\ \theta < \tfrac{1}{2}\pi\mathbb{D}$.

(2) $\mathbb{D} \geq 0,\ \mathbb{E} < 0,\ \theta = \tfrac{1}{2}\pi\mathbb{D},\ L \leq -\mathbb{E}w$.

(3) $\mathbb{D} > 0,\ \mathbb{E} = 0,\ \theta < \tfrac{1}{2}\pi\mathbb{D},\ 0 < K < 1$.

(4) $\mathbb{D} \geq 0,\ \mathbb{E} = 0,\ \theta = \tfrac{1}{2}\pi\mathbb{D},\ 0 < K < 1,\ L < 0$.

(5) $\mathbb{D} \geq 0,\ \mathbb{E} = 0,\ \theta \leq \tfrac{1}{2}\pi\mathbb{D},\ K = 1,\ L < -1$.

Next, consider the integral over the line CB, for which $s = x + Ni$ where $w \leq x \leq w + N$. Using equation (3.3) again,

$$\lim_{N\to\infty} \left|\int_C^B (\bullet)\, ds\right| \leq \lim_{N\to\infty} \int_{w+N}^{w} k\, N^{L+\mathbb{E}x} \exp(-\tfrac{1}{2}\pi\mathbb{D}N - N\theta)\, K^x\, dx$$

$$= k\, K^w \lim_{N\to\infty} \exp(-\tfrac{1}{2}\pi\mathbb{D}N - N\theta)\, N^{L+\mathbb{E}w}(1-(N^{\mathbb{E}}K)^N)/\log(N^{\mathbb{E}}K) \tag{3.5}$$

Using the same analysis as applied to (3.4), we find that (3.5) will equal zero under the same conditions as for (3.4), except $\theta > -\frac{1}{2}\pi\mathbb{D}$ and $\theta = -\frac{1}{2}\pi\mathbb{D}$ replace $\theta < \frac{1}{2}\pi\mathbb{D}$ and $\theta = \frac{1}{2}\pi\mathbb{D}$, respectively. Therefore, both (3.4) and (3.5), and thus both $\lim_{N\to\infty}\int_A^D (\bullet)\, ds$ and $\lim_{N\to\infty}\int_C^B (\bullet)\, ds$, will equal zero if:

CASE 1: $\mathbb{D} > 0,\ \mathbb{E} < 0,\ |\theta| < \frac{1}{2}\pi\mathbb{D}$.

CASE 2: $\mathbb{D} \geq 0,\ \mathbb{E} < 0,\ |\theta| = \frac{1}{2}\pi\mathbb{D},\ L \leq -\mathbb{E}w$.

CASE 3: $\mathbb{D} > 0,\ \mathbb{E} = 0,\ |\theta| < \frac{1}{2}\pi\mathbb{D},\ 0 < K < 1$.

CASE 4: $\mathbb{D} \geq 0,\ \mathbb{E} = 0,\ |\theta| = \frac{1}{2}\pi\mathbb{D},\ 0 < K < 1,\ L < 0$.

CASE 5: $\mathbb{D} \geq 0,\ \mathbb{E} = 0,\ |\theta| \leq \frac{1}{2}\pi\mathbb{D},\ K = 1,\ L < -1$.

Additionally, from (3.3), for all cases, $|\theta| < \pi$.

Finally, consider the integral over the line DC, for which $s = w + N + yi$ where $-N \leq y \leq N$. Using equation (3.3),

$$\lim_{N\to\infty} \left|\int_D^C (\bullet)\, ds\right| \leq \lim_{N\to\infty} \int_{-N}^{N} k|y|^{L+\mathbb{E}(w+N)} \exp(-\tfrac{1}{2}\pi\mathbb{D}\, y - y\theta)\, K^{w+N} dy$$

Substituting $t = -y$ for $-N \leq y < 0$ and $t = +y$ for $0 \leq y \leq N$,

$$\lim_{N\to\infty} \left|\int_D^C (\bullet)\, ds\right| \leq k \lim_{N\to\infty} K^{w+N} \int_0^N t^{\phi-1} e^{-\frac{1}{2}\pi\mathbb{D}t}(e^{t\theta} + e^{-t\theta})\, dt$$

$$= k \lim_{N\to\infty} K^{w+N} N^{\phi}\, \Gamma(\phi)(\gamma^*(\phi, \tfrac{1}{2}\pi\mathbb{D}N + \theta N) + \gamma^*(\phi, \tfrac{1}{2}\pi\mathbb{D}N - \theta N)), \tag{3.6}$$

where $\phi = L + \mathbb{E}(w+N) + 1$ and $\gamma^*(\phi, Y) = (Y^{-\phi}/\Gamma(\phi))\int_0^Y r^{\phi-1} e^{-r}\, dr.$

(3.6 cont.)

γ^* is the incomplete gamma function given in 6.5.4 of Abramowitz and Stegun (1:260 - 261).

When $\mathbb{E} \neq 0$, (3.6) is dominated by N^{ϕ} and thus diverges for $\mathbb{E} > 0$ but converges to zero for $\mathbb{E} < 0$. Therefore, (3.6) and $\lim_{N \to \infty} \int_D^C (\bullet)\, ds$ will equal zero and relation (3.1) is valid under the Case 1 and Case 2 conditions above for which both (3.4) and (3.5) equal zero.

When $\mathbb{D} > 0$ and $\mathbb{E} = 0$, (3.6) is dominated by K^{w+N} and converges to zero for $0 < K < 1$, or, if $L < -1$, for $K = 1$. For $L \geq -1$ and $K = 1$, (3.6) converges to a non-zero value. If $\mathbb{D} = \mathbb{E} = \theta = 0$, (3.6) reduces to

$$k \lim_{N \to \infty} K^{w+N} \int_0^N t^L\, dt = \frac{k}{L+1} \lim_{N \to \infty} K^{w+N} N^{L+1}, \quad L \neq -1,$$

which converges to zero if $0 < K < 1$ or if $K = 1$ and $L < -1$. Therefore, (3.6) and $\lim_{N \to \infty} \int_D^C (\bullet)\, ds$ will equal zero and relation (3.1) is valid under the conditions above for Cases 3, 4, and 5 for which both (3.4) and (3.5) equal zero.

Thus far, we have shown that the Mellin-Barnes integral may be evaluated as the negative of the sum of the RHP residues for Cases 1 through 5 above. For these cases, the last three integrals in the first equation in this section have been shown to approach zero as N tends to infinity.

3.2.3. Left Half Plane.

The conditions for which (3.2) is valid can be found in the same manner as those for (3.1). Or, better, we can note that substituting $\hat{s} = -s$ into a Mellin-Barnes integral yields another Mellin-Barnes integral for which RHP evaluation is equivalent to LHP evaluation of the original integral. This new integral has parameters $\hat{n} = m$, $\hat{m} = n$, $\hat{Q} = P$, $\hat{P} = Q$, $\hat{w} = -w$, $\hat{\mathbb{D}} = \mathbb{D}$, $\hat{\mathbb{E}} = -\mathbb{E}$, $\hat{L} = L$, $\hat{R} = 1/R$, and $\hat{z} = 1/z$. Applying the RHP results to these new parameters and then transforming back to the original parameters will yield the following conditions for which equation (3.2) is valid:

CASE $\hat{1}$: $\mathbb{D} > 0$, $\mathbb{E} > 0$, $|\theta| < \frac{1}{2}\pi\mathbb{D}$.

CASE $\hat{2}$: $\mathbb{D} \geq 0$, $\mathbb{E} > 0$, $|\theta| = \frac{1}{2}\pi\mathbb{D}$, $L \leq -\mathbb{E}w$.

CASE $\hat{3}$: $\mathbb{D} > 0$, $\mathbb{E} = 0$, $|\theta| < \frac{1}{2}\pi\mathbb{D}$, $K > 1$.

CASE $\hat{4}$: $\mathbb{D} \geq 0$, $\mathbb{E} = 0$, $|\theta| = \frac{1}{2}\pi\mathbb{D}$, $K > 1$, $L < 0$.

CASE $\hat{5}$: $\mathbb{D} \geq 0$, $\mathbb{E} = 0$, $|\theta| \leq \frac{1}{2}\pi\mathbb{D}$, $K = 1$, $L < -1$.

Additionally, for all cases, $|\theta| < \pi$.

Therefore, the Mellin-Barnes integral may be evaluated as the sum of the LHP residues for Cases $\hat{1}$ through $\hat{5}$. Note that when $\mathbb{D} = 0$ then $\arg(z) = \theta$ must equal zero. That is, $\mathbb{D} = 0$ limits the evaluation of the Mellin-Barnes integral to real positive values of the function variable. This is true also for the RHP evaluation.

3.2.4. Summary of Evaluation Guidelines.

Combining results for the RHP and the LHP, we have the following guidelines for evaluating a Mellin-Barnes integral $f(cz)$, where c is a positive real constant:

If $\mathbb{D} > 0$ and $L > -\mathbb{E}w$, $f(cz)$ may be evaluated for any $z \neq 0$ such that $|\arg(z)| < \min(\pi, \frac{1}{2}\pi\mathbb{D})$, except at $|z| = 1/(cR)$ when $\mathbb{E} = 0$.

If $\mathbb{D} \geq 0$ and $L < -\mathbb{E}w$, $f(cz)$ may be evaluated for any $z \neq 0$ such that $|\arg(z)| \leq \min(\pi, \frac{1}{2}\pi\mathbb{D})$, except at $|z| = 1/(cR)$ when $\mathbb{E} = 0$ and $L \geq -1$.

When $f(cz)$ may be evaluated, $f(cz) = -\sum$ RHP residues when $\mathbb{E} < 0$ or when $\mathbb{E} = 0$ and $|z| < 1/(cR)$, and $f(cz) = \sum$ LHP residues when $\mathbb{E} > 0$ or when $\mathbb{E} = 0$ and $|z| > 1/(cR)$. Either RHP or LHP residues may be used to find $f(cz)$ at $|z| = 1/(cR)$ when $\mathbb{D} \geq 0$, $\mathbb{E} = 0$, and $L < -1$.

These guidelines may be stated in terms of six basic evaluation types:

TYPE[1]	$\mathbb{D}$	$\mathbb{E}$	L	f(cz)	$\|z\|$	$\|\arg(z)\|$
I	>0	<0	$>-\mathbb{E}w$	$-\sum$ RHP res	>0	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
II	≥ 0	<0	$\leq -\mathbb{E}w$	$-\sum$ RHP res	>0	$<\pi, \leq \frac{1}{2}\pi\mathbb{D}$
III	>0	>0	$>-\mathbb{E}w$	$+\sum$ LHP res	>0	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
IV	≥ 0	>0	$\leq -\mathbb{E}w$	$+\sum$ LHP res	>0	$<\pi, \leq \frac{1}{2}\pi\mathbb{D}$
V	>0	$=0$	≥ 0	$-\sum$ RHP res	$<1/(cR)$	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
				$+\sum$ LHP res	$>1/(cR)$	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
VI[2]	≥ 0	$=0$	<0	$-\sum$ RHP res	$<1/(cR)$	$<\pi, \leq \frac{1}{2}\pi\mathbb{D}$
				$+\sum$ LHP res	$>1/(cR)$	$<\pi, \leq \frac{1}{2}\pi\mathbb{D}$

[1] Note that if $\mathbb{D}=0$ (Types II, IV, VI), then $\arg(z)=0$.

[2] For Type VI, $f(cz)$ is defined at $|z|=1/(cR)$ by the sum of residues in either half plane if $L<-1$.

Due to the treatment of $\mathbb{E}$ and of the limiting value $|\arg(z)|=\frac{1}{2}\pi\mathbb{D}$, none of the evaluation types given above is exactly equivalent to the convergence types given by Erdelyi (9:50). The first type of Erdelyi is divided among all six types above, the second is included in Types II and IV, and the third and fourth are included in Type VI with note 2.

When the Mellin-Barnes integral is expressed in terms of the Mellin transform inversion integral,

$$\frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty}\frac{\prod_{i=1}^{m}\Gamma(b_i+B_is)\prod_{i=1}^{n}\Gamma(a_i-A_is)}{\prod_{i=1}^{Q}\Gamma(c_i-C_is)\prod_{i=1}^{P}\Gamma(d_i+D_is)}\,z^{-s}\,ds\,,$$

then, using the same definitions given in section 3.2.1. for $\mathbb{D}$, $\mathbb{E}$, L, and R, the evaluation guidelines given above remain valid with the interchange of RHP and LHP wherever these occur, and $+\mathbb{E}w$ instead of $-\mathbb{E}w$ in the first four types. For example, Type I would become: $\mathbb{D}>0$, $\mathbb{E}<0$, $L>+\mathbb{E}w$, $\sum$ LHP res.

Overall, for all known convergence conditions except one, a Mellin-Barnes integral can be evaluated by summation of residues. The one situation for which summation of residues does not work is when $\mathbb{D}>0$, $\mathbb{E}=0$, $L\geq-1$, and $|z|=1/(cR)$. This is not a severe limitation since only a circular arc of complex z values and only one real z value for Types V and VI are involved.

3.3. CONVERGENCE CONDITIONS FOR THE H-FUNCTION

Since the H-function is a Mellin-Barnes integral, the results of section 3.2. may be used to determine convergence conditions and application of the residue theorem for the H-function. For an H-function as defined by form (2.1), the parameters of section 3.2.1. and evaluation types of 3.2.4 are:

$$\mathbb{D} = \sum_{i=1}^{n} A_i + \sum_{i=1}^{m} B_i - \sum_{i=n+1}^{p} A_i - \sum_{i=m+1}^{q} B_i$$

$$\mathbb{E} = \sum_{i=1}^{p} A_i - \sum_{i=1}^{q} B_i$$

$$L = \mathrm{Re}\left(\sum_{i=1}^{q} b_i - \tfrac{1}{2}q - \sum_{i=1}^{p} a_i + \tfrac{1}{2}p \right)$$

$$R = \prod_{i=1}^{p} A_i^{A_i} \Big/ \prod_{i=1}^{q} B_i^{B_i} .$$

TYPE	$\mathbb{D}$	$\mathbb{E}$	L	H(cz)	|z|	|arg(z)|
I	>0	<0	$>\mathbb{E}w$	$+\sum$ LHP res	>0	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
II	≥ 0	<0	$\leq\mathbb{E}w$	$+\sum$ LHP res	>0	$<\pi, \leq\frac{1}{2}\pi\mathbb{D}$
III	>0	>0	$>\mathbb{E}w$	$-\sum$ RHP res	>0	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
IV	≥ 0	>0	$\leq\mathbb{E}w$	$-\sum$ RHP res	>0	$<\pi, \leq\frac{1}{2}\pi\mathbb{D}$
V	>0	$=0$	≥ 0	$+\sum$ LHP res	$<1/(cR)$	$<\pi, <\frac{1}{2}\pi\mathbb{D}$
				$-\sum$ RHP res	$>1/(cR)$	
VI[1]	≥ 0	$=0$	<0	$+\sum$ LHP res	$<1/(cR)$	$<\pi, \leq\frac{1}{2}\pi\mathbb{D}$
				$-\sum$ RHP res	$>1/(cR)$	

[1] If $L<-1$, may use sum of either LHP or RHP res at $|z| = 1/(cR)$.

3.4. CONVERGENCE OF SPECIAL CASES OF THE H-FUNCTION

3.4.1. Exponential and Power Functions of Section 2.4.1.

For $H(z) = H^{1\,0}_{0\,1}[z : (b,B)] = B^{-1} z^{b/B} e^{-z^{1/B}}$, $\mathbb{D} = B > 0$

and $\mathbb{E} = -B < 0$. From section 3.3., H(z) is a Type I or Type II and converges by summing of LHP residues for all positive real z (and for complex $z \neq 0$ such that $|\arg(z)| < \min(\pi, \frac{1}{2}\pi B)$).

For $H(z) = H^{1\,0}_{1\,1}[z : (a+b+1,1) ; (b,1)] = z^b(1-z)^a/\Gamma(a+1)$,

$\mathbb{D} = \mathbb{E} = \arg(z) = 0$, $L = -a-1$ and $R = 1$. If $a > -1$, H(z) is a Type VI and converges by summing of LHP residues only for positive real $z < 1$ and, if $a > 0$, for $z = 1$. There are no RHP poles, so that $H(z) = 0$ for real $z > 1$. If $a < -1$, H(z) does not converge for any z.

For $H(z) = H^{1\,1}_{1\,1}[z : (b-a+1,1) ; (b,1)] = \Gamma(a)z^b(1+z)^{-a}$,

$\mathbb{D} = 2$, $\mathbb{E} = 0$, $L = a-1$ and $R = 1$. H(z) is Type V if $a \geq 1$ and Type VI if $a < 1$. Thus H(z) converges by summing of LHP residues for positive real $z < 1$ (and complex $z \neq 0$ where $|z| < 1$ and $|\arg(z)| < \pi$) and by the negative of the sum of RHP residues for real $z > 1$ (and complex z where $|z| > 1$ and $|\arg(z)| < \pi$). If $a < 0$, H(z) is not a properly defined H-function, because the LHP and RHP poles overlap.

3.4.2. Cases of Section 2.4.2.

For $H(kz) = H^{1\,0}_{0\,2}[kz : (a,\frac{1}{2}),(b,\frac{1}{2})]$, $\mathbb{D} = 0$, $\mathbb{E} = -1$ and $L = a+b-1$.

The contour parameter w must be greater than -2a, the rightmost pole of H(kz), by definition of an H-function. Also, H(kz) will converge (Type II) only if $L \leq \mathbb{E}w$ or, equivalently, $a+b-1 \leq -w$. That is,

$$-2a < w \leq -a-b+1 .$$

If $a+1>b$, such a w can be found. Then, H(kz) will converge by summing of LHP residues for all positive real kz. Therefore, when $a+1>b$, the H-functions representing sin(z), cos(z) and $J_v(z)$, having $k=\frac{1}{2}$, converge by summing of LHP residues for all positive real z. But, the H-functions for sinh(z) and cosh(z) have $k=\frac{1}{2}i$ and converge only for negative pure imaginary z, not for any real z.

Next, consider those H-functions representing the inverse functions of sin, tan, sinh and tanh, with form

$$H(kz) = H_{2\,2}^{1\,2}\left[kz : (a,\tfrac{1}{2}),(b,\tfrac{1}{2}) \;;\; (c,\tfrac{1}{2}),(d,\tfrac{1}{2})\right],$$

where $\mathbb{D}=1$, $\mathbb{E}=0$, $L=c+d-a-b$, and $R=1$. H(kz) is Type V if $L \geq 0$ and Type VI if $L<0$.

For the arctan(z) H-function, $L=-1$ and $k=1$ so that this H(kz) is Type VI and converges by summing of LHP residues when $0<|z|<1$ and $|\arg(z)| \leq \frac{1}{2}\pi$ and by the negative of the sum of RHP residues when $|z|>1$ and $|\arg(z)| \leq \frac{1}{2}\pi$. Thus, this H(kz) converges for all positive real $z \neq 1$.

With $L=-1.5$ and $k=1$, the arcsinh(z) H-function is Type VI and converges by summing of LHP residues when $0<|z| \leq 1$ and $|\arg(z)| \leq \frac{1}{2}\pi$ and by the negative of the sum of RHP residues when

$|z|>1$ and $|\arg(z)| \leq \frac{1}{2}\pi$.

For the arcsin(z) H-function, $L=-1.5$ and $k=i$. This $H(kz)$ is Type VI and converges with the sum of LHP residues for $0<|z|\leq 1$ and $-\pi \leq \arg(z) \leq 0$ and with the negative of the sum of RHP residues for $|z|\geq 1$ and $-\pi \leq \arg(z) \leq 0$. Thus, this $H(kz)$ converges for real $z \neq 0$.

The arctanh(z) H-function has $L=0$ and $k=i$. This $H(kz)$ is Type V and converges with the sum of LHP residues for $0<|z|<1$ and $-\pi<\arg(z)<0$ and with the negative of the sum of RHP residues for $|z|>1$ and $-\pi<\arg(z)<0$. Thus, this $H(kz)$ does not converge using summation of residues for any real z.

3.4.3. Logarithmic Function of Section 2.4.3.

$$H(z) = H_{2\,2}^{1\,2}[z \;;\; (1,1),(1,1) \;;\; (1,1),(0,1)] = \log(1+z) \;;$$

$\mathbb{D}=2$, $\mathbb{E}=0$, $L=-1$ and $R=1$. $H(z)$ is Type V and converges with the sum of LHP residues for $0<|z|<1$ and $|\arg(z)|<\pi$, and with the negative of the sum of RHP residues for $|z|>1$ and $|\arg(z)|<\pi$. Thus, $H(z)$ converges for all positive real $z \neq 1$.

3.4.4. Bessel Functions of Section 2.4.4.

$J_v(z)$ has been considered in the first paragraph of 3.4.2.

$K_v(z)$ has $\mathbb{D}=1$ and $\mathbb{E}=-1$, is Type I or II, and converges by summing of LHP residues for all positive real z (and for complex $z \neq 0$ such that $|\arg(z)|<\frac{1}{2}\pi$). $Y_v(z)$ has $\mathbb{D}=0$ and $\mathbb{E}=L=-1$ and will converge (Type II) only if $w \leq 1$. Since the rightmost pole of $Y_v(z)$ is v, a valid w exists only if $v<1$. Thus, $Y_v(z)$ converges by summing of

LHP residues only if $v < 1$ and then only for positive real z.

For $J_v^u(z)$, $\mathbb{D} = 1 - u$, $\mathbb{E} = -u - 1$ and $L = -v - 1$. Since $\mathbb{D}$ must be non-negative for convergence and u must be positive, then, for $0 < u \leq 1$, $J_v^u(z)$ as an H-function is Type I ($u < 1$ and $v \leq -1$) or Type II ($v > -1$) and converges by summing of LHP residues for all positive real z (and complex $z \neq 0$ such that $|\arg(z)| < \frac{1}{2}\pi(1-u) < \frac{1}{2}\pi$).

3.4.5. Confluent Hypergeometric Function of Section 2.4.5.

$\mathbb{D} = 1$ and $\mathbb{E} = -1$, so that the H-function representing $M(a,b;-z)$ is Type I or II and converges by summing of LHP residues for all positive real z (and complex $z \neq 0$ such that $|\arg(z)| < \frac{1}{2}\pi$).

3.4.6. Hypergeometric Function of Section 2.4.6.

$\mathbb{D} = 2$, $\mathbb{E} = 0$, $R = 1$, and $L = a+b-c-1$. This H-function is Type V or VI and converges by summing of LHP residues for $0 < |z| < 1$ and $|\arg(z)| < \pi$ and by the negative of the sum of RHP residues for $|z| > 1$ and $|\arg(z)| < \pi$. If $a + b < c$, then $L < -1$ and this H-function converges for $|z| = 1$ and $|\arg(z)| < \pi$, using either LHP or RHP residues.

3.4.7. Generalized Hypergeometric Function of Section 2.4.7.

The H-function that represents ${}_pF_q$ when $p \leq q + 1$ will converge only if $\mathbb{D} = p + 1 - q \geq 0$, that is, if $p \geq q - 1$. Thus, this H-function really represents ${}_pF_q$ only when p is $q - 1$, q or $q + 1$, a fact that has not been noted by those who have shown ${}_pF_q$ as an H-function (4:40; 7:101; 18:11,159; 21:197-198) or as a G-function (14: 143-147; 16:61). Since $\mathbb{E} = p - q - 1$, this H-function is Type V or VI if $p = q + 1$ and is Type I or II if $p < q + 1$.

3.4.8. Meijer's G-function of Section 2.4.8.

Because all A_i, $i = 1,\ldots,p$, and B_i, $i = 1,\ldots,q$, are equal to one, $\mathbb{D} = n + m - (p - n) - (q - m) = 2n + 2m - p - q$, $\mathbb{E} = p - q$ and $R = 1$. Using these values for $\mathbb{D}$, $\mathbb{E}$ and R, the six convergence types of section 3.3. agree with known convergence conditions given by Luke, where $\mathbb{D} = 2\delta$ and $L = \mathrm{Re}(v) - \frac{1}{2}q + \frac{1}{2}p$ (14:144).

3.4.9. Comment

In the above sections, the H-functions representing sinh(z), cosh(z) and arctanh(z) have been found not to converge for real values of z. Also, the H-function for the Bessel function $Y_v(z)$ does not converge for $v \geq 1$ and that for the generalized hypergeometric function ${}_pF_q$ does not converge for $p < q - 1$. These items have not been noted in the literature.

Throughout the literature on G- and H-functions, there are a number of important errors and omissions. Quite often such errors or omissions are due to failure to check that convergence conditions are met. In order to avoid using invalid H-functions or relations or arriving at invalid or improperly restricted results, convergence is verified and discussed throughout this dissertation. For example, derivatives and Laplace transforms of H-functions are used often; therefore, the next sections will treat convergence of both.

3.5. CONVERGENCE OF THE LAPLACE TRANSFORM OF AN H-FUNCTION

The following theorem is of utmost importance to finding the probablity density function of the sum of two or more H-function variates. Application of Theorem 1.5 depends on existence, meaning convergence, of the Laplace transform.

Theorem 3.1: Given that an H-function $H(cz)$, c a positive real constant, converges using the sum of LHP or RHP residues for some positive real values of z, then the Laplace transform of $H(cz)$, $L_r\{H(cz)\}$, converges using the sum of LHP or RHP residues for all complex $r \neq 0$ such that $|\arg(r)| < \frac{1}{2}\pi$, except at $|r| = cR$ when $L \geq -1.5$ and $\mathbb{E} = -1$.

From property (2.12),

$$L_r\{H(cz)\} = c^{-1} H_{q,p+1}^{n+1,m}\Big[r/c : \{(1-b_i-B_i, B_i)\} ; (0,1), \{(1-a_i-A_i, A_i)\}\Big] .$$

If $\mathbb{D}$, $\mathbb{E}$, L and R are the convergence parameters for $H(cz)$ as defined in section 3.3. and $\mathbb{D}_T$, $\mathbb{E}_T$, L_T, and R_T are the corresponding parameters for $L_r\{H(cz)\}$, then the following relations are immediately found:

$$\mathbb{D}_T = \mathbb{D}+1 , \quad \mathbb{E}_T = -\mathbb{E}-1 , \quad L_T = L-\mathbb{E}-\tfrac{1}{2} , \quad R_T = R^{-1} .$$

First, if $H(cz)$ is Type III, IV, V, or VI, or if $H(cz)$ is Type I or II with $\mathbb{E} > -1$, then we know that $\mathbb{D} \geq 0$ and $\mathbb{E} > -1$. This means that $\mathbb{D}_T \geq 1$ and $\mathbb{E}_T > 0$. By section 3.3., $L_r\{H(cz)\}$ is Type I or II and converges using the sum of LHP residues for all

$r \neq 0$ such that $|\arg(r)| < \frac{1}{2}\pi \mathbb{D}_T$. Since $\mathbb{D}_T \geq 1$, the convergence region includes $|\arg(r)| < \frac{1}{2}\pi$.

Second, if $H(cz)$ is Type I or II with $\mathbb{E} < -1$, then $\mathbb{D}_T \geq 1$ and $\mathbb{E}_T > 0$. In this case, $L_r\{H(cz)\}$ is Type III or IV and converges using the negative of the sum of RHP residues for all $r \neq 0$ such that $|\arg(r)| < \frac{1}{2}\pi \mathbb{D}_T$, which includes the region $|\arg(r)| < \frac{1}{2}\pi$.

Third, if $H(cz)$ is Type I or II with $\mathbb{E} = -1$, then $\mathbb{D}_T \geq 1$ and $\mathbb{E}_T = 0$. Thus, referring again to section 3.3., $L_r\{H(cz)\}$ is Type V or VI and converges using the sum of LHP residues for $r \neq 0$ such that $|\arg(r)| < \frac{1}{2}\pi \mathbb{D}_T$ and $|r| < 1/(R_T/c) = cR$, and using the negative of the sum of RHP residues for r such that $|\arg(r)| < \frac{1}{2}\pi \mathbb{D}_T$ and $|r| > cR$. Again, $|\arg(r)| < \frac{1}{2}\pi \mathbb{D}_T$ includes the region $|\arg(r)| < \frac{1}{2}\pi$, since $\mathbb{D}_T \geq 1$. $L_r\{H(cz)\}$ converges using LHP or RHP residues for $|r| = cR$ and $|\arg(r)| < \frac{1}{2}\pi$ only when $L_T < -1$ or, equivalently, $L < -1.5$.

The primary method used in this work to numerically evaluate the inverse Laplace transform of the product of Laplace transforms of H- functions requires finding the Laplace transform values at $r = a + kbi$ for $k = 0,1,2,\ldots$. Theorem 3.1 guarantees that, for some value $a > 0$, these Laplace transform values will all be calculable using residues. For example, a can be chosen to be greater than the largest value of cR for any of the H- functions for which $\mathbb{E} = -1$ and $L \geq -1.5$. Then, the Laplace transform values of all of the H- functions can be calculated at $r = a + kbi$, $k = 0$, 1,2,..., using residues, since $|\arg(a + kbi)| < \frac{1}{2}\pi$ for $a > 0$.

3.6. CONVERGENCE OF DERIVATIVES OF AN H-FUNCTION

By property (2.15), the r-th derivative of H(z) is given by:

$$H^{(r)}(z) = H_{p+1,q+1}^{m,n+1}\left[z : (-r,1),\{(a_i-rA_i,A_i)\} ; \{(b_i-rB_i,B_i)\},(0,1)\right].$$

From section 3.3., if $\mathbb{D}$, $\mathbb{E}$, L, and R are the convergence parameters for H(z), then the corresponding parameters for $H^{(r)}(z)$, $\mathbb{D}'$, $\mathbb{E}'$, L', and R', are seen to be related as follows:

$$\mathbb{D}' = \mathbb{D}\ , \quad \mathbb{E}' = \mathbb{E}\ , \quad L' = L + r(\mathbb{E}+1)\ , \text{ and } \quad R' = R\ .$$

If $D>0$, then $D'>0$ and $H^{(r)}(z)$ will be one of the six convergence types of section 3.3.

However, if $D = D' = 0$, then L' must be $\leq \mathbb{E}'w'$ when $\mathbb{E}' \neq 0$ and L' must be <0 when $\mathbb{E}' = 0$ in order that $H^{(r)}(z)$ converge (Type II, IV, or VI). That is,

$$L + r(\mathbb{E}+1) = L' \leq \mathbb{E}'w' = \mathbb{E}(w+r) \quad \text{or } L < \mathbb{E}w - r\ , \text{ when } \mathbb{E} \neq 0;$$

$$\text{and} \quad L + r = L' < 0 \quad \text{or } L < -r\ , \text{ when } \mathbb{E} = 0\ .$$

In summary, $H^{(r)}(z)$ converges when H(z) convergence parameters meet one of the following conditions:

CASE A: $\mathbb{D} > 0$.

CASE B: $\mathbb{D} = 0$, $\mathbb{E} \neq 0$, and $L \leq \mathbb{E}w - r$.

CASE C: $\mathbb{D} = \mathbb{E} = 0$, and $L < -r$.

From section 3.5., the Laplace transform of an H-function is an H-function with $\mathbb{D} \geq 1$, and, by **CASE A** above, $L_r^{(t)}\{H(cz)\}$ converges for all non-negative integer t. Moreover, since the Laplace transform has a (b_i,B_i) term equal to (0,1), Corollary 2.1 and equation (2.55) are applicable, giving the following theorem:

THEOREM 3.2: The Laplace transform of an H-function may be differentiated any number of times using equation (2.55) and all of its derivatives are convergent H-functions of the same order, or less, as the Laplace transform.

3.7. IMPROVED TRANSFORM AND DERIVATIVE FORMULAS

The right side of equations (2.12) through (2.15) with $k = r = 1$ does not give a valid H-function when any of the values $-b_i/B_i$ is not less than one, $i = 1,\dots,m$, and b_i real. This is because one or more of the poles associated with the (b_i, B_i) overlap one or more of the poles associated with the new $(-1,1)$ or $(0,1)$ term in the numerator, and no contour exists to properly separate the poles.

However, the Laplace and Fourier transforms and derivatives of an H-function are still able to be represented as valid H-functions. Simple modifications in the developments of these formulas can correct the problem.

