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TECHNICAL REPORT

WSRL-0197-TR

BEAM SPACE FORMULATION OF THE MAXIMUM SIGNAL-TO-NOISE RATIO ARRAY PROCESSOR

D. A. GRAY

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BEAM SPACE FORMULATION OF THE MAXIMUM SIGNAL-TO-NOISE RATIO ARRAY PROCESSOR

D.A. Gray

SUMMARY

The maximum signal-to-noise ratio array processor is formulated in beam space. Expressions for the optimum narrowband weight vectors and array gain are derived. Some general properties of the beamspace formulation are derived and conditions for the equivalence of the array and beam space formulations are proved.

Examples using both simulated and sonar data are given to compare the beam and array space formulations.

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1. INTRODUCTION

The performance of an array beamformer may often be improved by multiplying the receiver outputs by a set of weighting (or shading) coefficients prior to addition. In particular Edelblute, Fisk and Kinnison(ref.1) have derived a set of weights which for a known noise field, maximise the output signal-to-noise ratio of a beam steered in a given direction. These weights are determined by the crosspower spectral matrix of the receiver noise outputs and hence reflect the angular distribution of noise sources.

In practice the receiver outputs may not be accessible and so an estimate of the receiver noise crosspower spectral matrix cannot be directly obtained. One important practical case is where only beamformed outputs (ie conventional beams) are available. A set of weighting coefficients which maximise the output signal-to-noise ratio are derived in this paper from conventional beam rather than receiver outputs. Expressions for the array gains are also derived. The expressions derived are a generalisation of the equations obtained by Vural(ref.2) using a point constrained minimum power criterion.

In general the array space and beam space formulations are not equivalent. However as the number of independent beams used in the beam space formulation increases the beam space gains approach the array space gains. In Section 3 the equivalence of the two approaches is proved under some general conditions.

Some general properties of the beam space formulation are derived in Section 4. In Section 5 some theoretical and simulated examples are presented to illustrate the relationship of the conventional, the optimum array space and the optimum beam space processors. The simulation examples illustrate some practical limitations on the gains that can be achieved using the beam space methods. Finally in Section 5 some examples of the application of the beam space optimisation to sonar data are given.

This work is part of a continuing R&D programme in signal processing for underwater detection and has been carried out under task DST 79/069.

2. DERIVATION OF OPTIMUM BEAM WEIGHTS

2.1 Notation

Let y denote the vector of M conventional beam outputs at a frequency f, and let $\tilde{y}$ be a weighted scalar sum of these beam outputs, ie

$$\tilde{y} = w^H y$$

where w is the (in general complex) vector of weights.† If

$$y = s + n$$

where s_i and n_i (i=1,2,...M) are the signal and noise components in the i-th beam, then the crosspower spectral matrices of the signal and noise beam components are defined by

$$(Q)_{ij} = \langle s_i s_j^* \rangle$$

† The superscript ^{H} denotes the Hermitian transpose of a matrix or a vector.

and

$$(R)_{ij} = \langle n_i n_j^* \rangle$$

respectively, where $\langle \ \rangle$ denotes ensemble averaging and $i,j = 1,2,\ldots,M$ where M is the number of (preformed) beams used.

If the signal is a narrowband plane wave (of unity amplitude) then

$$Q = ss^H$$

where

$$s = V^H v(\theta),$$

$$(V)_{kj} = e^{-2\pi i f \tau_{kj}}$$

and

$$v_k(\theta) = e^{-2\pi i f \tau_k(\theta)}$$

where $k=1,2,\ldots,K$ and K is the number of receivers.

The time delay τ_{kj} is the time delay relative to some arbitrary reference point, at the k-th receiver of a plane wave from the j-th direction (corresponding to one of the chosen steer directions for the M preformed beams). The delays $\tau_k(\theta)$ are the time delays corresponding to the steering direction, θ, of the optimally weighted beam.

The signal-to-noise ratio in the derived beam, $\tilde{y}$, is defined as

$$SNR = \frac{w^H Q w}{w^H R w}$$

Array gain is defined by

$$g = \frac{\text{(SNR) beam}}{\text{(SNR) omnidirectional receivers.}}$$

2.2 Derivation

It can readily be shown that the SNR in the derived beam, $\tilde{y}$, is maximised by choosing w such that

$$Qw = \frac{w^H Q w}{w^H R w} Rw \qquad (1)$$

Equation (1) is satisfied if

$$w_o = \lambda R^{-1} s \tag{2}$$

where λ is any constant. Imposing the constraint of a fixed (ie unity) response in the steering direction defined by $v_j(\theta)$ implies that

$$w_o^H s = 1 .$$

This constraint is imposed to ensure that the signal power in the optimum beam increases as the signal power increases. Consequently equation (2) becomes

$$w_o = \frac{R^{-1} s}{s^H R^{-1} s}$$

$$= \frac{R^{-1} V^H v}{v^H V R^{-1} V^H v} . \tag{3}$$

As discussed in Appendix A the case where R is singular and V is of full rank can also be treated and an appropriate generalisation of equation (3) is to replace R^{-1} by R^+, the Moore Penrose pseudoinverse(ref.3) which, in addition to maximising the output SNR also minimises the superdirectivity of the weights.

It directly follows from the generalised version of equation (3) that the maximum output SNR for a unit amplitude signal is given by

$$(SNR)_{max} = s^H R^+ s .$$

For these optimally weighted beams the expression for array gain reduces to

$$g_o^y = \frac{s^H R^+ s}{N_o}$$

where N_o is the omnidirectional noise power at any receiver. (The assumption of a homogeneous medium and identical receivers, ie no shading, has been made).

Furthermore if R, the beam noise crosspower spectral matrix has been derived from normalised receiver outputs ie, the noise output of the j^{th} receiver, $n_j^{(x)}$, is transformed

$$n_j^{(x)} \longrightarrow n_j^{(x)} \Big/ \left< n_j^{(x)} \, n_j^{(x)*} \right>^{1/2}$$

then the optimum array gain in the steer direction specified by s is

$$g_o^y = s^H R^+ s.$$

3. SOME EQUIVALENCES OF THE ARRAY SPACE AND BEAM SPACE FORMULATIONS

For narrowband signals the conventional beam outputs y_i, i=1,2,...M can be related to the receiver outputs x_i, i=1,...,K by the equation

$$y = V^H x \tag{4}$$

where V is defined in the previous section. The crosspower spectral matrix of the M beam outputs, R_y, is defined by

$$(R_y)_{ij} = \langle n_i^{(y)}(f) n_j^{(y)*}(f) \rangle$$

for i,j=1,2,...,M and the crosspower spectral matrix of the K receivers is defined by

$$(R_x)_{ij} = \langle n_i^{(x)}(f) n_j^{(x)*}(f) \rangle$$

for i,j=1,2,...,K where $n_i^{(x)}$ and $n_j^{(y)}$ are the i-th normalised† receiver noise outputs and j-th beam noise outputs respectively.

