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DOMINATION PROBLEM FOR VECTOR MEASURES AND APPLICATIONS TO NONSTATIONARY PROCESSES.

by

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DOMINATION PROBLEM FOR VECTOR MEASURES AND APPLICATIONS TO NONSTATIONARY PROCESSES

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1. Introduction.

The domination problem of a signed measure, as commonly understood, is that of finding a positive (finite) measure with respect to which the given one is absolutely continuous. Hence the class of null sets of the given signed measure contains the class of null sets of the dominating measure, which can be taken as its (total) variation measure. For vector measures also, the dominating measure is usually taken to be the (total) variation measure if the latter is σ-finite, or at least locally finite. However, in a number of important applications a vector measure need not have a σ-finite total variation, and the last condition is a fundamental assumption for the Radon-Nikodým theory of these measures. But by an important theorem of Pettis, each vector measure into a Banach space has finite semi-variation and the determination of a dominating measure takes on an interest of its own. So one may consider weaker concepts of p-(semi-)variation of a vector measure for some $p \geq 1$, and then search for the existence of a dominating measure. It results that this existence problem depends both on such a $p \geq 1$ and the type of range space. This leads to the classification of (range) vector spaces which admit domination for each given p, and it is a nontrivial matter.

In this paper the question of finite p-variation and its representation (via the Radon-Nikodým theory) will not be considered beyond its comparison, even though it is useful in the integral representation of certain linear operations. On the other hand, the problem of finite p-semi-variation has immediate interest for certain stochastic process representations, and that will be treated in a reasonably detailed fashion for a class of vector measures. An outline of the content of this paper is as follows.

The next section is utilized to a precise formulation of the domination problem, and a solution of the general case. The generality of the result renders it somewhat ineffective for the special applications here. In Section 3, a class of spaces is thus isolated for which a

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complete solution of the domination problem is obtained for vector measures, which have p-semi-variation finite, $1 \le p \le 2$. The work here depends in part on an inequality of Grothendieck-Pietsch. The rest of the paper is devoted to some key applications of this theory to nonstationary processes. Thus Section 4 is utilized in showing that a large class of second order (nonstationary) processes, introduced by Cramér [2], admit a dilation to processes of the type considered by Karhunen [6] on an extended Hilbert space. Conversely, a continuous linear transformation of a Karhunen process is always of a Cramér process. Related study on stochastic measures is given in [10] and analogous results appear in [12]. These considerations also admit interesting applications to operator theory and the last section is devoted to this. There it is shown that a large family of bounded operators on a Hilbert space has self-adjoint dilations. This generalizes the classical results on unitary dilations of contractions in [14]. Let us now turn to details.

2. Domination problem for general vector measures.

If (Ω,Σ) is a measurable space, $\mathfrak{X}$ a Banach space with norm $\|\cdot\|$, and $\nu:\Sigma\to\mathfrak{X}$ is weakly (or equivalently strongly) σ-additive, called a vector measure, then the p-variation of ν relative to a measure $\mu:\Sigma\to\bar{\mathbb{R}}^+$ is defined on A as:

$$|\nu|_p(A) = \sup\{\sum_{i=1}^{n}\|\nu(A_i)\|\,|a_i| : A_i\in\Sigma(A),\ \text{disjoint},\ \|f\|_{q,\mu}\le 1\}\,, \tag{1}$$

where $f = \sum_{i=1}^{n} a_i\chi_{A_i}$, $q = p/p-1 \ge 1$, and $f\in L^q(\Omega,\Sigma,\mu) = L^q(\mu)$, $\Sigma(A)$ being the trace σ-algebra of Σ on A. If $|\nu|_p(A) < \infty$, then ν is said to have p-variation finite on A relative to μ. If $p = 1$, $L^\infty(\mu)$ is usually replaced by $B(\Omega,\Sigma)$, the vector space of bounded (Σ-) measurable scalar functions with uniform norm, without reference to μ, and the 1-variation is simply called variation. Then (1) reduces to:

$$|\nu|(A) = \sup\{\sum_{i=1}^{n}\|\nu(A_i)\| : A_i\in\Sigma(A),\ \text{disjoint}\}\,. \tag{2}$$

Also $|\nu|(\cdot)$ is additive or σ-additive accordingly as ν is, but this is obviously not true of $|\nu|_p(\cdot)$ for $p>1$.

A weaker concept is p-semi-variation relative to μ, defined as:

$$\|\nu\|_p(A) = \sup\{\|\sum_{i=1}^{n} a_i\nu(A_i)\| : A_i\in\Sigma(A),\ \text{disjoint},\ \|f\|_{q,\mu}\le 1\}\,, \tag{3}$$

where $f = \sum_{i=1}^{n} a_i\chi_{A_i}$, $q = p/p-1 \ge 1$, as before. If $\|\nu\|_p(A) < \infty$, then ν is said to have p-semi-variation finite on A relative to μ. If

$p = 1$, $L^\infty(\mu)$ is again replaced by $B(\Omega,\Sigma)$, and the 1-semi-variation is called semi-variation. In this case (3) becomes:

$$\|\nu\|(A) = \sup\{\|\sum_{i=1}^{n} a_i \nu(A_i)\| : |a_i| \le 1, A_i \in \Sigma(A), \text{ disjoint}\} . \tag{4}$$

Note that for 1-semi-variation also, the auxiliary measure is not necessary. The relations between the different definitions are:

$$|\nu|(A) \le |\nu|_1(A) \ , \quad \|\nu\|(A) \le \|\nu\|_1(A) \ ,$$

with equalities if ν is μ-continuous. Also $\|\nu\|_p(A) \le |\nu|_p(A)$ generally, with a strict inequality if X is infinite dimensional. An extended discussion of these variations can be found in [3].

It is convenient to restate the p-semi-variation definition (3) in the following integral form:

$$\|\nu\|_p(A) = \sup\{\|\int_A f d\nu\| : \|f\|_{q,\mu} \le 1\} \ , \tag{5}$$

where the integral of a measurable scalar function relative to a vector measure is taken in the sense of Dunford and Schwartz ([4], IV.10). Analogous formula does not obtain for the p-variation case.

With these concepts, the needed classical properties of vector measures can be quickly stated. It is a consequence of a theorem of Pettis (cf. [4], IV.10.2) that a vector measure is of finite semi-variation, for any Banach space X . Even though $|\nu|(\cdot)$ is σ-additive, it need not be finite on most sets of Σ . For the Radon-Nikodým theory however, the basic assumption is that $\mu = |\nu|(\cdot):\Sigma \to \bar{\mathbb{R}}^+$ is at least σ-finite, and then one seeks conditions on the spaces X such that the derivative $\frac{d\nu}{d\mu}$ exists. If X is reflexive or a separable adjoint space, such a result holds. In general even if X is a Hilbert space $\mathcal{H}$, $\mu = |\nu|(\cdot): \mathcal{B}(\mathbb{R}) \to \bar{\mathbb{R}}^+$ need not be σ-finite, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the line $\mathbb{R}$. For instance, if ν is defined by the Wiener process on $\mathbb{R}$ into $\mathcal{H}$, then $|\nu|(A) = +\infty$ for each nondegenerate open set $A \subset \mathbb{R}$. A similar phenomenon occurs in many other probabilistic applications involving integral representations of processes by stochastic measures such as those needed for the stationary or harmonizable processes, as well as the ones considered in Section 4 below.