For example, in the development of the Laplace transform of an H-function, the order of integrations is reversed and performing the inside integration introduces the term $\Gamma(1-s)$. If $-b_i/B_i \geq 1$ for any i, $i = 1,\dots,m$, then the poles of $\Gamma(b_i + B_i s)$ overlap those of $\Gamma(1-s)$. This overlap can be eliminated by replacing $\Gamma(1-s)$ by the equivalent expression

$$(-1)^I\, \Gamma(I-s+1)\, \Gamma(s-I) \,/\, \Gamma(s+1) \,,$$

where $I = \underset{i=1,\dots,m}{\text{maximum}}\,(\,0\,,\ \text{largest integer less than } -b_i/B_i\,)$. (3.7)

Now, no poles of $\Gamma(b_i + B_i s)$ and $\Gamma(s-I)$ are greater than or equal to any of the poles of $\Gamma(I-s+1)$ and $\Gamma(1-a_j-A_js)$, $i=1,\ldots,m$ and $j=1,\ldots,n$. The Laplace transform equation (2.12) and, by the same argument, the Fourier transform equation (2.13) are thereby changed to:

$$L_r\{H(cz)\} = \frac{(-1)^I}{c} H_{q+1,p+2}^{n+1,m+1}\Big[r/c : (I,1), \{(1-b_i-B_i,B_i)\};$$
$$(I,1), \{(1-a_i-A_i,A_i)\},(0,1)\Big], \tag{3.8}$$

and, $$F_t\{H(cz)\} = \frac{(-1)^I}{c} H_{q+1,p+2}^{n+1,m+1}\Big[\frac{-it}{c} : (I,1), \{(1-b_i-B_i,B_i)\};$$
$$(I,1),\{(1-a_i-A_i,A_i)\},(0,1)\Big], \tag{3.9}$$

where I is given by (3.7).

In the development of the formula for the derivative of an H-function, the order of differentiation and integration is reversed and performing the differentiation introduces the term $(-s)$ into the integrand. Equation (2.15) is obtained by replacing $(-s)$ by the equivalent form $\Gamma(1-s)/\Gamma(-s)$. When $-b_i/B_i \geq 1 =$ smallest pole of $\Gamma(1-s)$, use instead the equivalent form $-s = -\Gamma(1+s)/\Gamma(s)$. Since the largest pole of $\Gamma(b_i+B_is)$ is at least one and all poles of $\Gamma(1+s)$ are smaller, there is no overlap of poles of $\Gamma(1-a_i-A_is)$ with those of $\Gamma(1+s)$. Thus, the derivative formula (2.15) should be stated as two distinct cases, dependent on preserving the existence of a contour that properly separates the poles:

$$H^{(r)}(z) = \begin{cases} H^{m,n+1}_{p+1,q+1}\left[z : (-r,1), \{(a_i - rA_i, A_i)\} ; \{(b_i - rB_i, B_i)\}, (0,1)\right], & \text{if } I = 0; \\ -\,H^{m+1,n}_{p+1,q+1}\left[z : \{(a_i - rA_i, A_i)\}, (-r,1) ; (0,1), \{(b_i - rB_i, B_i)\}\right], & \text{if } I > 0, \end{cases} \tag{3.10}$$

or, alternatively,

$$H^{(r)}(z) = (-1)^I H^{m+1,n+1}_{p+2,q+2}\left[z : (-I-r,1), \{(a_i - rA_i, A_i)\}, (-r,1) ; (-I-r,1), \{(b_i - rB_i, B_i)\}, (0,1)\right], \tag{3.11}$$

where I is defined by (3.7).

Note that when $I = 0$, by property (2.9), equations (3.8), (3.9), and (3.11) reduce to the earlier formulas, respectively, (2.12), 2.13), and (2.15). Moreover, the H-functions for the Laplace transform, Fourier transform, and derivative, as given by these improved formulas, have convergence parameters that are identical to those given by the earlier formulas. Therefore, the results of sections 3.5. and 3.6. are unchanged, except that Theorem 3.2 should indicate that differentiation of the Laplace transform is done using (2.56) when $I > 0$.

CHAPTER 4

THE H-FUNCTION DISTRIBUTION

4.1. DEFINITION

Consider a random variable X with probability density function given by

$$f_X(x) = \begin{cases} K \cdot H(cx) \; , & cx \in S \\ 0 \; , & \text{otherwise} \end{cases}$$

where $H(cx)$ represents an H-function as defined in section 2.2., K and c are real constants such that

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \; ,$$

and S is a subset of the positive real values z for which $H(z)$ is convergent. The random variable X will then be called an <u>H-function variate</u> or a random variable with an <u>H-function distribution</u> (4:41; 7:103; 21:200).

4.2. KNOWN SPECIAL CASES

Carter (4:44-50; 21:202-206) and Eldred (7:103-108; 21:206) demonstrated that twelve of the classical non-negative probability distributions are H-function distributions. The standard form and the H-function form of the probability density functions for each of these distributions are given below:

Gamma Distribution:

$$f(x) = x^{\theta-1}\, e^{-x/\phi}/\phi^{\theta}\, \Gamma(\theta)$$

$$= (\phi\, \Gamma(\theta))^{-1}\, H_{0\,1}^{1\,0}[x/\phi : (\theta-1,1)],\ x>0,\ \theta,\phi>0. \qquad (4.1)$$

Exponential Distribution (Gamma distribution with $\theta = 1$):

$$f(x) = \phi^{-1}\, e^{-x/\phi}$$

$$= \phi^{-1}\, H_{0\,1}^{1\,0}[x/\phi : (0,1)],\ x>0,\ \phi>0. \qquad (4.2)$$

Chi - Square Distribution (Gamma distribution with $\phi = 2$ and $\theta = \frac{1}{2}\theta$):

$$f(x) = x^{\frac{1}{2}\theta-1}\, e^{-\frac{1}{2}x}/(2^{\frac{1}{2}\theta}\, \Gamma(\tfrac{1}{2}\theta))$$

$$= (2\, \Gamma(\tfrac{1}{2}\theta))^{-1}\, H_{0\,1}^{1\,0}[\tfrac{1}{2}x : (\tfrac{1}{2}\theta-1,1)],$$

$$x>0,\ \theta = \text{integer} > 0. \qquad (4.3)$$

Weibull Distribution:

$$f(x) = \theta\, \phi\, x^{\phi-1}\, e^{-\theta x^{\phi}}$$

$$= \theta^{1/\phi}\, H_{0\,1}^{1\,0}[\theta^{1/\phi}\, x : (1-\phi^{-1},\phi^{-1})],\ x>0,\ \theta,\phi>0. \qquad (4.4)$$

Rayleigh Distribution (Weibull distribution with $\phi = 2$):

$$f(x) = 2\,\theta\, x\, e^{-\theta\, x^2}$$

$$= \theta^{\frac{1}{2}}\, H_{0\,1}^{1\,0}[\theta^{\frac{1}{2}}\, x : (\tfrac{1}{2},\tfrac{1}{2})],\ x>0,\ \theta>0. \qquad (4.5)$$

Maxwell Distribution:

$$f(x) = 4\,\theta^{-3}\pi^{-\frac{1}{2}}\,x^2\,e^{-x^2/\theta^2}$$

$$= 2\,\theta^{-1}\pi^{-\frac{1}{2}}\,H_{0\,1}^{1\,0}[x/\theta : (1,\tfrac{1}{2})],\ x>0,\ \theta>0. \tag{4.6}$$

Half - Normal Distribution:

$$f(x) = 2\,\theta^{-1}(2\pi)^{-\frac{1}{2}}\,e^{-x^2/2\theta^2}$$

$$= \theta^{-1}(2\pi)^{-\frac{1}{2}}\,H_{0\,1}^{1\,0}[\theta^{-1}\,2^{-\frac{1}{2}}\,x : (0,\tfrac{1}{2})],\ x>0,\ \theta>0. \tag{4.7}$$

Beta Distribution:

$$f(x) = \begin{cases} x^{\theta-1}(1-x)^{\phi-1}/B(\theta,\phi)\ ,\ 0<x\leq 1,\ \theta,\phi>0 \\ 0\ ,\ x\leq 0 \text{ or } x>1 \end{cases}$$

$$= \begin{cases} \dfrac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\,H_{1\,1}^{1\,0}[x : (\theta+\phi-1,1)\ ;\ (\theta-1,1)],\ 0<x\leq 1 \\ 0\ ,\ x>1 \end{cases} \tag{4.8}$$

Half - Cauchy Distribution:

$$f(x) = 2\,\theta\pi^{-1}(\theta^2+x^2)^{-1}$$

$$= (\theta\pi)^{-1}\,H_{1\,1}^{1\,1}[x/\theta : (0,\tfrac{1}{2})\ ;\ (0,\tfrac{1}{2})],\ x>0,\ \theta>0. \tag{4.9}$$

Half - Student Distribution:

$$f(x) = 2\,k\,\Gamma(\theta+\tfrac{1}{2})\,(1+(x^2/2\theta))^{-(\theta+\frac{1}{2})}$$

$$= k\,H_{1\,1}^{1\,1}\left[x/\sqrt{2\theta} : (\tfrac{1}{2}-\theta,\tfrac{1}{2}) ; (0,\tfrac{1}{2})\right],$$

where $k = 1/(\sqrt{2\theta\pi}\,\Gamma(\theta))$, $x > 0$, $\theta > 0$.

(4.10)

F - Distribution:

$$f(x) = \frac{\theta_1^{\theta_1}\,\Gamma(\theta_1+\theta_2)\,x^{\theta_1-1}}{\theta_2^{\theta_1}\,\Gamma(\theta_1)\,\Gamma(\theta_2)\,(1+\theta_1 x/\theta_2)^{\theta_1+\theta_2}}$$

$$= \frac{\theta_1/\theta_2}{\Gamma(\theta_1)\,\Gamma(\theta_2)}\,H_{1\,1}^{1\,1}\left[\theta_1 x/\theta_2 : (-\theta_2,1) ; (\theta_1-1,1)\right],$$

$x > 0$, $\theta_1,\theta_2 > 0$. (4.11)

General Hypergeometric Distribution:

$$f(x) = d\,a^{c/d}\,\Gamma(b)\,k\,x^{c-1}\,M(b,r,-ax^d)/\Gamma(r)$$

$$= a^{1/d}\,k\,H_{1\,2}^{1\,1}\left[a^{1/d}x : (1-b+(c-1)/d,1/d) ; ((c-1)/d,1/d),(1-r+(c-1)/d,1/d)\right],$$

where $k = \Gamma(r-c/d)/(\Gamma(c/d)\,\Gamma(b-c/d))$, $x > 0$.

(4.12)

4.3. CONVERGENCE OF SPECIAL CASES

4.3.1. The gamma, exponential, chi-square, Weibull, Rayleigh, Maxwell, and half-normal distributions are all of the form:

$$f(x) = k\, H^{1\,0}_{0\,1}\left[cx : (b,B)\right].$$

Using the results of section 3.4.1., $f(x)$ is Type I or II and converges using the sum of LHP residues for all positive real x.

4.3.2. Beta distribution.

$$f(x) = k\, H^{1\,0}_{1\,1}\left[x : (a+b+1,1)\ ;\ (b,1)\right],$$

where $a = \phi - 1$ and $b = \theta - 1$. Using the results of section 3.4.1., $f(x)$ is Type VI if $a > -1$, or $\phi > 0$. Then $f(x)$ converges using the sum of LHP residues for positive real $x < 1$ and, if $\phi > 1$, for $x = 1$. There is no restriction on θ, and, $f(x) = 0$ for real $x > 1$. These results agree with known characteristics of the beta distribution.

4.3.3. Half-Cauchy distribution.

$$f(x) = (\theta\pi)^{-1}\, H^{1\,1}_{1\,1}\left[x/\theta : (0,\tfrac{1}{2})\ ;\ (0,\tfrac{1}{2})\right]$$

Using the convergence parameters defined in section 3.3., $D = 1$, $E = 0$, $L = 0$, and $R = 1$. Thus $f(x)$ is Type V and converges using the sum of LHP residues for $0 < x < \theta$ and using the negative of the sum of RHP residues for $x > \theta$. Most important, $f(x)$ does not converge using residues for $x = \theta$, since $L > -1$.

4.3.4. Half-Student distribution.

$$f(x) = k\, H_{1\,1}^{1\,1}\left[x/\sqrt{2\theta} : (\tfrac{1}{2}-\theta,\tfrac{1}{2}) ; (0,\tfrac{1}{2})\right], \ \theta > 0.$$

$\mathbb{D}=1$, $\mathbb{E}=0$, $L=\theta-\frac{1}{2}$ and $R=1$. Since $\theta>0$, $L>-1$ and $f(x)$ does not converge using residues for $x=\sqrt{2\theta}$. But, $f(x)$ converges using the sum of LHP residues for $0<x<\sqrt{2\theta}$ and using the negative of the sum of RHP residues for $x>\sqrt{2\theta}$, being a Type V or VI.

4.3.5. F-distribution.

$$f(x) = k\, H_{1\,1}^{1\,1}\left[\theta_1 x/\theta_2 : (-\theta_2,1) ; (\theta_1-1,1)\right], \ \theta_1,\theta_2 > 0.$$

$\mathbb{D}=2$, $\mathbb{E}=0$, $L=\theta_1+\theta_2-1>-1$, $R=1$. Like the Half-Student, $f(x)$ is Type V or VI and does not converge using residues for $x=\theta_2/\theta_1$. But, $f(x)$ converges using the sum of LHP residues for $0<x<\theta_2/\theta_1$ and using the negative of the sum of RHP residues for $x>\theta_2/\theta_1$.

4.3.6. General hypergeometric distribution.

For the H-function in formula (4.12), the convergence parameters of section 3.3. are $\mathbb{D}=1/d$ and $\mathbb{E}=-1/d<0$. Thus, $f(x)$ is Type I or II and converges using the sum of LHP residues for all positive real x.

All twelve of these classical distributions have probability density functions that can be expressed as H-functions that are validly defined with properly separated poles for all given ranges of the parameters. For a well-defined H-function, the condition a, b, $d>0$ must be added for the general hypergeometric distribution.

4.4. TRANSFORMATIONS OF H-FUNCTION VARIATES

Carter (4:52-65) proved that the product of independent H-function variates has an H-function distribution, that the quotient of two independent H-function variates has an H-function distribution, and that the rational power of an H-function variate has an H-function distribution. These theorems make the H-function distribution a very powerful tool for analyzing probability density functions of algebraic combinations of independent random variables, because none of the classical distributions has all of these closure properties. The theorems due to Carter are stated below (4:52-65; 21:208-217).

THEOREM 4.1: Distribution of Products. If $X_1, X_2, \ldots, X_N$ are independent H-function variates with probability density functions $f_1(x_1), f_2(x_2), \ldots, f_N(x_N)$, respectively, where, for $j=1,\ldots,N$, $x_j > 0$ and

$$f_j(x_j) = k_j \, H_{p_j,q_j}^{m_j,n_j}\left[c_j x_j : \{(a_{ij}, A_{ij})\} ; \{(b_{ij}, B_{ij})\}\right],$$

then the probability density function of $Y = \prod_{j=1}^{N} X_j$ is given by

$$f_Y(y) = \left(\prod_{j=1}^{N} k_j\right) H_{\sum_{j=1}^{N} p_j , \sum_{j=1}^{N} q_j}^{\sum_{j=1}^{N} m_j , \sum_{j=1}^{N} n_j}\Big[\left(\prod_{j=1}^{N} c_j\right) y :$$

$$\{(a_{ij}, A_{ij})\}, i=1,\ldots,n_j, j=1,\ldots,N, \{(a_{ij}, A_{ij})\}, i=n_j+1, \ldots, p_j, j=1,\ldots,N \; ; \{(b_{ij}, B_{ij})\}, i=1,\ldots,m_j, j=1,\ldots,N, \{(b_{ij}, B_{ij})\}, i=m_j+1,\ldots,q_j, j=1,\ldots,N \Big], \text{ for } y>0.$$

THEOREM 4.2: Distribution of a Quotient. If X_1 and X_2 are independent H-function variates with probability density functions $f_1(x_1)$ and $f_2(x_2)$, respectively, where, for $j = 1,2$, $x_j > 0$ and

$$f_j(x_j) = k_j\, H_{p_j,q_j}^{m_j,n_j}\left[c_j x_j : \{(a_{ij}, A_{ij})\} ; \{(b_{ij}, B_{ij})\}\right],$$

then the probability density function of $Y = X_1/X_2$ is given by

$$f_Y(y) = (k_1 k_2/c_2{}^2)\, H_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2}\left[(c_1/c_2)\, y : \right.$$

$$\{(a_{i1}, A_{i1})\}, i = 1,\ldots,n_1, \{(1 - b_{i2} - 2B_{i2}, B_{i2})\}, i = 1,\ldots,m_2,$$

$$\{(a_{i1}, A_{i1})\}, i = n_1+1,\ldots,p_1, \{(1-b_{i2}-2B_{i2}, B_{i2})\}, i = m_2+1,\ldots,q_2 ;$$

$$\{(b_{i1}, B_{i1})\}, i = 1,\ldots,m_1, \{(1 - a_{i2} - 2A_{i2}, A_{i2})\}, i = 1,\ldots,n_2,$$

$$\left.\{(b_{i1}, B_{i1})\}, i = m_1+1,\ldots,q_1, \{(1-a_{i2}-2A_{i2}, A_{i2})\}, i = n_2+1,\ldots,p_2\right]$$

for $y > 0$.

THEOREM 4.3: Distribution a Rational Power. If X is an H-function variate with probability density function

$$f_X(x) = k\, H_{p\,q}^{m\,n}\left[cx : \{(a_i, A_i)\} ; \{(b_i, B_i)\}\right], \quad x > 0,$$

then the probability density function of $Y = X^P$, for P rational, is given by

$$f_Y(y) = k\, c^{P-1}\, H_{p\,q}^{m\,n}\left[c^P y : \{(a_i - A_iP + A_i, A_iP)\} ; \{(b_i - B_iP + B_i, B_iP)\}\right], \quad P > 0;$$

and,

$$f_Y(y) = k\, c^{P-1}\, H_{q\,p}^{n\,m}\left[c^P y : \{(1 - b_i + B_iP - B_i, -B_iP)\} ; \{(1-a_i+A_iP-A_i, -A_iP)\}\right], \quad P < 0.$$

Noting that $X_1/X_2 = X_1(X_2^{-1})$, then Theorems 4.1, 4.2 and 4.3 can be combined into the following theorem.

* THEOREM 4.4: If $X_1, X_2, \ldots, X_U, X_{U+1}, \ldots, X_V$ are independent random variables with probability density functions $f_j(x_j)$, $j = 1, 2, \ldots, V$, respectively, where $x_j > 0$ and

$$f_j(x_j) = k_j \, H^{m_j, n_j}_{p_j, q_j}\left[c_j x_j : \{(a_{ij}, A_{ij})\} ; \{(b_{ij}, B_{ij})\}\right],$$

and if P_j are positive rational numbers for $j = 1, \ldots, U$ and are negative rational numbers for $j = U+1, \ldots, V$, then the probability density function of the random variable Y, where

$$Y = \prod_{j=1}^{V} X_j^{P_j} ,$$

is given by

$$f_Y(y) = \left(\prod_{j=1}^{V} k_j c_j^{P_j - 1}\right) H^{\sum_{j=1}^{U} m_j + \sum_{j=U+1}^{V} n_j ,\; \sum_{j=1}^{U} n_j + \sum_{j=U+1}^{V} m_j}_{\sum_{j=1}^{U} p_j + \sum_{j=U+1}^{V} q_j ,\; \sum_{j=1}^{U} q_j + \sum_{j=U+1}^{V} p_j}$$

$$\begin{aligned}
\Big[\Big(\prod_{j=1}^{V} c_j^{P_j}\Big) y : \quad & (a_{ij} - A_{ij}P_j + A_{ij}, A_{ij}P_j) \;, i=1,\ldots,n_j, j=1,\ldots,U, \\
& (1 - b_{ij} + B_{ij}P_j - B_{ij}, -B_{ij}P_j) \;, i=1,\ldots,m_j, j=U+1,\ldots,V, \\
& (a_{ij} - A_{ij}P_j + A_{ij}, A_{ij}P_j) \;, i=n_j+1,\ldots,p_j, j=1,\ldots,U, \\
& (1 - b_{ij} + B_{ij}P_j - B_{ij}, -B_{ij}P_j) \;, i=m_j+1,\ldots,q_j, j=U+1,\ldots,V; \\
& (b_{ij} - B_{ij}P_j + B_{ij}, B_{ij}P_j) \;, i=1,\ldots,m_j, j=1,\ldots,U, \\
& (1 - a_{ij} + A_{ij}P_j - A_{ij}, -A_{ij}P_j) \;, i=1,\ldots,n_j, j=U+1,\ldots,V, \\
& (b_{ij} - B_{ij}P_j + B_{ij}, B_{ij}P_j) \;, i=m_j+1,\ldots,q_j, j=1,\ldots,U, \\
& (1 - a_{ij} + A_{ij}P_j - A_{ij}, -A_{ij}P_j) \;, i=n_j+1,\ldots,p_j, j=U+1,\ldots,V \Big],
\end{aligned}$$

for $y > 0$.

Theorem 4.4 has been programmed as part of the computer program of Appendix B to implement the practical technique of section 4.5. Hereafter, Theorems 4.1 through 4.4 will be used extensively. Carter (4,5), Eldred (7), and Springer (21) provide many examples of usage of these theorems. In particular, Carter (4:57-65) derives the chi-square distribution of the square of a standard half-normal variate with Theorem 4.3 and the half-Cauchy distribution of the quotient of two independent half-normal variates with Theorem 4.2. Eldred (7:107-108) and Springer (21:207) indicate the use of Theorems 4.2 and 4.3 to derive both the half-Student and F distributions. The following examples also demonstrate the straight-forward, simple application of these theorems.

Applying Theorem 4.1 (or Theorem 4.4 with $U=V=2$ and $P_1=P_2=1$) to two half-normal variates with form (4.7) immediately gives the distribution of the product of two half-normal variates as

$$(2\pi\theta_1\theta_2)^{-1}\, H^{2\,0}_{0\,2}\left[z/(2\theta_1\theta_2) \;:\; (0,\tfrac{1}{2}),(0,\tfrac{1}{2}) \right].$$

By section 2.4.4., this equals the <u>Bessel distribution</u> given as $(2/(\pi\theta_1\theta_2))\cdot K_0\,(z/\theta_1\theta_2)$ and agrees with known results obtained without H-functions (21:160).

Similarly, applying Theorem 4.2 (or Theorem 4.4 with $U=1$, $V=2$, $P_1=1$, and $P_2=-1$) to two gamma variates with form (4.1) gives the quotient of two gamma variates as

$$(\phi_2/\phi_1\Gamma(\theta_1)\Gamma(\theta_2))\; H^{1\,1}_{1\,1}\left[\phi_2 z/\phi_1 \;:\; (-\theta_2,1) \;;\; (\theta_1-1,1) \right].$$

By section 2.4.1., this is equal to

$$(\phi_2/\phi_1)^{\theta_1}\ (\Gamma(\theta_1+\theta_2)/\Gamma(\theta_1)\Gamma(\theta_2))\ z^{\theta_1-1}(1+(\phi_2 z/\phi_1))^{-\theta_1-\theta_2}$$

which is known as the beta distribution of the second kind and agrees with known results obtained without H- functions (21:164). A special case of the above is the quotient of two exponential variates with the resulting probability density function

$$(\phi_2/\phi_1)\ H_{1\ 1}^{1\ 1}\left[\phi_2 z/\phi_1\ :\ (-1,1)\ ;\ (0,1)\right] = \frac{\phi_2/\phi_1}{(1+(\phi_2 z/\phi_1))^2}\ .$$

As a final example, consider the distribution $f_Y(y)$ for the quotient of two power variates that are independent but identically distributed with probability density function

$$f_{X_1}(x) = f_{X_2}(x) = (a+1)x^a = (a+1)\ H_{1\ 1}^{1\ 0}\left[x\ :\ (a+1,1);(a,1)\right]$$

for $0<x<1$. By Theorem 4.2, the probability density function of $Y=X_1/X_2$ is given directly as

$$f_Y(y) = (a+1)^2\ H_{2\ 2}^{1\ 1}\left[y\ :\ (-a-1,1),(a+1,1)\ ;\ (a,1),(-a-2,1)\right]$$

for $0<y<\infty$. From the section 3.3. convergence conditions, $D=E=0$, $L=-2$, and $R=1$, so that the H- function above is Type VI. It converges by summing of LHP residues for $0<y\leq 1$ and by summing of RHP residues for $y\geq 1$. There is only one LHP pole at $s=-a$ with residue $y^{-(-a)}/(2(a+1))$ and only one RHP pole at $s=a+2$ with residue $y^{-(a+2)}/(-2(a+1))$. Therefore, agreeing with known results (21:161),

$$f_Y(y) = \begin{cases} \frac{1}{2}(a+1)\ y^a\ , & 0<y\leq 1 \\ \frac{1}{2}(a+1)\ y^{-a-2}\ , & y\geq 1\ . \end{cases}$$

* 4.5. PRACTICAL TECHNIQUE FOR FINDING THE DISTRIBUTION OF A SUM

4.5.1. General Technique.

Thus far, only products, quotients, and rational powers of independent H-function variates have been considered (section 4.4). This section will demonstrate a practical technique for determining the probability density function of a sum of products, quotients and rational powers of independent H-function variates. This technique has been implemented and verified by an operational computer program, shown in Appendix B.

The general problem is to find the probability density function of the random variable Z given by

$$Z = \sum_{i=1}^{N} K_i \prod_{j=1}^{M_i} X_{ij}^{P_{ij}} , \quad K_i > 0, \tag{4.13}$$

where, for $j = 1,\ldots,M_i$, $i = 1,\ldots,N$, K_i are known constants, P_{ij} are known rational constants, and X_{ij} are independent random variables with known H-function distributions.

For $i = 1,\ldots,N$, let Y_i be the random variable such that

$$Y_i = \prod_{j=1}^{M_i} X_{ij}^{P_{ij}} .$$

The probability density function for each Y_i is immediately found as an H-function $H_i(y_i)$, $i = 1,\ldots,N$, by applying Theorem 4.4. This reduces the problem to that of finding the probability density function of the random variable Z given by

$$Z = \sum_{i=1}^{N} K_i \; Y_i \; , \quad K_i > 0,$$

where K_i are known constants and Y_i are independent random variables with known H-function probability density functions $H_i(y_i)$, $i = 1, \ldots, N$, respectively.

Since each of the $H_i(y_i)$ is an H-function, each of the corresponding Laplace transforms are also H-functions, $H_i'(r)$. These $H_i'(r)$ are immediately found using property (2.12) or equation (3.8).

Now, by Theorem 1.5, the probability density function $f(z)$ of the random variable Z is given by

$$f(z) = L_z^{-1}\left[\prod_{i=1}^{N} L_{K_i r}\left\{H_i(y_i)\right\}\right] = L_z^{-1}\left[\prod_{i=1}^{N} H_i'(K_i r)\right],$$

where L_z^{-1} is the inverse Laplace transform operation. Equation (5.14), with (5.15), (5.8), (5.9) and (5.10), can be used to find $H_i'(r)$, $i = 1, \ldots, N$, for any desired value of r for which the Laplace transform converges using the sum of LHP or RHP residues. Then, $f(z)$ can be found for specific values of z through any Laplace transform inversion technique that is based upon selected values of r.

By Theorem 3.1, $H_i'(K_i r)$ converges using the sum of LHP or RHP residues for all complex $r \neq 0$ such that $|\arg(K_i r)| < \frac{1}{2}\pi$, except at $|K_i r| = c_i R_i$ when $L_i \geq -1.5$ and $\mathbb{E}_i = -1$, where $\mathbb{E}_i$, L_i and R_i are the $\mathbb{E}$, L and R convergence parameters of section 3.3. for $H_i(z)$ and c_i is the constant in the argument of $H_i(z)$, $i = 1, \ldots, N$. A number of numerical inversion techniques exist that depend only upon values of r within this convergence region. For instance, the method by Dubner and Abate (232) uses only such complex values of r. An improvement on their method by Crump (231) uses complex values of r with $0 < \arg(r) < \frac{1}{2}\pi$. Crump's method is used in the computer program of Appendix B and is explained below after an example that will demonstrate the initial steps of the general technique.

4.5.2. Demonstration Example.

Determine the probability density function of the random variable $T = W \cdot X^2 + (Y/Z)$, where W, X, Y, and Z are independent random variables with the following probability density functions:

$$H_W(w) = w^{2.5}(1-w)^{0.5}$$

$$= (\Gamma(5)/\Gamma(3.5))\ H_{1\ 1}^{1\ 0}[w : (4,1) ; (2.5,1)],$$

$0 < w < 1$ (beta distribution, section 4.2.)

$$H_X(x) = 3\exp(-3\,x) = 3\ H_{0\ 1}^{1\ 0}[3\,x : (0,1)],\ x > 0$$

$$H_Y(y) = 0.5\exp(-0.5\,y) = 0.5\ H_{0\ 1}^{1\ 0}[0.5\,y : (0,1)],\ y > 0$$

$$H_Z(z) = 0.4\exp(-0.4\,z) = 0.4\ H_{0\ 1}^{1\ 0}[0.4\,z : (0,1)],\ z > 0$$

(exponential distributions, section 4.2.)

By applying Theorem 4.4, the probability density functions of $U = W \cdot X^2$ and $V = Y/Z$ are found to be:

$$H_U(u) = (9\,\Gamma(5)/\Gamma(3.5))\ H_{1\ 2}^{2\ 0}[9\,u : (4,1) ; (2.5,1),(-1,2)],\ u > 0$$

$$H_V(v) = 1.25\ H_{1\ 1}^{1\ 1}[1.25\,v : (-1,1) ; (0,1)],\ v > 0\ .$$

Using property (2.12), the Laplace transforms for $H_U(u)$ and $H_V(v)$ are:

$$H_U'(r) = (\Gamma(5)/\Gamma(3.5))\, H_{2\,2}^{1\,2}\left[r/9 : (-2.5,1),(0,2) ; (0,1),(-4,1)\right]$$

$$H_V'(r) = H_{1\,2}^{2\,1}\left[r/1.25 : (0,1) ; (0,1),(1,1)\right].$$

By Theorem 1.5, the probability density function of $T = U + V$ is given by:

$$f_T(t) = L_t^{-1}\left[H_U'(r) \cdot H_V'(r)\right],\ t>0,$$

where L_t^{-1} is the inverse Laplace transform operation.

4.5.3. Crump's Numerical Inversion of a Laplace Transform.

According to Crump's method (231), a function f(z) can be evaluated for $0<z<8$ and $C = \pi/.88$ by the convergent series:

$$f(z) = (e^{az}/.88)\left\{\tfrac{1}{2}L_r(a) + \sum_{k=1}^{\infty}\left\{\mathrm{Re}\left[L_r(a+kCi)\right]\cdot\cos(kCz) - \mathrm{Im}\left[L_r(a+kCi)\right]\cdot\sin(kCz)\right\}\right\},\ a>0,$$

where $L_r(r_o) = L_r\{f(z)\}\Big|_{r=r_o}$. (4.14)

In the general technique for determining the distribution of a sum of products, quotients and rational powers of independent H-function variates,

$$L_r(r_o) = \prod_{i=1}^{N} H_i'(K_i r_o),$$

and for the demonstration example above: $L_r(r_o) = H_U'(r_o)\cdot H_V'(r_o)$.

The rate of convergence of Crump's inversion series depends upon the choice of the constant a. Crump recommends

$$a = \log(1/E)/(1.6Z) ,$$

where E is the largest decimal error desired in the final f(z) value.

Computationally, the $L_r(r_o)$ terms for $r_o = a + kCi$, $k = 0,1,2,\ldots,$ are calculated once and stored. Then they are reused in computing f(z) for each different value of z that is desired.

Let A_{max} be the largest value of $c_i R_i/K_i$ for those $H_i(y_i)$ where the convergence parameters $E_i = -1$ and $L_i \geq -1.5$, $i = 1,\ldots,N$. Then the constant a should be chosen to be greater than A_{max}. This is to make sure that $H_i'(K_i r_o)$ can be evaluated by summing residues for all $r_o = a + kCi$ (per discussion in section 4.5.1.).

4.5.4. An Alternative Laplace Transform Inversion.

A numerical inversion of the Laplace transform by Jagerman (234) is based upon the Widder (228) inversion theorem:

$$f(z) = \lim_{k \to \infty} ((-1)^k/k!)((k+1)/z)^{k+1} L_r^{(k)}\{f(z)\}\Big|_{r=\frac{k+1}{z}} .$$

Because the Laplace transform and its derivatives are H-functions for an H-function, the Widder theorem leads to a formula that has no transforms when $L_r\{f(z)\}$ is a product of Laplace transforms of H-functions, as in the general technique here for finding the distribution of a sum.

For example, if X and Y are independent H-function variates, then the probability density function f(z) of Z = X + Y is given by:

$$f(z) = L_z^{-1}[L_r\{H_X(x)\} \cdot L_r\{H_Y(y)\}] .$$

Then, the k-th derivative of $L_r\{f(z)\}$ is:

$$\sum_{j=0}^{k} \binom{k}{j} L_r^{(k-j)}\{H_X(x)\} \cdot L_r^{(j)}\{H_Y(y)\} .$$

By Theorem 3.2 of section 3.6., each term above exists because the Laplace transform of an H- function can be differentiated any number of times with the resulting H- function still being convergent.