Assuming the signal and noise are uncorrelated it follows from equation (4) that

$$R_y = V^H R_x V.$$

From reference 1 the optimum weight vector, w_o^x, using receiver outputs is

$$w_o^x = \frac{R_x^{-1} v}{v^H R_x^{-1} v} \tag{5}$$

and the corresponding array gain, g_o^x, is given by

$$g_o^x = v^H R_x^{-1} v . \tag{6}$$

---

† The use of normalised receiver outputs as defined in the previous section is simply a convenience which allows simple expressions for the crosspower spectral matrices and a concise formulation of array gain within beam space.

From the previous section the optimum weight vector, w_o^y, using beam outputs is

$$w_o^y = \frac{R_y^+ s}{s^H R_y^+ s} \qquad (7)$$

where $s = V^H v$. Also the corresponding array gain, g_o^y, is given by

$$g_o^y = s^H R_y^+ s \qquad (8)$$

Two conditions under which equations (5) and (7) and (6) and (8) become identical will now be proved.

<u>Case A:</u> K = M and V nonsingular.

Since R_x is assumed nonsingular it follows that

$$R_y^+ = R_y^{-1} = V^{-1} R_x^{-1} V^{-1H}.$$

It then follows that

$$\begin{aligned} g_o^y &= s^H R_y^+ s \\ &= v^H V V^{-1} R_x^{-1} V^{-1H} V^H v \\ &= v^H R_x^{-1} v \\ &= g_o^x . \end{aligned}$$

Also the optimally weighted beam output is identical to the optimally weighted array output, ie

$$\begin{aligned} w_o^{yH} y &= \frac{v^H V R_y^{+H} y}{v^H V R_y^+ V^H v} \\ &= \frac{v^H V V^{-1} R_x^{-1} V^{-1H} V^H x}{v^H V V^{-1} R_x^{-1} V^{-1H} V^H v} \end{aligned}$$

$$= \frac{v^H R_x^{-1} x}{v^H R_x^{-1} v}$$

$$= w_o^{xH} x.$$

Thus for K = M and V nonsingular the two techniques are equivalent.

Case B: M > K and V is of rank K

In this case only K of the M beams formed are independent. For most arrays this can be satisfied by choosing the M steering directions to be distinct(ref.4). Since V has K independent rows it follows(ref.3) that

$$V^+ = V^H(VV^H)^{-1}$$

and consequently

$$VV^+ = V^{+H} V^H = I .$$

As a result it can readily be proved that

$$R_y^+ = V^+ R_x^{-1} V^{+H} .$$

It then follows that

$$g_o^y = s^H R_y^+ s$$

$$= v^H V V^+ R_x^{-1} V^{+H} V^H v$$

$$= v^H R_x^{-1} v$$

$$= g_o^x.$$

In a similar manner to the previous section the optimally weighted receiver and beam outputs can also be shown to be identical, ie

$$w_o^{yH} y = \frac{1}{g_o^y} v^H V R_y^{+H} y$$

$$= \frac{1}{g_o^x} v^H V V^+ R_x^{-1} V^{+H} V^H x$$

$$= \frac{1}{g_o^x} v^H R_x^{-1} x$$

$$= w_o^{xH} x$$

These examples imply that if the number of independent beams is greater than or equal to the number of receivers then in principle it is possible to achieve the full array space gains using the beam space approach. Furthermore as discussed in reference 4 choosing the steering directions to be distinct implies linear independence of the steering vectors. This implies that at any frequency the full array gains can be achieved by increasing the number of beams used. In practice the finite precision used in the calculations prevents this and in Section 5 some examples are given to illustrate this difficulty.

4. PROPERTIES

In this section some general properties of the beam space formulation will be detailed.

4.1 Gains

The array gains using the beam space formulation ie the g_o^y are always less than or equal to those derived from the array space formulation ie g_o^x.

The proof follows from Rao(p.48 ref.5):

$$g_o^x = v^H R_x^{-1} v = \max_{x \in E_K} \frac{x^H v v^H x}{x^H R_x x}$$

where E_K is the vector space of dimension K. Thus for any vector $y \in E_K$ it follows that

$$\frac{y^H v v^H y}{y^H R_x y} \leq g_o^x$$

In particular take

$$y = V R_y^+ V^H v$$

it then follows that

$$\frac{(v^H V R_y^+ V^H v)^2}{v^H V R_y^+ V^H R_x V R_y^+ V^H v} \leq g_o^x .$$

From the definition of R_y and the Moore-Penrose pseudoinverse it follows that the LHS of the inequality reduces to $v^H V R_y^+ V^H v$ and hence $g_o^y \leq g_o^x$. The conditions for the equality to hold have been investigated in the previous section.

4.2 Gain as a function of number of input beams

In a previous classified report the author has shown that in array space the optimum array gain estimates always increase with the addition of extra receivers. This is not the case in conventional beamforming where at low frequencies the addition of extra receivers may, provided the array aperture is not increased, actually result in a poorer array gain.

A similar result proved in Appendix II and generalising the previous derivation, holds for the beam space formulation ie, optimum beam space gains in general increase when additional input beams are used. However the array gains attain a maximum when the number of beams used equals the number of receivers and all these beams are independent. As shown in Appendix II when R_y formed from using n+1 beams becomes singular (or when the rank of R_y is not increased by the additional beam) then the optimum beam space gain is no longer increased by the addition of an extra beam.

5. EXAMPLES

In this section, array gain estimates for a conventional beamformer will be denoted as g_c and can readily be proved to be given* by

$$g_c = \frac{K^2}{v^H R_x v}$$

5.1 Uncorrelated receiver noise

In this case the receiver noise covariance matrix is defined by

$$R_x = I.$$

It trivially follows that

$$g_c = g_o^x = K$$

for all directions.