Thus the main technical problem of this paper is the following. If $\nu:\Sigma \to X$ is σ-additive, does there exist a σ-finite $\mu:\Sigma \to \bar{\mathbb{R}}^+$ such that for some $1 \le p < \infty$ one has (with $q = p/p-1$)

$$\|\int_\Omega f(\omega)\nu(d\omega)\|_X \le \|f\|_{q,\mu}, \quad f \in L^q(\mu) \ ? \tag{6}$$

In other words, does ν have finite p-semi-variation for some $1 \le p < \infty$ and some measure μ ? This is referred to as the domination problem. A solution of this problem is important for applications. A related question is to classify the spaces $\mathcal{X}$ for which the existence (or non-existence) of such a μ is to be determined for each given p. Some partial solutions are obtained to these questions, and they will be given here. These results have already proved useful for important applications.

For the general case, it is convenient to restate (5) in a somewhat extended form. Recall that if $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a symmetric convex function, $\varphi(0) = 0$ and $\varphi \neq 0$, it is called a Young function with $\psi : \mathbb{R} \to \mathbb{R}^+$ as its conjugate where $\psi(x) = \sup\{|x|y - \varphi(y) : y \geq 0\}$. Then ψ is also convex with similar properties and the <u>gauge norm</u> of a measurable f is defined as:

$$\|f\|_{\psi,\mu} = \inf\{\alpha > 0 : \int_\Omega \psi(\tfrac{f}{\alpha})d\mu \leq 1\} .$$

The φ-<u>semi-variation</u> of $\nu : \Sigma \to \mathcal{X}$ is then defined as in (5), i.e.,

$$\|\nu\|_\varphi(A) = \sup\{\|\int_\Omega f(\omega)\nu(d\omega)\| : \|f\|_{\psi,\mu} \leq 1\} , \tag{7}$$

where the vector integral is in the sense of Dunford-Schwartz, and other symbols are as defined before. If $\varphi(x) = |x|^p$, $p \geq 1$, then (7) becomes (5).

A solution of the general case is given by the following:

<u>THEOREM 1</u>. <u>Let</u> (Ω,Σ) <u>be a measurable space</u>, $\mathcal{X}$ <u>a Banach space and</u> $\nu : \Sigma \to \mathcal{X}$ <u>a vector measure</u>. <u>Then there exists a finite positive</u> $\mu : \Sigma \to \mathbb{R}^+$, <u>a continuous Young function</u> $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\frac{\varphi(x)}{x} \uparrow \infty$ <u>as</u> $x \uparrow \infty$, <u>such that</u> $\|\nu\|_\varphi(\Omega) < \infty$. <u>Thus</u> ν <u>is dominated by the pair</u> (φ,μ). <u>The pair in general is not unique</u>.

<u>Proof</u>: As noted already, the weak and strong σ-additivity of a vector measure are equivalent. Let S^* be the unit sphere of the adjoint space $\mathcal{X}^*$ of $\mathcal{X}$, and $\{A_n, n \geq 1\} \subset \Sigma$ be a disjoint sequence. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|\nu(\bigcup_{n\geq 1} A_n) - \sum_{k=1}^n \nu(A_k)\| \\ &= \lim_{n\to\infty} \sup\{|(x^*\circ\nu)(\bigcup_{n\geq 1} A_n) - \sum_{k=1}^n (x^*\circ\nu)(A_k)| : x^* \in S^*\} . \end{aligned} \tag{8}$$

So the scalar (signed) measures $\{x^*\circ\nu : x^* \in S^*\}$ are uniformly σ-additive on Σ. By a result of Bartle-Dunford-Schwartz (cf. [4], IV.10.5) there exists a positive finite (sometimes called a "control") measure $\mu : \Sigma \to \mathbb{R}^+$ such that $x^*\circ\nu$ is μ-continuous for all $x^* \in S^*$. Hence by the scalar Radon-Nikodým theorem, $g_{x^*} = \frac{d(x^*\circ\nu)}{d\mu}$ exists and by (8) one has

$$0 = \lim_{\mu(A)\to 0} |x^*\circ\nu(A)| = \lim_{\mu(A)\to 0} \int_A g_{x^*}(\omega)\mu(d\omega) . \tag{9}$$

uniformly in $x^*\in S^*$. Hence $\{g_{x^*}:x^*\in S^*\}\subset L^1(\mu)$ is bounded (since a vector measure is bounded) and uniformly μ-integrable. Remembering the fact that $\mu(\Omega)<\infty$, one can invoke the classical de la Vallée Poussin's theorem (cf. e.g. [11], Thm. I.4.4, for the form used here), there exists a convex function $\varphi:\mathbb{R}\to\mathbb{R}^+$ of the given description such that

$$\int_\Omega \varphi(|g_{x^*}|(\omega))\mu(d\omega)\le k_0<\infty, \qquad x^*\in S^*. \tag{10}$$

Let $\psi:\mathbb{R}\to\mathbb{R}^+$ be the conjugate function to φ. Then one has

$$\begin{aligned}\|\nu\|_\varphi(\Omega) &= \sup\{\|\int_\Omega f(\omega)\nu(d\omega)\|:\|f\|_{\psi,\mu}\le 1\}\\ &= \sup\{\sup[|\int_\Omega f(\omega)(x^*\circ\nu)(d\omega)|:x^*\in S^*]:\|f\|_{\psi,\mu}\le 1\}\\ &= \sup\{\sup[|\int_\Omega f(\omega)g_{x^*}(\omega)\mu(d\omega)|:x^*\in S^*]:\|f\|_{\psi,\mu}\le 1\}\\ &\le 2\sup\{\sup[\|g_{x^*}\|_{\varphi,\mu}\|f\|_{\psi,\mu}:x^*\in S^*]:\|f\|_{\psi,\mu}\le 1\}\text{, by the Hölder inequality for Orlicz spaces,}\\ &\le 2\sup[\|g_{x^*}\|_{\varphi,\mu}:x^*\in S^*]\le 2k_0<\infty\text{, by (10).}\end{aligned}$$

This completes the proof.

<u>Discussion 2</u>. By the earlier remarks, since ν is μ-continuous, it follows that $\|\nu\|(\Omega)=\|\nu\|_1(\Omega)<\infty$ relative to μ. By the support line property of the convex function φ, and the fact that $\mu(\Omega)<\infty$, it is seen that $L^\varphi(\mu)\subset L^1(\mu)$ where $L^\varphi(\mu)$ is the Orlicz space defined as $L^\varphi(\mu)=\{f:\|f\|_{\varphi,\mu}<\infty\}$ with norm $\|\cdot\|_{\varphi,\mu}$. This is a Banach space and the inclusion into $L^1(\mu)$ is topological. From this one deduces that there is a constant $0<C<\infty$ such that $\|\nu\|_\varphi(\Omega)\le C\|\nu\|(\Omega)<\infty$. However φ may (in general does) grow faster than any polynomial, and it also depends on the space $\mathfrak{X}$. If $\varphi(x)=|x|^p$, it is nontrivial to classify Banach spaces $\mathfrak{X}$ for each given $p\ge 1$. This problem has interest in applications and is essentially open.

If $\varphi(x)=x^2$, an important aspect of the corresponding problem can be solved, so that ν is dominated by a pair $(2,\mu)$.

3. <u>Domination problem for a special class of spaces</u>.

It is convenient to introduce the following concept:

<u>Definition 3</u>. Let $p\ge 1$, $\lambda\ge 1$ be numbers, and $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}$ is termed an $\mathcal{L}_{p,\lambda}$<u>-space</u> if for each n-dimensional subspace E of $\mathfrak{X}$, $1\le n<\infty$, there is an m-dimensional subspace F of $\mathfrak{X}$ ($1\le m<\infty$), $E\subset F$, such that $d(F,\ell_p^m)\le\lambda$ where ℓ_p^m is the m-dimensional Lebesgue sequence space, and where for any pair of normed vector spaces E_1,E_2,

$d(E_1,E_2) = \inf\{\|T\|\cdot\|T^{-1}\|: T\in B(E_1,E_2)\}$. Here and below $B(E_1,E_2)$ stands for the space of bounded linear mappings on E_1 into E_2 . The space is an $\mathcal{L}_p$-*space* if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$. (An operator means a linear operator in this paper.)