Before proceeding further, let us derive an expression for the t-th derivative of the Laplace transform at $r = (k+1)/z$ for the H- function

$$H(z) = H_{p\ q}^{m\ n}\left[cz : \{(a_i, A_i)\} ; \{(b_i, B_i)\}\right].$$

For simplicity, assume $I = 0$ in formula (3.7) of section 3.7.; then, by using properties (2.12) and (2.8) and the Laplace transform property 1.3.2.5.,

$$\begin{aligned} L_r^{(t)}\{H(z)\} &= (-1)^t(1/c)(r/c)^{-t-1}\, H_{q,p+1}^{n+1,m}\left[r/c :\right. \\ &\qquad \left.\{(1-b_i, B_i)\} ; (t+1,1), \{(1-a_i, A_i)\}\right] \cdot c^{-t} \\ &= (-1)^t\, r^{-t-1}\, H_{q,p+1}^{n+1,m}\left[r/c : \{(1-b_i, B_i)\} ;\right. \\ &\qquad \left.(t+1,1), \{(1-a_i, A_i)\}\right]. \end{aligned}$$

And, at $r = (k+1)/z$, $L_r^{(t)}\{H(z)\}\Big|_{r=(k+1)/z}$

$$\begin{aligned} &= (-1)^t(z/(k+1))^{t+1}\, H_{p+1,q}^{m,n+1}\left[cz/(k+1) : (-t,1), \{(a_i, A_i)\} ;\right. \\ &\qquad \left.\{(b_i, B_i)\}\right] . \end{aligned}$$

Note the similarity to the original H- function H(z).

Thus, the k-th derivative of $L_r\{f(z)\}$ at $r = (k+1)/z$ is:

$$\sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}(z/(k+1))^{k-j+1}(-1)^{j}(z/(k+1))^{j+1}$$

$$\cdot\, H_{p_X+1,q_X}^{m_X,n_X+1}\left[\frac{c_X\, z}{k+1} : (-k+j,1), \{(a_{iX},A_{iX})\} ; \{(b_{iX},B_{iX})\}\right]$$

$$\cdot\, H_{p_Y+1,q_Y}^{m_Y,n_Y+1}\left[\frac{c_Y\, z}{k+1} : (-j,1), \{(a_{iY},A_{iY})\} ; \{(b_{iY},B_{iY})\}\right] .$$

Denote the H-functions above by $H_X{}^{+}(z/(k+1):(-k+j,1))$ and $H_Y{}^{+}(z/(k+1):(-j,1))$, then $f(z)$ is given by the Widder inversion theorem as:

$$f(z) = \lim_{k \to \infty} (z/k!(k+1))\sum_{j=0}^{k} \binom{k}{j}\, H_X{}^{+}(z/(k+1):(-k+j,1))$$

$$\cdot\, H_Y{}^{+}(z/(k+1):(-j,1)) .$$

As the number of terms in the original problem increases, the complexity of this inversion technique increases considerably. The probability density function of the random variable $W = X + Y + Z$, where X, Y, and Z are independent H-function variates, is given by:

$$f(w) = \lim_{k \to \infty} (z^2/k!(k+1)^2) \sum_{j=0}^{k} \binom{k}{j}\, H_X{}^{+}(z/(k+1):(-k+j,1))$$

$$\cdot \sum_{i=0}^{j} \binom{j}{i}\, H_Y{}^{+}(z/(k+1):(-j+i,1))\cdot H_Z{}^{+}(z/(k+1):(-i,1)).$$

This inversion technique has not been programmed for computer implementation and cannot really be compared to Crump's as yet. Both techniques have the disadvantage now of there being no criteria for selecting the upper limit on the summations.

4.6. CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function $H_C(x)$ of a probability density function $H(x)$ is defined as

$$H_C(x) = \int_0^x H(t)\, dt .$$

Using a well-known Mellin transform relation from Erdelyi (9:307), Eldred (7:139) and Springer (21:243) derive the expression

$$H_C(x) = 1 - (2\pi i)^{-1} \int_{w-i\infty}^{w+i\infty} s^{-1}\, x^{-s}\, M_{s+1}\{H(x)\}\, ds \tag{4.15}$$

Eldred (7) developed a computer program that evaluates an H-function probability density function and its cumulative distribution function by summing residues. In the first pass through the basic program, for the desired values of x, he finds the corresponding values of a probability density function given by

$$H(x) = K \cdot H_{p\,q}^{m\,n}\left[cx : \{(a_i, A_i)\} ; \{(b_i, B_i)\}\right], \text{ K a constant.}$$

Then, in the second pass through the basic program, he determines the values of $H_C(x)$ by summing the residues of

$$(K/c)\, H_{p\,q}^{m\,n}\left[cx : \{(a_i + A_i, A_i)\} ; \{(b_i + B_i, B_i)\}\right], \tag{4.16}$$

but (1) multiplying each residue by $1/s_k$, where s_k is the pole for the residue and (2) adding the pole $s_k = 0$ (or increasing by 1 the order of an existing pole at $s_k = 0$ for (4.16)) (7:140-141). Once the result of the second pass is subtracted from one, Eldred completes the implementation of (4.15) for H(x) an H-function density.

$H_C(x)$ can be found more efficiently. First, substitute in (4.15) for s^{-1} the equivalent form $\Gamma(s)/\Gamma(s+1)$ so that

$$H_C(x) = 1 - (K/c)\, H^{m+1,n}_{p+1,q+1}\left[cx : \{(a_i + A_i, A_i)\}, (1,1) ; (0,1), \{(b_i + B_i, B_i)\}\right],$$

and then apply property (2.8) with $k = 1$ so that

$$H_C(x) = 1 - Kx\, H^{m+1,n}_{p+1,q+1}\left[cx : \{(a_i, A_i)\}, (0,1) ; (-1,1), \{(b_i, B_i)\}\right]. \tag{4.17}$$

As indicated by (4.17), $H_C(x)$ can be found at the same time as $H(x)$ by using the calculations for the residues of $H(x)$, multiplying each residue by $1/(s_k - 1)$, and then adding the pole $s_k = 1$ (or increasing by 1 the order of an existing pole at $s_k = 1$). This single pass procedure is used in the computer program of Appendix B.

Another formula for the cumulative distribution function of an H-function probability density can be derived using the Laplace transform.

<u>THEOREM 4.5</u>: The cumulative distribution function for an H-function probability density function is an H-function.

Using Laplace transform property 1.3.2.7., that is,

$$L_r\left\{\int_0^x f(u)\, du\right\} = L_r\{f(x)\}/r ,$$

and the Laplace transform formulas (2.12) and (3.8), with $f(x) = H(x)$, the H-function defined above, then

$$L_r\{H_C(x)\} = \begin{cases} (K/rc)\, H^{n+1,m}_{q,p+1}\Big[r/c : \{(1-b_i-B_i,B_i)\};\ (0,1), \\ \qquad \{(1-a_i-A_i,A_i)\}\Big],\ I=0\ ; \\ (K/rc)(-1)^I\, H^{n+1,m+1}_{q+1,p+2}\Big[r/c : (I,1),\ \{(1-b_i-B_i,B_i)\}; \\ \qquad (I,1),\ \{(1-a_i-A_i,A_i)\},\ (0,1)\Big],\ I>0. \end{cases}$$

And, using properties (2.8) to (2.10), with $k=-1$,

$$L_r\{H_C(x)\} = \begin{cases} (K/c^2)\, H^{n+2,m}_{q+1,p+2}\Big[r/c : \{(1-b_i-2B_i,B_i)\},(0,1)\ ; \\ \qquad (0,1),(-1,1),\ \{(1-a_i-2A_i,A_i)\}\Big],\ I=0\ ; \\ (K/c^2)(-1)^I\, H^{n+1,m+2}_{q+2,p+3}\Big[r/c : (0,1),(I-1,1), \\ \{(1-b_i-2B_i,B_i)\}\ ;\ (I-1,1),\ \{(1-a_i-2A_i,A_i)\}, \\ \qquad (-1,1),(0,1)\Big],\ I>0\ . \end{cases}$$

Then, using formulas (2.12) and (3.8) to find the inverse transforms,

$$* \ H_C(x) = \begin{cases} (K/c)\, H^{m,n+1}_{p+1,q+1}\Big[cx : (1,1),\ \{(a_i+A_i,A_i)\}; \\ \qquad \{(b_i+B_i,B_i)\},(0,1)\Big],\ I=0\ ; \\ (-K/c)\, H^{m+1,n}_{p+1,q+1}\Big[cx : \{(a_i+A_i,A_i)\},(1,1)\ ; \\ \qquad (0,1),\ \{(b_i+B_i,B_i)\}\Big],\ I>0\ . \end{cases} \tag{4.18}$$

Also, from property (2.8) with $k=1$, and equation (3.7) for I,

$$* \ H_C(x) = (Kx)\, H^{m,n+1}_{p+1,q+1}\Big[cx : (0,1),\ \{(a_i,A_i)\}\ ;\ \{(b_i,B_i)\},(-1,1)\Big]$$

$$\text{all } -b_i/B_i < 1,\ i=1,\ldots,m\ ; \tag{4.19}$$

$$*\ H_C(x) = (-Kx)\ H^{m+1,n}_{p+1,q+1}\left[cx : \{(a_i,A_i)\},(0,1)\ ;\ (-1,1),\{(b_i,B_i)\}\right]$$
$$\text{if any } -b_i/B_i \geq 1,\ i = 1,\ldots,m\ . \qquad (4.20)$$

GAMMA CUMULATIVE DISTRIBUTION FUNCTION: Applying (4.18) to the gamma probability density function given by (4.1),

$$*\qquad H_C(x) = (\phi^{\theta}\,\Gamma(\theta))^{-1}\int_0^x t^{\theta-1}\,e^{-t/\phi}\,dt$$
$$= (\Gamma(\theta))^{-1}\ H^{1\,1}_{1\,2}\left[x/\phi : (1,1)\ ;\ (\theta,1),(0,1)\right],$$
$$x>0,\ \theta,\phi>0\ . \qquad (4.21)$$

Also, $\Gamma(\theta)\cdot H_C(z)$ is called the incomplete gamma function, $\gamma(\theta,z)$, (1:260) so that

$$*\qquad \gamma(\theta,z) = H^{1\,1}_{1\,2}\left[z : (1,1)\ ;\ (\theta,1),(0,1)\right],\ \theta>0\ ,\ z>0\ . \qquad (4.22)$$

HALF-NORMAL CUMULATIVE DISTRIBUTION FUNCTION: Applying (4.18) to the half-normal probability density function given by (4.7),

$$*\qquad H_C(x) = 2\,\theta^{-1}(2\pi)^{-\frac{1}{2}}\int_0^x \exp(-t^2/2\theta^2)\,dt$$
$$= \pi^{-\frac{1}{2}}\ H^{1\,1}_{1\,2}\left[x/(\theta\sqrt{2}) : (1,1)\ ;\ (\tfrac{1}{2},\tfrac{1}{2}),(0,1)\right],$$
$$x>0,\ \theta>0\ . \qquad (4.23)$$

Also, $H_C(x)$ is equal to $\mathrm{erf}(x/\theta\sqrt{2})$ (18:140), so that

$$*\qquad \mathrm{erf}(z) = \pi^{-\frac{1}{2}}\ H^{1\,1}_{1\,2}\left[z : (1,1)\ ;\ (\tfrac{1}{2},\tfrac{1}{2}),(0,1)\right],\ z>0\ . \qquad (4.24)$$

BETA CUMULATIVE DISTRIBUTION FUNCTION: Applying (4.18) to the beta probability density function given by (4.8),

$$* \quad H_C(x) = (B(\theta,\phi))^{-1}\int_0^x t^{\theta-1}(1-t)^{\phi-1}\,dt$$

$$= (\Gamma(\theta+\phi)/\Gamma(\theta))\, H_{2\,2}^{1\,1}\Big[x : \begin{matrix}(1,1),(\theta+\phi,1) ; \\ (\theta,1),(0,1)\end{matrix}\Big],$$

$$0<x<1,\ \theta,\phi<0 . \qquad (4.25)$$

The incomplete beta function (1:263) is given by

$$B_x(\theta,\phi) = B(\theta,\phi)\cdot H_C(x)\ ,\ H_C(x) \text{ given by } (4.25).$$

Although the above results, equations (4.18) to (4.20), were found using the Laplace transform, they could also be achieved by switching the order of integration of the cumulative distribution function integral and the H-function contour integral. Let

$$H(x) = K\int_{w-i\infty}^{w+i\infty} (\bullet)\,(cx)^{-s}\,ds\ ,$$

where $(\bullet)$ represents the gamma products in the H-function definition (2.1), which do not depend on the variable x. Then,

$$H_C(x) = \int_0^x K\int_{w-i\infty}^{w+i\infty} (\bullet)\,(ct)^{-s}\,ds\,dt$$

$$= K\int_{w-i\infty}^{w+i\infty} (\bullet)\int_0^x (ct)^{-s}\,dt\,ds$$

$$= K\int_{w-i\infty}^{w+i\infty} (\bullet)\,(cx)^{1-s}\,(c(1-s))^{-1}\,ds$$

$$= Kx\int_{w-i\infty}^{w+i\infty} (\bullet)\,(1-s)^{-1}\,(cx)^{-s}\,ds\ . \qquad (4.26)$$

In equation (4.26), if $\Gamma(1-s)/\Gamma(2-s)$ is substituted for $(1-s)^{-1}$, then (4.26) becomes (4.19). And, if $-\Gamma(s-1)/\Gamma(s)$ is substituted for $(1-s)^{-1}$, then (4.26) becomes (4.20).

The probability density function $f(z)$ for the random variable Z, given in the general problem form (4.13), can be evaluated by using equation (4.14). The cumulative distribution function $F_C(z)$ can be found in the same manner by replacing $L_r(r_o)$ in (4.14) by $L_r(r_o)/r_o$ for $r_o = a + kCi$, $k = 0, 1, \ldots.$ With this procedure, all calculations for $f(z)$ apply to $F_C(z)$ and both are determined in the same pass through the computer program in Appendix B. Thus, Crump's method for numerical inversion of a Laplace transform has the added advantage of simultaneous inversion of transforms that are closely related, as $f(z)$ and $F_C(z)$ are by Laplace transform property 1.3.2.7.

4.7. EVALUATION OF THE H-FUNCTION DISTRIBUTION CONSTANT

Carter (4), Eldred (7), and Springer (21) presented special cases of the H-function distribution and the definition of the general H-function distribution. However, they gave no method to determine the constant K in definition 4.1. One approach to finding K is to investigate $H_C(x)$ for large x, since

$$\lim_{x \to \infty} H_C(x) = \lim_{x \to \infty} K(\text{H-function given by (4.18)}) = 1 .$$

That is, if $K \cdot H(cx)$ is a proposed H-function probability density, use (4.18) to find the associated H-function for the cumulative distribution function, which for large x will approach 1/K.

The numerical approach to finding K is not nearly as appealing as an exact method that has been found. Setting the right sides of (4.17) and (4.19) equal to each other immediately yields:

$$* \quad H^{m,n+1}_{p+1,q+1}\left[cx : \{(a_i,A_i)\},(0,1)\ ;\ (-1,1),\{(b_i,B_i)\}\right]$$

$$+ H^{m+1,n}_{p+1,q+1}\left[cx : (0,1),\{(a_i,A_i)\}\ ;\ \{(b_i,B_i)\},(-1,1)\right] = \frac{1}{Kx},$$

where all $-b_i/B_i < 1$, $i = 1,\ldots,m$, $(1-a_i)/A_i > 1$, $i = 1,\ldots,n$. (4.27)

Compare the residues of the two H- functions in (4.27). Each RHP residue of the first H- function has a matching RHP residue of the second H- function that is exactly equal but opposite in sign, except the residue at $s_k = 1$. Similarly, each LHP residue of the second has a matching residue of the first that is exactly equal but opposite in sign, except again the residue at $s_k = 1$. Therefore, whether (4.27) is evaluated by summation of LHP or of RHP residues, it reduces to only one term on the left side:

(- RHP residue at $s_k = 1$) = (+ LHP residue at $s_k = 1$) = $1/(Kx)$.

If the probability density function H(x) has no pole at $s_k = 1$, then the cumulative distribution function $H_C(x)$ has a pole of order 1 at $s_k = 1$ and equation (4.27) reduces to:

$$* \quad \frac{\prod_{i=1}^{m} \Gamma(b_i + B_i) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i) \prod_{i=m+1}^{q} \Gamma(1 - b_i - B_i)} \cdot \frac{1}{cx} = \frac{1}{Kx} \qquad (4.28)$$

Solving for K and noting property (2.11), $K = (1/M_s\{H(x)\})\Big|_{s=1}$.

The above result is summarized in the following theorem.

THEOREM 4.6: If $H(x)$ is an H-function probability density defined by

$$H(x) = K \cdot H_{p\,q}^{m\,n}\left[cx : \{(a_i,A_i)\} ; \{(b_i,B_i)\}\right],$$

such that $-b_i/B_i < 1$, $i = 1,\ldots,m$, and $(1-a_i)/A_i > 1$, $i = 1,\ldots,n$, (which implies that $H(x)$ has no pole at $s = 1$)

then

$$K = (1/M_s\{H(x)\})\Big|_{s=1}$$

$$= c\,\frac{\prod_{i=n+1}^{p}\Gamma(a_i + A_i)\ \prod_{i=m+1}^{q}\Gamma(1 - b_i - B_i)}{\prod_{i=1}^{m}\Gamma(b_i + B_i)\ \prod_{i=1}^{n}\Gamma(1 - a_i - A_i)} \qquad (4.29)$$

The twelve classical distributions given as special cases of the H-function distribution in section 4.2., equations (4.1) to (4.12), all meet the conditions of Theorem 4.6 and their constants agree with (4.29).

Another way to arrive at equation (4.29) is to consider Theorem 1.10 with $k = 0$, that is, the zero moment of a probability density function $K \cdot H(x)$, where $P\{X \leq 0\} = 0$ and $M_s\{H(x)\}$ has no pole at $s = 1$:

$$E(X^0) = \int_0^\infty K \cdot H(x)\,dx = 1 = M_s\{K \cdot H(x)\}\Big|_{s=0+1}$$

$$= K \cdot M_s\{H(x)\}\Big|_{s=1}\ .$$

* 4.8. CONVERGENCE OF THE CUMULATIVE DISTRIBUTION FUNCTION

If $\mathbb{D}$, $\mathbb{E}$, L, R, and w are convergence parameters for a given H-function probability density H(x) and $\mathbb{D}_C$, $\mathbb{E}_C$, L_C, R_C and w_C are the corresponding parameters for the cumulative distribution function $H_C(x)$, then the application of the section 3.3. formulas to (4.18) yields the following relations:

$$\mathbb{D}_C = \mathbb{D}\,,\quad \mathbb{E}_C = \mathbb{E}\,,\quad L_C = L-1-\mathbb{E}\,,\quad R_C = R\,,$$

$$w_C = w-1, \text{ and if } L<\mathbb{E}w \text{ then } L_C<\mathbb{E}_C \cdot w_C - 1\,.$$

Therefore, if H(x) is Type I or II then $H_C(x)$ is also Type I or II. If H(x) is Type III or IV then $H_C(x)$ is also Type III or IV. If H(x) is Type V then $H_C(x)$ is Type V or Type VI without convergence at $x = 1/(cR)$ by summation of residues. And, if H(x) is Type VI then $H_C(x)$ is also Type VI with convergence at $x = 1/(cR)$ by summation of either LHP or RHP residues. Overall, if H(x) converges then $H_C(x)$ also converges.

CHAPTER 5

EVALUATION OF THE H-FUNCTION

5.1. MATHAI AND SAXENA FORMULATIONS

In 1973, Mathai and Saxena presented a theoretical computable representation of a G-function which involves a series expansion and the summation of residues at LHP poles, using psi and polygamma functions (16:177-185; 15). The psi and polygamma functions are the first and higher order derivatives of the gamma function (1:258-260). Due to the series expansion and no obvious simplifications, this formulation is lengthy and complicated. One of the two terms in the series formula contains 9 nested levels of summation and the other term has 11, where the fourth level also involves an infinite sum.

Mathai and Saxena presented some details on handling poles for the H-function and stated that their G-function formulation is extendable to the H-function. They later gave more details for their H-function representation (18:70-75).

No indication is given that the Mathai and Saxena formulations have actually been programmed for computer usage. Such an effort will not be an easy task. Moreover, their G- and H-function representations are limited to cases where no denominator singularity coincides with any pole. This is a severe limitation since such coinciding occurs quite often.

5.2. <u>ELDRED FORMULATION</u> (7:119-136; 21:227-241)

In 1979, Eldred presented a simpler formulation for the numerical evaluation of the H-function, accompanied by an operational computer program. Eldred treats LHP and RHP evaluation separately. Following his LHP derivation, assume the poles s_k, $k = 1, 2, 3, \ldots$, are ordered from largest, most positive or least negative, to smallest, most negative. And, assume r_k is the order of pole s_k and r_{dk} is the number of singularities for $s = s_k$ in the denominator of the H-function integrand. Then,

$$H(z) = H_{p\,q}^{m\,n}\left[z : \{(a_i, A_i)\} ; \{(b_i, B_i)\}\right]$$

$$= \sum_k \frac{1}{(r_k - 1)!}\left[\frac{d^{r_k - 1}}{ds^{r_k - 1}}\left(C^{(0)}(s)\, U^{(0)}(s)\, z^{-s}\right)\right]\Bigg|_{s = s_k}$$

where
$$C^{(0)}(s) = \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s) / \left[(s - s_k)^{r_{dk}} \cdot \prod_{i=1}^{q} \Gamma(1 - b_i - B_i s) \prod_{i=1}^{p} \Gamma(a_i + A_i s)\right]$$

and
$$U^{(0)}(s) = (s - s_k)^{r_k + r_{dk}} \prod_{i=1}^{m} \Gamma(b_i + B_i s) \,. \qquad (5.1)$$

Eldred applies Leibnitz's rule for differentiation of products and obtains:

$$H(z) = \sum_k \frac{1}{(r_k - 1)!}\left[\sum_{w=0}^{r_k - 1} \binom{r_k - 1}{w} C^{(r_k - 1 - w)}(s) \sum_{v=0}^{w} \binom{w}{v} U^{(v)}(s) \cdot \left(\frac{d^{w-v}}{ds^{w-v}} z^{-s}\right)\right]\Bigg|_{s = s_k} . \qquad (5.2)$$

The next step is the main contribution of Eldred. He notes that $C^{(0)}(s)$ and $U^{(0)}(s)$ are products of terms $f_i(s)$ whose derivatives can be expressed in the form $f_i'(s) = f_i(s) \cdot g_i(s)$. Thus, the first derivatives with respect to s, $C^{(1)}(s)$ and $U^{(1)}(s)$, are of the form

$$d\left(\prod_i f_i(s) \right)/ds = \sum_i g_i(s) \cdot \prod_i f_i(s) .$$

Eldred then uses this simpler product rule to develop recursive formulas for finding higher order derivatives, $C^{(t)}(s)$ and $U^{(t)}(s)$, in terms of $C^{(0)}(s)$, $U^{(0)}(s)$, psi functions and polygamma functions:

$$C^{(0)}(s_k) = \prod_{i=1}^{n} \Gamma(1-a_i-A_i s_k) \prod_{\substack{i=m+1 \\ 1-b_i-B_i s_k = -J_{ik}}}^{q} (-B_i)(-1)^{J_{ik}} J_{ik}! \cdot$$

$$\cdot \prod_{\substack{i=n+1 \\ a_i+A_i s_k = -J_{ik}}}^{p} A_i(-1)^{J_{ik}} J_{ik}! \Big/ \Big(\prod_{\substack{i=m+1 \\ 1-b_i-B_i s_k \neq -J_{ik}}}^{q} \Gamma(1-b_i-B_i s_k) \prod_{\substack{i=n+1 \\ a_i+A_i s_k \neq -J_{ik}}}^{p} \Gamma(a_i+A_i s_k) \Big)$$

$$U^{(0)}(s_k) = \prod_{\substack{i=1 \\ b_i+B_i s_k \neq -J_{ik}}}^{m} \Gamma(b_i+B_i s_k) \Big/ \prod_{\substack{i=1 \\ b_i+B_i s_k = -J_{ik}}}^{m} (B_i(-1)^{J_{ik}} J_{ik}!)$$

$$C^{(r)}(s_k) = \sum_{t=0}^{r-1} \binom{r-1}{t} C^{(r-1-t)}(s_k)\, X^{(t+1)}(s_k)$$

$$U^{(r)}(s_k) = \sum_{t=0}^{r-1} \binom{r-1}{t} U^{(r-1-t)}(s_k)\, V^{(t+1)}(s_k)$$

$$\left.\frac{d^r(z^{-s})}{ds^r}\right|_{s=s_k} = (-\log z)^r z^{-s_k}$$

(next page)

$$X^{(t+1)}(s_k) = \sum_{i=1}^{n} (-A_i)^{t+1}\, \psi^{(t)}(1-a_i-A_i s_k)$$

$$- \sum_{\substack{i=m+1 \\ 1-b_i-B_i s_k \neq -J_{ik}}}^{q} (-B_i)^{t+1}\, \psi^{(t)}(1-b_i-B_i s_k) + \sum_{\substack{i=n+1 \\ a_i+A_i s_k \neq -J_{ik}}}^{p} A_i^{\,t+1}\, \psi^{(t)}(a_i+A_i s_k)$$

$$- \sum_{\substack{i=m+1 \\ 1-b_i-B_i s_k = -J_{ik}}}^{q} \left[(-B_i)^{t+1}\, \psi^{(t)}(1) + \sum_{j=0}^{J_{ik}-1} B_i^{\,t+1}\, t!\, (j-J_{ik})^{-t-1}\right]$$

$$- \sum_{\substack{i=n+1 \\ a_i+A_i s_k = -J_{ik}}}^{p} \left[A_i^{\,t+1}\, \psi^{(t)}(1) + \sum_{j=0}^{J_{ik}-1} (-A_i)^{t+1}\, t!\, (j-J_{ik})^{-t-1}\right]$$

$$V^{(t+1)}(s_k) = \sum_{\substack{i=1 \\ b_i+B_i s_k \neq -J_{ik}}}^{m} B_i^{\,t+1}\, \psi^{(t)}(b_i+B_i s_k)$$

$$+ \sum_{\substack{i=1 \\ b_i+B_i s_k = -J_{ik}}}^{m} \left[B_i^{\,t+1}\, \psi^{(t)}(1) + \sum_{j=0}^{J_{ik}-1} (-B_i)^{t+1}\, t!\, (j-J_{ik})^{-t-1}\right]$$

where $\psi^{(0)} = \psi$ and $\psi^{(r)}$ are the standard psi and polygamma functions (1:258 - 260). The conditional notations beneath the product and summation signs, $f(i,k) = -J_{ik}$ and $f(i,k) \neq -J_{ik}$, are read "equal to any negative integer or zero" and "not equal to any negative integer or zero." Eldred derives a similar formulation for the summation of RHP residues, where $U^{(0)}$ is composed of the terms $\Gamma(1-a_i-A_i s)$ which give the RHP poles.

Eldred deserves considerable credit for significantly simplifying the H-function evaluation and providing a working computer program to implement his formulation.

* 5.3. NEW FORMULATION

A simplified version of Eldred's formulation can be obtained by applying the simpler product rule used by Eldred immediately to equation (5.1), instead of applying Leibnitz's rule.

Define $V^{(0)}(s) \equiv C^{(0)}(s) \cdot U^{(0)}(s) \cdot z^{-s}$. $V^{(0)}(s)$ can be expressed as a product of terms $f_i(s)$ with derivatives of the form $f_i'(s) = f_i(s) \cdot g_i(s)$, as shown below. Let $I_i = 0$ when $c_i + d_i s_k$ is not a negative integer, and let $I_i = 1$ when $c_i + d_i s_k = -J_{ik}$, for some non-negative integer J_{ik}. Then, near any pole s_k of the H-function integrand, $V^{(0)}(s)$ may be considered the product of the following $(p+q+1)$ functions $f_i(s)$:

$$f_{p+q+1}(s) = z^{-s} \quad \text{with} \quad g_{p+q+1}(s) = -\log(z) \tag{5.3}$$

For $i = 1, \ldots, m+n$,

$$f_i(s) = F_i(s) \equiv \begin{cases} \Gamma(c_i + d_i s) \text{ , if } I_i = 0 \\ (s - s_k)\, \Gamma(c_i + d_i s) = \\ \qquad \dfrac{\Gamma(J_{ik}+1+c_i+d_i s)}{d_i(c_i+d_i s)(1+c_i+d_i s)\cdots(J_{ik}-1+c_i+d_i s)} \text{ ,} \\ \quad \text{if } I_i = 1 \text{ .} \end{cases} \tag{5.4}$$

$$g_i(s) = G_i^{(0)}(s) \equiv \begin{cases} d_i \Psi(c_i+d_i s) \text{ , } I_i = 0 \\ d_i \left[\Psi(J_{ik}+1+c_i+d_i s) + \sum_{j=0}^{J_{ik}-1} (-c_i-d_i s-j)^{-1} \right] \text{ ,} \\ \quad \text{if } I_i = 1 \text{ .} \end{cases} \tag{5.5}$$

$$G_i^{(r)}(s) \equiv \begin{cases} d_i^{\,r+1} \Psi^{(r)}(c_i + d_i s) \text{ , if } I_i = 0 \\ d_i^{\,r+1} \left[\Psi^{(r)}(J_{ik}+1+c_i+d_i s) + r! \sum_{j=0}^{J_{ik}-1} (-c_i-d_i s-j)^{-r-1} \right] \text{,} \\ \quad \text{if } I_i = 1 \text{ .} \end{cases} \tag{5.6}$$

For $i = m + n + 1, \ldots, p + q$,

$$f_i(s) = 1/F_i(s) \quad \text{with} \quad g_i(s) = -\, G_i^{(0)}(s) \;, \tag{5.7}$$

where F_i and $G_i^{(0)}$ as defined above in (5.4) and (5.5).

There will be $r_d + r_{dk}$ occasions when $I_i = 1$, for $i = 1, \ldots, m{+}n$, since there must be this many numerator singularities for the order of the pole s_k to be r_k with r_{dk} denominator singularities. And, for $i = m{+}n{+}1, \ldots, p{+}q$, there will be r_{dk} occasions when $I_i = 1$, since there are r_{dk} singularities for $s = s_k$ in the denominator.

Evaluation of (5.4), (5.5) and (5.6) at $s = s_k$ yields:

$$F_i(s_k) = \begin{cases} \Gamma(c_i + d_i s_k) \;, & \text{if } I_i = 0 \\ (d_i (-1)^{J_{ik}} J_{ik}!)^{-1} \;, & \text{if } I_i = 1 \end{cases} \tag{5.8}$$

$$G_i^{(0)}(s_k) = \begin{cases} d_i \Psi(c_i + d_i s_k) \;, & \text{if } I_i = 0 \\ d_i \Psi(J_{ik} + 1) \;, & \text{if } I_i = 1 \end{cases} \tag{5.9}$$

$$G_i^{(r)}(s_k) = \begin{cases} d_i^{\,r+1} \psi^{(r)}(c_i + d_i s_k) \;, & \text{if } I_i = 0 \\ d_i^{\,r+1} \Big[\psi^{(r)}(1) + (-1)^r \big(\psi^{(r)}(J_{ik}+1) \\ \quad - \psi^{(r)}(1) \big) \Big] \;, & \text{if } I_i = 1 \end{cases} \tag{5.10}$$

To complete the development, the derivatives of $V^{(0)}(s)$ are needed. Paralleling Eldred's formulation, $V^{(t)}$ is found recursively with a formula of the same form as that for $C^{(t)}$ or $U^{(t)}$:

If we define $W^{(0)}(s) = \sum_{i=1}^{m+n} G_i^{(0)}(s) - \sum_{i=m+n+1}^{p+q} G_i^{(0)}(s) - \log(z)$,

$$\tag{5.11}$$

then, $r > 0$, $W^{(r)}(s) = \sum_{i=1}^{m+n} G_i^{(r)}(s) - \sum_{i=m+n+1}^{p+q} G_i^{(r)}(s)$.

$$\tag{5.12}$$

Now, $V^{(1)}(s) = V^{(0)}(s) \cdot W^{(0)}(s)$, and

$$V^{(t)}(s) = \sum_{r=0}^{t-1} \binom{t-1}{r} V^{(t-1-r)}(s) \cdot W^{(r)}(s) \ . \qquad (5.13)$$

Combining the results of (5.3) through (5.13) provides the following new formulation for the sum of LHP residues:

$$H(z) = H_{p\,q}^{m\,n}\left[z : (a_i, A_i) \ ; \ (b_i, B_i) \right]$$

$$= \sum_k V^{(r_k - 1)}(s_k)/(r_k - 1)! \ ,$$

where $$V^{(0)}(s_k) = \prod_{i=1}^{p+q+1} f_i(s_k) = z^{-s_k} \prod_{i=1}^{m+n} F_i(s_k) \Big/ \prod_{i=m+n+1}^{p+q} F_i(s_k) \ ,$$

$F_i(s_k)$ are defined by (5.8), and

for $i = 1,\ldots,m$: $c_i = b_i$, $d_i = B_i$;

for $i = m+1,\ldots,m+n$: $c_i = 1 - a_{i-m}$, $d_i = -A_{i-m}$;

for $i = m+n+1,\ldots,q+n$: $c_i = 1 - b_{i-n}$, $d_i = -B_{i-n}$;

for $i = q+n+1,\ldots,p+q$: $c_i = a_{i-q}$, $d_i = A_{i-q}$.