Furthermore g_o^y reduces to

$$v^H V (V^H V)^+ V^H v$$

* The assumption that the signal lies in the look direction is made henceforth.

and consequently the optimum beam space gains will, as would be expected, depend on the number of beams used and the steer direction.

(a) Single beam

Using just a single beam steered in a direction corresponding to the i-th column of V then

$$R_y = v_i^H v_i = K$$

where v_i is the i-th column of V.

Thus

$$g_o^y = v^H(\theta)\ v_i v_i^H\ v(\theta)$$

which is just the polar diagram of the array when steered in the i-th direction. Since only one beam has been used to effect the optimisation this result is hardly unexpected.

(b) M orthogonal beams ($M < K$)

In this case

$$V^H V = K\ I_M \Rightarrow (V^H V)^{-1} = \frac{1}{K} I_M ,$$

and hence

$$g_o^y = \frac{1}{K} v^H(\theta)\ VV^H v(\theta)$$

$$= \frac{1}{K} \sum_{j=1}^{M} v^H(\theta)\ v_j v_j^H\ v(\theta)$$

ie the sum of the M polar diagrams.

(c) Gain in beam directions

Let θ be chosen such that $v(\theta) = v_i$ where v_i is some column of V.

Rearranging such that v_i is the first column of V it follows that

$$(V^H V)^{-1}\ V^H v_i = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

Thus

$$g_o^y = v^H V \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

$$= K.$$

Thus the optimum beam space gains in the M directions determined by the v_i's, i = 1, 2, ... M will be equal to the optimum array space and conventional array gains.

5.2 Simulation results

To investigate the dependance of the beam space gains on the number of input beams used the crosspower spectral matrix was simulated for a number of noise fields and the gains were evaluated. The conventional array gain and the optimum array space gain were also evaluated. An array of two concentric rings each of 5 receivers (see figure 1) was used for the simulation.

(a) The array gains for a unit receiver noise covariance matrix (corresponding to uncorrelated receiver noise) are plotted in figure 2 where $r/\lambda = 1/8$. The number of beams used in deriving the beamspace gains was varied from 1 to 5; in each case the beams were equispaced from 0 to 2 π. For M=1 the polar diagram of the array is apparent. For $M > 1$ a gain of 10 log 10 (see previous section) is obtained whenever the steering direction coincides with one of the preformed beam directions. Furthermore at this frequency only 5 beams are needed to achieve the full array gain of 10 log 10.

The limiting number of beams needed is a function of frequency and are plotted in figure 3. This limiting number was calculated as the minimum number of beams used such that the beamspace gains lay within an average* of ¼ dB of the limiting array space gains. For such a noise field this gives the maximum number of beams that need be used at any given frequency.

(b) Uncorrelated receiver noise and a single interference

The array gains for the case where the above noise field has also a 10 dB interference at 180° are plotted in figure 4 where the analysis parameters are the same as figure 2. The number of beams necessary to achieve the limiting array gains are plotted as a function of frequency in figure 3. Comparing the two plots of figure 3 it can be seen that the interference requires roughly an additional beam (particularly at very low values of r/λ) to be used in order to achieve the full array gains.

(c) Isotropic noise

For an isotropic noise field the cross-correlation between any two receivers separated by a distance of d is given by

$$\frac{\sin 2\pi d/\lambda}{2\pi d/\lambda} .$$

* averaged over the azimuth angles 0-2π

The array gains for the array of figure 1 in such a noise field are plotted in figure 5 where the number of beams used was varied from 1 to 5 and $r/\lambda = 1/8$. The number of beams necessary to achieve the limiting array gains are plotted as a function of r/λ in figure 3. Apart from very low values of r/λ the values are the same as for the uncorrelated receiver noise case.

(d) Isotropic noise and a single interference

As discussed earlier it is in principle always possible to obtain the full array gains by increasing the number of beams used in the beamspace approach. However in this case it was found that as the number of beams increased the factorisation† of the beam crosspower spectral matrix failed.

A method that was partially successful in overcoming this difficulty was to perturb the diagonal elements of R_y by a small positive number (10^{-5}). This permitted factorisation when $r/\lambda > 1/6$ and in these cases the full array gains were achieved using either 9 or 10 beams depending on frequency. Increasing the perturbation (10^{-4}) permitted factorisation at most lower values of r/λ. However in general, the limiting beamspace gains (at $r/\lambda = 1/8$) were significantly lower than those achieved in the array space and often as the example of figure 6 shows the limits could be achieved using less than 10 beams (eg 5 in figure 6).

The inference from these results is that all 10 beams are needed at all frequencies to achieve the maximum gains. At lower frequencies the ill-conditioning of the matrices limits the gains that can be achieved in practice. However if uncorrelated receiver noise is also present (and in practice this certainly will always be the case) then the limiting gains may be achievable. If -10 dB (re the isotropic noise power) of uncorrelated receiver noise is added in the above example then figure 7 is obtained and it can be seen that using 10 beams allows the full array gains to be achieved.

5.3 Application to real data

The methods described have been used to estimate the optimum gains from the array described in the previous examples when processing sonar data. A block diagram of the processing is given in figure 8. After narrowband filtering the data (by means of the FFT) to a 5 Hz binwidth, the receiver crosspower spectral matrix, R_x, was estimated and normalised at each frequency of interest. This was factored and the factored matrix was used to estimate the conventional and optimum array space gains. The factored matrix was also premultiplied by the matrix (V) of phase delays whose columns corresponded to the steering directions. From this R_y was derived (see figure 8), factored and inverted and hence the array gain estimates g_o^y were evaluated.

The array gain estimates as a function of azimuthal angle for the array of figure 1 are shown in figures 9 and 10 for r/λ equal 0.125 and 0.42 respectively. Since one of the outer ring hydrophones was faulty only 9 hydrophones were used in the analysis. The data also contained random phase errors and this tended to bias the optimum array gain estimates positively. In both these examples no problems were encountered in the beam crosspower matrix factorisation although at other lower values of r/λ and also when using arrays with a larger number of hydrophones on the same data the matrix factorisation failed and the perturbation technique of the previous section had to be used.

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† The matrix factorisation arithmetic was effected in double precision on single precision inputs on a 4 byte word machine.