It is known that each $L^p(\mu')$ on a measure space (Ω',Σ',μ') is an $\mathcal{L}_{p,\lambda}$-space for every $\lambda > 1$, and an abstract (M)-space is an $\mathcal{L}_{\infty,\lambda}$-space for every $\lambda > 1$. Further the class of $\mathcal{L}_2$-spaces coincides with the class of Banach spaces isomorphic with Hilbert spaces. For instance, a Banach space, such that its norm and the norm of its adjoint space are both twice continuously Fréchet differentiable, is of class $\mathcal{L}_2$. Several properties of $\mathcal{L}_p$-spaces, some of which will be needed here, can be found in [7].

In terms of the above notation and concepts one has:

THEOREM 4. *Let* (Ω,Σ) *be a measurable space,* $B(\Omega,\Sigma)$ *the Banach space of scalar* $(\Sigma\text{-})$*measurable bounded functions with uniform norm, and* $\mathcal{Y}$ *an* $\mathcal{L}_p$-*space,* $1 \le p \le 2$. *Let* $\nu:\Sigma \to \mathcal{Y}$ *be a vector measure. Then* ν *is* $(2,\mu)$-*dominated. More explicitly, there exists a finite positive measure* μ *on* Σ *such that*

$$\left\|\int_\Omega f(\omega)\nu(d\omega)\right\|_{\mathcal{Y}} \le \|f\|_{2,\mu} , \qquad f\in B(\Omega,\Sigma) , \tag{11}$$

and ν *has 2-semi-variation finite relative to* μ .

Proof: For the following relatively short argument, some auxiliary results from Functional Analysis are needed, and they will be given with references. Let $T:f \mapsto \int_\Omega f(\omega)\nu(d\omega)\in\mathcal{Y}$, $f\in\mathfrak{X} = B(\Omega,\Sigma)$, so that T is a well-defined operator and since ν is a vector measure it is also sequentially continuous for bounded pointwise limits, by ([4], IV.10.10). This means if $f_n\in\mathfrak{X}$, $f_n \to f$ pointwise and boundedly, then $\|Tf_n\|_{\mathcal{Y}} \to \|Tf\|_{\mathcal{Y}}$, and of course T is bounded. Now (11) will be established in three steps.

I. First assume that $\mathfrak{X} = C(S)$, the space of real continuous functions on a compact Hausdorff space S . Let $q:s \mapsto \ell_s\in\mathfrak{X}^*$, where $\ell_s(f) = f(s)$, $f\in\mathfrak{X}$, the evaluation functional on $\mathfrak{X} = C(S)$. If $K \subset \mathfrak{X}^*$ is the set of all extreme points of the unit ball, then by the Mil'man's theorem (cf. [4], V.8, pp. 440-442), since S is compact Hausdorff, K is closed and equals $q(S)\cup(-q(S))$, the extreme points being of the form $\alpha\ell_s$ with $|\alpha| = 1$. Thus if $T\in B(C(S),\mathcal{Y})$ where $\mathcal{Y}$ is an $\mathcal{L}_p$-space $1 \le p \le 2$, then by ([7], Corol. 2 to Thm. 4.3 and Prop. 3.1, the latter is the Grothendieck-Pietsch inequality alluded to in the Introduction), the space $\mathfrak{X} = C(S)$ being an $\mathcal{L}_\infty$-space, there exists a regular probability measure μ_0 on K, hence on $q(S)\cup(-q(S))$, absolute constants c_1, c_2 such that

$$\|Tf\|_{\mathcal{Y}}^2 \le c_1 \int_{q(S)} |\ell_s(f)|^2 \mu_0(d\ell_s) + c_2 \int_{-q(S)} |\ell_s(f)|^2 \mu_0(d\ell_s), \quad 1 \le p \le 2,$$

$$\le c_3 \int_S |f(s)|^2 \mu_0(ds), \quad f \in \mathcal{X}. \tag{12}$$

Here S and $q(S)$ are identified (as they can be) and $c_3 = 2\max(c_1, c_2)$. For the complex case $C(S) = C_r(S) + iC_r(S)$ so that the inequality (12) holds if c_3 is replaced by $c_4 = 2c_3$. This is (11) if $\mathcal{X} = C(S)$ there, and if one defines the measure μ as $c_4\mu_0$.

II. Suppose $\mathcal{X} = B(\Omega, \Sigma)$, and $\mathcal{Y}$ an $\mathcal{L}_p$-space, $1 \le p \le 2$ as before. Since $\mathcal{X}$ is a closed subalgebra of $B(\Omega)$ $(= B(\Omega, 2^{\Omega}))$, it follows by the isomorphism theorem (cf. [4], IV.6.18) that there is a compact (extremally disconnected) Hausdorff space S_0 and an isometric algebraic isomorphism I between $\mathcal{X}$ and $\mathcal{X}_0 = C(S_0)$ which maps real elements of $\mathcal{X}$ into real functions of $\mathcal{X}_0$, complex conjugate *functions into* complex conjugate ones and preserves order relation between real functions. Let $\tilde{T} = T \cdot I^{-1} : \mathcal{X}_0 \to \mathcal{Y}$. Then $\tilde{T} \in B(\mathcal{X}_0, \mathcal{Y})$ and $\tilde{T}$ satisfies the hypothesis of Step I. Hence there is a regular Borel measure μ_1 on S_0 into $\mathbb{R}^+$ such that

$$\|\tilde{T}f\|_{\mathcal{Y}} \le \|f\|_{2,\mu_1}, \qquad f \in \mathcal{X}_0. \tag{13}$$

Now $f \in \mathcal{X}$ *implies* $\tilde{f} = I(f) \in \mathcal{X}_0$. Consequently (13) can be simplified as follows:

$$\|Tf\|_{\mathcal{Y}} = \|\tilde{T}\tilde{f}\|_{\mathcal{Y}} \le \|\tilde{f}\|_{2,\mu_1}, \quad f \in \mathcal{X},$$

$= \langle \tilde{f}\bar{\tilde{f}}, \mu_1 \rangle$, since $\mu_1 \in \mathcal{X}_0^*$ and $\langle \cdot, \cdot \rangle$ is the duality pairing,

$= \langle I(f) I(\bar{f}), \mu_1 \rangle$

$= \langle I(f\bar{f}), \mu_1 \rangle$, by the algebraic properties of I,

$= \langle f\bar{f}, I^*(\mu_1) \rangle$, $I^* : \mathcal{X}_0^* \to \mathcal{X}^*$ is the adjoint mapping of I,

$$= \int_\Omega |f|^2 \mu_2(d\omega), \tag{14}$$

where $\mu_2 = I^*(\mu_1) \in \mathcal{X}^* = ba(\Omega, \Sigma)$, the space of bounded additive set functions on Σ with total variation as norm. Here the integral relative to a finitely additive μ_2 is defined in the standard manner (cf. [4], p. 108ff). It thus remains to show that, in (14), μ_2 may be replaced by a σ-additive measure.