(5.14)

Additionally,

$$V^{(1)}(s_k) = V^{(0)}(s_k) \cdot W^{(0)}(s_k)$$

$$= V^{(0)}(s_k) \left[\sum_{i=1}^{m+n} G_i^{(0)}(s_k) - \sum_{i=m+n+1}^{p+q} G_i^{(0)}(s_k) - \log(z) \right]$$

where $G_i^{(0)}(s_k)$ are defined by (5.9); and, for $r_k > 2$, $V^{(r_k - 1)}(s_k)$ is found recursively, using

$$V^{(t)}(s_k) = \sum_{r=0}^{t-1} \binom{t-1}{r} V^{(t-1-r)}(s_k) \cdot W^{(r)}(s_k) \ ,$$

(next page)

where, for $r > 0$,

$$W^{(r)}(s_k) = \sum_{i=1}^{m+n} G_i^{(r)}(s_k) - \sum_{i=m+n+1}^{p+q} G_i^{(r)}(s_k)$$

and $G_i^{(r)}(s_k)$ are defined by (5.10). (5.15)

One advantage of this formulation over others is that it may be used for either LHP or RHP evaluation without changes. For RHP evaluation, the poles are ordered from smallest to largest and the negative of the final result is taken.

Computationally, this formulation has the advantage that all F_i, $G_i^{(0)}$, $G_i^{(r)}$, and $W^{(r)}$ for $r > 0$ depend only upon the pole s_k and not upon z. Thus, for a given pole, these values are computed once and used to find $V^{(0)}$ without (z^{-s_k}), $W^{(0)}$ without $(-\log z)$, and the other $W^{(r)}$, $r = 1,\ldots,r_k - 2$, which are stored. Then, $V^{(r_k - 1)}$ is found recursively using z, $V^{(0)}$, and the $W^{(r)}$, for as many values of z as desired.

Comparing the number of computer manipulations required, Appendix A shows that the total number of operations saved by the new formulation over Eldred's is

$$4 \sum_{k=1}^{NP} r_k(r_k - 1) + (8\ NZ + 3) \sum_{k=1}^{NP} r_k - NZ \cdot N1 ,$$

where NP is the number of poles evaluated, NZ is the number of z values considered, and N1 is the number of poles evaluated where $r_k > 1$. If all poles are order 1, then NP(8 NZ + 3) additions, subtractions, multiplications, and divisions are saved, and if all poles are order $r > 1$, NP(NZ(8r - 1) + 3r) calculations are saved.

5.4. EXAMPLES OF NEW FORMULATION

5.4.1. $H(z) = H^{1\,0}_{0\,1}[z : (b,B)]$, $B>0$.

From section 3.4.1., $H(z)$ converges for all $z \neq 0$, $|\arg z| < \min(\pi, \frac{1}{2}\pi B)$, using the sum of the LHP residues. There are an infinite number of LHP poles of order 1 at $s_k = -(k+b)/B$, $k = 0,1,\ldots$, so that equation (5.14) yields:

$$H(z) = \sum_{k=0}^{\infty} F_1(s_k)\, z^{-s_k}$$

$$I_1 = 1\,,\ F_1 = (B(-1)^k \cdot k!)^{-1}$$

$$H(z) = \sum_{k=0}^{\infty} (B(-1)^k \cdot k!)^{-1}\, z^{(k+b)/B} = B^{-1} z^{b/B} \sum_{k=0}^{\infty} (-z^{1/B})^k/k!$$

Recognizing, the well-known series for an exponential function,

$$H(z) = B^{-1}\, z^{b/B} \exp(-z^{1/B})\,,$$

which agrees with the known formula in section 2.4.1.

5.4.2. $H(z) = H^{0\,1}_{1\,0}[z : (a,A)]$, $A>0$.

By section 3.3. convergence conditions, $H(z)$ converges for all $z \neq 0$, $|\arg z| < \min(\pi, \frac{1}{2}\pi A)$, using the negative of the sum of the RHP residues. There are an infinite number of RHP poles of order 1 at $s_k = (k+1-a)/A$ for $k = 0,1,\ldots$, so that equation (5.14) yields:

$$H(z) = -\sum_{k=0}^{\infty} F_1(s_k)\, z^{-s_k}\,,\ I = 1,\ F_1 = (-A(-1)^k \cdot k!)^{-1}$$

$$H(z) = A^{-1}\, z^{(a-1)/A} \sum_{k=0}^{\infty} (-z^{-1/A})^k/k!$$

Recognizing once again the series for an exponential function,

$$H(z) = A^{-1}\, z^{(a-1)/A} \exp(-z^{-1/A}) \; .$$

Another way to reach this result is to use property (2.3) to change

$$H^{0\,1}_{1\,0}[z : (a,A)] \quad \text{to} \quad H^{1\,0}_{0\,1}[z^{-1} : (1-a,A)],$$

and then apply the result of section 5.4.1.

5.4.3. $H^{1\,0}_{1\,1}[z : (a,B) \; ; \; (b,B)]$, $B>0$.

By section 3.3. convergence conditions, with $\mathbb{D} = \mathbb{E} = 0$ and $L = b-a$, $H(z)$ is Type VI only if $L<0$, that is, $a>b$, and then $H(z)$ converges using the sum of LHP residues for real z, $0<z<1$. There are no RHP poles.

If $a-b$ equals some integer I, there are I poles in the LHP, each of order 1; else, there are an infinite number of LHP poles of order 1 at $s_k = -(k+b)/B$, $k=0,1,\ldots$. In either case, by equation (5.14):

$$H(z) = \sum_k z^{-s_k} F_1(s_k)/F_2(s_k)$$

$$= \sum_k z^{(k+b)/B} (B(-1)^k \cdot k!)^{-1}/\Gamma(a-b-k)$$

$$= \frac{z^{b/B}}{B\cdot\Gamma(a-b)} \sum_k (a-b-1)\cdots(a-b-k)\cdot(-z^{1/B})^k/k!$$

Whether finite or infinite, the series above equals $(1-z^{1/B})^{a-b-1}$, so that

$$H(z) = (B\cdot\Gamma(a-b))^{-1}\, z^{b/B}\, (1-z^{1/B})^{a-b-1} \; , \; a>b.$$

5.4.4. $H^{0\,1}_{1\,1}[z : (a,A) ; (b,A)]$, $A>0$.

Using property (2.3), this H-function is also equal to

$$H(z) = H^{1\,0}_{1\,1}[z^{-1} : (1-b,A) ; (1-a,A)] .$$

Then, using the result of section 5.4.3.,

$$H(z) = (A\cdot\Gamma(a-b))^{-1}\, z^{(a-1)/A}\, (1-z^{-1/A})^{a-b-1} , \quad a>b.$$

Since $(1-z^{-1/A}) = z^{-1/A}\,(z^{1/A}-1)$, the result simplifies to

$$* \qquad H(z) = (A\cdot\Gamma(a-b))^{-1}\, z^{b/A}\, (z^{1/A}-1)^{a-b-1} , \quad a>b.$$

5.4.5. $H^{1\,1}_{1\,1}[z : (a,B) ; (b,B)]$, $B>0$.

By section 3.3. convergence conditions, with $\mathbb{D} = 2$, $\mathbb{E} = 0$ and $L = b-a$, H(z) is Type V or VI with LHP convergence for $0<|z|<1$ and $|\arg z|<\pi$ and RHP convergence for $|z|>1$ and $|\arg z|<\pi$. There is not convergence using residues for $|z|=1$, because L must be >-1 for the LHP poles of this H-function to be properly separated from the RHP poles.

Using equation (5.14) to sum LHP residues yields:

$$H(z) = \sum_k z^{-s_k}\, F_1(s_k)\, F_2(s_k)$$

$$= \sum_k z^{(k+b)/B}\, (B\cdot(-1)^k\cdot k!)^{-1}\, \Gamma(1-a+b+k)$$

$$= B^{-1}\, z^{b/B}\, \Gamma(1-a+b) \sum_k z^{k/B}\, (-1)^k\, (1-a+b)\cdots(k-a+b)/k!$$

$$= B^{-1}\, z^{b/B}\, \Gamma(1-a+b) \sum_k (z^{1/B})^k\, (a-b-1)\cdots(a-b-k)/k! .$$

The above series is a binomial series that equals $(1+z^{1/B})^{a-b-1}$, so that

$$H(z) = B^{-1}\,\Gamma(1-a-b)\,z^{b/B}\,(1+z^{1/B})^{-(1-a+b)}.$$

Using equation (5.14) to sum RHP residues yields:

$$H(z) = -\sum_k z^{-s_k}\,F_1(s_k)\,F_2(s_k)$$

$$= -\sum_k z^{-(k-1+a)/B}\,\Gamma(1-a+b+k)\,(-B\cdot(-1)^k\cdot k!)^{-1}$$

$$= B^{-1}\,z^{(1-a)/B}\,\Gamma(1-a+b)\sum_{k=0}^{\infty}(z^{-1/B})^k\,(a-b-1)\cdots(a-b-k)/k!\ .$$

The above series is a binomial series that equals

$$(1+z^{-1/B})^{-(1-a+b)} = (z^{-1/B}\,(z^{1/B}+1))^{-(1-a+b)}\ .$$

After substitution for the series and simplification, the result for the RHP is exactly the same as that above for the LHP.

Therefore, for all z such that $z\neq 0$, $|z|\neq 1$, and $|\arg z|<\pi$, and for $(1-a+b)>0$:

$$* \quad H^{1\,1}_{1\,1}[z : (a,B)\ ;\ (b,B)]$$

$$= B^{-1}\,\Gamma(1-a+b)\,z^{b/B}\,(1+z^{1/B})^{-(1-a+b)}\ .$$

5.4.6. $H^{1\,0}_{0\,2}[z : (b,B),(b+\tfrac{1}{2},B)]$, $B>0$.

By the argument in section 3.4.2., $H(z)$ converges using LHP residues for all real $z>0$. There are an infinite number of LHP poles of order 1 at $s_k=-(k+b)/B$, $k=0,1,\ldots$, so that equation (5.14) yields:

$$H(z) = \sum_{k=0}^{\infty} z^{(k+b)/B}\,(B\cdot(-1)^k\cdot k!)^{-1}/\,\Gamma(k+\tfrac{1}{2})$$

Replacing $k!\cdot\Gamma(k+\frac{1}{2})$ with the equivalent expression $\Gamma(\frac{1}{2})(2k)!/2^{2k}$,

$$H(z) = (B\cdot\Gamma(\tfrac{1}{2}))^{-1}\, z^{b/B} \sum_{k=0}^{\infty} (-1)^k\, (2z^{1/2B})^{2k}/(2k)!\ .$$

Recognizing a series for a cosine function,

$$* \qquad H(z) = (B\cdot\Gamma(\tfrac{1}{2}))^{-1}\, z^{b/B} \cos(2z^{1/2B})\ .$$

Following the same steps,

$$* \qquad H_{0\,2}^{1\,0}\left[z : (b,B),(b-\tfrac{1}{2},B)\right] = (B\cdot\Gamma(\tfrac{1}{2}))^{-1}\, z^{(b-\frac{1}{2})/B} \sin(2z^{1/2B})\ .$$

5.4.7. $H_{N\,N}^{N\,0}\left[z : \{(1,1)\} ; \{(0,1)\}\right]$.

By Theorem 4.1, $H(z)$ is the probability density function of the product of N identical, independent, uniformly-distributed random variables. The uniform distribution is a beta distribution with $\theta = \phi = 1$, which, from section 4.2., is thus given by:

$$H_{1\,1}^{1\,0}\left[x : (1,1) ; (0,1)\right].$$

By section 3.3. convergence conditions, with $\mathbb{D} = \mathbb{E} = 0$ and $L = -N$, $H(z)$ can be evaluated by summing LHP residues for real z, 0 z 1. There are no RHP poles, and there is only one LHP pole of order N at $s = 0$. Equation (5.14) is evaluated as follows:

$$H(z) = V^{(N-1)}(0)/(N-1)!$$

$$F_i(0) = 1 \text{ for } i = 1,\ldots,2N \quad \text{and } V^{(0)}(0) = 1$$

$$W^{(0)}(0) = N\Psi(1) - N\Psi(1) - \log(z) = -\log(z)$$

$$W^{(r)}(0) = 0 = N[\psi^{(r)}(1) + (-1)^r(\psi^{(r)}(1) - \psi^{(r)}(1))] - N\psi^{(r)}(1)$$

$$V^{(1)}(0) = -\log(z),\ V^{(2)}(0) = (-\log(z))^2,\ \ldots,$$

$$V^{(N-1)}(0) = (-\log(z))^{N-1}$$

* $$H(z) = (-\log(z))^{N-1}/(N-1)! = (\log(1/z))^{N-1}/(N-1)!\ ,$$

for $0 < z \leq 1$, which agrees with known results obtained without using H-functions (21:101-102). This also provides a set of H-function identities not found in the literature:

* $$\log(z) = -H_{2\,2}^{2\,0}[z : (1,1),(1,1) ; (0,1),(0,1)],\ 0 < z \leq 1.$$

* $$(\log(z))^N = (-1)^N N!\ H_{N+1,N+1}^{N+1,0}[z : \{(1,1)\} ; \{(0,1)\}],\quad 0 < z \leq 1.$$

And, by property (2.3) for a reciprocal argument,

* $$\log(z) = H_{2\,2}^{0\,2}[z : (1,1),(1,1) ; (0,1),(0,1)],\ z > 1.$$

5.4.8. $H_{2\,2}^{2\,2}[z : (0,\tfrac{1}{2}),(0,\tfrac{1}{2}) ; (0,\tfrac{1}{2}),(0,\tfrac{1}{2})]$.

By section 3.3. convergence conditions, with $D = 2$ and $E = L = 0$, $H(z)$ converges for all z such that $z \neq 0$, $|z| \neq 1$ and $|\arg z| < \pi$, using LHP residues for $|z| < 1$ and RHP residues for $|z| > 1$.

There are an infinite number of LHP poles of order 2 at $s_k = -2k$, $k = 0,1,\ldots$, so that equation (5.14) yields:

$$H(z) = \sum_{k=0}^{\infty} V^{(1)}(-2k) = \sum_{k=0}^{\infty} V^{(0)}(-2k) \cdot W^{(0)}(-2k)$$

$$F_1(s_k) = F_2(s_k) = (\tfrac{1}{2}\cdot(-1)^k\cdot k!)^{-1}$$

$$F_3(s_k) = F_4(s_k) = \Gamma(k+1) = k!$$

$$G_1^{(0)}(s_k) = G_2^{(0)}(s_k) = \tfrac{1}{2}\Psi(k+1)$$

$$G_3^{(0)}(s_k) = G_4^{(0)}(s_k) = -\tfrac{1}{2}\Psi(k+1)$$

$$V^{(0)}(-2k) = (k!)^2\ z^{2k}/(\tfrac{1}{2}\cdot(-1)^k\cdot k!)^2 = 4z^{2k}$$

$$W^{(0)}(-2k) = \Psi(k+1) - \Psi(k+1) - \log(z) = -\log(z)$$

$$H(z) = \sum_{k=0}^{\infty} 4z^{2k}(-\log(z)) = -4\log(z)\sum_{k=0}^{\infty}(z^2)^k$$

* $$H(z) = -4\log(z)/(1-z^2) = 2\log(z^2)/(z^2-1)\ .$$

There are an infinite number of RHP poles of order 2 at $s_k = 2k$, $k = 1,2,\ldots$, so that the negative of equation (5.14) yields:

$$H(z) = -\sum_{k=1}^{\infty} V^{(1)}(2k)$$

$$F_1(s_k) = F_2(s_k) = (k-1)! \text{ and } G_1^{(0)}(s_k) = G_2^{(0)}(s_k) = \tfrac{1}{2}\Psi(k)$$

$$F_3(s_k) = F_4(s_k) = (-\tfrac{1}{2}\cdot(-1)^{k-1}\cdot(k-1)!)^{-1};\ G_3^{(0)}(s_k) = G_4^{(0)}(s_k) = -\tfrac{1}{2}\Psi(k)$$

$$V^{(0)}(2k) = 4z^{-2k} \quad \text{and} \quad W^{(0)}(2k) = -\log(z)$$

$$H(z) = -\sum_{k=1}^{\infty} 4z^{-2k}(-\log(z) = +4\log(z)\ z^2\sum_{k=0}^{\infty}(z^{-2})^k$$

* $$H(z) = 4\log(z)/(z^2(1-z^{-2})) = 2\log(z^2)/(z^2-1)\ .$$

$H(z)$ converges to the same function for $|z| < 1$ and $|z| > 1$. This particular H-function also is the probability density function of either the product or the quotient of two Half-Cauchy variates, and the above result agrees with the known result obtained without using H-functions (21:158).

5.4.9. $H^{3\,3}_{3\,3}\left[z : \{(0,\tfrac{1}{2})\} ; \{(0,\tfrac{1}{2})\}\right]$, $z \neq 0$, $z \neq 1$, $|\arg z| < \pi$.

This H(z) is the probability density function of the product of three Half-Cauchy variates, by Theorem 4.1, and, as in section 5.4.8., LHP evaluation for $|z| < 1$ and RHP evaluation for $|z| > 1$ converge to the same elementary function. For example, there are an infinite number of LHP poles of order 3 at $s_k = -2k$, $k = 0, 1, \ldots$, so that equation (5.14) gives the following:

$$H(z) = \sum_{k=0}^{\infty} \frac{V^{(2)}(s_k)}{2!} = \sum_{k=0}^{\infty} \sum_{r=0}^{1} V^{(1-r)}(s_k) \cdot W^{(r)}(s_k)/2!$$

$$V^{(0)}(s_k) = (k!)^3\, z^{2k} \;/\; (\tfrac{1}{2} \cdot (-1)^k \cdot k!)^3 = 8(-z^2)^k$$

$$W^{(0)}(s_k) = 3 \cdot \tfrac{1}{2}\,\psi(k+1) - 3 \cdot \tfrac{1}{2}\,\psi(k+1) - \log(z) = -\log(z)$$

$$\begin{aligned} W^{(1)}(s_k) &= 3(\psi^{(1)}(1)/4) - 3(\psi^{(1)}(k+1)/4) \\ &\qquad + 3(\psi^{(1)}(1)/4) + 3(\psi^{(1)}(k+1)/4) \\ &= (3/2)\,\psi^{(1)}(1) \end{aligned}$$

$$V^{(1)}(s_k) = (-8 \log(z))(-z^2)^k$$

$$V^{(2)}(s_k) = (-8 \log(z))(-z^2)^k(-\log(z) + 8(-z^2)^k(3/2)\,\psi^{(1)}(1)$$

$$H(z) = 8\left[(\log(z))^2 + (3/2)\,\psi^{(1)}(1)\right] \cdot \sum_{k=0}^{\infty} (-z^2)^k \;/2!$$

$$H(z) = ((\log(z^2))^2 + \pi^2)/(1 + z^2)$$, as expected (21:159).

* 5.5. SECOND NEW FORMULATION

The Laplace transform of an H-function is another H-function. This fact combined with any numerical inversion technique for the Laplace transform provides a second formulation for evaluating an H-function inversion integral.

Instead of using equations (5.14) and (5.15) directly to find $H(z)$, one can do the following steps, providing $H(z)$ is of exponential order, that is, $|H(z)| \leq M \exp(Az)$, M and A constants:

1. Use equation (2.12) or (3.8) to determine the form for the Laplace transform of $H(z)$, $H'(r)$.

2. Use equations (5.14) and (5.15) to evaluate $H'(r)$ at the values $r = a + k\pi i/.8Z$, $k = 0,1,2,\ldots,N$, where Z is the maximum value of z for which $H(z)$ is desired.

3. Using these values of $H'(r)$ in equation (4.14), find $H(z)$ for desired values of z with Crump's method for numerical inversion of the Laplace transform, section 4.5.3., where

$$a = A + \log(1/E)/(1.6Z)$$

and E is the maximum desired decimal error. If $H(z)$ is a probability density function or a cumulative distribution function, then $A = 0$.

4. To find values for $\int_0^z H(u)\,du$, use the values $H'(r)/r$ in equation (4.14) in place of the values $H'(r)$. If $H(z)$ is a probability density function, then this represents the cumulative distribution function.

CHAPTER 6

APPLICATIONS FOR PRACTICAL TECHNIQUE

6.1. GENERAL REMARKS

Determining the distributions of algebraic combinations of independent random variables has applications in virtually every area of probability and statistics. Therefore, the applications in this chapter are intended to prod the imagination as to the many potential usages and not to delineate the extent of possible usages.

The practical technique presented in section 4.5. and implemented by the computer program of Appendix B can be used to find the probability density function and the cumulative distribution function for any of the following cases:

1. A single H- function variate. In fact, the computer program may be used to evaluate any H- function.

2. Any combination of products, quotients, and rational powers of any number of independent H- function variates. It is not required that the variates have identical or even similar distributions.

3. The sum of any number of independent H- function variates. Again, the variates do not have to have identical or similar distributions.

4. The sum of any number of terms that each have the form above for case 2.

Eldred (7) has treated distributions of the present worth of probabilistic cash flow profiles that involve products, quotients, and powers. He also indicated the need for treating such distributions that involve sums. Mathai and Saxena (18:82-91) and Springer (21:6-9) present a number of potential applications in statistics that involve algebraic combinations of random variables. For example, from queueing theory, consider a service facility which is an N-step process, where the distributions are known for the service time X_i of the i-th step, $i = 1,\ldots,N$. Then, with no queues between steps, the total service time for the service facility is given by

$$X_t = \sum_{i=1}^{N} X_i .$$

For a Monte Carlo simulation of this service facility, knowing the exact cumulative distribution function of X_t is more desirable than handling the N X_i separately. If the distributions of the X_i are H-function distributions, this is case 3 above and the practical technique of section 4.5. can be used to assist in the simulation.

An especially useful characteristic of the practical technique is that the variates need not have identical or even similar types of H-function distributions. This characteristic and the general properties of the H-function can be exploited in the application areas of the sections that follow. Each of these areas are vast fields of study so that only a few examples are given to indicate the potential applications. However, even the examples cover a wide range of special cases that until now had to be treated individually.

6.2. DEVELOPMENT AND STUDY OF GENERALIZED DISTRIBUTIONS

Although the H-function distribution is the most general distribution, there is still much study that can be done for given types of H-functions. For instance, by Theorem 4.6, the function

$$H(x) = K \cdot H_{0\,1}^{1\,0}\left[cx : (b,B)\right] \tag{6.1}$$

is a probability density function if $K = c/\Gamma(b+B)$. Letting $a = 1/B$, $p = (b+B)/B$, and $q = c^{-1/B}$, (6.1) becomes

$$a(q^{p/a}\,\Gamma(p/a))^{-1}\,x^{p-1}\exp(-x^a/q) \,,$$

which is the generalized gamma statistical distribution introduced by Stacy (22). The H-function form is not only "nicer", but much more can be done using H-function properties and theorems. This generalized gamma distribution is easily seen to have the following special cases:

Gamma distribution (θ,ϕ) when $c = 1/\phi$, $b = \theta - 1$, $B = 1$.

Exponential distribution when $c = 1/\phi$, $b = 0$, $B = 1$.

Chi-square, θ degrees of freedom, when $c = \frac{1}{2}$, $b = \frac{1}{2}\theta - 1$, $B = 1$.

Weibull distribution for $c = \theta^{1/\phi}$, $b = 1 - \phi^{-1}$, $B = \phi^{-1}$.

Rayleigh distribution for $c = \theta^{\frac{1}{2}}$, $b = \frac{1}{2}$, $B = \frac{1}{2}$.

Maxwell distribution for $c = 1/\theta$, $b = 1$, $B = \frac{1}{2}$.

Half-normal distribution for $c = (\theta\sqrt{2})^{-1}$, $b = 0$, $B = \frac{1}{2}$.

By systematically varying the inputs b, c, and B into the computer program of Appendix B, the shapes for the family of distributions represented by (6.1) can be studied.

Also, the computer program can be used systematically to study the distributions of algebraic combinations of independent random variables with densities of the form given by (6.1). For example, investigation of the class of H-functions of the form

$$(c/\Gamma(b+B))^N \cdot H_{0\ N}^{N\ 0}[c^N z : \{(b,B)\}] \tag{6.2}$$

is equivalent to investigation of the distribution of the product of N independent, identically-distributed generalized gamma variates, using Theorem 4.1.

Or, by Theorem 4.3, if X_i is a random variable with (6.1) for its probability density function, then $Y_i = X_i^2$ has the probability density function

$$(c^2/\Gamma(b+B))\ H_{0\ 1}^{1\ 0}[c^2\ y_i : (b-B,2B)]. \tag{6.3}$$

If X_i is a standard half-normal variate, (6.3) represents a chi-square distribution with one degree of freedom. Thus, (6.3) can be used in a "generalized chi-square test", which is just like the well-known chi-square test except that the deviations X_i may be from any common generalized gamma distribution. Then, the generalized chi-square test statistic becomes

$$W = \sum_{i=1}^{N} X_i^2 = \sum_{i=1}^{N} Y_i$$

with probability density function, from section 4.5., given by

$$f_W(w) = L_w^{-1}\{(H_{1\ 1}^{1\ 1}[r/c^2 : (1-b-B,2B) ; (0,1)]/\Gamma(b+B))^N\}. \tag{6.4}$$

And, similarly, an even more generalized chi-square test is possible, using the practical technique of section 4.5. and the computer program of Appendix B, by allowing the X_i to have any H-function distribution.

6.3. CHARACTERIZATION OF PROBABILITY LAWS

The simple, straight-forward rules for finding distributions of algebraic combinations of H-function variates permits rather easy construction of examples to check proposed probability laws. For instance, consider the proposal that only the quotient of two normal variates will follow a Cauchy distribution. Much work has been done by various authors to construct counter-examples (292,294,295,296, 300,305,307,342). Such counter-examples are easily found and checked using H-functions.

By equation (4.9), the half-Cauchy distribution is

$$(\theta\pi)^{-1} H_{1\,1}^{1\,1}\left[y/\theta : (0,\tfrac{1}{2}) ; (0,\tfrac{1}{2})\right], \tag{6.5}$$

and, by Theorem 4.2, the quotient $Y = X_1/X_2$ of two generalized gamma variates is

$$(c_1/(c_2\Gamma(b_1+B_1)\Gamma(b_2+B_2))) \cdot$$
$$\cdot\, H_{1\,1}^{1\,1}\left[c_1 y/c_2 : (1-b_2-2B_2, B_2) ; (b_1, B_1)\right]. \tag{6.6}$$

Equating (6.5) and (6.6) immediately gives $\theta = c_2/c_1$, $b_1 = b_2 = 0$, and $B_1 = B_2 = \frac{1}{2}$. Therefore, the only two generalized gamma variates whose quotient is Cauchy-distributed are half-normal variates.

However, one can find many other types of H- function variates whose quotient has a half-Cauchy distribution. Consider X_1 and X_2 having probability density functions, respectively,

$$(c_1\Gamma(1-b-B)/\Gamma(\tfrac{1}{2}))\ H_{0\ 2}^{1\ 0}\left[c_1x_1\ :\ (0,\tfrac{1}{2}),(b,B)\right],\ \text{and}$$

$$(c_2/(\Gamma(\tfrac{1}{2})\Gamma(1-b-B)))\ H_{0\ 2}^{2\ 0}\left[c_2x_2\ :\ (0,\tfrac{1}{2}),(1-b-2B,B)\right].$$

Then, by Theorem 4.2, the probability density function of $Y=X_1/X_2$ is

$$(c_1/c_2\pi)\ H_{2\ 2}^{1\ 2}\left[c_1y/c_2\ :\ (0,\tfrac{1}{2}),(b,B)\ ;\ (0,\tfrac{1}{2}),(b,B)\right],$$

which, using property (2.9), reduces to the half-Cauchy distribution.

Or, consider the case where X_1 and X_2 have probability density functions that are the same type of H- functions,

$$(c_1\Gamma(b+\tfrac{1}{2})/(\Gamma(\tfrac{1}{2})\Gamma(1-a-\tfrac{1}{2})))\ H_{2\ 1}^{1\ 1}\left[c_1x_1\ :\ (a,\tfrac{1}{2}),(b,\tfrac{1}{2})\ ;\ (0,\tfrac{1}{2})\right],\ \text{and}$$

$$(c_2\Gamma(-a+\tfrac{1}{2})/(\Gamma(\tfrac{1}{2})\Gamma(1+b-\tfrac{1}{2})))\ H_{2\ 1}^{1\ 1}\left[c_2x_2\ :\ (-b,\tfrac{1}{2}),(-a,\tfrac{1}{2})\ ;\ (0,\tfrac{1}{2})\right].$$

Then, by Theorem 4.2, the probability density function of $Y=X_1/X_2$ is

$$(c_1/c_2\pi)\ H_{3\ 3}^{2\ 2}\left[c_1y/c_2\ :\ (a,\tfrac{1}{2}),(0,\tfrac{1}{2}),(b,\tfrac{1}{2})\ ;\ (b,\tfrac{1}{2}),(0,\tfrac{1}{2}),(a,\tfrac{1}{2})\right],$$

which, using properties (2.9) and (2.10), reduces to the half-Cauchy distribution. If $a=-b$ and $c_1=c_2$, then X_1 and X_2 are identically distributed. If $a=b=0$, then the distributions for X_1 and X_2 reduce to the form $(c_1/\Gamma(\tfrac{1}{2}))\cdot H_{1\ 0}^{0\ 1}\left[c_1x_1\ :\ (0,\tfrac{1}{2})\right]$.

As another example of using H- functions to study probability laws, consider the question of whether a distribution exists such that, if the random variables X_1 and X_2 follow this distribution, then the distribution of $X_1 \cdot X_2$ is identical to that of X_1/X_2. From Theorems 4.1 and 4.2, it is evident that the desired distribution must have the following form, with $m = n$, $p = q$, and $c = 1$:

$$K \cdot H_{q\ q}^{m\ m} \left[x : \{(a_i, A_i)\} ; \{(b_i, B_i)\} \right].$$

Next, equating the product and quotient formulas in those theorems,

$$\begin{aligned} K \cdot H_{2q\ 2q}^{2m\ 2m} \Big[y : & \{(a_i, A_i)\}, i = 1,\ldots,m, \{(a_i, A_i)\}, i = 1,\ldots,m, \\ & \{(a_i, A_i)\}, i = m+1,\ldots,q, \{(a_i, A_i)\}, i = m+1,\ldots,q ; \\ & \{(b_i, B_i)\}, i = 1,\ldots,m, \{(b_i, B_i)\}, i = 1,\ldots,m, \\ & \{(b_i, B_i)\}, i = m+1,\ldots,q, \{(b_i, B_i)\}, i = m+1,\ldots,q \Big] \\ = K \cdot H_{2q\ 2q}^{2m\ 2m} \Big[y : & \{(a_i, A_i)\}, i = 1,\ldots,m, \{(1-b_i-2B_i, B_i)\}, i = 1,\ldots,m, \\ & \{(a_i, A_i)\}, i = m+1,\ldots,q, \{(1-b_i-2B_i, B_i)\}, i = m+1,\ldots,q ; \\ & \{(b_i, B_i)\}, i = 1,\ldots,m, \{(1-a_i-2A_i, A_i)\}, i = 1,\ldots,m, \\ & \{(b_i, B_i)\}, i = m+1,\ldots,q, \{(1-a_i-2A_i, A_i)\}, i = m+1,\ldots,q \Big]. \end{aligned}$$

Thus, for all $i = 1,\ldots,q$, $a_i = 1-b_i-2B_i$ or, equivalently, $b_i = 1-a_i-2A_i$, so that the desired distribution is

$$K \cdot H_{q\ q}^{m\ m} \left[y : \{(a_i, A_i)\} ; \{(1 - a_i - A_i, A_i)\} \right].$$

For $m = q = 1$ and $(a, A) = (1-a-2A, A) = (0, \frac{1}{2})$, this is the half-Cauchy distribution, for which this probability law was known (298).