The two examples were chosen to illustrate two different environments; in the first example (figure 9) the noise field exhibited only a weak azimuthal dependance whereas in figure 10 the presence of a strong interference at 264° implied a strong azimuthal dependance of the noise field. Both results showed an improvement in the beamspace array gain estimates as the number of beams used was increased. When the number of beams equalled the number of receivers (figure 9) then the full optimum array space gains were achieved in both cases. Both examples showed distinct dips in the gain estimates when only a small number of beams were used (eg 3 in figure 9 and 4 in figure 10). Care should be taken to avoid associating these dips with an azimuthal variation of the noise field; they are simply a result of using an insufficient number of beams. Figure 10 differs from figure 9 in that significant improvement over the conventional gains can be achieved by using only a few beams, eg 4, whereas in figure 9, the gains resulting from using just 3 beams are often significantly lower than the conventional array gains. This is because only a single beam pointed at or near the interfering source is sufficient to partially null it and hence produce the significant array gains of figure 10.

Two further features are of particular interest in figure 10. The first is the depth of the null as a function of the number of beams used; this depth is a good indication of the sensitivity of the technique to receiver errors. As the number of beams increases the depth of the null decreases indicating increased sensitivity as the number of beams used increases. The second feature also of interest is the distinct dip at 68° in the array gain estimate for the 6 beam case. It is common to associate such dips in array gain estimates with interfering targets, however as indicated by both the conventional and optimum array space gains there is no interference at 68°. Thus as discussed above for the three and four beam case this dip must be a result of using an insufficient number of beams. This again illustrates the importance of choosing the number of beams correctly if the array gain estimates are to be used for the detection of interferences.

6. SUMMARY

The optimum signal-to-noise ratio processor has been formulated using only narrowband preformed beam outputs. The relationship of this to the array space formulation has been investigated, and in priciple the full array space gains can be recovered from the beamspace formulation provided the number of beams is chosen appropriately. A number of theoretical properties together with examples illustrating these properties have been given. Some practical limitations of the beamspace methods are also illustrated. The method has been applied to sonar data from an array of hydrophones to verify its viability and demonstrate some of the theoretical predictions. The results indicate that a trade-off of increased sensitivity versus more uniform (in an angular sense) array gain estimates must be made. This problem of choosing the correct number of beams is receiving further attention.

7. ACKNOWLEDGEMENTS

The software was developed from that supplied by Dr A.K. Steele who also contributed to a number of technical discussions.

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APPENDIX I

DERIVATION OF MAXIMUM SIGNAL-TO-NOISE RATIO PROCESSOR FOR A SINGULAR BEAM COVARIANCE MATRIX

Let R_x be a positive definite matrix and define R_y by

$$R_y = V^H R_x V \tag{I.1}$$

where V is an K x M matrix and M > K.

Since R_x is a nonsingular it follows that

$$u^H R_x u = 0 \Rightarrow u = 0 . \tag{I.2}$$

However from equation (I.1) the rank of R_y is K and hence there exist (M-K) vectors u' such that

$$u'^H R_y u' = 0 .$$

Substituting equation (I.1) in the above expression it follows from equation (I.2) that

$$Vu' = 0 .$$

The general solution to the above equation is(ref.3)

$$u' = (I - V^+ V)z$$

where z is arbitrary and V^+ is the Moore-Penrose pseudoinverse of V.

Let $s(\theta) = V^H v(\theta)$ be the steering vector, it thus follows that

$$s^H u' = v^H(\theta) V(I-V^+ V)z$$

$$= 0$$

since $$VV^+V = I .$$

Thus it follows that any vector which completely nulls out the noise power also nulls out the signal power.

This important conclusion now allows a derivation of the maximum SNR processor for R_y singular to be effected. That is the expression for the SNR in a given output beam ie

$$(SNR)_{beam} = \frac{w^H s s^H w}{w^H R_y w}$$

can be maximised subject to the constraint of a finite response (say unity) in the look direction.

This is a formulated equivalently as minimising $w^H R_y w$ subject to the constraint

$$s^H w = 1 \qquad (I.3)$$

Solving in the usual way by introducing a Lagrangian multiplier and differentiating with respect to w^H implies that

$$R_y w = \lambda s .$$

Any solution of this equation can be expressed in the form

$$w = \lambda R_y^+ s + (I - R_y^+ R_y) z$$

where z is arbitrary and R_y^+ is the pseudoinverse of R_y.

For the case of $M > K$ but V of full rank it can be shown that (see Section 3)

$$R_y^+ = V^+ R_x^{-1} V^{+H}$$

and consequently that

$$R_y^+ R_y = V^+ V$$

Thus $w = \lambda R_y^+ s + (I - V^+ V) z$ and so

$$s^H w = \lambda s^H R_y^+ s + v^H V (I - V^+ V) x$$

$$= \lambda s^H R_y^+ s .$$

Choosing λ such that equation (I.3) holds then implies that a possible solution for w_o is

$$w_o = \frac{R^+ s}{s^H R^+ s}$$

and consequently

$$(SNR)_{max} = s^H R^+ s .$$

APPENDIX II

INCREASE IN OPTIMUM ARRAY GAINS BY THE ADDITION OF AN EXTRA INPUT

Let R_n denote the crosspower spectral matrix formed using n 'normalised' input channels. These could be either array elements or conventional beams. Let R_{n+1} be the crosspower spectral matrix formed using the same, n, input channels plus an additional channel. It follows that R_{n+1} can be written in the form:

$$R_{n+1} = \begin{pmatrix} R_n & b \\ b^H & d \end{pmatrix}$$

where

$$b_i = < X_{n+1} X_i^* > ,$$

$$d = < X_{n+1} X_{n+1}^* >$$

and the X_i's are the Fourier transforms of the 'normalised' i^{th} channel outputs.

If u is any 'steering vector' for n input channels then the optimum array gain is given by

$$g_o^n = u^H R_n^+ u$$

where R_n^+ is the Moore-Penrose pseudoinverse of R_n. Similarly for the n+1 input channels the array gain is

$$g_o^{n+1} = (u^H \overset{*}{a}) R_{n+1}^+ \begin{pmatrix} u \\ a \end{pmatrix} \qquad (II.1)$$

where a is the complex weight required by the additional channel.