III. To extend the result for a bounded σ-additive measure, let μ be the Carathéodory generated measure by the pair (Σ,μ). Let Σ_μ be the class of μ-measurable sets. Then the classical theory implies (cf., e.g. [13], pp. 66-67) that $\Sigma_\mu \supset \Sigma$, and μ is σ-additive on Σ_μ, $\mu(A) \le \mu_2(A)$, $A \in \Sigma$ (equality holds iff μ_2 is also σ-additive on Σ). Now (11) will follow if (14) is shown to be true with μ in place of μ_2 and f a step function, since step functions are uniformly dense in $B(\Omega,\Sigma)$ (cf. [4], p. 259). This is verified by a direct computation below.

So let $f = \sum_{i=1}^{m} a_i \chi_{A_i}$, $A_i \in \Sigma_0$, disjoint, and $a_i \ne 0$. By definition of μ and the boundedness of μ_2, given $\epsilon > 0$, there exist $A_{in}^\epsilon \in \Sigma$ such that $A_i \subset \bigcup_{n=1}^{\infty} A_{in}^\epsilon$ and

$$\mu(A_i) + \frac{\epsilon}{m|a_i|^2} > \sum_{n=1}^{\infty} \mu_2(A_{in}^\epsilon) . \tag{15}$$

Replacing A_{in}^ϵ by $A_i \cap A_{in}^\epsilon$ in Σ, if necessary, one may assume in the above that $A_i = \bigcup_{n\ge 1} A_{in}^\epsilon$ also, without changing the inequality (15). Let $f_N^\epsilon = \sum_{i=1}^{m} a_i \chi_{\bigcup_{k=1}^{N} A_{ik}^\epsilon}$ with the stated modifications. Then $f_N^\epsilon \in \Sigma$, and $f_N^\epsilon \to f$ pointwise and boundedly. Consequently (14) simplifies to:

$$\begin{aligned}
\|Tf_N^\epsilon\|_\psi^2 &= \|\int_\Omega f_N^\epsilon(\omega)\nu(d\omega)\|_\psi^2 \le \int_\Omega |f_N^\epsilon(\omega)|^2 \mu_2(d\omega) \\
&= \sum_{i=1}^{m} |a_i|^2 \mu_2\Big(\bigcup_{k=1}^{N} A_{ik}^\epsilon\Big) \\
&= \sum_{i=1}^{m} |a_i|^2 \sum_{k=1}^{N} \mu_2(A_{ik}^\epsilon) , \text{ since } \mu_2 \text{ is additive.}
\end{aligned}$$

Letting $N \to \infty$ on both sides and using ([4], IV.10.10) one has

$$\begin{aligned}
\|Tf\|_\psi^2 &= \|\int_\Omega f(\omega)\nu(d\omega)\|_\psi^2 \\
&\le \sum_{i=1}^{m} |a_i|^2 \Big[\mu(A_i) + \frac{\epsilon}{m|a_i|^2}\Big] , \quad \text{by (15)}, \\
&= \int_\Omega |f(\omega)|^2 \mu(d\omega) + \epsilon .
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, (11) is proved for all step functions $f \in \Sigma$ and hence, by the earlier comment, generally. This completes the proof.

In the rest of the paper some important stochastic and operator applications of this result with ψ as an L^2-space will be presented.

4. Application to Cramér and Karhunen processes.

One of the most interesting applications of the domination problem, especially the special case treated in the preding section, is in relating two general classes of nonstationary second order processes, to be called *Cramér and Karhunen classes here. It will be shown essentially* that the projection of each Karhunen class is of Cramér class and many, but not all, Cramér classes are projections of some Karhunen classes on enlarged probability spaces, depending on the process under consideration.

To introduce these processes, let $X:\mathbb{R} \to L_0^2(\Omega,\Sigma,P) = L_0^2(P)$ be a mapping where $L_0^2(P)$ is the L^2-space on a probability triple (Ω,Σ,P), where $f \in L_0^2(P)$ iff $\int_\Omega f dP = E(f) = 0$. Then X is called a Karhunen process (or class) if the covariance function $r(\cdot,\cdot):(s,t) \mapsto E(X(s)\overline{X(t)}) = (X(s),X(t))$, the inner product, can be represented as the Lebesgue-Stieltjes (LS-) integral (cf. [6]):

$$r(s,t) = \int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda)}F(d\lambda), \qquad s,t\in\mathbb{R}, \tag{16}$$

relative to a class of Borel functions $\{g(s,\cdot),s\in\mathbb{R}\}$ and a σ-finite Borel measure F on $\mathcal{B}(\mathbb{R})$. It can be shown that such a process is representable as:

$$X(t) = \int_{\mathbb{R}} g(t,\lambda)Z(d\lambda), \qquad t\in\mathbb{R}, \tag{17}$$

where $Z:\mathcal{B}_0(\mathbb{R}) \to L_0^2(P)$, satisfies $(Z(A),Z(B)) = F(A\cap B)$, $\mathcal{B}_0(\mathbb{R})$ being the δ-ring of bounded Borel sets of $\mathbb{R}$. Thus $Z(\cdot)$ has orthogonal values. The mapping X, instead, is called a Cramér process (or class) if its covariance is expressible as the strong Morse-Transue (or MT-) integral (cf. [9] for the basic theory of this nonabsolute integral):

$$r(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda')}\tilde{F}(d\lambda,d\lambda'), \qquad s,t\in\mathbb{R} \tag{18}$$

relative to a class of Borel functions $\{g(s,\cdot),s\in\mathbb{R}\}$ and a covariance bimeasure of finite Fréchet variation on $\mathcal{B}_0(\mathbb{R})\times\mathcal{B}_0(\mathbb{R})$, such that

$$0 \le \int_{\mathbb{R}}\int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda')}\tilde{F}(d\lambda,d\lambda') < \infty .$$

If $\tilde{F}$ is of finite Vitali variation on $\mathcal{B}_0(\mathbb{R})\times\mathcal{B}_0(\mathbb{R})$, then the above integrals become LS-integrals. The latter case is the one actually considered in [2], but the present generality is needed. This will be called a Cramér process. Note that if $\tilde{F}$ concentrates on the diagonal, then (18) reduces to (16) so that the Karhunen class is a subset of the Cramér class.

Again it can be proved that the Cramér process also admits an integral representation as:

$$X(t) = \int_{\mathbb{R}} g(t,\lambda)\tilde{Z}(d\lambda)\ , \qquad t\in\mathbb{R}\ , \tag{19}$$

for a σ-additive $\tilde{Z}:\mathcal{B}_0(\mathbb{R}) \to L_0^2(P)$ such that $(\tilde{Z}(A),\tilde{Z}(B)) = \tilde{F}(A,B)$ and the integrals in (17) and (19) are in the Dunford-Schwartz sense. (I have verified both these representations for the work in [12].) For simplicity, it will be assumed hereafter that, for the work of (16)-(19), $\mathcal{B}_0(\mathbb{R})$ can be replaced by $\mathcal{B}(\mathbb{R})$ itself, so that $Z,\tilde{Z}$ are vector measures on $\mathcal{B}(\mathbb{R})$ into $L_0^2(P)$.

If $g(t,\lambda) = e^{it\lambda}$, then the above defined Cramér process becomes a <u>weakly harmonizable process</u> and it is <u>strongly harmonizable</u> if the MT-integral in (18) is replaced by the LS-integral. The latter concept was first introduced by Loève ([8], p. 474). The general dilation result stated at the beginning of this section will now be demonstrated.