* Also, consider the use of the practical technique of section 4.5. to determine the probability density function h(w) for

$$W = \sum_{i=1}^{N} (c_i X_i)^{1/B_i}/k ,$$

where the independent random variables X_i all have generalized gamma distributions of the form (6.1), that is,

$$f_i(x_i) = (c_i/\Gamma(b_i+B_i))\, H_{0\,1}^{1\,0}\left[c_i x_i : (b_i, B_i)\right] .$$

By Theorem 4.3, the probability density function of $Y_i = X_i^{1/B_i}$ is

$$g_i(y_i) = (c_i^{1/B_i}/\Gamma(b_i+B_i))\, H_{0\,1}^{1\,0}\left[c_i^{1/B_i} y_i : (b_i+B_i-1, 1)\right] .$$

Then, from (2.12) and section 5.4.5.,

$$L_{(c_i^{1/B_i} r/k)}\{g_i(y_i)\} = L_{K_i r}\{g_i(y_i)\}$$

$$= (\Gamma(b_i+B_i))^{-1} H_{1\,1}^{1\,1}\left[r/k : (1-b_i-B_i, 1) ; (0,1)\right]$$

$$= (r/k)^0 \, (1+(r/k))^{-b_i-B_i} .$$

Letting $S = \sum_{i=1}^{N} (b_i+B_i)$,

$$\prod_{i=1}^{N} L_{K_i r}\{g_i(y_i)\} = (1+(r/k))^{-S}$$

$$= (\Gamma(S))^{-1} H_{1\,1}^{1\,1}\left[r/k : (1-S, 1) ; (0,1)\right] ,$$

so that $h(w) = L_w^{-1}\{(1+(r/k))^{-S}\} = (k/\Gamma(S))\, H_{0\,1}^{1\,0}\left[kw : (S-1, 1)\right]$.

Therefore, W as defined above represents a class of sums of independent generalized gamma variates that are generalized gamma variates. If $B_i = 1$ and $c_i = k$, $i = 1,\dots,N$, then we have the well-known fact that the sum of independent gamma variates with common parameter ϕ is gamma-distributed with parameter ϕ, where $\phi = 1/k$ here. If $B_i = B$ and $c_i = k^B$, $i = 1,\dots,N$, then we have a probability law that the sum of the p-th powers of independent generalized gamma variates with parameter $B = 1/p$ is a generalized gamma variate with parameter $B = 1$. A well-known case of this is the chi-square variate, where the X_i are normal variates with $b_i = 0$ and $B_i = B = \frac{1}{2}$ (or $p = 2$). And, if $B_i = 1$ and $c_i = k/N$, then $f(w)$ gives the distribution of the mean of N independent gamma variates with common parameter $\phi = N/k$.

H-functions do not simply assist in a more general characterization of a probability law, but may also lead to a different type of characterization. By first defining a <u>generalized beta variate</u> as a random variable with probability density function of the form

$$H(x) = (c\,\Gamma(a+A)/\Gamma(b+B))\, H_{1\,1}^{1\,0}\left[cx : (a,A)\ ;\ (b,B)\right]$$

(see section 5.4.3. and (4.8) for known special cases), then an H-function distribution with parameters $m+n \geq p$ and $m+n \geq q$ can be characterized as a distribution for a product of $p-n$ generalized beta variates times $m+n-p$ generalized gamma variates divided by the product of $q-m$ generalized beta variates times $m+n-q$ generalized gamma variates. For example, while not derived in terms of products of such variates, the multivariate test criteria of the next section appear so.

6.4. MULTIVARIATE TEST CRITERIA

To test various hypotheses on the parameters of a multivariate statistical distribution, one of the standard procedures is the likelihood ratio principle. A number of multivariate test criteria based upon likelihood ratios, first introduced by Wilks (354), have the property that their moments are expressible in terms of products and quotients of gamma functions. Using the formula

$$f(x) = (2\pi i)^{-1} \int_C E(x^{s-1})\, x^{-s}\, ds, \tag{6.7}$$

the probability density function $f(x)$ of such a test criteria can often be expressed as an H- function.

Wilks defined the determinant of the covariance matrix as a scalar measure of scatter in a multivariate distribution, using $U = |nS|$ where S is the sample sum of product matrix with n degrees of freedom. Then, following Mathai (338), if the sample is from a multivariate normal distribution (central case), then

$$E(U^{s-1}) = 2^{p(s-1)} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}(n+1-i) + s - 1)/\Gamma(\tfrac{1}{2}(n+1-i)) .$$

Thus, from (6.7),

$$f(U) = K \cdot H_{0\ p}^{p\ 0}\left[U/2^p : (\tfrac{1}{2}(n-1-1),1),\ldots,(\tfrac{1}{2}(n-1-p),1)\right],$$

where K is given by (4.29).

To test the hypothesis that the diagonal elements are equal given that the covariance matrix in a multinormal distribution is diagonal, the statistic W is used. W is the likelihood ratio to the power $2/N$, where N is the sample size. From Mathai (339), with $n=N-1$,

$$E(W^{s-1}) = \frac{p^{p(s-1)}\,\Gamma^p(\frac{1}{2}n+s-1)\,\Gamma(\frac{1}{2}np)}{\Gamma^p(\frac{1}{2}n)\,\Gamma(\frac{1}{2}np+p(s-1))} ,$$

and, from (6.7), with K again given by (4.29),

$$f(W) = K\cdot H^{p\,0}_{1\,p}\left[W/p^p : (\tfrac{1}{2}np-p,p) ; \{(\tfrac{1}{2}n-1,1)\}\right].$$

Similarly, from Mathai and Rathie (340), for the criterion W_1 to test the hypothesis that the covariance matrix of a multinormal distribution is diagonal,

$$E(W_1{}^{s-1}) = \frac{p^{p(s-1)}\,\Gamma(\frac{1}{2}np)}{\Gamma(\frac{1}{2}np+p(s-1))}\prod_{i=1}^{p}\frac{\Gamma(\frac{1}{2}(n+1-i)+s-1)}{\Gamma(\frac{1}{2}(n+1-i))}$$

and, $f(W_1) = K\cdot H^{p\,0}_{1\,p}\left[W_1/p^p : (\tfrac{1}{2}np-p,p) ; \{(\tfrac{1}{2}(n-1-i),1)\}, i=1,\ldots,p\right]$,

where K is given by (4.29).

Consul (327) showed that, for Wilk's likelihood ratio criteria for testing the linear hypothesis about regression coefficients,

$$E(U^{s-1}) = K\cdot\Gamma(\tfrac{1}{2}(n+1-i)+s-1)/\Gamma(\tfrac{1}{2}(n+m+1-i)+s-1).$$

Nair (349) referred to U as Wilk's generalized correlation ratio. Applying (6.7), with K again given by (4.29),

$$f(U) = K\cdot H^{p\,0}_{p\,p}\left[U : \{(\tfrac{1}{2}(n+m-1-i),1)\} ; \{(\tfrac{1}{2}(n-1-i),1)\}\right],$$

where $i=1,\ldots,p$. Consul (327) expressed f(U) in terms of other known special functions for $p=1$, $p=2$, $p=3$ with $m=1,\ldots,8$, and $p=4$ with $m=1,\ldots,8$. For examples:

If $p=1$, then $f(U) = C_1\,U^{\frac{1}{2}n-1}(1-U)^{\frac{1}{2}m-1}$, $0<U<1$;

If $p=2$, then $f(U) = C_2\,U^{\frac{1}{2}(n-3)}(1-U^{\frac{1}{2}})^{m-1}$, $0<U<1$;

If $p = 3$ and $m = 3$, then, for $0 < U < 1$, $f(U) =$

$$C_3\, U^{\frac{1}{2}n-2}\left((1-U)^{3/2} - 3U^{\frac{1}{2}}\arcsin(1-U)^{\frac{1}{2}} + 3U\cdot\log\left(U^{-\frac{1}{2}} + (U^{-1}-1)^{\frac{1}{2}}\right)\right) ;$$

And, if $p = 3$ and $m = 4$, then, for $0 < U < 1$,

$$f(U) = C_4\, U^{\frac{1}{2}n-2}\left(1 - U^2 + 8U^{\frac{1}{2}}(1-U) - 6U\cdot\log U\right) ,$$

where C_1, C_2, C_3, and C_4 are constants.

Also expressible as H- functions are the distributions of Votaw's criteria for testing compound symmetry of a covariance matrix (329) and Bartlett's criteria for testing equality of the covariance matrices in a set of independent multinormal populations (18:87). The sum of independent variates defined by likelihood ratios was the subject of a recent study on the detection of radar targets of unknown Doppler frequency (21:6).

6.5. SYSTEM EFFECTIVENESS

An extremely important area of application for determining distributions of algebraic combinations of independent random variables is the usage of hypothesis testing and other statistical methods in analyzing system effectiveness, particularly for research and development or for operational testing and evaluation within the U. S. Department of Defense. The problems associated with evaluating weapon system effectiveness were sufficiently important for the Air Force Systems Command to form the Weapon System Effectiveness Industry Advisory Committee (WSEIAC) in 1963, comprised of both industry and Department of Defense personnel, each approved by the Secretary of the Air Force. A primary objective of the WSEIAC effort was to recommend a methodology for measuring and predicting system effectiveness in all phases of the life of a weapon system. The WSEIAC findings were published in eleven volumes in 1965 (353:v.65-6,1-3).

The WSEIAC methodology was based upon defining effectiveness E as the product of three random variables: availability A, dependability D, and capability C. A is a measure of the condition of the system at the start of a mission, when the mission is required at an unknown (random) point in time. D is a measure of system condition during the performance of the mission, given its condition at the start. And, C is a measure of the results of the mission, given the system condition during the mission. Thus, knowing the effectiveness of a weapon system for accomplishing a given mission depends on being able to determining the distribution of a product of three variates, $E = A \cdot D \cdot C$ (353:v.65-6,8-9). The WSEIAC volumes contain many examples based upon this effectiveness model, including a tactical fighter bomber system, a radar surveillance system, a spacecraft system, and an intercontinental ballistic missile system (353:v.65-2-2,72-97; 353:v.65-2-3,22-50,67-132).

Each of the random variables A, D, and C may also be expressed as a product of the corresponding factors, A_i, D_i, or C_i, for the sub-systems or components of the system. For example, the effectiveness of the fighter bomber system depends upon the effectiveness of many avionics sub-systems, including communications, navigation, engine, flight controls, target identification, penetration, and delivery method. The effectiveness of a system with N sub-systems is given by

$$E = A \cdot D \cdot C = \prod_{i=1}^{N} A_i \cdot \prod_{i=1}^{N} D_i \cdot \prod_{i=1}^{N} C_i = \prod_{i=1}^{N} A_i D_i C_i = \prod_{i=1}^{N} E_i \ .$$

Thus, if the distributions of availability, dependability, and capability are known for the sub-systems, then the overall system effectiveness distribution is equal to the distribution of the product of 3N random variables.

Another WSEIAC consideration is total mission effectiveness. If a mission can be accomplished by more than one weapon system or by more than one method of using the same system, then the total mission effectiveness can be expressed as a linear combination of the effectivenesses of the different systems or methods. That is,

$$E = \sum_k P_k \cdot E_k \ , \text{ where } \sum_k P_k = 1 \ .$$

Using the fighter bomber system as an example (353:v.65-2-3,22-50), there are three possible bomb delivery systems: lay-down with effectiveness E_L, visual-toss with E_V, and blind-toss with E_B. Then the delivery mission effectiveness is given by

$$E = P_L \cdot E_L + P_V \cdot E_V + P_B \cdot E_B, \text{ where } P_L + P_V + P_B = 1 \ ,$$

and P_L, P_V, and P_B are constants as defined below:

P_L = probability of daytime mission x probability of visual flight conditions x probability that the lay-down system is preferred over visual-toss.

P_V = P(daytime mission) x P(visual flight conditions) x P(visual-toss preferred over lay-down).

P_B = P(night mission) x P(instrument flight required).

The WSEIAC study also addressed the treatment of costs as variables. Cost effectiveness of a system or a sub-system can be defined as the quotient of system effectiveness and the total system cost. Many different models with system cost equal to algebraic combinations of variates are presented. For example, the total cost of a small, mobile, short-range weapons launcher is given by

$$C_T = M \cdot C_i + C_f + C_s + C_m ,$$

where C_i = incremental cost separate from fixed costs, for producing, supporting, and maintaining one unit,

C_f = fixed or sunk costs for production,

C_s = total system support cost,

C_m = total system maintenance cost,

M = number of units, may be considered as a constant or as a variate (353:v.65-6,128-136).

Abraham and Prasad (325) also consider cost as a random variable equal to an algebraic combination of independent random variables and provide examples with some simple distributions for estimating manufacturing cost. Other aerospace applications involving products and quotients of independent random variables are discussed by Donahue (291,330).

CHAPTER 7

CONCLUSION AND RECOMMENDATIONS

The main purpose of this dissertation has been to demonstrate a practical technique for determining the probability density function and the cumulative distribution function of a sum of any number of terms involving any combination of products, quotients and powers of H- function variates. This has been accomplished in section 4.5. and the implementing computer program of Appendix B. On the road to accomplishing this purpose, other contributions have resulted.

In trying to learn everything now known about H- functions and the H- function distribution, one quickly becomes aware that this study area has tremendous potential for new discoveries. Just from an effort to understand basic H- function properties, many new formulas have been found, including relations between H- functions and known named functions or lower order H- functions in sections 2.3. through 2.7., 4.6. and 5.4., derivative formulas for special cases in section 2.7., and improved transform and derivative formulas in section 3.7.

Similarly, investigation of the evaluation of the H- function by summing residues has led to an improved formulation, given in section 5.3., and has pointed out the need to formally establish guidelines for when left-half-plane versus right-half-plane summation of residues will converge. Hence, in chapter 3, evaluation guidelines have been established for the general Mellin-Barnes integral and the H-function and have been applied to known special cases, the Laplace transform, and the derivatives of the H-function. For a convergent H-function, the Laplace transform and all derivatives of the Laplace transform have been shown to converge. Since the Laplace transform of an H-function is also an H-function of readily known form, a second new formulation for evaluating the H-function has been addressed in section 5.5. This consists of using the first new formulation to find values for the Laplace transform, which are in turn used to numerically invert the Laplace transform. The first new formulation can also be used to find more relations between H-functions and other named functions, such as the new relations found in section 5.4.

A remarkably rewarding area has been the study of the cumulative distribution function of an H-function distribution. First, by section 4.6., the cumulative distribution function has been shown to be an H-function, and, by section 4.8., it has been shown to converge. Second, a more efficient way to compute the cumulative distribution function of an H-function variate or of a sum of H-function variates has been employed, by using those calculations made for the probability density function. Third, expressing the cumulative distribution function as an H-function not only has led to new relations in section 4.6. between particular H-functions and other named functions, but has also filled in a gap for understanding certain orders of H- functions. And, fourth, the study of the cumulative distribution function has led to a formula for finding the constant for the H- function distribution, given in section 4.7.

The following recommendations are made for directions of future work on H- function distributions:

1. A practical technique needs to be developed for finding the probability density function of the difference of two H- function variates. A method is also needed to find the probability density function of the product or quotient of sums of H- function variates.

2. H- functions have not yet been applied to the study of combinations of dependent variates. It is possible that multi-variable H- functions or H- functions of matrix argument might be useful here. Mathai and Saxena (18) list references for such types of H- functions and devote a few chapters to them.

3. Usage of H- functions to treat probability density functions defined over the entire real line is another unexplored realm. The positive-negative component methods developed by Epstein (8) and Springer and Thompson (320,321) should accommodate such usage, particularly for distributions symmetric about zero.

4. For evaluation of an H- function by summation of residues, analysis is needed to relate the number of poles evaluated to the error in the H- function value, especially for large values of the argument. Error analysis could also be done for numerical inversion of the Laplace transform, where the Laplace transform is a product of H- functions.

5. Including the two methods presented in section 4.5., the various methods for numerical inversion of the Laplace transform (230 - 242) could be compared with respect to their appropriateness and computational efficiency for inverting a product of Laplace transforms of H- functions.

6. The more one studies the H- function, the more suited it seems for analyzing probability distributions. Some additional theoretical advances would be most welcome. For example, for the distribution of a sum of H- function variates, a closed-form solution would be very desirable, since this would eliminate the numerical inversion requirement and possibly lead to a method for handling products and quotients of sums of H- function variates. Also very worthwhile would be the establishment of the conditions on the H- function parameters that are necessary for the sum of H- function variates to have an H- function distribution. At present only certain special cases are known, such as the sum of gamma variates being a gamma variate and the sum of normal variates being a normal variate. The H- function parameter conditions necessary for an H- function to be a probability density function also need to be established.

7. From the section 3.3. convergence conditions, when an H- function is Type VI with $L \geq -1$ or is Type V, then it cannot be evaluated at the point $1/(cR)$ by summation of residues. Another means should be found to compute the H- function value at this point. Many H- functions, including the half-Cauchy, half-Student, and F

distributions, fall into this category.

8. The application of H- functions to the fitting of curves to data should definitely be studied. Being the most general of the special functions and having easily-determined derivatives and moments, the H- function appears to be as suitable for curve-fitting as it has been for analyzing probability density functions of algebraic combinations of independent random variables.

During this research effort, to quote Isaac Newton, "I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." I must thank God, the Creator, for this new world to which He has introduced me. This research area should be equally exciting and rewarding to everyone that pursues it. Hopefully, each new pursuer will discover, as I have, that there is just as much excitement and reward in the pursuit as in the actual attainment of any objective.

APPENDIX A

COMPUTER MANIPULATIONS REQUIRED FOR EVALUATION OF AN H- FUNCTION

* APPENDIX A: COMPUTER MANIPULATIONS REQUIRED FOR EVALUATION OF AN H-FUNCTION

let NP = number of poles evaluated

NZ = number of values of z (independent variable)

s_k = k-th pole

r_k = order of the k-th pole

r_{nk} = order of H(z) numerator poles at s_k

r_{dk} = order of H(z) denominator poles at s_k

A.1. ELDRED'S FORMULATION (section 5.2.)

To compute s_k, $k = 1,\ldots,NP$:

$NP(q+p)$ +'s (additions and subtractions)

$NP(q+p)$ *'s (multiplications)

To compute $V^{(t)}(s_k)$ and $X^{(t)}(s_k)$ for $t = 1,\ldots,r_k-1$, $k = 1,\ldots,NP$:

$$(q+p)\sum_{k=1}^{NP}(r_k-1) + \sum_{k=1}^{NP} \mathrm{INT}((r_k-1)/2)\cdot(r_{nk}+r_{dk})$$

+'s, *'s, and /'s (divisions),

where INT(x) is the integer part of x.

$$(q+p)\sum_{k=1}^{NP}(r_k-1) + \mathrm{INT}((r_k-1)/2) \qquad \Psi\text{'s (psi functions)}$$

To compute $C^{(0)}(s_k)$ and $U^{(0)}(s_k)$, $k = 1,\ldots,NP$:

$$NP(m+n) + \sum_{k=1}^{NP}(r_{nk}+3\,r_{dk}) \qquad *\text{'s}$$

$$NP(q-m+p-n) + \sum_{k=1}^{NP} r_k \qquad /\text{'s}$$

$NP(q+p)$ Γ's (gamma functions and factorials)

$$\sum_{k=1}^{NP} (r_{nk} + r_{dk}) \quad \text{**'s (powers)}$$

To compute $C^{(t)}(s_k)$ and $U^{(t)}(s_k)$ for $t = 1,\ldots,r_k-1$, $k = 1,\ldots,NP$:

$$\sum_{k=1}^{NP} r_k(r_k-1) \quad \text{+'s}$$

$$\frac{5}{2}\sum_{k=1}^{NP} r_k(r_k-1) \quad \text{*'s}$$

$$\frac{1}{2}\sum_{k=1}^{NP} r_k(r_k-1) \quad \text{/'s}$$

To compute $WSUM(w,z) = \sum_{v=0}^{w} \binom{w}{v}(-\log(z))^{w-v}\, U^{(v)}(s_k)$ for

$w = 0,\ldots,r_k-1$, $k = 1,\ldots,NP$:

$$NZ\left(\sum_{k=1}^{NP} \tfrac{1}{2}r_k(r_k+1) \right) \quad \text{+'s and /'s}$$

$$3\ NZ\left(\sum_{k=1}^{NP} \tfrac{1}{2}r_k(r_k+1) \right) + NZ\sum_{k=1}^{NP} r_k \quad \text{*'s}$$

To compute $H(z) = \pm\sum_{k=1}^{NP} (z^{-s_k}/(r_k-1)!)\left[\binom{r_k-1}{w}\, C^{(r_k-1-w)}(s_k)\ \cdot \right.$

$\left. \cdot\ WSUM(w,z) \right]$:

NP !'s (factorials)

$NP \cdot NZ$ **'s

$$NP\cdot NZ + NZ\sum_{k=1}^{NP} r_k \quad \text{+'s}$$

$$NP\cdot NZ + (NZ+2)\sum_{k=1}^{NP} r_k \quad \text{*'s}$$

$$NP\cdot NZ + \sum_{k=1}^{NP} r_k \quad \text{/'s}$$

A.2. NEW FORMULATION (section 5.3.)

To compute s_k, $k = 1,\ldots,NP$: same as Eldred's for s_k.

To compute $W^{(t)}(s_k)$ for $t = 0,\ldots,r_k-2$, $k = 1,\ldots,NP$, without the $(-\log(z))$ term on $W^{(0)}(s_k)$:

Same as Eldred's for $V^{(t)}(s_k)$ and $X^{(t)}(s_k)$.

To compute $V^{(0)}(s_k)$ without the z^{-s_k} term, $k = 1,\ldots,NP$:

Same as Eldred's for $C^{(0)}(s_k)$ and $V^{(0)}(s_k)$, except that, when $r_k > 1$, there are an additional NZ +'s for adding $(-\log(z))$ in $W^{(0)}(s_k)$.

To compute $V^{(r_k-1)}(s_k)$:

NP·NZ **'s

$NZ \sum_{k=1}^{NP} \frac{1}{2} r_k(r_k-1)$ +'s and /'s

$NP \cdot NZ + 3\ NZ \sum_{k=1}^{NP} \frac{1}{2} r_k(r_k-1)$ *'s

To compute $H(z) = \pm \sum_k V^{(r_k-1)}(s_k)/(r_k-1)!$:

NP·NZ +'s and /'s

NP !'s

A.3. NUMBER OF OPERATIONS SAVED BY NEW FORMULATION

+'s : $\sum_{k=1}^{NP} r_k(r_k-1) + NZ \sum_{k=1}^{NP} (2r_k - I)$,

where $I = 1$ if $r_k > 1$ and $I = 0$ if $r_k = 1$;

minimum saving occurs when all $r_k = 1$ and then is $2 \cdot NP \cdot NZ$.

*'s : $\frac{5}{2} \sum_{k=1}^{NP} r_k(r_k-1) + (5 \cdot NZ + 2) \sum_{k=1}^{NP} r_k$;

minimum saving when all $r_k = 1$ is $NP(5 \cdot NZ + 2)$.

/'s : $\frac{1}{2} \sum_{k=1}^{NP} r_k(r_k-1) + (NZ + 1) \sum_{k=1}^{NP} r_k$;

minimum saving when all $r_k = 1$ is $NP(NZ + 1)$.

Total operations saved:

$$4 \sum_{k=1}^{NP} r_k(r_k-1) + (8 \cdot NZ + 3) \sum_{k=1}^{NP} r_k - NZ \cdot NPG1 ,$$

where NPG1 is the number of poles evaluated with r_k 1 .

If all poles are order 1, then $NP(8 \cdot NZ + 3)$ additions, subtractions, multiplications and divisions are saved. If all poles are order $r' > 1$, then $NP(NZ(8r' - 1) + 3r')$ operations are saved. For large NZ compared to r', the total number of operations saved by the new formulation is of the order $NP \cdot NZ(8r' - 1)$.

If all poles are order r, the number of operations saved will increase linearly with r, but savings will be a decreasing percentage of total operations, which increase as r^2. For example, with $q + p = 20$ and $NZ = 100$, the percent savings for +'s, *'s and /'s combined for $r = 1,2,5,10$ are 62.4%, 56.5%, 40.2%, 25.7%, respectively.