Now since R_{n+1} is positive semidefinite then

$$R_{n+1} = \tilde{A}^H \tilde{A}$$

and $\tilde{A}$ can be partitioned $\tilde{A} = (A : a)$ where A is n x n-1 and a is n x 1. Thus

$$R_{n+1} = \begin{pmatrix} A^H A & A^H a \\ a^H A & a^H a \end{pmatrix}$$

It can easily be verified that theorem† 3.6.3 (p.66 of Rao and Mitra) also holds for the pseudoinverse and hence

$$R_{n+1}^{+} = \begin{pmatrix} (A^H A)^{+} + \delta h h^H & -\delta h \\ -\delta h^H & \delta \end{pmatrix}$$

where

$$h = (A^H A)^{+} A^H a$$

$$= A^{+} a,$$

and

$$\delta = (a^H a - a^H A A^{+} a)^{-1} \text{ if } a \notin M(A)$$

$$= 0 \text{ if } a \in M(A)$$

where M(A) denotes the column space spanned by the columns of A.

Thus it follows that the pseudoinverse of R_{n+1} is given by

$$R_{n+1}^{+} = \begin{pmatrix} R_n^{+} + \delta R_n^{+} b b^H R_n^{+} & -\delta R_n^{+} b \\ -\delta b^H R_n^{+} & \delta \end{pmatrix}$$

Then substituting in equation (II.1) it can be shown that

$$g_o^{n+1} = u^H R_n^{+} u + \delta (w - a^*) (w^* - a) \qquad \text{(II.2)}$$

where $w = -u^H R_n^{+} b$. Thus provided $\delta \geqslant 0$ then $g_o^{n+1} \geqslant u^H R_n^{+} u$ and so the array gain is never decreased by the addition of an extra input channel.

Now since R_{n+1} is non-negative definite it follows that

$$R_{n+1} = U^H \Lambda_{n+1} U$$

where U is a unitary matrix and Λ_{n+1} is a diagonal matrix such that

$$\lambda_i = (\Lambda_{n+1})_{ii} \geqslant 0.$$

† Two typographical errors occur on p.66 of reference 6, a is an m vector and $a \in M(A)$ should read $a \notin M(A)$.

It holds that

$$R^{+}_{n+1} = U^{H} \Lambda^{+}_{n+1} U$$

where Λ^{+}_{n+1} is diagonal and

$$(\Lambda^{+}_{n+1})_{ii} = \frac{1}{\lambda_i} \text{ if } \lambda_i \neq 0$$

$$= 0 \text{ if } \lambda_i = 0.$$

Thus R^{+}_{n+1} is non-negative definite and so

$$x^{H} R^{+}_{n+1} x \geqslant 0 \;\forall\; x.$$

Taking $x^{H} = (0, 0, \ldots, 1)$ it follows that

$$(R^{+}_{n+1})_{n+1,\, n+1} \geqslant 0,$$

ie $$\delta \geqslant 0.$$

The case of $\delta=0$ corresponds to the case of where the rank of R_{n+1} equals that of R_n. In this case the array gain is not increased because no additional information is contained in R_{n+1} that was contained in R_n.

If the rank of R_{n+1} is greater than that of R_n then $\delta > 0$ and hence, in general, the array gain will increase since $w \neq \alpha^*$ for all steering vectors u.