Let $X:\mathbb{R} \to L_0^2(P)$ be a Karhunen process relative to a family $\{g(s,\cdot),s\in\mathbb{R}\}$ and a σ-finite measure F on $\mathcal{B}(\mathbb{R})$ as in (16). If $T:L_0^2(P) \to L_0^2(P)$ is any bounded linear operator, consider $Y(t) = TX(t)$, $t\in\mathbb{R}$. Using the representation (17), one has

$$Y(t) = T\int_{\mathbb{R}} g(t,\lambda)Z(d\lambda) = \int_{\mathbb{R}} g(t,\lambda)(T\cdot Z)(d\lambda)\ , \tag{20}$$

by a classical theorem (cf. [4], p. 324), since $g(t,\cdot)$ is Z-integrable implies it is also $T\cdot Z$-integrable (cf. [15], p. 79). Letting $\tilde{Z} = T\cdot Z$, which is a vector measure on $\mathcal{B}(\mathbb{R})$ into $L_0^2(P)$, it is seen that the covariance of the process Y is expressible as in (18) relative to the bimeasure function $\tilde{F}:(A,B) \mapsto (\tilde{Z}(A),\tilde{Z}(B))$, $A,B\in\mathcal{B}(\mathbb{R})$. Thus $Y:\mathbb{R} \to L_0^2(P)$ is a Cramér process.

The result in the opposite direction is harder. It uses Theorem 4 in a crucial manner. Thus let $\{X(t),t\in\mathbb{R}\}$ be a Cramér process relative to $\{g(t,\cdot),t\in\mathbb{R}\}$ and $\tilde{F}$ as in (18), and then by (19)

$$X(t) = \int_{\mathbb{R}} g(t,\lambda)\tilde{Z}(d\lambda)\ , \qquad t\in\mathbb{R}\ ,$$

with $\tilde{Z}:\mathcal{B}(\mathbb{R}) \to L_0^2(P)$ as a vector measure, by the current assumption. Taking $\mathcal{V} = L_0^2(P)$ in Theorem 4, it follows that there is a finite regular Borel measure μ on $\mathcal{B}(\mathbb{R})$ such that

$$\left\|\int_{\mathbb{R}} f(\lambda)\tilde{Z}(d\lambda)\right\|_{\mathcal{V}} \le \|f\|_{2,\mu}\ , \qquad f\in B(\mathbb{R},\mathcal{B}(\mathbb{R}))\ . \tag{21}$$

What if f is not bounded in (21)? If f is $\tilde{Z}$-integrable, then there exists $f_n\in B(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that $f_n \to f$ pointwise and by the vector dominated convergence theorem (cf. [4], IV.10.10) one has

$$\left\|\int_{\mathbb{R}} f(\lambda)\tilde{Z}(d\lambda)\right\|^2 = \lim_{n\to\infty}\left\|\int_{\mathbb{R}} f_n(\lambda)\tilde{Z}(d\lambda)\right\|^2$$

$$\le \varliminf_{n\to\infty} \int_{\mathbb{R}} |f_n(\lambda)|^2 \mu(d\lambda) \text{ , by (21),}$$

$$\le \int_{\mathbb{R}} \varliminf_{n\to\infty} |f_n(\lambda)|^2 \mu(d\lambda) \text{ , by Fatou's inequality,}$$

$$= \int_{\mathbb{R}} |f(\lambda)|^2 \mu(d\lambda) \text{ .} \tag{22}$$

However, while the left side of (22) is finite, the right side can be infinite when f is not bounded. Nevertheless, (22) is of interest. If μ is either a Lebesgue measure, or is dominated by the Lebesgue measure with a bounded density, then (the process determined by) the vector measure $\tilde{Z}$ for which (22) is true is called an $L^{2,2}$-bounded measure (and process, respectively) by Bochner (cf. [1], p. 25) who emphasized the importance of this concept. The Wiener process is a particular example of this. So hereafter $\{g(t,\cdot), t\in\mathbb{R}\}$ will also be considered as contained in $L^2(\mu)$ for any μ satisfying (22). In particular $g(t,\cdot)\in B(\mathbb{R},\mathcal{B}(\mathbb{R})) \subset L^2(\mu)$, $t\in\mathbb{R}$, for every such finite dominating μ, verifies this assumption.

Define a bimeasure $v:\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}) \to \mathbb{C}$ as $v(A,B) = \mu(A\cap B)$. Hence one has

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_1(\lambda,\lambda')v(d\lambda,d\lambda') = \int_{\mathbb{R}} f_1(\lambda,\lambda)\mu(d\lambda) \text{ ,} \quad f_1\in L^1(\mu) \text{ .} \tag{23}$$

Setting $\alpha = v - \tilde{F}:\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}) \to \mathbb{C}$, (22) implies with $f_1(\lambda,\lambda') = f(\lambda)\overline{f(\lambda')}$ in (23),

$$0 \le \int_{\mathbb{R}} |f(\lambda)|^2 \mu(d\lambda) - \left\|\int_{\mathbb{R}} f(\lambda)\tilde{Z}(d\lambda)\right\|^2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f(\lambda')}v(d\lambda,d\lambda') - \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f(\lambda')}\tilde{F}(d\lambda,d\lambda')$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f(\lambda')}\alpha(d\lambda,d\lambda') = I(f,f) \quad \text{(say),} \quad f\in L^2(\mu) \text{ .} \tag{24}$$

Thus $\alpha(\cdot,\cdot)$ is a covariance bimeasure on $\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R})$. Considering $I(f,g)$ as the MT-integral relative to α which is clearly of finite Fréchet variation (since v and $\tilde{F}$ are), it follows that $I(f,g) = \overline{I(g,f)}$, $0 \le I(f,f) < \infty$, so that $I:L^2(\mu)\times L^2(\mu) \to \mathbb{C}$ qualifies to be a positive hermitian kernel. Then by the theory of Aronszajn, there is a Hilbert space $\mathcal{H}$ determined by $I(\cdot,\cdot)$, such that $I(f,g) = (h_f,h_g)$, $h_f,h_g\in\mathcal{H}$. But for concreteness, a short explicit construction of $\mathcal{H}$ will be included here so that a Karhunen process in $\mathcal{H}$ with α as its covariance bimeasure (in the representation (16)) can be exhibited.

Let $[\cdot,\cdot]':L^2(\mu)\times L^2(\mu) \to \mathbb{C}$ be defined by $[f,g]' = I(f,g)$. Then by (24), $[\cdot,\cdot]'$ is a semi-inner product on $L^2(\mu)$. If $\eta_0 = \{f:[f,f] = 0\}$ and $\mathcal{H}_1$ is the set $L^2(\mu)/\eta_0$, define $[\cdot,\cdot]:\mathcal{H}_1\times\mathcal{H}_1 \to \mathbb{C}$ by

$$[(f),(g)] = [f,g]', \qquad f\in(f)\in\mathcal{H}_1,\ g\in(g)\in\mathcal{H}_1. \tag{25}$$

Then $[\cdot,\cdot]$ is an inner product on $\mathcal{H}_1$ and let $\mathcal{H}_0$ be its completion in this induced metric. Let $\pi_0 : L^2(\mu)\to\mathcal{H}_0$ be the canonical projection. Note that $\mathcal{H}_0$ may not be separable. Consider the subspaces $\mathcal{H}', \mathcal{H}''$ defined by $\mathcal{H}' = \overline{sp}\{X(t), t\in\mathbb{R}\}\subset L_0^2(P)$, $\mathcal{H}'' = \overline{sp}\{X_1(t)=\pi_0(g(t,\cdot)), t\in\mathbb{R}\}\subset\mathcal{H}_0$ and set up the direct sum $\mathcal{H} = \mathcal{H}'\oplus\mathcal{H}''$ whose inner product is the sum of the inner products of $\mathcal{H}'$ and $\mathcal{H}''$, identified as $\mathcal{H}'\oplus\{0\}$, $\{0\}\oplus\mathcal{H}''$.