APPENDIX B

COMPUTER PROGRAM

APPENDIX B: COMPUTER PROGRAM - B.1. FORTRAN LISTING

```
C...THIS PROGRAM MAY BE USED TO EVALUATE A GIVEN H-FUNCTION
C...OR TO DETERMINE THE PROBABILITY DENSITY FUNCTION (P.D.F.)
C...AND THE CUMULATIVE DISTRIBUTION FUNCTION (C.D.F.) FOR
C...THE SUM OF ANY NUMBER OF TERMS INVOLVING ANY COMBINATION
C...OF PRODUCTS, QUOTIENTS, AND RATIONAL POWERS OF ANY NUMBER
C...OF INDEPENDENT RANDOM VARIABLES WITH H-FUNCTION DISTRIBUTIONS.
C
C
C...IVY D. COOK, JR., PH.D. DISSERTATION, MAY, 1981
C...THE UNIVERSITY OF TEXAS AT AUSTIN
C
C
C...INPUT DATA CARDS (FREE FORMAT)
C......CARD 1: ZO,ZN,DZ,NS,IDT,MP,MI,PCT,NY,NP
C
C    ZO = FIRST POINT FOR EVALUATION, MUST BE A NON-NEGATIVE
C         REAL VALUE OR THE PROGRAM WILL GIVE A DEFAULT VALUE
C         OF ZO=DZ.  INPUT OF ZO=0.0 IS PERMITTED FOR USER
C         CONVENIENCE, BUT FIRST Z VALUE EVALUATED WILL BE DZ.
C    ZN = LAST POINT FOR EVALUATION, MUST BE A POSITIVE REAL
C         VALUE GREATER THAN ZO BY AT LEAST 1.E-10.
C    DZ = STEP SIZE, MUST BE A POSITIVE REAL VALUE NOT LESS
C         THAN 1.E-10 OR THE PROGRAM WILL GIVE A DEFAULT
C         VALUE THAT RESULTS IN 100 STEPS.  IF DZ IS SUCH
C         THAT THERE WOULD BE MORE THAN 1000 STEPS, THEN THE
C         PROGRAM LOWERS ZN TO THE VALUE FOR 1000 STEPS
C         (DUE TO DIMENSION OF PROGRAM ARRAYS).
C    NS = NUMBER OF TERMS IN THE SUM, MUST BE A POSITIVE
C         INTEGER.
C    IDT= 1, IF ALL TERMS IN THE SUM ARE IDENTICALLY DISTRIBUTED;
C         OTHERWISE, LET IDT=0 (OR ANY INTEGER VALUE EXCEPT 1).
C         IF NS=1, PROGRAM WILL SET IDT=0.
C    MP = MAXIMUM NUMBER OF POLES TO BE EVALUATED, MUST BE A
C         POSITIVE INTEGER.  IF MP IS GREATER THAN 100, THE
C         PROGRAM WILL GIVE A DEFAULT VALUE OF 100.
C    MI = NUMBER OF COMPLEX VALUES EVALUATED IN THE CRUMP METHOD
C         FOR NUMERICAL INVERSION OF A LAPLACE TRANSFORM, ANY
C         INTEGER VALUE CAN BE ENTERED IF NS=1 AND NY=1, BUT
C         MUST BE A POSITIVE INTEGER VALUE IF NS IS GREATER
C         THAN 1.  IF NS=1 AND MI IS LESS THAN 1, THE PROGRAM
C         WILL SET NY=1.  IF MI IS GREATER THAN 1001, THEN THE
C         PROGRAM WILL GIVE A DEFAULT VALUE OF 1001 (DUE TO
C         DIMENSION OF PROGRAM ARRAYS).
C    PCT= PROPORTION OF MAXIMUM Z VALUE TO BE USED TO FIND THE
C         VALUE FOR THE CRUMP AXIS POINT, MUST BE A POSITIVE
C         REAL VALUE BETWEEN 0.05 AND 2.1 OR THE PROGRAM WILL
C         GIVE A DEFAULT VALUE OF 1.0.
C    NY = 1, IF THE P.D.F. AND C.D.F. FOR EACH TERM OF THE SUM
C         ARE DESIRED; OTHERWISE, LET NY=0 (OR ANY INTEGER
C         VALUE EXCEPT 1).
C
C         FOR MI GREATER THAN 0 AND NS=1, THE PROGRAM EVALUATES
C         THE H-FUNCTION BY SUMMATION OF RESIDUES IF NY=1 AND
C         BY THE CRUMP METHOD FOR NUMERICAL INVERSION OF A
C         LAPLACE TRANSFORM IF NY IS NOT 1 (WHERE THE LAPLACE
C         TRANSFORM VALUES ARE FOUND BY SUMMATION OF RESIDUES).
C
C    NP = 1, IF PLOTS OF THE P.D.F. AND THE C.D.F. ARE DESIRED;
C         OTHERWISE, LET NY=0 (OR ANY INTEGER VALUE EXCEPT 1).
```

```
C
C......CARD 2: NLT(IS),IS=1,NS
C
C   NLT(IS) = THE NUMBER OF THE LAST VARIATE (COUNTING ANY
C             CONSTANTS) IN THE IS-TH TERM OF THE SUM,
C             MUST BE POSITIVE INTEGERS SUCH THAT
C             NLT(IS) IS GREATER THAN NLT(IS-1),IS=2,NS.
C
C             IF IDT=1, ONLY NLT(1) NEEDS TO BE ENTERED.
C
C......CARD 3 TO CARD NLT(NS)+2: NV,THETA,PHI,POW
C
C   NV = TYPE OF VARIATE (SEE BELOW), MUST BE A POSITIVE
C        INTEGER FROM 1 TO 14.
C   THETA = VARIATE PARAMETER, MUST BE A POSITIVE REAL VALUE
C        GREATER THAN 1.E-10 UNLESS NV=4 OR NV=5.
C   PHI= VARIATE PARAMETER, MUST BE A NON-NEGATIVE REAL VALUE
C        AND, IF NV= 2, 5, 7, OR 10, MUST BE GREATER THAN
C        1.E-10.
C   POW= POWER TO WHICH VARIATE IS TO BE RAISED, MUST BE A
C        POSITIVE OR A NEGATIVE (QUOTIENT) NON-ZERO REAL
C        VALUE.  IF THE MAGNITUDE OF POW IS LESS THAN 1.E-10,
C        THE PROGRAM WILL GIVE A DEFAULT VALUE OF 1.0.
C
C...TYPES OF VARIATES
C
C   NV = 1, RAYLEIGH VARIATE (WEIBULL WITH PHI=2)
C
C      PDF(X) = 2 * THETA * X * EXP(- THETA * X**2)
C
C   NV = 2, WEIBULL VARIATE
C
C      PDF(X) = THETA * PHI * X**(PHI-1) * EXP(- THETA * X**PHI)
C
C   NV = 3, CONSTANT THETA
C
C   NV = 4, H-FUNCTION VARIATE
C
C      PDF(X) = THETA * H(PHI * X)
C
C      IF NV=4, THE REMAINING H-FUNCTION PARAMETERS MUST BE
C      ENTERED AS ADDITIONAL CARDS:
C         EXTRA CARD 1: M,N,P,Q (NON-NEGATIVE INTEGERS)
C            P NOT LESS THAN N, Q NOT LESS THAN M,
C            M+N AND P+Q NOT ZERO AND NOT GREATER THAN 20
C            (DUE TO DIMENSION OF PROGRAM ARRAYS, IN FACT THE
C            SUM OF P+Q FOR ALL VARIATES IN A TERM MUST NOT
C            EXCEED 20).
C         EXTRA CARD 2: (A(I),GA(I)),I=1,P
C            ALL REAL, GA(I) POSTTIVE.
C            DELETE THIS CARD IF P=0.
C         EXTRA CARD 3: (B(I),GB(I)),I=1,Q
C            ALL REAL, GB(I) POSITIVE.
C            DELETE THIS CARD IF Q=0.
C
C      FOR NV=4, IF DESIRED, ENTER THETA=0 AND THE PROGRAM
C      WILL COMPUTE THE H-FUNCTION DISTRIBUTION CONSTANT
C
C      FOR NV=4, A NEGATIVE THETA MAY BE ENTERED.
```

```
C
C   NV = 5, EXPONENTIAL VARIATE (GAMMA WITH THETA=1)
C
C      PDF(X) = (1/PHI) * EXP(- X / PHI)
C
C   NV = 6, CHI-SQUARE VARIATE WITH THETA DEGREES OF FREEDOM
C          (GAMMA WITH PHI=2 AND THETA=THETA/2)
C
C   NV = 7, GAMMA VARIATE
C
C      PDF(X) = X**(THETA-1) * EXP(- X / PHI)
C               /(PHI**THETA * GAMMA(THETA))
C
C   NV = 8, UNIFORM VARIATE
C
C      PDF(X) = 1/THETA, FOR X IN (0,THETA); = 0, OTHERWISE.
C
C   NV = 9, BETA VARIATE
C
C      PDF(X) = X**(THETA-1) * (1-X)**(PHI-1) / BETA(THETA,PHI)
C               FOR X IN (0,1); = 0, OTHERWISE.
C
C      FOR A BETA VARIATE ON (0,K), USE THE PRODUCT OF AN NV=9
C      AND AN NV=3 WITH THETA=1/K.
C
C   NV = 10, F DISTRIBUTION WITH DEGREES OF FREEDOM (THETA,PHI)
C
C   NV = 11, MAXWELL VARIATE
C
C      PDF(X) = 4 * X**2 * EXP(- (X/THETA)**2)
C               /(THETA**3 * SQRT(PI))
C
C   NV = 12, HALF-NORMAL VARIATE
C
C      PDF(X) = 2 * EXP(- (X/THETA)**2 /2) / (THETA * SQRT(2 * PI))
C
C   NV = 13, HALF-CAUCHY VARIATE
C
C      PDF(X) = 2 * THETA / (PI * (THETA**2 + X**2))
C
C   NV = 14, HALF-STUDENT VARIATE
C
C      PDF(X) = 2 * GAMMA(THETA + .5) / ( SQRT(2 * THETA * PI)
C               * GAMMA(THETA) * (1 + X**2/(2*THETA))**(THETA + .5) )

C
C.......
C...
C
      COMMON/CHPDF/BA(21),CD(21),GBA(21),GCD(21),CA,IT,LR,M,MN,QP
      COMMON/CHONLY/IP,IQ,N,NAY,TR
      COMMON/PDF1/HPDF(1001),HCDF(1001),ZK1(1001)
      COMMON/PDF2/CC,CN,DZ,KEY,KPM,MI,MP,NP,ZM,PSI1(20)
      DIMENSION NLI(50),XL(21),XG(21),TLR(1001),TLI(1001),GC1(21)
      DIMENSION A(21),B(21),GA(21),GB(21),A1(21),C1(21),GA1(21)
      DIMENSION IFM(190)
      INTEGER P,PP,Q,QP,QQ
```

```
C
C...INPUT AND CHECK THE PROBLEM LIMITS AND REQUIREMENTS.
C
      READ(5,    ) ZO,ZN,DZ,NS,IDT,MP,MI,PCT,NY,NP
      IF (NS.LT.1.OR.MP.LT.1) GO TO 46
      IF (NS.GT.1.AND.MI.LT.1) GO TO 46
      IF (NS.EQ.1.AND.MI.LT.1) NY=1
      IF (MP.GT.100) MP=100
      IF (MI.GT.1001) MI=1001
      IF (NS.EQ.1) IDT=0
      IF (ZO.LT.0.0) ZO=0.0
      ZT=ZN-ZO
      IF (ZT.LT.1.E-10) GO TO 48
      IF (DZ.LT.1.E-10) DZ=(ZN-ZO)/1.E+2
      ZT=ZO+DZ*1.E+3
      IF (ZT.LT.ZN) ZN=ZT
      IF (PCT.LT.0.05.OR.PCT.GT.2.1) PCT=1.0
      T=PCT*ZN
      C=ALOG(2.E+8)/(2.0*T)
      ZC=3.14159265358979/T
      IF (ZO.LT.1.E-10) ZO=DZ
      WRITE(6,910) ZO,ZN,DZ,NS
  910 FORMAT(1H1 ,//,*  DETERMINE P.D.F.(Z) AND C.D.F.(Z)*,/,
     1*  FOR VALUES OF Z FROM *,F8.4,* TO *,F8.4,* WITH STEP SIZE *,
     2F8.4,/,*  FOR THE SUM OF *,I2,* TERMS, WHERE*,//)
      IF (IDT.EQ.1) WRITE(6,912)
  912 FORMAT(44H  THE TERMS ARE IDENTICALLY DISTRIBUTED, AND )
      WRITE(6,915) MP,MI,PCT,C
  915 FORMAT(//,*  THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS *,
     1I4,*.*,//,*  CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = *,
     2I4,*,*,//,7X,*PERCENT OF HIGHEST Z VALUE = *,F4.2,
     3*, AXIS POINT A = *,F8.4,//)
C
C...INPUT THE NUMBER OF THE LAST ELEMENT FOR EACH TERM OF THE SUM
C...AND CHECK FOR AN ASCENDING ORDER.
C
      NT=NS
      IF (IDT.NE.1) GO TO 8
      NT=1
      WRITE(6,    ) 41H FORM FOR EACH TERM (WHERE YJ = XJ**PJ):
      GO TO 9
    8 WRITE(6,    ) 47H FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
    9 READ(5,    ) (NLT(I),I=1,NT)
      IF (NLT(1).LT.1) GO TO 50
      IF (NS.EQ.1.OR.IDT.EQ.1) GO TO 13
      DO 10 IS=2,NS
         NLTL=NLT(IS)
         NLTS=NLT(IS-1)
   10    IF (NLTS.GE.NLTL) GO TO 50
   13 IS=1
      DO 6 J=1,5
         J1=J-1
         IJ=10*J1
      DO 5 I=1,10
         I1=I-1
         IF (IJ.EQ.0) GO TO 5
         IF (IJ.GT.NLT(NT)) GO TO 7
```

```
            IF (J1.EQ.0) IE=3*IJ
            IF (J1.GT.0) IE=(4*IJ)-9
            IF (IJ.LT.NLT(IS)) GO TO 2
            IF (IJ.EQ.NLT(NT)) IL=IE-1
            IFM(IE)=1H+
            IS=IS+1
            GO TO 3
    2 IFM(IE)=1H*
    3 IE=IE-1
      IFM(IE)=I1
      IF (J1.EQ.0) GO TO 4
      IE=IE-1
      IFM(IE)=J1
    4 IE=IE-1
      IFM(IE)=1HY
    5 IJ=IJ+1
    6 CONTINUE
      IF (NLT(NT).LT.50) GO TO 7
      IL=190
      IFM(190)=0
      IFM(189)=5
      IFM(188)=1HY
    7 WRITE(6,920) (IFM(I),I=1,IL)
  920 FORMAT(/,*  Z = *,9(A1,I1,A1),16(A1,2I1,A1),//,6X,25(A1,2I1,A1))
C
C...INITIALIZE THE OVERALL PROBLEM CONSTANT AND RANGE PARAMETERS.
C
      CNF=1.0
      KPM=1
      NF=1
      KR=0
      RU=0.0
      RL=0.0
C
C...FIND THE PSI FUNCTION VALUES TO BE USED
C...WHEN THE ORDER OF A POLE IS MORE THAN 2.
C
      DO 15 I=2,20,2
   15    PSI1(I)=2.0*PSI(I-1,1.0)
      DO 44 IS=1,NS
C
C...INITIALIZE THE PARAMETERS FOR THE IS-TH TERM OF THE SUM.
C
      NL=NLT(IS)
      CC=1.0
      CN=1.0
      M=0
      N=0
      P=0
      Q=0
      LM=0
      LN=0
      LP=0
      LQ=0
      NAY=0
```

```
      DO 144 NH=NF,NL
C
C...INPUT THE H-FUNCTION PARAMETERS FOR THE NH-TH VARIATE,
C...AND SET UP THIS H-FUNCTION.
C
      READ(5,    ) NV,THETA,PHI,POW
      IF (ABS(POW).LT.1.E-10) POW=1.0
      WRITE(6,930) NH,NV,THETA,PHI,POW
  930 FORMAT(//,*  VARIATE X*,I2,* IS TYPE NUMBER *,I2,//,
     1*  INPUT PARAMETERS ARE THETA =*,F10.5,*, PHI =*,F10.5,
     2*, AND POWER =*,F10.5)
      IF (THETA.LT.1.E-10.AND.NV.LT.4) GO TO 148
      IF (THETA.LT.1.E-10.AND.NV.GT.5) GO TO 148
      IF (PHI.LE.-1.E-10) GO TO 148
      A(1)=1.E+3
      GA(1)=1.E+3
      MM=1
      NN=0
      PP=0
      QQ=1
      GO TO (102,104,132,134,80,80,80,80,80,80,80,80,80,80),NV
   80 GB(1)=1.0
      NV=NV-4
      GO TO (108,110,112,116,118,122,90,90,90,90),NV
   90 B(1)=0.0
      GB(1)=0.5
      NV=NV-6
      GO TO (124,126,100,100),NV
  100 NN=1
      PP=1
      GA(1)=0.5
      NV=NV-2
      GO TO (128,130),NV
  102 PHI=2.0
      GO TO 106
  104 IF (PHI.LT.1.E-10) GO TO 148
  106 GB(1)=1.0/PHI
      B(1)=1.0-GB(1)
      TN=THETA**GB(1)
      TC=TN
      GO TO 136
  108 IF (PHI.LT.1.E-10) GO TO 148
      TN=1.0/PHI
      TC=TN
      B(1)=0.0
      GO TO 136
  110 THETA=THETA*0.5
      PHI=2.0
      GO TO 114
  112 IF (PHI.LT.1.E-10) GO TO 148
  114 TC=1.0/PHI
      TN=TC/DGAMMA(THETA)
      B(1)=THETA-1.0
      GO TO 136
  116 TN=1.0/THETA
      TC=TN
      A(1)=1.0
      B(1)=0.0
      GO TO 120
```

```
  118 TN=DGAMMA(THETA+PHI)/DGAMMA(THETA)
      TC=1.0
      A(1)=THETA+PHI-1.0
      B(1)=THETA-1.0
  120 GA(1)=1.0
      PP=1
      GO TO 136
  122 IF (PHI.LT.1.E-10) GO TO 148
      TC=THETA/PHI
      TN=TC/(DGAMMA(THETA)*DGAMMA(PHI))
      NN=1
      PP=1
      A(1)=-PHI
      GA(1)=1.0
      B(1)=THETA-1.0
      GO TO 136
  124 TC=1.0/THETA
      TN=TC/.886226925452758
      B(1)=1.0
      GO TO 136
  126 TN=.398942280401433/THETA
      TC=.707106781186548/THETA
      GO TO 136
  128 TN=.318309886183791/THETA
      TC=1.0/THETA
      A(1)=0.0
      GO TO 136
  130 TN=(THETA*6.28318530717958)**(-0.5)/DGAMMA(THETA)
      TC=(2.0*THETA)**(-0.5)
      A(1)=0.5-THETA
      GO TO 136
  132 CN=CN/THETA
      CC=CC/THETA
      GO TO 144
C
C...IF NV=4, INPUT THE REMAINING H-FUNCTION PARAMETERS.
C
  134 READ(5,    ) MM,NN,PP,QQ
      IF (MM.LT.0.OR.NN.LT.0) GO TO 149
      IF (QQ.LT.MM.OR.PP.LT.NN) GO TO 150
      IF (PP.LT.1.AND.QQ.LT.1) GO TO 152
      IF (PP.LT.1) READ(5,    ) (B(I),GB(I),I=1,QQ)
      IF (QQ.LT.1) READ(5,    ) (A(I),GA(I),I=1,PP)
      IF (PP.GT.0.AND.QQ.GT.0) READ(5,    )
     1   (A(I),GA(I),I=1,PP),(B(I),GB(I),I=1,QQ)
      TN=THETA
      TC=PHI
C
C...FOR NV=4 AND THETA=0, THE PROGRAM COMPUTES
C...THE CONSTANT FOR THE H-FUNCTION DISTRIBUTION.
C
      IF (ABS(TN).GT.1.E-10) GO TO 136
      TN=TC
      IF (QQ.LT.1) GO TO 730
      DO 720 I=1,QQ
```

```
         IF (I.GT.MM) GO TO 710
         TN=TN/DGAMMA(B(I)+GB(I))
         GO TO 720
  710    TN=TN*DGAMMA(1.0-B(I)-GB(I))
  720 CONTINUE
  730 IF (PP.LT.1) GO TO 136
      DO 750 I=1,PP
         IF (I.GT.NN) GO TO 740
         TN=TN/DGAMMA(1.0-A(I)-GA(I))
         GO TO 750
  740    TN=TN*DGAMMA(A(I)+GA(I))
  750 CONTINUE
C
C...PRINT THE H-FUNCTION FOR THE NH-TH VARIATE
C
  136 WRITE(6,940) NH
  940 FORMAT(//,*  THE P.D.F. FOR VARIATE X*,I2,* IS GIVEN BY:*,/)
      IF (PP.EQ.0) WRITE(6,947) MM,NN
      IF (PP.GT.0) WRITE(6,945) MM,NN,((A(I),GA(I)),I=1,PP)
      WRITE(6,942) TN,TC
  942 FORMAT(F10.5,2X,*H*,7X,*(*,F10.5,* X):*)
      IF (QQ.EQ.0) WRITE(6,947) PP,QQ
      IF (QQ.GT.0) WRITE(6,945) PP,QQ,((B(I),GB(I)),I=1,QQ)
  945 FORMAT(14X,2I3,15X,(* (*,F8.3,*,*,F8.3,*)*))
  947 FORMAT(14X,2I3)
      IF (ABS(TN).LT.1.E-10.OR.TC.LT.1.E-10) GO TO 153
C
C...BASED UPON THE POWER TO WHICH THE NH-TH VARIATE IS RAISED,
C...ITS H-FUNCTION PARAMETERS ARE ADJUSTED AND USED TO FIND
C...THE PARAMETERS OF THE H-FUNCTION DISTRIBUTION OF THE IS-TH TERM.
C
      PS1=POW-1.0
      IF (ABS(PS1).LT.1.E-10) GO TO 140
      TN=TN*TC**(PS1)
      TC=TC**POW
  140 CN=CN*TN
      CC=CC*TC
      IF (POW.LT.0.0) GO TO 142
      CALL SETUP(0,MM,M,LM,1.0,POW,B,GB,BA,GBA)
      CALL SETUP(MM,QQ,Q,LQ,-1.0,POW,B,GB,CD,GCD)
      CALL SETUP(0,NN,N,LN,-1.0,POW,A,GA,A1,GA1)
      CALL SETUP(NN,PP,P,LP,1.0,POW,A,GA,C1,GC1)
      GO TO 144
  142 CALL SETUP(0,MM,N,LN,1.0,POW,B,GB,A1,GA1)
      CALL SETUP(MM,QQ,P,LP,-1.0,POW,B,GB,C1,GC1)
      CALL SETUP(0,NN,M,LM,-1.0,POW,A,GA,BA,GBA)
      CALL SETUP(NN,PP,Q,LQ,1.0,POW,A,GA,CD,GCD)
  144 CONTINUE
      IP=P-N
      IQ=Q-M
      IF (IP.NE.LP) WRITE(6,    )  17H  SETUP ERROR, LP
      IF (IQ.NE.LQ) WRITE(6,    )  17H  SETUP ERROR, LQ
      MN=M
      QP=IQ
      CALL SETUP(0,N,MN,LM,1.0,1.0,A1,GA1,BA,GBA)
      CALL SETUP(0,IP,QP,LQ,1.0,1.0,C1,GC1,CD,GCD)
```

```
C
C...PRINT THE H-FUNCTION FOR THE IS-TH TERM
C
      WRITE(6,950) IS
  950 FORMAT(//,*  THE P.D.F. FOR TERM *,I2,
     1* OF THE SUM IS GIVEN BY:*,/)
      WRITE(6,951) M,N,CN,CC,P,Q
  951 FORMAT(14X,2I3,/,F10.5,2X,*H*,7X,*(*,F10.5,* Z), WHERE*,
     1/,14X,2I3,/)
      IF (MN.GT.0) WRITE(6,952) ((BA(I),GBA(I)),I=1,MN)
  952 FORMAT(* (BA(I),GBA(I)):*,(6(* (*,F8.3,*,*,F8.3,*)*),/,16X))
      IF (QP.GT.0) WRITE(6,953) ((CD(I),GCD(I)),I=1,QP)
  953 FORMAT(* (CD(I),GCD(I)):*,(6(* (*,F8.3,*,*,F8.3,*)*),/,16X))
      GO TO 156
  148 WRITE(6,    ) 27H  THETA OR PHI NOT POSITIVE
      GO TO 52
  149 WRITE(6,    ) 17H  M OR N NEGATIVE
      GO TO 52
  150 WRITE(6,    ) 27H  Q(OR P) LESS THAN M(OR N)
      GO TO 52
  152 WRITE(6,    ) 30H  PARAMETERS P AND Q BOTH ZERO
      GO TO 52
  153 WRITE(6,    ) 27H  ZERO OR NEGATIVE CONSTANT
      GO TO 52
C
C...CHECK THAT THE H-FUNCTION IS VALIDLY DEFINED, DETERMINE ITS
C...CONVERGENCE TYPE, AND, IF NEEDED, ADJUST THE RANGE PARAMETERS.
C
  156 CALL CHECK
      IF (NAY.EQ.1) GO TO 52
      ZT=1.0/(CC*TR)
      IF (LR.NE.0) GO TO 157
      IF (N.LT.1) RU=RU+ZT
      IF (N.LT.1) KR=KR+1
      IF (M.LT.1) RL=RL+ZT
C
C...IF DESIRED, THE P.D.F. AND C.D.F. OF THE IS-TH TERM ARE FOUND.
C...IF NS=1 AND NY=0, THESE ARE FOUND BY LAPLACE TRANSFORM INVERSION.

C
  157 IF (NY.NE.1) GO TO 18
      ZM=ZT
      KEY=0
      IF (NS.NE.1) GO TO 12
      IF (LR.NE.0) GO TO 11
      IF (N.LT.1.AND.ZT.LT.ZN) ZN=ZT
      IF (M.LT.1.AND.ZT.GT.ZO) ZO=ZT
      IF (ZN.LT.ZO) GO TO 48
   11 CALL PDFCDF(ZO,ZN)
      GO TO 52
   12 IF (LR.EQ.0.AND.N.LT.1) GO TO 14
      IF (LR.EQ.0.AND.M.LT.1) GO TO 16
      ZT=DZ*1.E+3
   14 IF (ZN.LT.ZT) ZT=ZN
      CALL PDFCDF(DZ,ZT)
      GO TO 18
   16 IF (ZO.GT.ZT) ZT=ZO
      IF (ZN.LT.ZT) GO TO 18
      CALL PDFCDF(ZT,ZN)
```

```
C
C...SET UP THE LAPLACE TRANSFORM H-FUNCTION FOR THE IS-TH TERM.
C
   18 JD=N+1
      IF (MN.LT.1) GO TO 28
      DO 20 I=1,MN
         XL(I)=BA(I)
   20    XG(I)=GBA(I)
      IF (N.LT.1) GO TO 24
      DO 22 I=2,JD
         J=M+I-1
         BA(I)=XL(J)+XG(J)
   22    GBA(I)=-XG(J)
   24 IF (M.LT.1) GO TO 28
      DO 26 I=1,M
         J=JD+I
         BA(J)=XL(I)+XG(I)
   26    GBA(J)=-XG(I)
   28 BA(1)=0.0
      GBA(1)=1.0
      IF (QP.LT.1) GO TO 38
      DO 30 I=1,QP
         XL(I)=CD(I)
   30    XG(I)=GCD(I)
      IF (IP.LT.1) GO TO 34
      DO 32 I=1,IP
         J=IQ+I
         CD(I)=XL(J)+XG(J)
   32    GCD(I)=-XG(J)
   34 IF (IQ.LT.1) GO TO 38
      DO 36 I=1,IQ
         J=IP+I
         CD(J)=XL(I)+XG(I)
   36    GCD(J)=-XG(I)
   38 N=M
      M=JD
      JD=IP
      IP=IQ
      IQ=JD
      P=N+IP
      Q=M+IQ
      MN=MN+1
      CN=CN/CC
      CC=1.0/CC
      WRITE(6,954)
  954 FORMAT(1H1 ,* LAPLACE TRANSFORM:*,/)
      WRITE(6,951) M,N,CN,CC,P,Q
      IF (MN.GT.0) WRITE(6,952) ((BA(I),GBA(I)),I=1,MN)
      IF (QP.GT.0) WRITE(6,953) ((CD(I),GCD(I)),I=1,QP)
      CALL CHECK
      IF (NAY.EQ.1) GO TO 52
      CNF=CNF*CN
      ZM=1.0/(CC*IR)
C
C...USE PDFCDF TO FIND THE MI VALUES OF THE LAPLACE TRANSFORM
C...OF THE IS-TH TERM:  HPDF(I) HAS THE REAL PARTS AND
C...HCDF(I) HAS THE IMAGINARY PARTS OF THE TRANSFORM VALUES.
C
```

```
      KEY=1
      CALL PDFCDF(C,ZC)
C
C...FIND THE PRODUCT OF THE LAPLACE TRANSFORMS AT EACH VALUE.
C
      LK=1
      IF (IDT.EQ.1) LK=NS-1
      DO 42 I=1,MI
         IF (IS.NE.1) GO TO 40
         TLR(I)=HPDF(I)
         TLI(I)=HCDF(I)
         IF (IDT.NE.1) GO TO 42
   40    DO 41 J=1,LK
         Y=TLR(I)
         TLR(I)=Y*HPDF(I)-TLI(I)*HCDF(I)
   41    TLI(I)=TLI(I)*HPDF(I)+Y*HCDF(I)
   42 CONTINUE
      IF (IDT.EQ.1) GO TO 45
   44 NF=NL+1
C
C...CRUMP METHOD FOR NUMERICAL INVERSION OF A LAPLACE TRANSFORM
C
   45 IF (KR.EQ.NS.AND.RU.LT.ZN) ZN=RU
      IF (ZO.LT.RL) ZO=RL
      IF (IDT.NE.1) GO TO 43
      IF (KR.EQ.1.AND.RU.LT.ZN) ZN=RU
      CNF=CNF**NS
   43 CNF=CNF/T
      K=1
      WRITE(6,955) KPM
  955 FORMAT(38H1 MAXIMUM NUMBER OF POLES EVALUATED =  ,I5,//,
     1*        Z          PDF(Z)        CDF(Z)*,/)
  602 ZK=ZO+FLOAT(K-1)*DZ
      IF (ZK.GT.ZN) GO TO 606
      HPDF(K)=0.5*TLR(1)
      HCDF(K)=HPDF(K)/C
      IF (MI.EQ.1) GO TO 605
      DO 604 I=2,MI
         ZI=ZC*FLOAT(I-1)
         Z1=COS(ZI*ZK)
         Z2=SIN(ZI*ZK)
         T1=TLR(I)
         T2=TLI(I)
         HPDF(K)=HPDF(K)+T1*Z1-T2*Z2
         HCDF(K)=HCDF(K)+((T1*C+T2*ZI)*Z1+(T1*ZI-T2*C)*Z2)/(C*C+ZI*ZI)

  604 CONTINUE
  605 CNFE=CNF*EXP(C*ZK)
      HPDF(K)=HPDF(K)*CNFE
      HCDF(K)=HCDF(K)*CNFE
      ZK1(K)=ZK
      WRITE(6,960) ZK1(K),HPDF(K),HCDF(K)
  960 FORMAT(F11.4,2F12.6)
      IF (HPDF(K).LT.0.0) HPDF(K)=-HPDF(K)
      IF (HCDF(K).LT.0.0) HCDF(K)=-HCDF(K)
      K=K+1
      GO TO 602
```

```
606 IF (NP.NE.1) GO TO 52
    K=K-1
    WRITE(6,961)
961 FORMAT(1H1 ,*              PROBABILITY DENSITY FUNCTION*,/)
    CALL PLOT(ZK1,HPDF,K)
    WRITE(6,962)
962 FORMAT(1H1 ,*              CUMULATIVE DISTRIBUTION FUNCTION*,/)

    CALL PLOT(ZK1,HCDF,K)
    GO TO 52
 46 WRITE(6,    ) 27H  NS,MP,OR MI LESS THAN ONE
    GO TO 52
 48 WRITE(6,    ) 42H  LAST Z-VALUE LESS THAN OR EQUAL TO FIRST
    GO TO 52
 50 WRITE(6,    ) 34H  NLT(I+1) NOT GREATER THAN NLT(I)
 52 STOP
    END
```

```
      SUBROUTINE SETUP(I1,J1,K1,L1,S1,PWR,E,GE,F,GF)
C
C...ADJUST AND ORDER THE H-FUNCTION PARAMETERS FOR THE IS-TH TERM
C...INTO ARRAYS THAT ARE CONVENIENT FOR CALCULATING RESIDUES.
C
      DIMENSION E(21),GE(21),F(21),GF(21)
      I2=I1+1
      J2=J1
      K1=K1+J2
      IF (J2.LT.I2) GO TO 160
      S2=S1
      S3=1.0-S2
      S4=S2*PWR
      PM1=PWR-1.0
      DO 158 II=I2,J2
         L1=L1+1
         F(L1)=S3/2.0 + S2*(E(II)-GE(II)*PM1)
  158    GF(L1)=S4*GE(II)
  160 RETURN
      END
```

```
      SUBROUTINE CHECK
C
C...CHECK FOR VALIDLY DEFINED H-FUNCTION AND FIND CONVERGENCE TYPE
C
      COMMON/CHPDF/BA(21),CD(21),GBA(21),GCD(21),CA,IT,LR,M,MN,QP
      COMMON/CHONLY/IP,IQ,N,NAY,TR
      INTEGER QP
      IT=0
      LR=0
      TD1=0.0
      TD2=0.0
      TL=FLOAT(QP-MN)/2.0
      TR=1.0
      CA=1.E+3
      CB=-1.E+3
      IF (N.LT.1) GO TO 204
      J=M+1
      DO 202 I=J,MN
         G=-GBA(I)
         IF (G.LT.1.E-10) GO TO 228
         CH=BA(I)/G
         IF (CH.LT.CA) CA=CH
         TD1=TD1+G
         TL=TL+BA(I)
  202    TR=TR*(G**G)
  204 IF (M.LT.1) GO TO 208
      DO 206 I=1,M
         G=GBA(I)
         IF (G.LT.1.E-10) GO TO 228
         CH=-BA(I)/G
         IF (CH.GT.CB) CB=CH
         TD2=TD2+G
         TL=TL+BA(I)
  206    TR=TR/(G**G)
  208 IF (IP.LT.1) GO TO 212
      J=IQ+1
      DO 210 I=J,QP
         G=GCD(I)
         IF (G.LT.1.E-10) GO TO 228
         TD2=TD2-G
         TL=TL-CD(I)
  210    TR=TR*(G**G)
  212 IF (IQ.LT.1) GO TO 216
      DO 214 I=1,IQ
         G=-GCD(I)
         IF (G.LT.1.E-10) GO TO 228
         TD1=TD1-G
         TL=TL-CD(I)
  214    TR=TR/(G**G)
  216 TD=TD1+TD2
      TP=TD1-TD2
```

```
      IF (TD.GE.-1.E-10) GO TO 218
      WRITE(6,    ) 37H  D IS LESS THAN ZERO, NO CONVERGENCE
      GO TO 232
  218 IF (CA.GT.CB) GO TO 220
      WRITE(6,    ) 38H  NUMERATOR POLES NOT PROPERLY DIVIDED
      GO TO 232
  220 IF (TP.LE.-1.E-10) GO TO 222
      IF (TP.GE.1.E-10) GO TO 224
      CH=0.0
      IT=5
      GO TO 226
  222 CH=TP*CA
      IT=1
      LR=-1
      GO TO 226
  224 CH=TP*CB
      IT=3
      LR=1
  226 IF (TD.LT.1.E-10.AND.TL.GE.CH) GO TO 230
      IF (TL.LT.CH) IT=IT+1
      IF (IT.EQ.6.AND.TL.LT.-1.0) IT=7
      GO TO 234
  228 WRITE(6,    ) 38H  ALPHA OR BETA PARAMETER NOT POSITIVE
      GO TO 232
  230 WRITE(6,    ) 33H  D IS ZERO AND L IS GREATER THAN
      WRITE(6,    ) 29H  E TIMES W , NO CONVERGENCE
      IT=0
  232 NAY=1
  234 WRITE(6,965) IT
  965 FORMAT(//,*  CONVERGENCE TYPE = *,I1)
      WRITE(6,970) TD,TP,TL,TR
  970 FORMAT(/,7X,*D =*,F6.2,5X,*E =*,F6.2,5X,*L =*,F6.2,
     15X,*R =*,F7.4)
      RETURN
      END
```

```
      SUBROUTINE PDFCDF(ZFP,ZLP)
C
C...CALCULATION OF H-FUNCTION VALUES BY SUMMATION OF RESIDUES:
C...THE P.D.F. AND C.D.F. ARE FOUND IF KEY=0. THE REAL AND
C...IMAGINARY PARTS OF THE LAPLACE TRANSFORM IF KEY=1.
C
      COMMON/CHPDF/BA(21),CD(21),GBA(21),GCD(21),CA,IT,LR,M,MN,QP
      COMMON/PDF1/HPDF(1001),HCDF(1001),ZK1(1001)
      COMMON/PDF2/CC,CN,DZ,KEY,KPM,MI,MP,NP,ZM,PSI1(20)
      DIMENSION ID(21),JS(21),PL(21),V(22,2),W(21,2),ZLN(1001)
      DIMENSION GARGN(21),GARGD(21),ZK2(1001)
      INTEGER QP,ERROR
      ZF=ZFP
      ZL=ZLP
      DO 301 I=1,MN
         ID(I)=0
         JS(I)=0
  301    PL(I)=0.0
      K2=2-KEY
      KKM=0
      KS1=0
      KF=1
      KMX=KEY*MI+(1-KEY)*(INT((ZL-ZF)/DZ+1.E-10)+1)
      KL=KMX
      DO 304 K=1,KL
         HPDF(K)=0.0
         HCDF(K)=0.0
         IF (KEY.EQ.1) GO TO 302
         ZK1(K)=ZF+DZ*FLOAT(K-1)
         ZK2(K)=ALOG(ZK1(K)*CC)
         GO TO 304
  302    ZIMK=ZL*FLOAT(K-1)
         ZK1(K)=SQRT(ZF*ZF+ZIMK*ZIMK)
         ZK2(K)=ATAN(ZIMK/ZF)
         ZLN(K)=ALOG(ZK1(K)*CC)
  304 CONTINUE
C
C...SETUP FOR LHP OR RHP EVALUATION
C
      IF (LR.EQ.-1) GO TO 312
      IF (LR.EQ.1) GO TO 306
      IF (ZM.GT.ZF) GO TO 308
      LR=1
  306 MF=M+1
      ML=MN
      SV=-1.0
      GO TO 314
  308 IF (KEY.EQ.1) GO TO 310
      IF (ZM.LT.ZL) KL=INT((ZM-ZF)/DZ+1.E-10)+1
      IF (ZM.GE.ZL) LR=-1
      GO TO 311
  310 KM=INT(SQRT(ZM*ZM-ZF*ZF)/ZL+1.E-10)+1
      IF (KEY.EQ.1) WRITE(6,    ) 31H  AXIS POINT A IS LESS THAN ZM
      IF (KM.LT.KL) KL=KM
      IF (KM.GE.KL) LR=-1
  311 IF (MN.EQ.M) LR=-1
      IF (M.EQ.0) GO TO 360
  312 MF=1
      ML=M
      SV=1.0
```

```
C
C...FIND THE RIGHTMOST(LHP) OR LEFTMOST(RHP) UNEVALUATED NUMERATOR
C...POLE S AND ITS ORDER KN IN THE NUMERATOR
C
  314 DO 316 I=MF,ML
         JS(I)=0
  316    PL(I)=-BA(I)/ABS(GBA(I))
      KP=0
      KPZ=0
  318 KP=KP+1
      DO 319 I=1,MN
  319    ID(I)=0
      DO 326 I=MF,ML
         IF (I.EQ.MF) GO TO 322
         SMPL=S-PL(I)
         IF (ABS(SMPL).GT.1.E-10) GO TO 320
         KN=KN+1
         GO TO 324
  320    IF (SMPL.GT.1.E-10) GO TO 326
  322    S=PL(I)
         KN=1
  324    ID(KN)=I
  326 CONTINUE
      IF (LR.EQ.1) S=-S
      IF (KEY.EQ.1) GO TO 328
      SM1=S-1.0
      IF (ABS(SM1).GT.1.E-10) GO TO 328
      KN=KN+1
      KS1=1
C
C...CALCULATE VZERO(WITHOUT POWER TERM) AND THE ORDER KT OF POLE S.
C
  328 PROD2=1.0
      PROD4=1.0
      KD=0
      IF (QP.LT.1) GO TO 334
      DO 332 I=1,QP
         X=CD(I)+GCD(I)*S
         GARGD(I)=X
         IF (ERROR(X).EQ.0.AND.X.LT.0.5) GO TO 330
         PROD4=PROD4*DGAMMA(X)
         GO TO 332
  330    KD=KD+1
         IF (KD.EQ.KN) GO TO 342
         JX=INT(-X+0.1)
         PROD2=PROD2*DFACT(JX)*((-1.0)**JX)*GCD(I)
  332 CONTINUE
  334 PROD1=1.0
      PROD3=1.0
      KS=1
      IF (MN.LT.1) GO TO 339
      DO 338 I=1,MN
         IDKS=ID(KS)
         IF (I.EQ.IDKS) GO TO 336
         GARGN(I)=BA(I)+GBA(I)*S
         PROD1=PROD1*DGAMMA(GARGN(I))
         GO TO 338
  336    PROD3=PROD3*DFACT(JS(I))*((-1.0)**JS(I))*GBA(I)
         KS=KS+1
  338 CONTINUE
```

```
  339 KNP1=KN+1
      IF (KS1.EQ.1.AND.KS.NE.KN) GO TO 701
      IF (KS1.NE.1.AND.KS.NE.KNP1) GO TO 701
      GO TO 702
  701 WRITE(6,    ) 28H  ERROR IN VZERO CALCULATION
  702 VP1=PROD1*PROD2/(PROD3*PROD4)
      VP2=VP1
      IF (KEY.EQ.0.AND.KS1.NE.1) VP2=VP2/SM1
      KT=KN-KD
      II=KT-1
      DF1=DFACT(II)*SV
      DF2=DFACT(KT-2)*SV
      VMX=0.0
C
C...CALCULATE WZERO(WITHOUT LOG TERM) THROUGH W SUPERSCRIPT KT-2.
C
      IF (KT.LT.2) GO TO 416
      VZ1=1.0
      DO 414 L=1,II
         VZ1=1.0-VZ1
         LM1=L-1
         W(L,1)=0.0
         IF (MN.LT.1) GO TO 406
         KS=1
         DO 404 I=1,MN
            IDKS=ID(KS)
            IF (I.EQ.IDKS) GO TO 402
            W(L,1)=W(L,1)+(GBA(I)**L)*PSI(LM1,GARGN(I))
            GO TO 404
  402       W(L,1)=W(L,1)-((-GBA(I))**L)*PSI(LM1,FLOAT(JS(I)+1))
         IF (VZ1.GT.0.5) W(L,1)=W(L,1)+((GBA(I))**L)*PSI1(L)
            KS=KS+1
  404    CONTINUE
  406    IF (QP.LT.1) GO TO 412
         DO 410 I=1,QP
            X=GARGD(I)
            IF (ERROR(X).EQ.0.AND.X.LT.0.5) GO TO 408
            W(L,1)=W(L,1)-(GCD(I)**L)*PSI(LM1,X)
            GO TO 410
  408       W(L,1)=W(L,1)+((-GCD(I))**L)*PSI(LM1,1.0-X)
         IF (VZ1.GT.0.5) W(L,1)=W(L,1)-((GCD(I))**L)*PSI1(L)
  410    CONTINUE
  412    IF (KEY.EQ.1) GO TO 414
         W(L,2)=W(L,1)
         IF (KS1.NE.1) W(L,2)=W(L,2)+DFACT(LM1)/((1.0-S)**L)
  414 CONTINUE
      WR1=W(1,1)
      WR2=W(1,2)
C
C...ADD THE POWER TERM TO VZERO AND THE LOG TERM TO WZERO.
C...COMPUTE V SUPERSCRIPT KT-1 AND THE RESIDUE VDK.
C
  416 K1=0
      IF (ABS(ZK1(KL)-ZM).LT.1.E-5) K1=1
      KKM=KKM+(K1*KEY*KL)
      DO 340 K=KF,KL
         Z1=ZK1(K)*CC
         Z2=ZK2(K)
      V(1,1)=VP1*(Z1**(-S))
      V(1,2)=VP2*(Z1**(-S))
```

```
      IF (KEY.EQ.1) GO TO 502
      IF (KT.EQ.1) GO TO 510
      W(1,1)=WR1-Z2
      W(1,2)=WR2-Z2
      WI=0.0
      GO TO 504
  502 V(1,1)=V(1,1)*COS(-S*Z2)
      V(1,2)=V(1,2)*SIN(-S*Z2)
      IF (KT.EQ.1) GO TO 510
      W(1,1)=WR1-ZLN(K)
      WI=-Z2
  504 DO 508 I=1,II
         IP1=I+1
         V(IP1,1)=-V(I,2)*WI
         V(IP1,2)=V(I,1)*WI
         BN=1.0
         DO 506 J=1,I
            L=I-J+1
            IF (J.EQ.1.OR.I.EQ.1) GO TO 505
            BN=BN*FLOAT(L)/FLOAT(J-1)
  505       V(IP1,1)=V(IP1,1)+BN*V(L,1)*W(J,1)
  506       V(IP1,2)=V(IP1,2)+BN*V(L,2)*W(J,K2)
  508 CONTINUE
  510 IF (K.LT.KL.OR.K1.EQ.0) GO TO 512
      IF (IT.EQ.5) GO TO 341
      IF (IT.EQ.6.AND.KEY.EQ.1) GO TO 341
  512    VDK=V(KT,2)/DF1
         IF (ABS(VDK).GT.ABS(VMX)) VMX=VDK
         HCDF(K)=HCDF(K)+VDK
      IF (K.LT.KL.OR.K1.EQ.0) GO TO 514
      IF (IT.EQ.6) GO TO 341
  514    VDK=0.0
         IF (KS1.NE.1) VDK=V(KT,1)/DF1
         IF (KS1.EQ.1.AND.KT.GT.1) VDK=V(II,1)/DF2
         IF (ABS(VDK).GT.ABS(VMX)) VMX=VDK
  340    HPDF(K)=HPDF(K)+VDK
C
C...CHECK TERMINATION CONDITIONS, UPDATE THE ARRAYS FOR THE
C...NUMERATOR POLES, AND DETERMINE WHETHER THE RESIDUE AT
C...S=1 IS NEEDED FOR THE C.D.F.
C
  341 IF (ABS(VMX).GE.1.E-15) KPZ=0
      IF (ABS(VMX).LT.1.E-15) KPZ=KPZ+1
      IF (KPZ.GT.10) GO TO 348
  342 IF (KP.EQ.MP) GO TO 348
      IF (KS1.NE.1) GO TO 344
      KN=KN-1
      KS1=2
  344 DO 346 KS=1,KN
         I=ID(KS)
         JS(I)=JS(I)+1
  346    PL(I)=-(BA(I)+FLOAT(JS(I)))/ABS(GBA(I))
      GO TO 318
```

```
348 IF (KEY.EQ.1) GO TO 360
    RES=0.0
    IF (QP.LT.1) GO TO 352
    DO 350 I=1,QP
       X=CD(I)+GCD(I)
       IF (ERROR(X).NE.0) GO TO 350
       IF (X.GT.0.5) GO TO 350
       RES=1.0
       GO TO 352
350 CONTINUE
352 IF (KS1.EQ.2) RES=1.0
    IF (LR.LT.1.AND.CA.LT.1.0) RES=1.0
    IF (LR.EQ.1.AND.CA.GT.1.0) RES=1.0
    WRITE(6,975)
975 FORMAT(1H1 ,*      Z          PDF(Z)       CDF(Z)*,/)
    DO 358 K=KF,KL
       HPDF(K)=HPDF(K)*CN
       HCDF(K)=RES-(ZK1(K)*CN*HCDF(K))
       WRITE(6,980) ZK1(K),HPDF(K),HCDF(K)
       IF (HPDF(K).LT.0.0) HPDF(K)=-HPDF(K)
358    IF (HCDF(K).LT.0.0) HCDF(K)=-HCDF(K)
980 FORMAT(F11.4,2F12.6)
    WRITE(6,985) KP
985 FORMAT(//,*  NUMBER OF POLES EVALUATED = *,I4)
360 IF (KP.GT.KPM) KPM=KP
    IF (LR.NE.0) GO TO 362
    LR=1
    KF=KL+1
    KL=KMX
    GO TO 306
362 IF (IT.EQ.7.OR.IT.LT.5) GO TO 364
    IF (KKM.LT.2) GO TO 364
    HPDF(KKM)=(HPDF(KKM+1)+HPDF(KKM-1))/2.0
    HCDF(KKM)=(HCDF(KKM+1)+HCDF(KKM-1))/2.0
364 IF (KEY.EQ.1) GO TO 399
    IF (NP.NE.1) GO TO 399
    WRITE(6,990)
990 FORMAT(1H1 ,*              PROBABILITY DENSITY FUNCTION*,/)
    CALL PLOT(ZK1,HPDF,KMX)
    WRITE(6,995)
995 FORMAT(1H1 ,*              CUMULATIVE DISTRIBUTION FUNCTION*,/)

    CALL PLOT(ZK1,HCDF,KMX)
399 RETURN
    END


      FUNCTION DFACT (N)
C
C...CALCULATE N FACTORIAL
C
      DFACT=1.0
      IF (N.LE.1) GO TO 1204
      DO 1202 I=1,N
1202  DFACT=DFACT*FLOAT(I)
1204  RETURN
      END
```

```
      INTEGER FUNCTION ERROR (X)
C
C...AN INTEGER FUNCTION WHICH RETURNS THE CODE ERROR=0 IF X IS AN
C...INTEGER, AND ERROR=1 OTHERWISE (A CHECK FOR PRECISION PROBLEMS)
C
      ERROR=1
      IF (X.LT.0.E+0) J=X-5.E-1
      IF (X.GE.0.E+0) J=X+5.E-1
      Z=ABS(X-FLOAT(J))
      IF (Z.LT.1.E-10) ERROR=0
      RETURN
      END


      FUNCTION DGAMMA (XFP)
      DIMENSION CK(26)
      INTEGER ERROR
      DATA CK/1.E+0,0.5772156649015 33,-0.655878071520254,-0.042002635034
     1095,.166538611332292,-4.21977345555443E-2,-.00962197152787 7,0.0072
     218943246663,-1.16516759185 91E-3,-2.15241674114 9E-4,1.280502823882E
     3-4, -2.01348547807E-5, -1.25049348 21E-6,1.1330272320E-6,-2.0563384
     417E-7,6.1160950E-9,5.0020075E-9,-1.1812746E-9,1.043427E-10,7.7823E
     5-12,-3.6968E-12,5.100E-13,-2.06E-14,-5.4E-15,1.4E-15,1.E-16/
      X=XFP
      Z=X
      IF (X.LT.0.E+0) J=X-5.E-1
      IF (X.GE.0.E+0) J=X+5.E-1
      IF ((ERROR(X).EQ.0).AND.X.LT.5.E-1) GO TO 1614
      IF (ERROR(X).EQ.0) GO TO 1616
      PROD=1.E+0
      IF (X.GT.1.E+0) GO TO 1602
      IF (X.LT.0.E+0) GO TO 1606
      GO TO 1610
 1602 M=INT(X)
      DO 1604 I=1,M
 1604 PROD=PROD*(X-FLOAT(I))
      Z=X-FLOAT(M)
      GO TO 1610
 1606 M=INT(ABS(X))+1
      DO 1608 I=1,M
 1608 PROD=PROD/(X+FLOAT(I-1))
      Z=X+FLOAT(M)
 1610 SUM=0.E+0
      DO 1612 K=1,26
 1612 SUM=SUM+CK(K)*Z**K
      DGAMMA=PROD/SUM
      RETURN
 1614 WRITE (6,1618)
      RETURN
 1616 DGAMMA=DFACT(J-1)
      RETURN
 1618 FORMAT (  1H-, 30HATTEMPT TO FIND GAMMA(X) FOR NONPOSITIVE INTEGER
     1 ARGUMENT X IN SUBPROGRAM DGAMMA)
      END
```

```
      FUNCTION PSI (NFP,ZFP)
      DIMENSION ZETA(100)
      INTEGER ERROR
      DATA ZETA/0.E+0,.644934066848226, .202056903159594, .823232337111
     1382E-1, .369277551433699E-1, .173430619844491E-1,  .834927738192283
     2E-2,.407735619794434E-2, .200839282608221E-2,  .994575127818085E-3
     3,.494188604119465E-3, .246086553308048E-3, .122713347578489E-3, .6
     412481350587048E-4, .305882363070205E-4, .152822594086519E-4, .7637
     519763789976E-5,.381729326499984E-5,.190821271655394E-5,.9539620338
     672800E-6,.476932986787781E-6,.238450502727773E-6,.119219925965311E-6,.
     7596081890512 6E-7,.298035035146 5E-7,.149015548283 7E-7,.74507117898 4
     8E-8,.372533402479E-8,.186265972351E-8,.931327432 42E-9,.4656629065 0
     9E-9,.232831183 37E-9,.116415501 73E-9,.58207720 88E-10,.291038504 4E-1
     *0,.1455192189E-10,.727595984E-11,.363797955E-11,.181898965E-11,.90
     *949478E-12,.454747 38E-12,.227373 68E-12,58*0.E+0/
      N=NFP
      Z=ZFP
      DO 1702 I=42,100
 1702 ZETA(I)=5.E-1*ZETA(I-1)
      PSI=0.E+0
      IF (N.EQ.0) PSI=-.577215664901533
      TERM=0.D+0
      X=1.E+0
      RN1=((-X)**(N+1))*DFACT(N)
      SUM=0.E+0
      IF (Z.LT.0.E+0) J=Z-5.E-1
      IF (Z.GE.0.E+0) J=Z+5.E-1
      IF (ERROR(Z).EQ.0) GO TO 1726
      IF (ABS(Z-1.E+0).LE.5.E-1) GO TO 1706
      IF (Z.LT.5.E-1) GO TO 1712
 1704 SUM=SUM+(Z-X)**(-N-1)
      X=X+1.0
      IF (ABS(Z-X).GT.5.E-1) GO TO 1704
      TERM=-RN1*SUM
 1705 SUM=0.E+0
 1706 I=1
      XZ=X-Z
      W=DFACT(N)
      IF (N.NE.0) GO TO 1710
      W=XZ
 1708 I=I+1
 1710 DUMMY=SUM
      SUM=SUM+W+W*ZETA(N+I)
      W=W*XZ*FLOAT(N+I)/FLOAT(I)
      IF (ABS(SUM-DUMMY).GE.1.E-15) GO TO 1708
      PSI=PSI+((-1.0)**(N+1))*SUM+TERM
      RETURN
 1712 X=-1.0
 1714 X=X+1.0
      SUM=SUM+(Z+X)**(-N-1)
      IF (ABS(Z+X).GT.5.E-1) GO TO 1714
      TERM=RN1*SUM
      X=-X
      GO TO 1705
```

```
 1724 WRITE (6,1736)
      RETURN
 1726 IF (J.LE.0) GO TO 1724
      JJ=J-1
      IF (N.EQ.0) GO TO 1731
      IF (J.EQ.1) GO TO 1729
      DO 1728 I=1,JJ
 1728 SUM=SUM+FLOAT(I)**(-N-1)
 1729 PSI=PN1*(1.E+0-SUM+ZETA(N+1))
      RETURN
 1731 IF (J.EQ.1) GO TO 1733
      DO 1732 I=1,JJ
 1732 SUM=SUM+1.E+0/FLOAT(I)
      PSI=PSI+SUM
 1733 RETURN
 1736 FORMAT (  1H-, 89HATTEMPT TO FIND PSI(Z) OR DERIVATIVE FOR NONPOSI
     1TIVE INTEGER ARGUMENT Z IN SUBPROGRAM PSI)
      END
      SUBROUTINE PLOT (Z,H,NPOINTS)
C
C...PLOT THE P.D.F. OR C.D.F. VALUES CALCULATED  BY PDFCDF
C
      REAL Z(NPOINTS),H(NPOINTS),SCALE(11)
      INTEGER GRAPH(51,101),CHAR(5),LINE(101)
      DATA CHAR/' ','*','=','.','-'/LINE/'*',9*'-','*',9*'-','*',9*'-','*
     1',9*'-','*',9*'-','*',9*'-','*',9*'-','*',9*'-','*',9*'-','*',9*'-
     2','*'/
      II=NPOINTS
      KK=101
      LL=51
      DO 1802 I=1,KK
      DO 1802 J=1,LL
         GRAPH(J,I)=' '
 1802 CONTINUE
      XMIN=2.0*Z(1)-Z(2)
      IF (ABS(XMIN).GT.1.E-10) XMIN=Z(1)
      XMAX=Z(II)
      YMAX=0.0
      YMIN=H(II)
      DO 1804 I=1,II
         IF (H(I).GT.YMAX) YMAX=H(I)
         IF (H(I).LT.YMIN) YMIN=H(I)
 1804 CONTINUE
      XSTEP=(XMAX-XMIN)/FLOAT(KK-1)
      IF (YMAX.GT.0.9.AND.YMAX.LT.1.0) YMAX=1.0
      IF (YMIN.LT.0.1) YMIN=0.0
      YSTEP=(YMAX-YMIN)/FLOAT(LL-1)
      IF (0.0.GT.YMIN) GO TO 1812
      DO 1806 I=1,II
         K=1+INT((Z(I)-XMIN)/XSTEP+0.45)
         L=1+INT((YMAX-H(I))/YSTEP+0.50)
         GRAPH(L,K)=CHAR(2)
 1806 CONTINUE
```

```
      K=-1+INT(ALOG10(AMAX1(ABS(XMAX),ABS(XMIN))))
      IF (-2.LE.K.AND.2.GE.K) K=0
      L=-1+INT(ALOG10(YMAX))
      IF (-2.LE.L.AND.2.GE.L) L=0
      MM=0
      DO 1808 I=1,LL
         SCALE(1)=(YMAX-FLOAT(I-1)*YSTEP)/(10.0**L)
         IF (MM.EQ.0) WRITE (6,1814) SCALE(1),(GRAPH(I,J),J=1,KK)
         IF (MM.NE.0) WRITE (6,1816) (GRAPH(I,J),J=1,KK)
         MM=MM+1
         IF (MM.EQ.5) MM=0
 1808 CONTINUE
      SCALE(1)=YMIN/(10.0**L)
      WRITE (6,1816)(LINE(I),I=1,KK)
      DO 1810 I=1,11
         SCALE(I)=(XMIN+(FLOAT(I-1)*10.0*XSTEP))/(10.0**K)
 1810 CONTINUE
      WRITE (6,1818) (SCALE(I),I=1,11)
      WRITE (6,1820) 10.0**K,10.0**L
      RETURN
 1812 WRITE (6,) 52H  NEGATIVE VALUE OF H OCCURS, PLOT TERMINATES
      RETURN
 1814 FORMAT (2X,F8.4,  2H *,101A1)
 1816 FORMAT (10X,  2H I,101A1)
 1818 FORMAT (9X,11(F6.2,4X))
 1820 FORMAT (//,10X, 25HHORIZONTAL SCALE FACTOR= ,E8.1,/,10X, 25HVERTIC
     1AL SCALE FACTOR=    ,E8.1)
      END
```