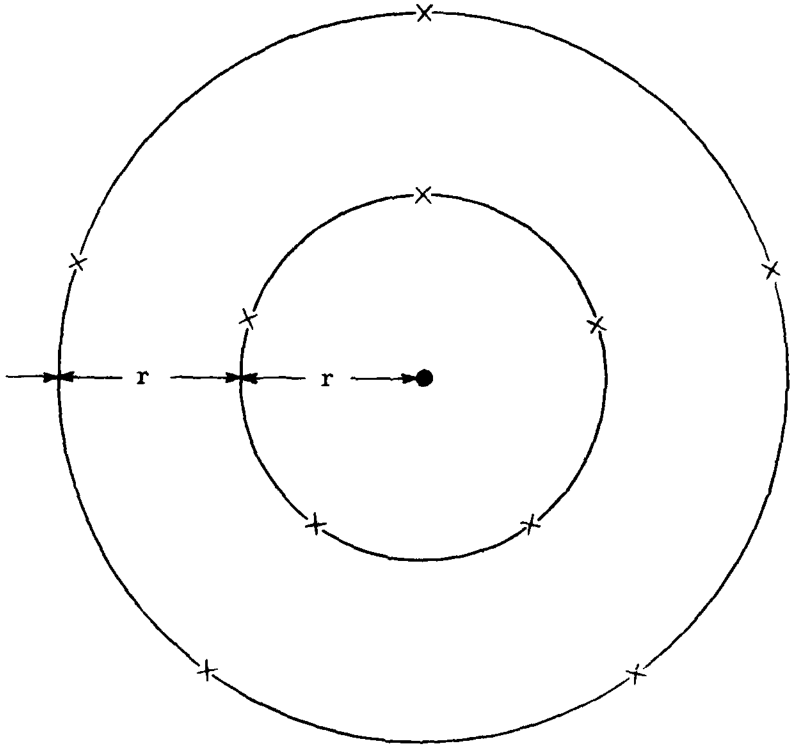

Figure 1. 2-ring array of 10 elements

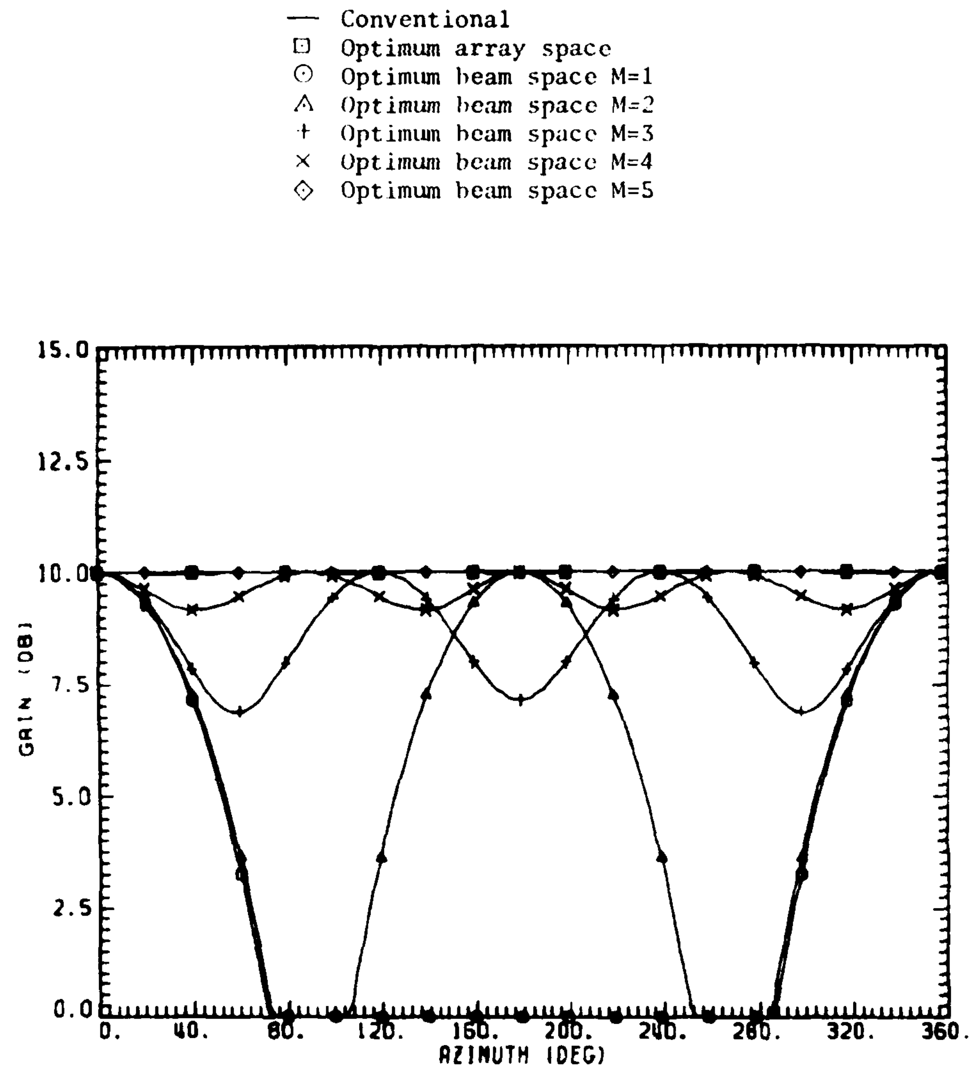

Figure 2. Array gains in uncorrelated receiver noise when r/λ = 1/8

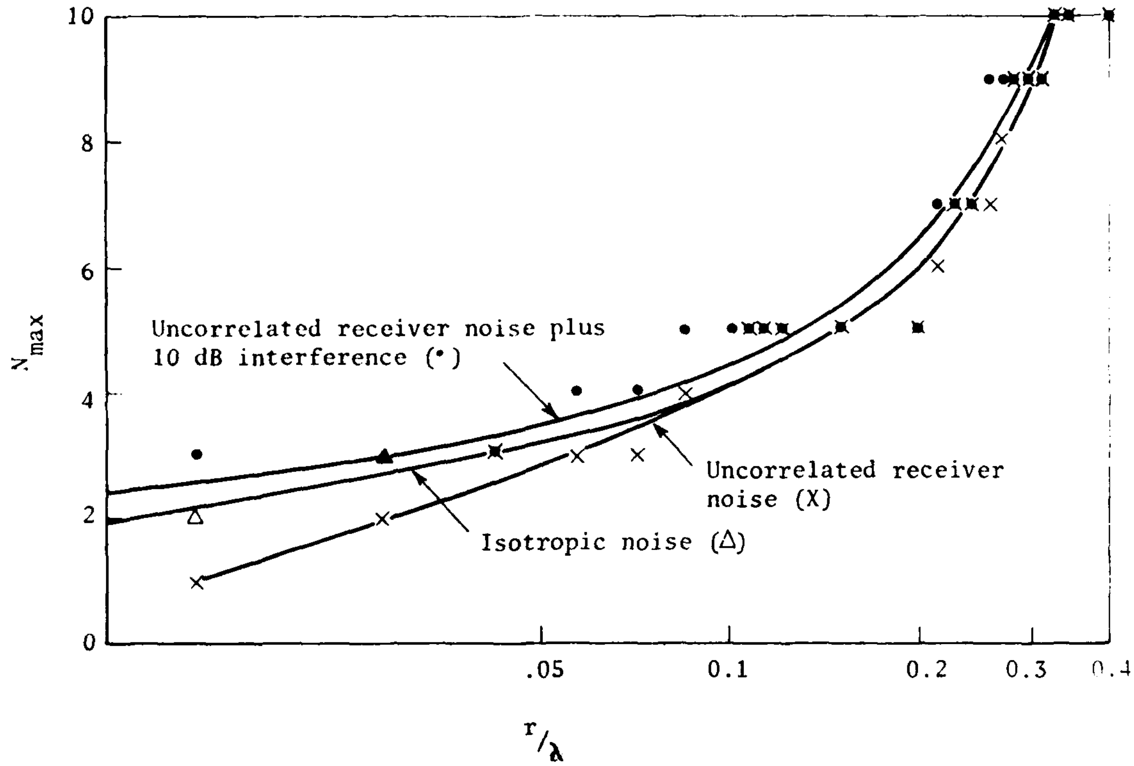