Since $\mathcal{H}'\subset L_0^2(P)$, let us realise $\mathcal{H}''$ as a subspace of some $L_0^2(P')$ on a probability space (Ω',Σ',P') so that one can enlarge the original (Ω,Σ,P) by adjoining this new triple (i.e., $(\tilde\Omega,\tilde\Sigma,\tilde P) = (\Omega,\Sigma,P)\otimes(\Omega',\Sigma',P')$) and then one can realize $\mathcal{H}$ as a subspace of $L_0^2(\tilde P)$. Thus both X and X_1-processes will be independent and take values in $\mathcal{H}$, $Y = X+X_1$, $Y:\mathbb{R}\to L_0^2(\tilde P)$ will be shown to be the desired Karhunen process. Such a realization as noted above is classical, but may not be as well known. So a brief sketch will be included here.

Let $\{h_j, j\in J\}\subset\mathcal{H}''$ be a complete orthonormal set. If $\Omega_j = \mathbb{C}$, Σ_j = Borel σ-algebra of $\mathbb{C}$, and $P_j(A) = (2\pi)^{-1}\int_A \exp(-|t|^2/2)\,dt_1 dt_2$, $A\in\Sigma_j$, $t = t_1+it_2\in\mathbb{C}$, $j\in J$, let $(\Omega',\Sigma',P') = \bigotimes_{j\in J}(\Omega_j,\Sigma_j,P_j)$, the product space given by the Fubini-Jessen theorem (cf. [4], III.11.20). If $\tilde X(j) = \omega(j)$, $j\in J$, $\omega\in\Omega' = \mathbb{C}^J$, the coordinate function, then it follows that $E(\tilde X(j)) = 0$, $E(|\tilde X(j)|^2) = 1$ and $\{\tilde X(j), j\in J\}$ is a set of independent standard Gaussian random variables in $L_0^2(P')$. The correspondence $\tau : h_i\mapsto\tilde X(i)$, extended linearly, sets up an isomorphism of $\mathcal{H}''$ onto $\mathcal{L} = \overline{sp}\{\tilde X(j), j\in J\}\subset L_0^2(P')$, and $\|\tau(h_j)\|^2 = E(|X(j)|^2) = [h_j,h_j,] = 1$, $j\in J$. Thus τ is an isometric isomorphism. Corresponding to $X_1 : \mathbb{R}\to\mathcal{H}''$, let $\tilde X_1 = \tau(X_1) : \mathbb{R}\to L_0^2(P')$, so that realizing $\mathcal{H} = \mathcal{H}'\oplus\mathcal{H}''\subset L_0^2(\tilde P)$, if $Y(t) = X(t)+\tilde X_1(t)$, $t\in\mathbb{R}$, then $(X(s),\tilde X_1(t)) = 0$, $s,t\in\mathbb{R}$, $Y(t)\in L_0^2(\tilde P)$. It is claimed that $Y:\mathbb{R}\to L_0^2(\tilde P)$ is the desired Karhunen process.

Identifying $L_0^2(P)$ as a subspace of $L_0^2(\tilde P)$, if Q is the orthogonal projection of $L_0^2(\tilde P)$ onto $\mathcal{H}'$ ($\subset L_0^2(\tilde P)$), then $QY(t) = X(t)$, $t\in\mathbb{R}$. Also

$$\begin{aligned}
\tilde r(s,t) &= (Y(s),Y(t)) = (X(s),X(t)) + (\tilde X_1(s),\tilde X_1(t)), \quad \text{since } X\perp\tilde X_1 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda')}\,\tilde F(d\lambda,d\lambda') + \int_{\mathbb{R}}\int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda')}\,\nu_2(d\lambda,d\lambda'), \text{ by (24) and (25)}, \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda')}\,\nu(d\lambda,d\lambda') \\
&= \int_{\mathbb{R}} g(s,\lambda)\overline{g(t,\lambda)}\,\mu(d\lambda), \quad \text{by (23)}.
\end{aligned}$$

Hence $\{Y(t), t \in \mathbb{R}\} \subset L_0^2(\tilde{P})$ is a Karhunen process relative to the same family $\{g(t,\cdot), t \in \mathbb{R}\}$ and μ.

The preceding work thus proves the following comprehensive result:

THEOREM 5. Let $X: \mathbb{R} \to L_0^2(P)$ be a process and $\{g(t,\cdot), t \in \mathbb{R}\}$ be a family of Borel functions. If X is a Karhunen process relative to this $g(t,\cdot)$-family and a σ-finite measure F on $\mathcal{B}(\mathbb{R})$, and $T: L_0^2(P) \to L_0^2(P)$ is any continuous linear mapping, then $\{Y(t) = TX(t), t \in \mathbb{R}\}$ is a Cramér process relative to the same g-family and a suitable covariance bimeasure. Conversely, if $\{g(t,\cdot), t \in \mathbb{R}\}$ is a bounded Borel family and $X: \mathbb{R} \to L_0^2(P)$ is a Cramér process relative to this g-family and a suitable covariance bimeasure, then there exists an extension space $L_0^2(\tilde{P})$ ($\supset L_0^2(P)$) determined by the given process, a Karhunen process $Y: \mathbb{R} \to L_0^2(\tilde{P})$ relative to the same g-family and a suitable finite Borel measure on $\mathbb{R}$, such that $X(t) = QY(t)$, $t \in \mathbb{R}$, where Q is the orthogonal projection of $L_0^2(\tilde{P})$ onto $L_0^2(P)$.

Some comments on this general result are now in order.

Remarks 6. (a) One of the important queries raised by this result is that whether every Cramér process is obtainable as a projection of some Karhunen process on a sufficiently large super Hilbert space. In general, the answer is in the negative. Indeed, if the result were true where the $g(t,\cdot)$-family is merely $\tilde{Z}$-integrable in (19), then it must also be Z-integrable for (17). Since $\tilde{Z} = Q \cdot Z$ is then true, one has $\mathcal{L}^2(\tilde{Z}) \supset \mathcal{L}^2(Z)$. But now there must be equality here. However, a counter-example for this equality has been constructed at the Oberwolfach meetings by Erik Thomas. I wish to acknowledge an enlightening discussion with him on this matter.

(b) The preceding remark and theorem show that the class of Cramér processes is quite large and some of its members cannot be dilated to Karhunen processes. Since each $g(t,\cdot)$-function which is integrable relative to a vector measure ν is also integrable relative to $\tilde{\nu} = T \cdot \nu$ for each continuous linear mapping $T: L_0^2(P) \to L_0^2(P)$ (cf. [15], p. 79), it follows that the Cramér class is closed under all such transformations.

(c) In the above work, the fact that the processes are indexed by $\mathbb{R}$ is not really used. Hence the result holds if $\mathbb{R}$ is replaced by a locally compact space and F or $\tilde{F}$ is a Radon positive definite bimeasure on such a space, since the LS- and MT-integration theories are available on these spaces.

In the case of $\mathbb{R}$, taking $g(t,\lambda) = e^{it\lambda} = e_t(\lambda)$, a character of $\mathbb{R}$, the Cramér process becomes a weakly harmonizable process, and the above result thus implies the following one. Now $\mathbb{R}$ can be replaced by an LCA group by the last remark, and the $r(\cdot,\cdot)$ is assumed _continuous_ for the next result.