B.2. GLOSSARY OF IDENTIFIERS IN COMPUTER PROGRAM

SUBROUTINES:

CHECK	= subroutine that checks for a validly defined H-function and finds the convergence type.
DFACT(N)	= function that computes N factorial.
DGAMMA(XFP)	= function that computes gamma of XFP.
ERROR(X)	= integer function that returns the value zero if X is integer and one if X is not integer.
PDFCDF	= subroutine for calculation of H-function values by summation of residues.
PLOT	= subroutine that plots the PDF and CDF.
PSI(NFP,ZFP)	= function that computes the NFP-th derivative of the psi function evaluated at ZFP.
SETUP	= subroutine that adjusts and orders H-function parameters into arrays convenient for calculating residues.

A(21)	= Array for a_i, i=1,...,n, the first elements of ordered pairs in an H-function parameter list, for a single variate.
A1(21)	= Temporary holding array for second part of array BA.
B(21)	= Array for b_i, i=1,...,m, the first elements of ordered pairs in an H-function parameter list, for a single variate.
BA(21)	= Array for a term of the sum, arranged for convenient computing of poles and residues; BA(i) = b_i for i=1,...,m and BA(m+i) = 1 - a_i for i=1,...,n.
BN	= Binomial coefficient(PDFCDF only).
C	= Crump constant A.
CA	= Upper bound for W, the intercept of the contour integral.
CB	= Lower bound for W(CHECK only).
CC	= Coefficient in H-function argument for term IS of sum.
CD(21)	= Array for a term of the sum, arranged for convenient computing of poles and residues; CD(i) = 1 - b_{m+i}, i=1,..., q - m = IQ and CD(i+IQ) = a_{n+i}, i=1,...,p - n = IP.
CH	= Check value used to find CA and CB and to compare convergence parameter L and E*W (CHECK only).
CN	= Distribution constant for term IS of the sum.
CNF	= Leading constant for Crump method, without exponential part.
CNFE	= Leading constant for Crump method, with exponential part.
C1(21)	= Temporary holding array for second part of array CD.
DF1	= ± the factorial of KT-1 (PDFCDF only).
DF2	= ± the factorial of KT-2, used for PDF with pole at s = 1 (PDFCDF only).
DZ	= Step size for Z (see input data card 1).

E(21)	=	Input array of first elements of H- function ordered pairs for SETUP.
F(21)	=	Output array of first elements of ordered pairs(SETUP).
FM(190)	=	Alphanumeric array used to print the form of the problem (limit of 50 variates and constants).
G	=	Second element of an H- function ordered pair(CHECK).
GA(21)	=	Array for A_i, i=1,...,n, the second elements of ordered pairs in an H- function parameter list, for a single variate.
GARGD(21)	=	Array for the values of denominator gamma arguments at a given pole(PDFCDF only).
GARGN(21)	=	Array for the values of numerator gamma arguments at a given pole(PDFCDF only).
GA1(21)	=	Temporary holding array for second part of array GBA.
GB(21)	=	Array for B_i, i=1,...,m, the second elements of ordered pairs in an H- function parameter list, for a single variate.
GBA(21)	=	Array for a term of the sum, arranged for convenient computing of poles and residues; GBA(i) = B_i for i=1,..., m, and GBA(m + i) = $-A_i$ for i=1,...,n.
GC1(21)	=	Temporary holding array for second part of array GCD.
GCD(21)	=	Array for a term of the sum, arranged for convenient computing of poles and residues; GCD(i) = $-B_{m+i}$ for i=1, ...,q-m = IQ, and GCD(IQ + i) = A_{n+i} for i=1,...,p-n = IP.
GE(21)	=	Input array of second elements of H- function ordered pairs for SETUP.
GF(21)	=	Output array of second elements of ordered pairs(SETUP).
HCDF(1001)	=	Array for intermediate and final answers, CDF if KEY = 0.
HPDF(1001)	=	Array for intermediate and final answers, PDF if KEY = 0.
I	=	Generally used counter.
ID(21)	=	Array giving the locations of numerator singularities for a given pole(PDFCDF only).
IDKS	=	ID(KS).
IDT	=	Indicator for identically distributed terms (see input data card 1).
IE	=	Counter used to fill in array FM.
II	=	Counter in SETUP, = KT-1 in PDFCDF.
IJ	=	Counter used to fill in array FM.
IL	=	Last element in FM.
IP	=	p - n, for a term of the sum.
IP1	=	I + 1 (PDFCDF only).
IQ	=	q - m, for a term of the sum.
IS	=	Counter for terms of the sum, IS = 1,...,NS.
IT	=	Convergence type, IT = 0,1,...,7 (CHECK).
I1	=	Counter in SETUP, = I-1 to fill in array FM.
I2	=	Counter in SETUP.
J	=	Generally used counter.
JD	=	Temporary holding place for an integer value.
JS(21)	=	Array for J_{ik}, where $-J_{ik}$ is the next singularity of the i-th gamma term in numerator(PDFCDF only).

JX = Nearest integer value of -X (PDFCDF only).
J1 = Counter in SETUP, = J-1 to fill in array FM.
J2 = Counter in SETUP.
K = Counter for Z, HPDF and HCDF values.
KD = Order of denominator singularities for a given pole (PDFCDF only).
KEY = 0 if PDFCDF is used to find the PDF and CDF of a term;
= 1 if PDFCDF is used to find the real and imaginary parts of a Laplace transform.
KF = 1 or KM+1, first value of K for a LHP or a RHP evaluation(PDFCDF only).
KKM = 0 if HPDF(KL) is to be evaluated by summing residues;
= KL otherwise (PDFCDF only).
KL = KM or KMX, last value of K for a LHP or a RHP evaluation (PDFCDF only).
KM = Last value of K for LHP evaluation when both LHP and RHP evaluations are made(PDFCDF only).
KMX = Maximum possible value of K (PDFCDF only).
KN = Order of numerator singularities for a given pole (PDFCDF only).
KNP1 = KN + 1 (PDFCDF only).
KP = Counter for number of poles evaluated (PDFCDF only).
KPM = Maximum number of poles evaluated for all terms in a sum.
KPZ = Counter for number of consecutive times that there is a negligible value for a residue(PDFCDF only).
KR = Counter for the number of terms in the sum with an upper bound on range; if the final value is not equal to NS, there is no upper bound on range for the sum.
KS = Counter for number of singularities considered in the numerator for a given pole(PDFCDF only).
KS1 = Indicator for status of pole at s = 1; 0 if not yet considered, 1 if under consideration, 2 if has been considered(PDFCDF only).
KT = Order of a given pole(PDFCDF only).
K1 = Counter in SETUP; = 0 in PDFCDF if ZK1(KL) ≠ ZM, else =1.
K2 = 2 - KEY (PDFCDF only).
L = Counter in PDFCDF.
LK = NS - 1 if IDT = 1, = 1 otherwise; number of required products of Laplace transform values.
LM,LN,LP,LQ= Counters used to check SETUP.
LM1 = L - 1 (PDFCDF only).
LR = 0, if both LHP and RHP evaluation is required;
= 1, if only RHP evaluation is required;
= -1, if only LHP evaluation is required.
L1 = Counter in SETUP.
M = m for a term of the sum.
MF = 1 or M+1, first value for a counter on BA or GBA(PDFCDF).
MI = Number of Crump values (see input data card 1).
ML = M or M+N, last value for a counter on BA or GBA(PDFCDF).

MM	=	m for a single variate.
MN	=	M + N.
MP	=	Maximum number of poles (see input data card 1).
N	=	n for a term of the sum.
NAY	=	Indicator for an error requiring program termination.
NF	=	Number of first variate/constant for term IS of the sum.
NH	=	Counter for variate/constant under consideration; = 1,...,NLT(NS).
NL	=	Number of last variate/constant for term IS of the sum.
NLT(50)	=	Array for values of NL (see input data card 2).
NLTL,NLTS	=	Temporary values used to check ascending order of NLT.
NN	=	n for a single variate.
NP	=	Indicator for plotting requirement (see input data card 1).
NS	=	Number of terms in the sum (see input data card 1).
NT	=	1 if IDT = 1, = NS otherwise.
NV	=	Type of variate (see input data cards).
P	=	p for a term of the sum.
PCT	=	Proportion of maximum Z (see input data card 1).
PHI	=	Variate parameter (see input data cards).
PL(21)	=	Array for the next value of the pole for the i-th term in the numerator(PDFCDF only).
PM1	=	PWR - 1 (SETUP only).
POW	=	Power to which variate is raised (see input data cards).
PP	=	p for a single variate.
PROD1,PROD2, PROD3,PROD4	=	Products used in computing VZERO(PDFCDF only).
PSI1(20)	=	Psi function values (I,1), used when order of a pole is more than 2.
PS1	=	POW - 1.
PWR	=	POW (SETUP only).
Q	=	q for a term of the sum.
QP	=	IP + IQ = p - n + q - m, number of denominator gamma terms.
QQ	=	q for a single variate.
RES	=	1, if residue at s = 1 is not included in finding CDF; = 0, otherwise (PDFCDF only).
RL	=	Lower bound on range of Z, if one exists that is > ZO.
RU	=	Upper bound on range of Z, if one exists that is < ZN.
S	=	Value of the pole under consideration(PDFCDF only).
SMPL	=	Difference between poles of gamma functions, used to determine the unevaluated pole with smallest magnitude and its order in the numerator(PDFCDF only).
SM1	=	S - 1 (PDFCDF only).
SV	=	1 for LHP evaluation, = - 1 for RHP (PDFCDF only).
S1,S2,S3,S4	=	Values used in SETUP.
T	=	Crump constant T.
TC	=	Coefficient in H- function argument, for a single variate.
TD	=	Convergence parameter TD (CHECK only).
TD1,TD2	=	Values used to compute TD and TP (CHECK only).

THETA	= Variate parameter (see input data cards).
TL	= Convergence parameter L (CHECK only).
TLI(1001)	= Imaginary parts of Laplace transform values.
TLR(1001)	= Real parts of Laplace transform values.
TN	= Distribution constant for a single variate.
TP	= Convergence parameter E (CHECK only).
TR	= Convergence parameter R .
T1,T2	= Temporary values of TLR(I) and TLI(I).
V(22,2)	= VZERO through V(KT - 1) array(PDFCDF only).
VDK	= Final value of the residue for a given pole and a given K (PDFCDF only).
VMX	= Maximum residue for a given pole and for all K (PDFCDF only).
VP1,VP2	= V(1,1) and V(1,2) without Z**-S term(PDFCDF only).
VZ1	= 0.0 if counter L is odd, PSI1 term not added to W(L,1); = 1.0 if L is even, PSI1 term added to W(L,1) (PDFCDF).
W(21,2)	= WZERO through W(KT - 2) (PDFCDF only).
WI	= Imaginary part of WZERO (PDFCDF only).
WR1,WR2	= WZERO, or W(1,1) and W(1,2), without -logZ term(PDFCDF).
X	= Argument of a gamma function for a given pole(PDFCDF).
XG(21)	= Temporary holding array for GBA and GCD elements, used to set up GBA and GCD for the Laplace transform H- function for term IS of the sum.
XL(21)	= Temporary holding array for BA and CD elements, used to set up BA and CD for the Laplace transform H- function for term IS of the sum.
Y	= Temporary holding place for old TLR(I) value when computing new TLR(I) and TLI(I) values.
ZC	= Constant for argument increments in Crump method.
ZF	= Starting Z value(PDFCDF only).
ZI	= Imaginary part of Crump complex number.
ZIMK	= Imaginary part of Crump complex number(PDFCDF only).
ZK	= ZK1(K).
ZK1(1001)	= Array for values of Z.
ZK2(1001)	= Array for arctan or log values(PDFCDF only).
ZL	= Final Z value if KEY = 0, = ZC if KEY = 1 (PDFCDF only).
ZLN(1001)	= Array for log values(PDFCDF only).
ZM	= Z value that separates LHP and RHP evaluations.
ZN	= Last Z value (see input data card 1).
ZO	= First Z value (see input data card 1).
ZT	= Value used to change or test reasonableness of ZO, ZN, and DZ inputs and ZF, ZL, and ZM values.
Z1,Z2	= Temporary values used in Crump method.

APPENDIX C

EXAMPLES OF COMPUTER PROGRAM OUTPUT

* APPENDIX C: EXAMPLES OF COMPUTER PROGRAM OUTPUT

The following examples were run at The University of Texas at Austin on a CYBER 170/750B, using the computer program of Appendix B.