Figure 3. Limiting number of beams as a function of r/λ

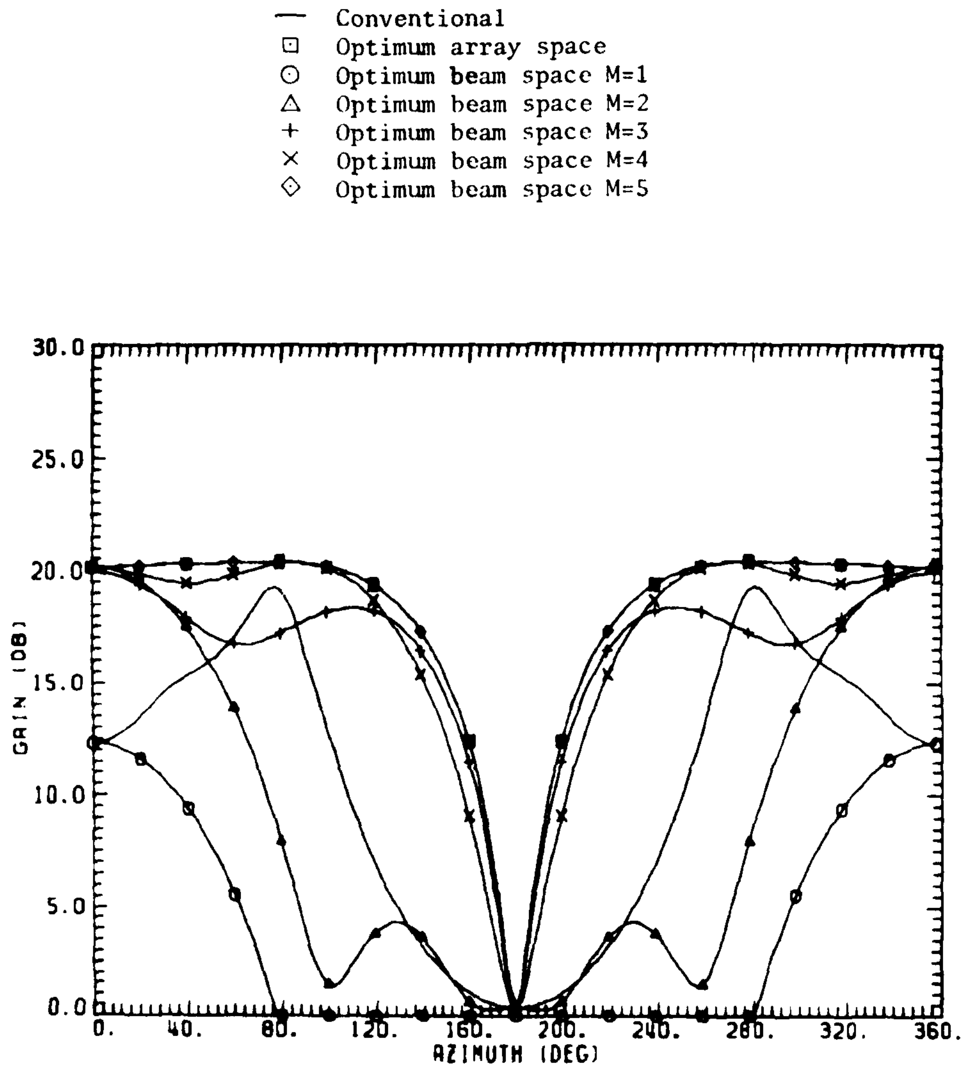

Figure 4. Array gains in uncorrelated receiver noise plus 10 dB interference when $r/\lambda = 1/8$