THEOREM 7. _Let_ G _be a locally compact abelian group and_ $X:G \to L_0^2(P)$ _be a process_. _If_ X _is weakly stationary in that_ $r(s,t) = r(st^{-1})$, _then_ $Y(t) = TX(t)$, $t \in G$ _and_ $T \in \mathcal{B}(L_0^2(P))$ _defines a weakly harmonizable process_, $Y:G \to L_0^2(P)$. _Conversely, given a weakly harmonizable process_ $Y:G \to L_0^2(P)$, _there exists an extension space_ $L_0^2(\tilde{P}) \supset L_0^2(P)$ _and a weakly stationary process_ $X:G \to L_0^2(\tilde{P})$ _such that_ $Y(t) = QX(t)$, $t \in G$, _where_ Q _is the orthogonal projection of_ $L_0^2(\tilde{P})$ _onto_ $L_0^2(P)$.

Since each character of the LCA group G is a bounded continuous function, all weakly harmonizable processes are accounted for in this construction. However, each extension space may be different for each process and all those super spaces may have nothing in common, except the given $L_0^2(P)$ as a subspace.

5. _Application to self-adjoint operator dilations_.

It is of interest to present an operator theoretic characterization of Theorem 5. Though it is essentially a translation of language, it nevertheless gives further insight into the structure of these processes.

Definition 8. Let $\mathfrak{X}$ be a Banach space, $A:\mathfrak{X} \to \mathfrak{X}$ be a linear (perhaps unbounded) operator, and $\sigma(A)$, the spectrum of A, be a proper subset of $\mathbb{C}$. Let $\mathfrak{F}(A)$ be the collection of all mappings $f:\mathbb{C} \to \mathbb{C}$, analytic in some neighborhood of $\sigma(A)$ and at "∞," where "∞" may be in $\sigma(A)$ in the unbounded case. The neighborhood need not be connected, and can depend on f. If $f \in \mathfrak{F}(A)$, let Γ be the boundary of an open $V \supset \sigma(A)$ consisting of a finite number of Jordan arcs such that f is analytic on $V \cup \Gamma$. Then one defines the operator $f(A)$ as:

$$f(A) = f(\infty)I + \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A)\,d\lambda\ , \tag{26}$$

where $R(\lambda,A) = (A-\lambda I)^{-1}$, is the resolvent of A. (26) is the Bochner integral.

The operator $f(A)$ is well defined and is closed. When $\mathfrak{X} = \mathcal{H}$, a Hilbert space, one can show that an equivalent definition is as follows: for each Borel f on $\mathbb{R}$ and self-adjoint A, if $f_n = f\chi_{[|f(\cdot)|\le n]}$, and $\mathcal{D}_{f(A)} = \{x \in \mathcal{H} : \lim_n f_n(A)x \text{ exists}\}$, then $f(A)x = \lim_n f_n(A)x$, $x \in \mathcal{D}_{f(A)}$. Thus (26) specializes to this when all the conditions are met. This formulation will be used in the next result.

In what follows, it will be assumed that $g(0,\cdot) = 1$ so that $r(0,0) = k < \infty$. This will force F to be a finite measure for the Karhunen process case (cf. (16)). It is a convenient normalization. Then the following result obtains:

THEOREM 9. *Let* $X:\mathbb{R}\to L_0^2(P)$ *be a Cramér process, relative to a class* $\{g(t,\cdot),t\in\mathbb{R}\}$ *of bounded Borel functions with* $g(0,\cdot)\equiv 1$. *Then there is an extension space* $L_0^2(P')\supset L_0^2(P)$, *an element* $Y_0\in L_0^2(P')$ *and an unbounded linear operator* A *from* $L_0^2(P')$ *into* $L_0^2(P)$ *such that* $A|_{L_0^2(P)}$ *is symmetric with domain dense in* $\overline{sp}\{X(t),t\in\mathbb{R}\}\subset L_0^2(P)$, *and that* ($g(t,A)$ *is defined as in the above definition and comment*)

$$X(t) = g(t,A)Y_0 , \quad t\in\mathbb{R} . \tag{27}$$

Conversely, if A *is a symmetric densely defined operator in* $L_0^2(P)$, $X_0\in L^2(P)$ *and the* $g(t,\cdot)$'s *are as above, then the process* $Y(t) = g(t,A)X_0$, $t\in\mathbb{R}$, *is always a Cramér process relative to the g-family.*

Proof: Let $X:\mathbb{R}\to L_0^2(P)$ be a Cramér process relative to the given g-family. Then by Theorem 5, there is an extension space $L_0^2(P')\supset L_0^2(P)$, and a Karhunen process $Y:\mathbb{R}\to L_0^2(P')$ such that $X(t) = QY(t)$, $t\in\mathbb{R}$, where Q is the orthogonal projection of $L_0^2(P')$ onto $L_0^2(P)$. Since $g(0,\cdot)\equiv 1$, the representing measure (of (16)) is finite. But then a Karhunen process can also be given in an operator theoretic form as:

$$Y(t) = g(t,\tilde{A})Y(0) , \quad t\in\mathbb{R} , \tag{28}$$

where $\tilde{A}$ is an unbounded self-adjoint operator with dense domain in $\overline{sp}\{Y(t),t\in\mathbb{R}\}\subset L_0^2(P')$. This version of the representation (17) has been proved by Getoor ([5], Thm. 3A). Note that $g(t,\tilde{A})$ is actually defined by the spectral theorem for $\tilde{A}$. Consequently

$$g(t,\tilde{A})Y(0) = \int_{\mathbb{R}} g(t,\lambda)\tilde{E}(d\lambda)Y(0) \tag{29}$$

where $\{\tilde{E}(t),t\in\mathbb{R}\}$ is the resolution of the identity of $\tilde{A}$. Since $g(t,\cdot)\in\mathcal{L}(\tilde{Z})$, where $\tilde{Z} = \tilde{E}(\cdot)Y(0)$ is a vector measure, it follows also that $g(t,\cdot)\in\mathcal{L}(Q\tilde{Z})$, by ([15], p. 79). Thus

$$\begin{aligned} X(t) = QY(t) &= \int_{\mathbb{R}} g(t,\lambda)(Q\cdot\tilde{E})(d\lambda)Y(0) , \quad \text{by ([4], p. 324)}, \\ &= g(t,A)Y(0) , \end{aligned} \tag{30}$$

where $A = \int_{\mathbb{R}} \lambda E(d\lambda)$, with $E(\lambda) = Q\tilde{E}(\lambda)$, $\lambda\in\mathbb{R}$, as a *generalized spectral family* (i.e., its increments are positive but not necessarily projections) (cf. [14], p. 6). It is known and easily verified that $A|_{L_0^2(P)}$ is a symmetric and densely defined operator of the stated kind. Note that $g(0,A) = Q$ and so $X(0) = QY(0)$.

For the converse part, if A is a symmetric densely defined operator as in the statement, then by Naĭmark's theorem ([14], Thm. I), it extends to a self-adjoint operator $\tilde{A}$ on an extension space $L_0^2(P')$, such that $A = Q\tilde{A}$. Hence $g(t,A) = Qg(t,\tilde{A})$. But $Y(t) = g(t,\tilde{A})Y_0$, $t\in\mathbb{R}$, is a Karhunen process in $L_0^2(P')$ by (12) and [5] relative to the g-family. So by the corresponding part of Theorem 5, $X(t) = QY(t)$, $t\in\mathbb{R}$, is a Cramér process in $L_0^2(P)$ relative to the same g-family. This completes the proof.

The preceding result has an interesting consequence:

Remark. Each vector measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ into a Hilbert space is derived from a generalized spectral family.