C.1. SUM OF TWO IDENTICALLY DISTRIBUTED EXPONENTIAL VARIATES

Problem requirements:

Variates have exponential distributions with PHI = 2.0

ZO = 0.0   ZN = 10.0   DZ = 0.2

Distribution of individual variate is desired

Plots are desired

Input data cards:

```
0.0  10.0  0.2  2  1  100  1001  1.0  1  1
1
5  0.0  2.0  1.0
```

Computer Time:

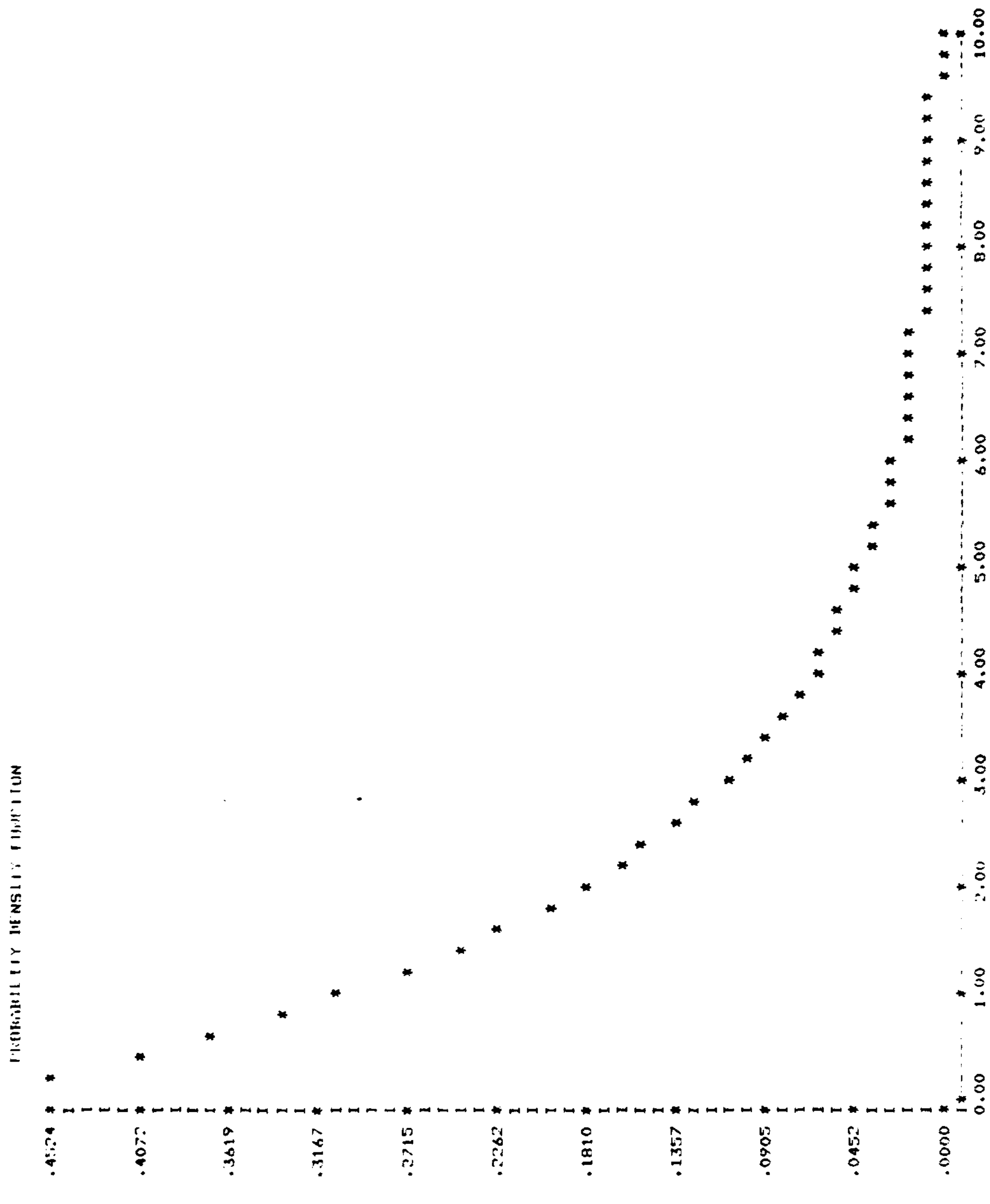
I/O Time = 3.939 * .8 =	3.151 seconds
CPU Time = 7.792 * 1.6 =	12.467 seconds
TM Time =	15.618 seconds

```
DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM     .2000 TO  10.0000 WITH STEP SIZE    .2000
FOR THE SUM OF   2 TERMS, WHERE


THE TERMS ARE IDENTICALLY DISTRIBUTED, AND


THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.

CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,

     PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =    .9557


FORM FOR EACH TERM (WHERE YJ = XJ**PJ):

Z = Y1


VARIATE X 1 IS TYPE NUMBER  5

INPUT PARAMETERS ARE THETA =   0.00000, PHI =   2.00000, AND POWER =   1.00000


THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:

              1  0
  .50000  H         (     .50000 X):
              0  1                   (   0.000,   1.000)
```

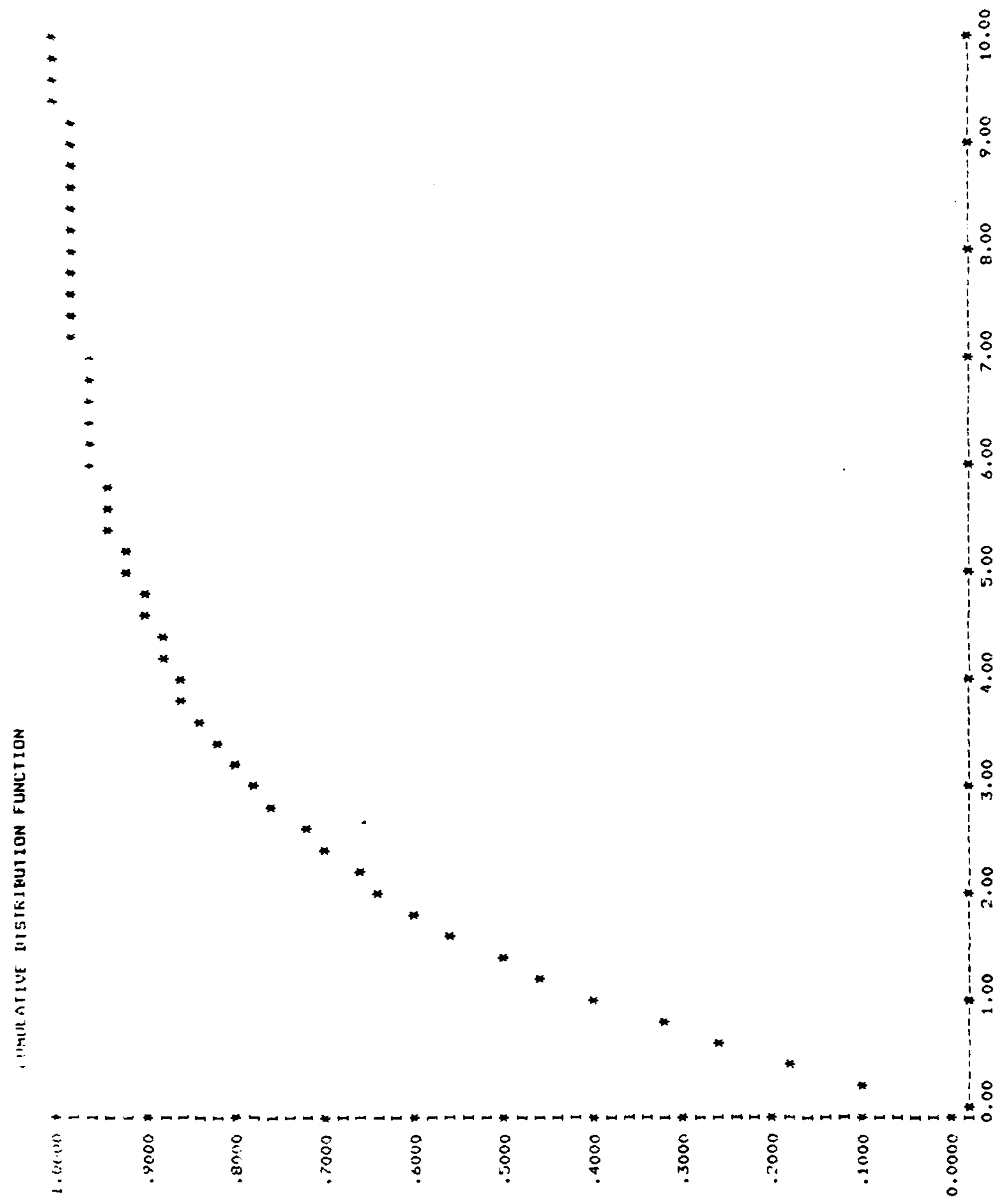
```
THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:

                  1  0
  .50000  H           (     .50000 Z), WHERE
                  0  1

(BA(I),GBA(I)): (   0.000,   1.000) (

 CONVERGENCE TYPE = 1

      D =  1.00      E = -1.00     L =  -.50     R = 1.0000

 LAPLACE TRANSFORM:

                  1  1
 1.00000  H           (    2.00000 Z), WHERE
                  1  1

(BA(I),GBA(I)): (   0.000,   1.000) (   1.000,  -1.000) (

 CONVERGENCE TYPE = 5

      D =  2.00      E =  0.00     L =  0.00     R = 1.0000
```

Z	PDF(Z)	CDF(Z)
.2000	.452419	.095163
.4000	.409365	.181269
.6000	.370409	.259182
.8000	.335160	.329680
1.0000	.303265	.393469
1.2000	.274406	.451188
1.4000	.248293	.503415
1.6000	.224664	.550671
1.8000	.203285	.593430
2.0000	.183940	.632121
2.2000	.166436	.667129
2.4000	.150597	.698806
2.6000	.136266	.727468
2.8000	.123298	.753403
3.0000	.111565	.776870
3.2000	.100948	.798103
3.4000	.091342	.817316
3.6000	.082649	.834701
3.8000	.074784	.850431
4.0000	.067668	.864665
4.2000	.061228	.877544
4.4000	.055402	.889197
4.6000	.050129	.899741
4.8000	.045359	.909282
5.0000	.041042	.917915
5.2000	.037137	.925726
5.4000	.033603	.932794
5.6000	.030405	.939190
5.8000	.027512	.944977
6.0000	.024894	.950213
6.2000	.022525	.954951
6.4000	.020381	.959238
6.6000	.018442	.963117
6.8000	.016687	.966627
7.0000	.015099	.969803
7.2000	.013662	.972676
7.4000	.012362	.975276
7.6000	.011185	.977629
7.8000	.010121	.979758
8.0000	.009158	.981684
8.2000	.008286	.983427
8.4000	.007498	.985004
8.6000	.006784	.986431
8.8000	.006139	.987723
9.0000	.005554	.988891
9.2000	.005026	.989948
9.4000	.004548	.990905
9.6000	.004115	.991770
9.8000	.003723	.992553
10.0000	.003369	.993262

NUMBER OF POLES EVALUATED = 46

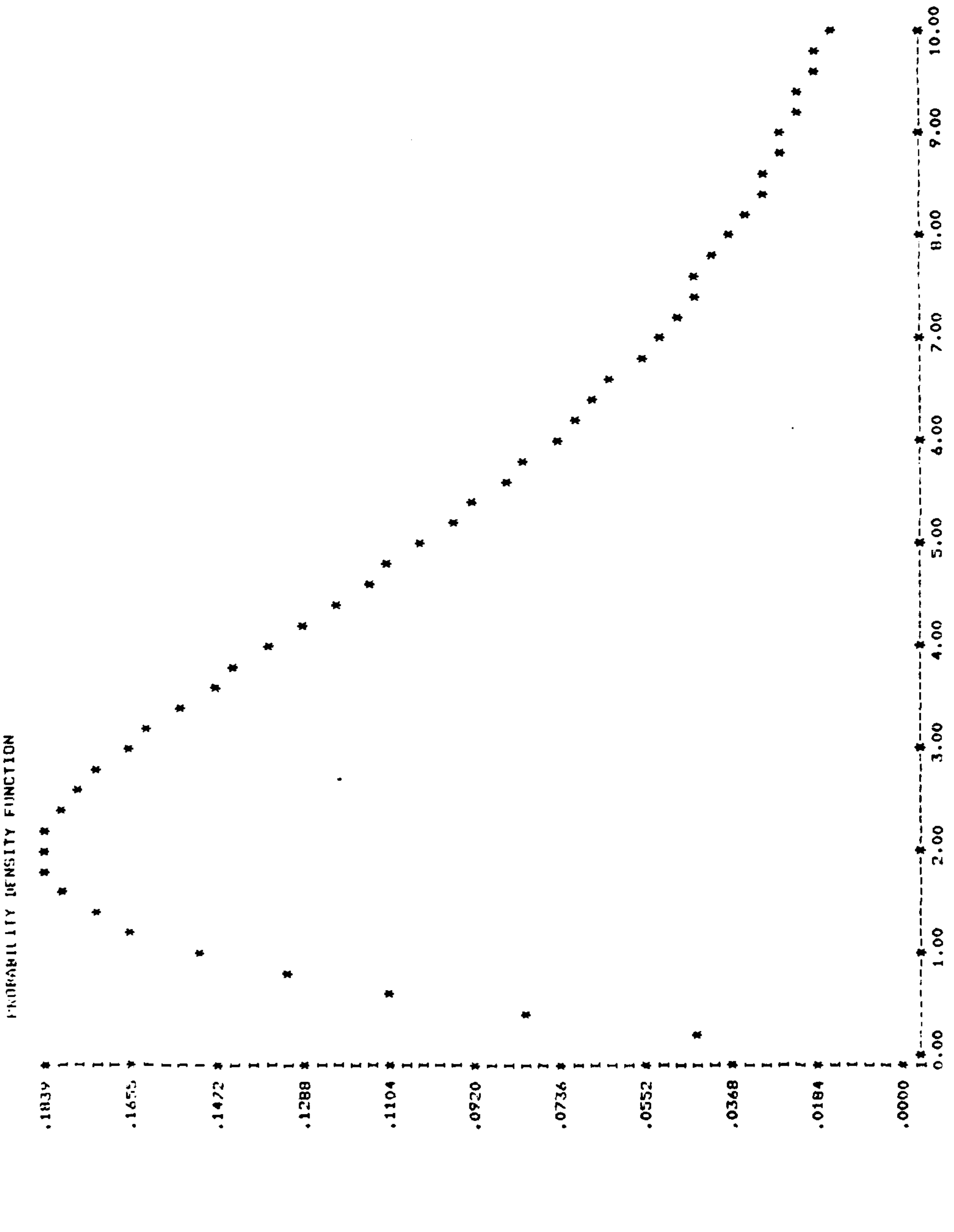


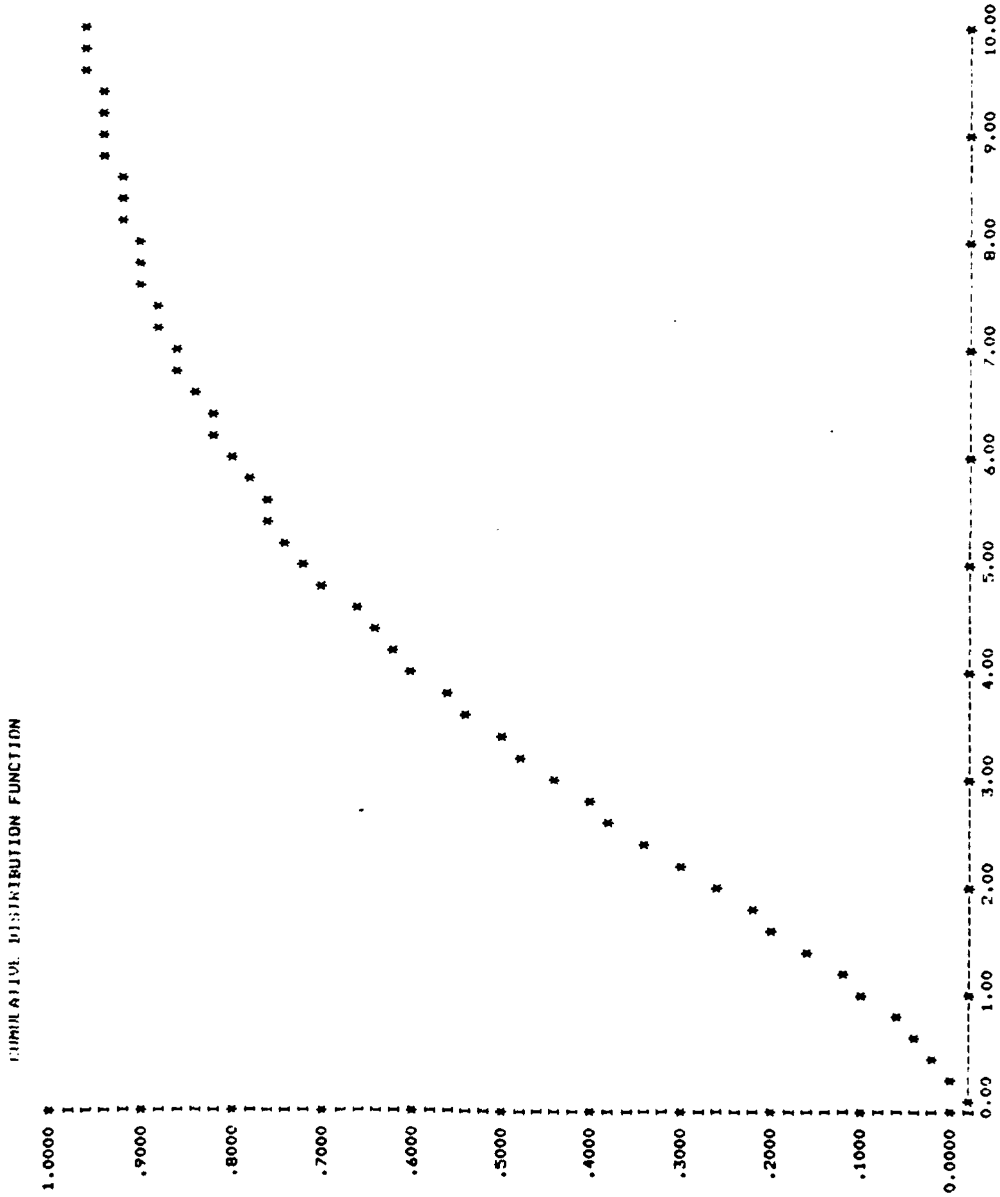
PROBABILITY DENSITY FUNCTION
.4524
.4072
.3619
.3167
.2715
.2262
.1810
.1357
.0905
.0452
.0000
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00


CUMULATIVE DISTRIBUTION FUNCTION
1.0000
.9000
.8000
.7000
.6000
.5000
.4000
.3000
.2000
.1000
0.0000
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00

MAXIMUM NUMBER OF POLES EVALUATED = 64

Z	PDF(Z)	CDF(Z)			
.2000	.045242	.004679	5.2000	.096537	.732615
.4000	.081873	.017523	5.4000	.090705	.751340
.6000	.111123	.036936	5.6000	.085107	.768922
.8000	.134064	.061552	5.8000	.079751	.785410
1.0000	.151632	.090204	6.0000	.074641	.800852
1.2000	.164643	.121901	6.2000	.069779	.815298
1.4000	.173804	.155805	6.4000	.065162	.828799
1.6000	.179731	.191208	6.6000	.060787	.841403
1.8000	.182956	.227518	6.8000	.056650	.853158
2.0000	.183939	.264241	7.0000	.052743	.864112
2.2000	.183078	.300971	7.2000	.049059	.874311
2.4000	.180715	.337373	7.4000	.045589	.883800
2.6000	.177144	.373177	7.6000	.042323	.892621
2.8000	.172616	.408167	7.8000	.039252	.900815
3.0000	.167345	.442175	8.0000	.036366	.908422
3.2000	.161514	.475069	8.2000	.033653	.915480
3.4000	.155278	.506755	8.4000	.031102	.922023
3.6000	.148765	.537163	8.6000	.028702	.928087
3.8000	.142085	.566251	8.8000	.026441	.933703
4.0000	.135329	.593994	9.0000	.024306	.938901
4.2000	.128572	.620785	9.2000	.022285	.943710
4.4000	.121875	.645430	9.4000	.020365	.948157
4.6000	.115287	.669146	9.6000	.018530	.952268
4.8000	.108849	.691559	9.8000	.016766	.956065
5.0000	.102591	.712703	10.0000	.015056	.959572


PROBABILITY DENSITY FUNCTION
.1839
.1655
.1472
.1288
.1104
.0920
.0736
.0552
.0368
.0184
.0000
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00


CUMULATIVE DISTRIBUTION FUNCTION
1.0000
.9000
.8000
.7000
.6000
.5000
.4000
.3000
.2000
.1000
0.0000
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00

C.2. SUM OF TWO EXPONENTIAL VARIATES, NOT IDENTICALLY DISTRIBUTED

Problem requirements:

Exponential distributions have parameters ϕ_1=2.0 and ϕ_2=3.0

ZO = 0.0   ZN = 15.0   DZ = 0.2

Distributions of individual variates are not desired

Plots are desired

Input data cards:

0.0 15.0 0.2 2 0 100 1001 1.0 0 1

1 2

5 0.0 2.0 1.0

5 0.0 3.0 1.0

Computer Time:

I/O Time = 4.003 * .8 = 3.202 seconds

CPU Time = 16.204 * 1.6 = 25.926 seconds

TM Time = 29.128 seconds

```
DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM    .2000 TO  15.0000 WITH STEP SIZE    .2000
FOR THE SUM OF  2 TERMS, WHERE



THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.

CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,

     PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =    .6371


FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):

Z = Y1+Y2


VARIATE X 1 IS TYPE NUMBER  5

INPUT PARAMETERS ARE THETA =   0.00000, PHI =   2.00000, AND POWER =   1.00000


THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:

                1  0
  .50000  H        (     .50000 X):
                0  1                 (    0.000,    1.000)
```

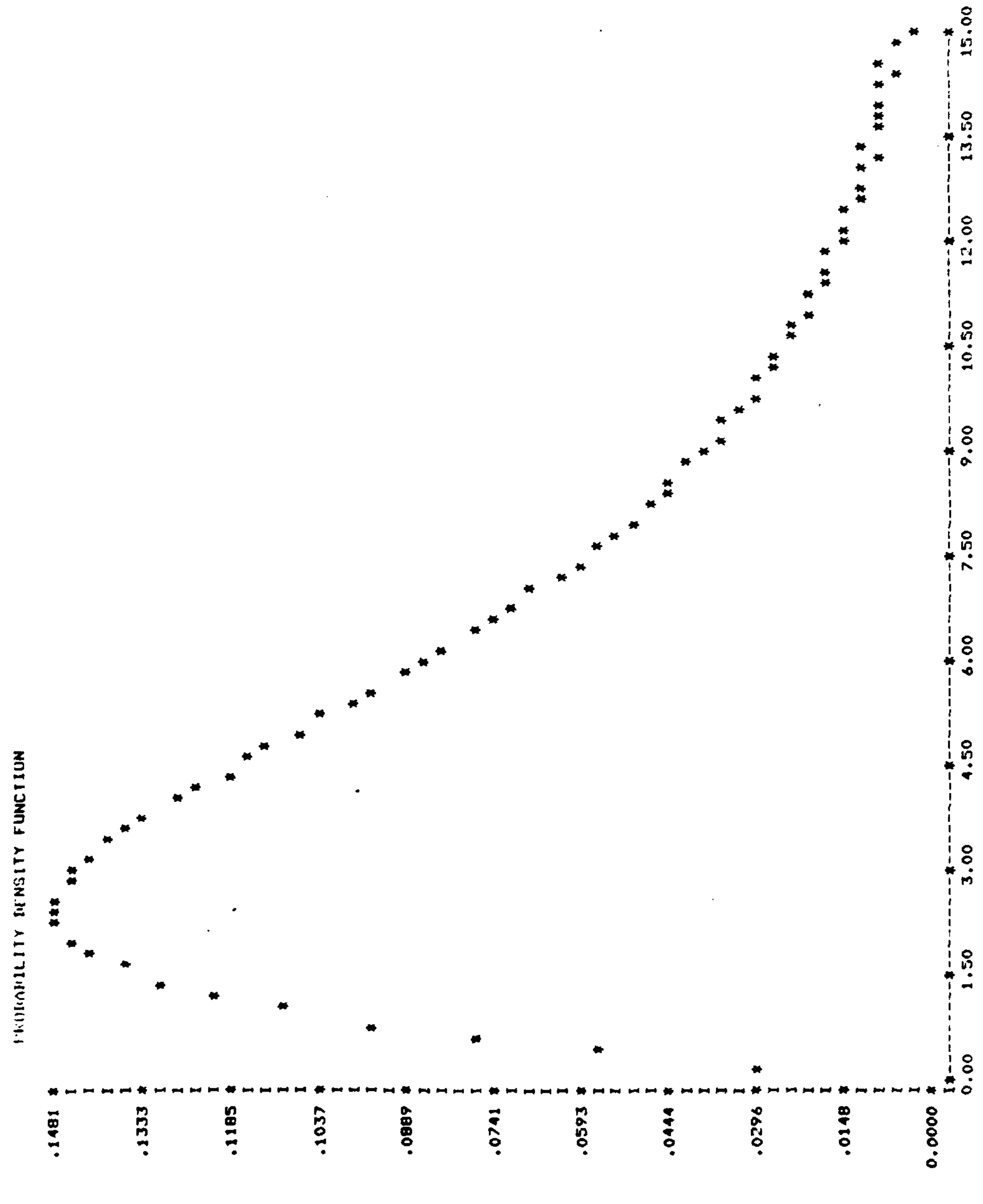
```
THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:

                 1  0
  .50000  H          (    .50000 Z), WHERE
                 0  1

(BA(I),GBA(I)): (   0.000,   1.000) (

CONVERGENCE TYPE = 1

      D =  1.00      E = -1.00      L =  -.50      R = 1.0000

LAPLACE TRANSFORM:

                 1  1
 1.00000  H          (   2.00000 Z), WHERE
                 1  1

(BA(I),GBA(I)): (   0.000,   1.000) (   1.000,  -1.000) (

CONVERGENCE TYPE = 5

      D =  2.00      E =  0.00      L =  0.00      R = 1.0000

VARIATE X 2 IS TYPE NUMBER  5

INPUT PARAMETERS ARE THETA =   0.00000, PHI =   3.00000, AND POWER =   1.00000
```

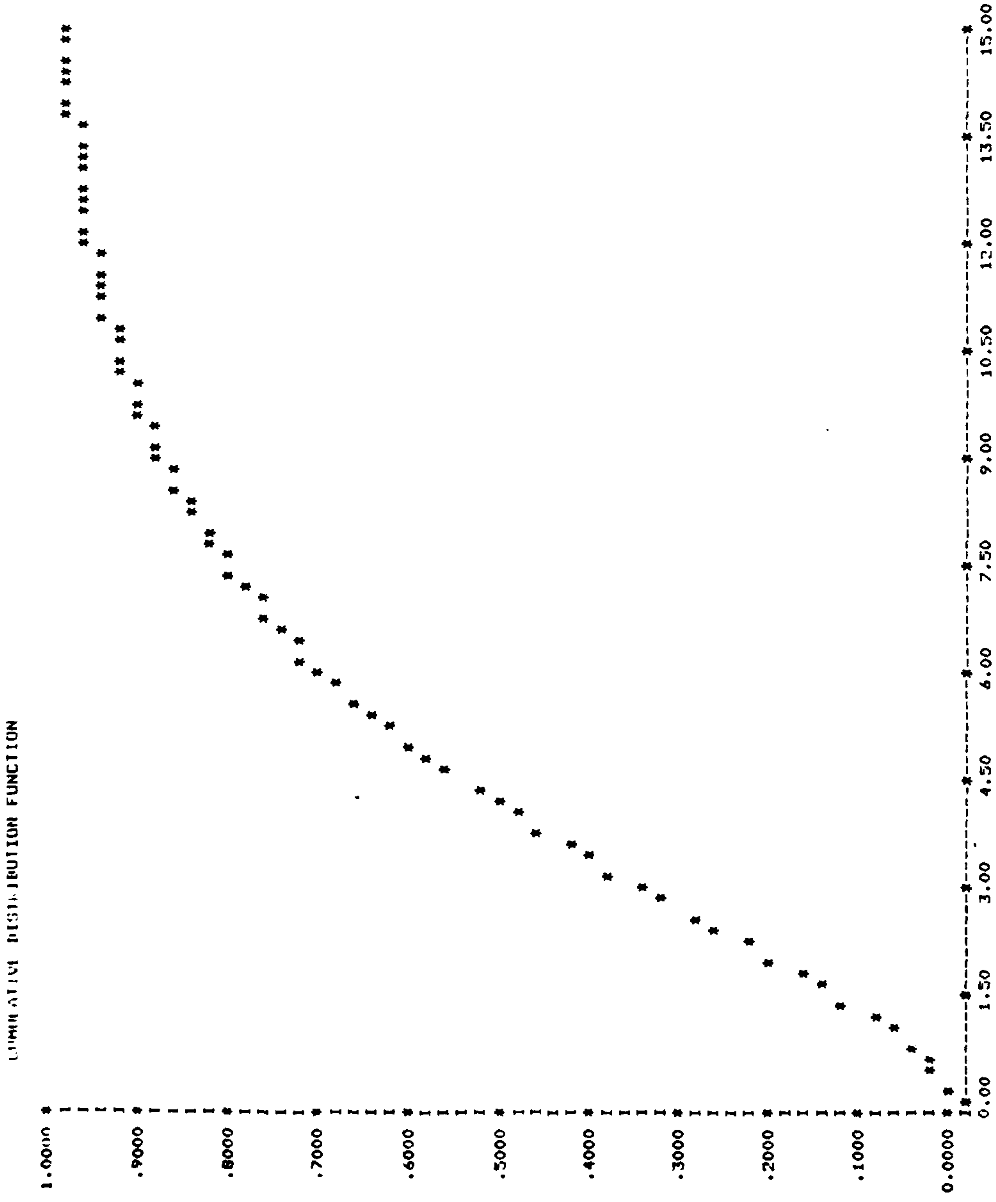
```
THE P.D.F. FOR VARIATE X 2 IS GIVEN BY:

               1  0
  .33333  H        (     .33333 X):
               0  1                  (    0.000,    1.000)

THE P.D.F. FOR TERM  2 OF THE SUM IS GIVEN BY:

               1  0
  .33333  H        (     .33333 Z), WHERE
               0  1

(BA(I),GBA(I)): (    0.000,    1.000) (

CONVERGENCE TYPE = 1

      D =  1.00      E = -1.00      L =  -.50      R = 1.0000
LAPLACE TRANSFORM:

               1  1
 1.00000  H        (    3.00000 Z), WHERE
               1  1

(BA(I),GBA(I)): (    0.000,    1.000) (    1.000,   -1.000) (

CONVERGENCE TYPE = 5

      D =  2.00      E =  0.00      L =  0.00      R = 1.0000
```

MAXIMUM NUMBER OF POLES EVALUATED = 100

Z	PDF(Z)	CDF(Z)
.2000	.030675	.003154
.4000	.056439	.011942
.6000	.077912	.025444
.8000	.095611	.042855
1.0000	.109999	.063467
1.2000	.121508	.086663
1.4000	.130506	.111903
1.6000	.137316	.138719
1.8000	.142242	.166704
2.0000	.145540	.195508
2.2000	.147433	.224826
2.4000	.148134	.254402
2.6000	.147821	.284012
2.8000	.146642	.313472
3.0000	.144748	.342622
3.2000	.142260	.371332
3.4000	.139273	.399492
3.6000	.135894	.427015
3.8000	.132204	.453829
4.0000	.128260	.479879
4.2000	.124139	.505122
4.4000	.119895	.529527
4.6000	.115554	.553072
4.8000	.111176	.575746
5.0000	.106797	.597543
5.2000	.102418	.618464
5.4000	.098089	.638514
5.6000	.093836	.657705
5.8000	.089638	.676051
6.0000	.085542	.693568
6.2000	.081569	.710277
6.4000	.077675	.726199
6.6000	.073911	.741357
6.8000	.070298	.755775
7.0000	.066770	.769479
7.2000	.063382	.782494
7.4000	.060163	.794845

7.6000	.057018	.806560
7.8000	.054013	.817663
8.0000	.051194	.828181
8.2000	.048424	.838138
8.4000	.045788	.847561
8.6000	.043356	.856472
8.8000	.040937	.864896
9.0000	.038639	.872857
9.2000	.036573	.880375
9.4000	.034474	.887473
9.6000	.032475	.894173
9.8000	.030753	.900493
10.0000	.028937	.906454
10.2000	.027192	.912074
10.4000	.025795	.917371
10.6000	.024225	.922360
10.8000	.022683	.927062
11.0000	.021598	.931489
11.2000	.020238	.935656
11.4000	.018844	.939580
11.6000	.018068	.943272
11.8000	.016887	.946743
12.0000	.015571	.950011
12.2000	.015118	.953084
12.4000	.014091	.955969
12.6000	.012771	.958686
12.8000	.012673	.961239
13.0000	.011783	.963633
13.2000	.010348	.965889
13.4000	.010667	.968008
13.6000	.009911	.969990
13.8000	.008210	.971861
14.0000	.009043	.973617
14.2000	.008441	.975254
14.4000	.006262	.976804
14.6000	.007755	.978260
14.8000	.007364	.979609
15.0000	.004396	.980892


PROBABILITY DENSITY FUNCTION
.1481
.1333
.1185
.1037
.0889
.0741
.0593
.0444
.0296
.0148
0.0000
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00


CUMULATIVE DISTRIBUTION FUNCTION
1.0000
.9000
.8000
.7000
.6000
.5000
.4000
.3000
.2000
.1000
0.0000
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00

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