For, let $\nu:\mathcal{B}(\mathbb{R})\to L_0^2(P)$ be a vector measure. Then $X(t) = \int_{\mathbb{R}} e^{it\lambda}\nu(d\lambda)$ is weakly harmonizable by Theorem 7, and hence, by the above result, there is an extension space $L_0^2(P') \supset L_0^2(P)$ a self-adjoint operator $\tilde{A}$ on it, and an element $\tilde{X}_0\in L_0^2(P')$ such that, with $g(t,\lambda) = e^{it\lambda}$,

$$\int_{\mathbb{R}} e^{it\lambda}\nu(d\lambda) = X(t) = Qg(t,\tilde{A})\tilde{X}_0 = \int_{\mathbb{R}} e^{it\lambda}(Q\cdot\tilde{E})(d\lambda)\tilde{X}_0\ , \quad t\in\mathbb{R}, \tag{31}$$

where $\{\tilde{E}(t),t\in\mathbb{R}\}$ is the resolution of the identity of $\tilde{A}$ in $L_0^2(P')$. If $\{E(t) = Q\tilde{E}(t),t\in\mathbb{R}\}$ is the generalized resolution, then (31) implies $\nu(\cdot) = E(\cdot)\tilde{X}_0$. This establishes the assertion.

The last theorem also yields the next result of interest on self-adjoint dilations of certain operators:

THEOREM 10. *Let A be a symmetric operator with dense domain in $\mathcal{H}$, and $\{g_t,t\in\mathbb{R}\}$ be a family of bounded Borel functions with $g_0 = 1$. Then $T_t = g_t(A)$, $t\in\mathbb{R}$, defines a family of bounded operators for which there exists an extension Hilbert space $\mathcal{K}\supset\mathcal{H}$, a self-adjoint operator $\tilde{A}$ on $\mathcal{K}$ extending A such that $T_t = Qg_t(\tilde{A})$, where Q is the orthogonal projection on $\mathcal{K}\to\mathcal{H}$. Conversely, every densely defined self-adjoint operator $\tilde{A}$ on a Hilbert space $\mathcal{K}$, and a family $\{g_t,t\in\mathbb{R}\}$ of Borel functions define a class of closed operators $T_t = Qg_t(\tilde{A})|_{\mathcal{H}} = g_t(A)|_{\mathcal{H}}$ where $A = Q\tilde{A}$ and $\mathcal{H} = Q(\mathcal{K})$, Q being an orthogonal projection on $\mathcal{K}$.*

If, in the above, $g_t(\lambda) = e^{it\lambda}$, then $g_t(\tilde{A}) = e^{it\tilde{A}} = U_t$ is a unitary operator and $T_t = Qg_t(\tilde{A}) = \int_{\mathbb{R}} e^{it\lambda}(Q\cdot\tilde{E})(d\lambda)$, defines a weakly continuous family of positive definite contractive operators in $\mathcal{H} = Q\mathcal{K}$. Hence the following result of Sz.-Nagy ([14], Thm. IV) is obtained from the above result, which depends only on Naĭmark's theorem.

THEOREM 11. *If $\{T_t,t\in\mathbb{R}\}$ is a weakly continuous positive definite contractive family of operators on a Hilbert space $\mathcal{H}$ with T_0 = identity, then there is an extension Hilbert space $\mathcal{K}\supset\mathcal{H}$, a unitary group*

$\{U_t, t\in\mathbb{R}\}$ *of operators on* $\mathcal{K}$ *such that* $T_t = QU_t$, $t\in\mathbb{R}$. *Conversely, every weakly continuous group of unitary operators* $\{U_t, t\in\mathbb{R}\}$ *defines a weakly continuous positive definite contractive family of operators* $\{T_t = QU_t, t\in\mathbb{R}\}$ *on* $\mathcal{H} = Q(\mathcal{K})$ *for each orthogonal projection* Q *on* $\mathcal{K}$.

It may be noted that the above theorem was independently proved in [14], and then Naimark's result was deduced from this one. The above work shows that the converse implication is valid as well. Thus both these results are essentially equivalent, though this equivalence lies somewhat deeper. Actually Sz.-Nagy has proved a more general result, the "Principal Theorem," in [14] for a suitable semi-group of operators and then deduced several of these results including Theorem 11. By a suitable choice of the g_t-family, it appears that one can obtain this general theorem with the above work plus Naimark's result, which will then imply the equivalence of all the results of [14] with the above point of view.

REFERENCES

1. Bochner, S., "Stationarity, boundedness, almost periodicity of random valued functions," *Proc. 3d. Berkeley Symp. Math. Statist. & Prob.* 2(1956), 7-27, Univ. of Calif. Press, Berkeley.

2. Cramér, H., "A contribution to the theory of stochastic processes," *Proc. 2d Berkeley Symp. Math. Statist. & Prob.* (1951), 329-339, Univ. of Calif. Press, Berkeley.

3. Dinculeanu, N., *Vector Measures*, Pergamon Press, London, 1966.

4. Dunford, N. and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, 1958.

5. Getoor, R. K., "The shift operator for nonstationary stochastic processes," Duke Math. J. 23(1956), 175-187.

6. Karhunen, K., "Über lineare Methoden in der Wahrscheinlichkeitsrechnung," Ann. Acad. Sci. Fenn., Ser. AI Math. 37(1947), 3-79.

7. Lindenstrauss, J. and A. Pełczyński, "Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications," Studia Math. 29(1968), 275-326.

8. Loève, M., *Probability Theory* (3d. ed.), D. Van Nostrand Co., Princeton, 1963.

9. Morse, M. and W. Transue, "$\mathbb{C}$-bimeasures and their integral extensions," Ann. Math. 64(1956), 480-504.

10. Niemi, H., "Stochastic processes as Fourier transformations of stochastic measures," Ann. Acad. Sci. Fenn. AI Math. 591(1975), 1-47.

11. Rao, M. M., *Foundations of Stochastic Analysis*, Academic Press, New York, 1981.

12. Rao, M. M., "Representation of weakly harmonizable processes," Proc. Nat. Acad. Sci. 79(1981), Sept. issue.

13. Sion, M., Introduction to Methods of Real Analysis, Holt, Rinehart, and Winston, San Francisco, 1968.

14. Sz.-Nagy, B., "Extensions of linear transformations in Hilbert space which extend beyond this space," Appendix to F. Riesz and B. Sz.-Nagy: Functional Analysis, Unger, New York, 1955.

15. Thomas, E., "L'integration par raport a une mesure de Radon vectrielle," Ann. Inst. Fourier Grenoble 20(1970), 55-191.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The domination problem for vector measures with φ-semi-variation finite is formulated and a general solution given. If these are stochastic measures having two moments, then a more tractable form of the result implying 2-domination is obtained. The latter is used in showing that a large class (but not all) of Cramér processes, which are nonstationary, admit dilations to Karhunen processes on an enlarged space. This generalizes the weakly harmonizable case.

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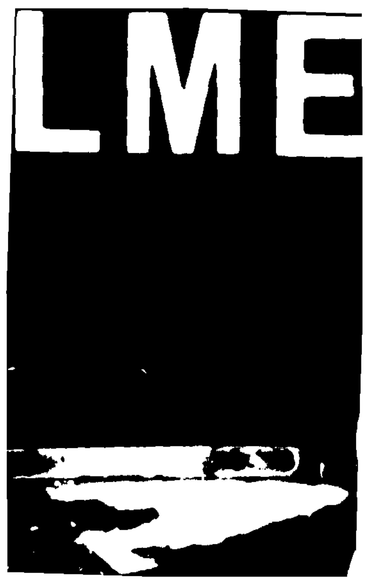
As a consequence, a large class of bounded operators on a Hilbert space is shown to admit self-adjoint dilations. This exhibits an equivalence between certain classical results of M. Naĭmark and B. Sz.-Nagy on operator dilations in Hilbert spaces.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