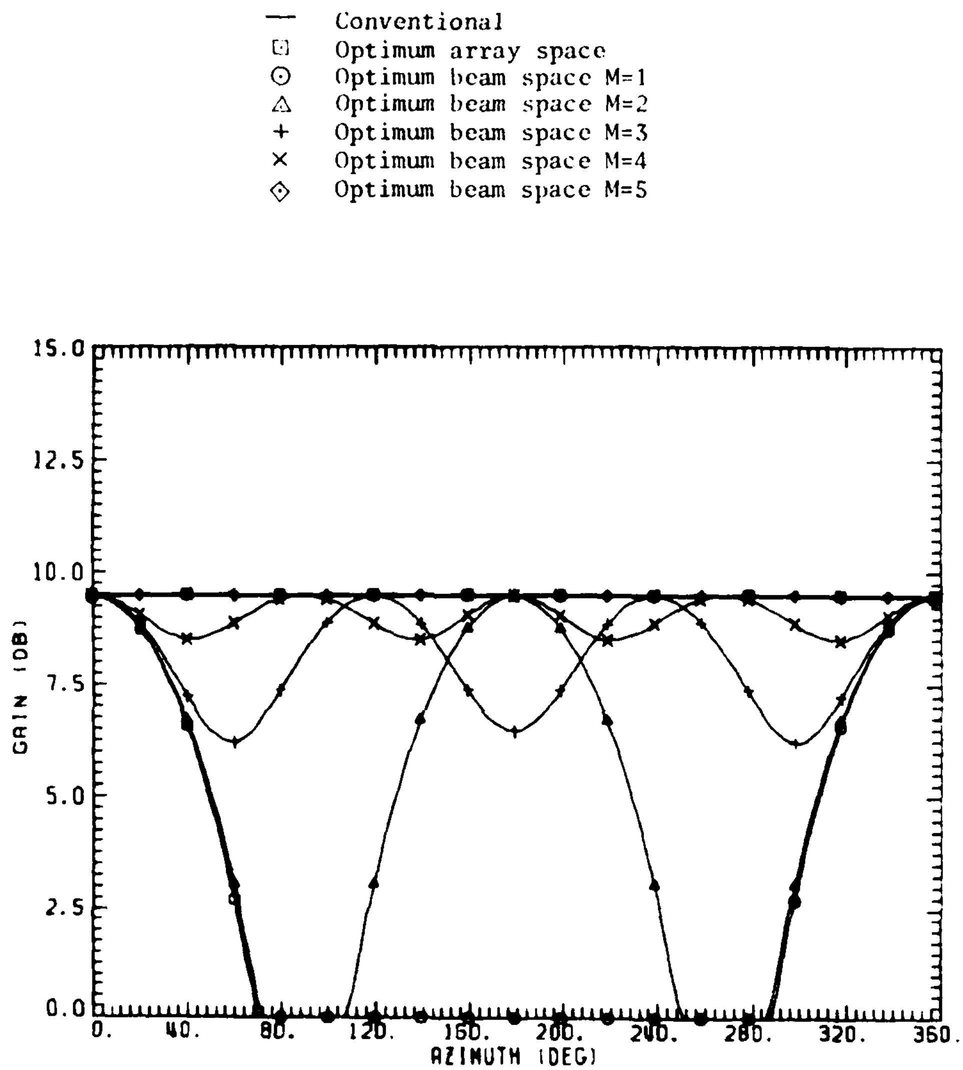

Figure 5. Array gains in isotropic noise when $r/\lambda = 1/8$

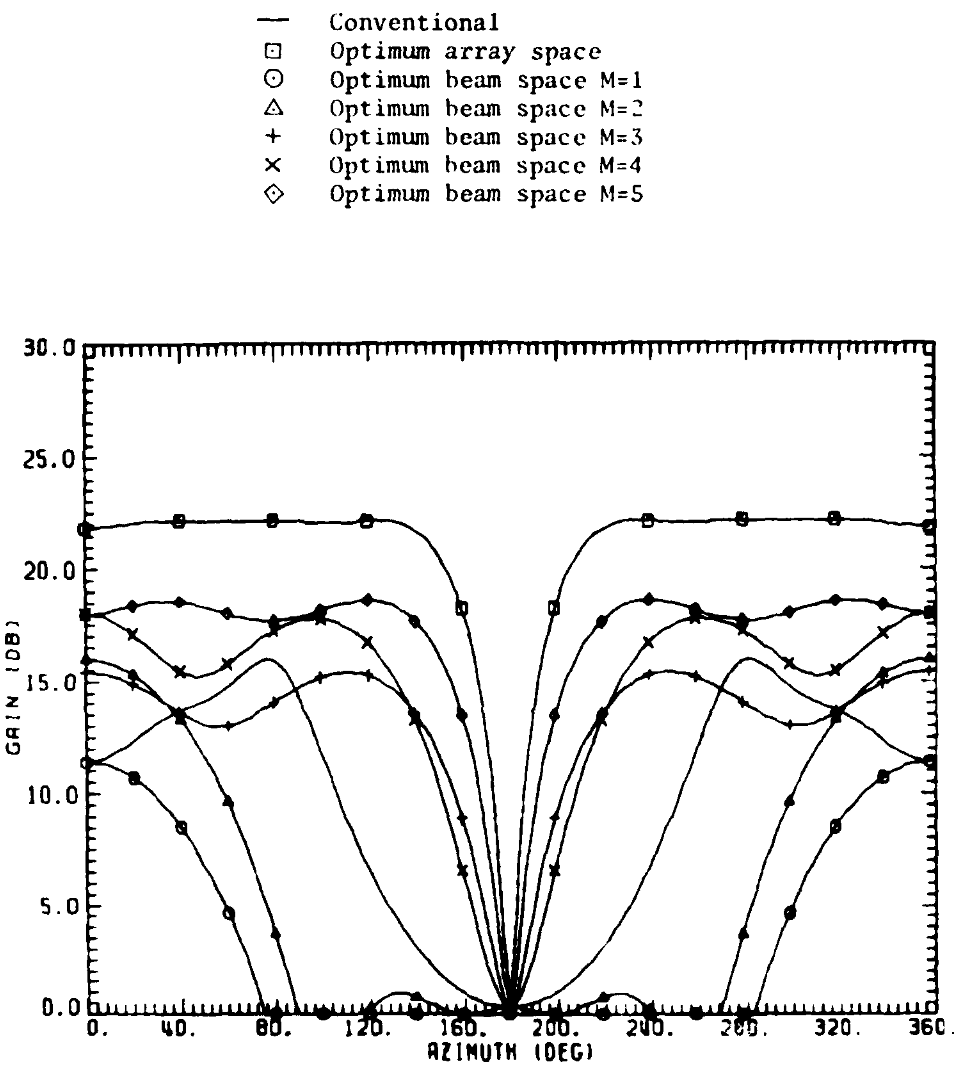

Figure 6. Array gains in isotropic noise plus 10 dB interference when $r/\lambda = 1/8$

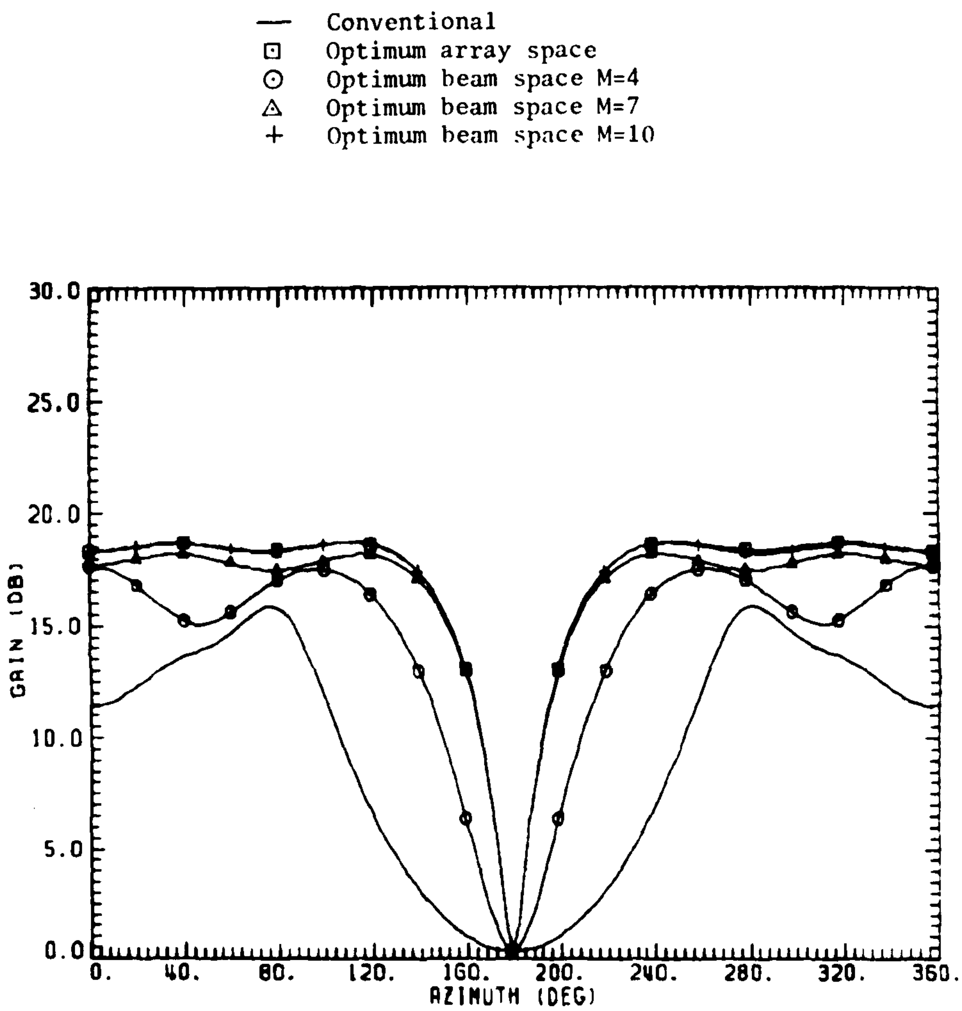

Figure 7. Array gains in isotropic noise plus 10 dB interference plus -10 dB receiver self noise at $r/\lambda = 1/8$

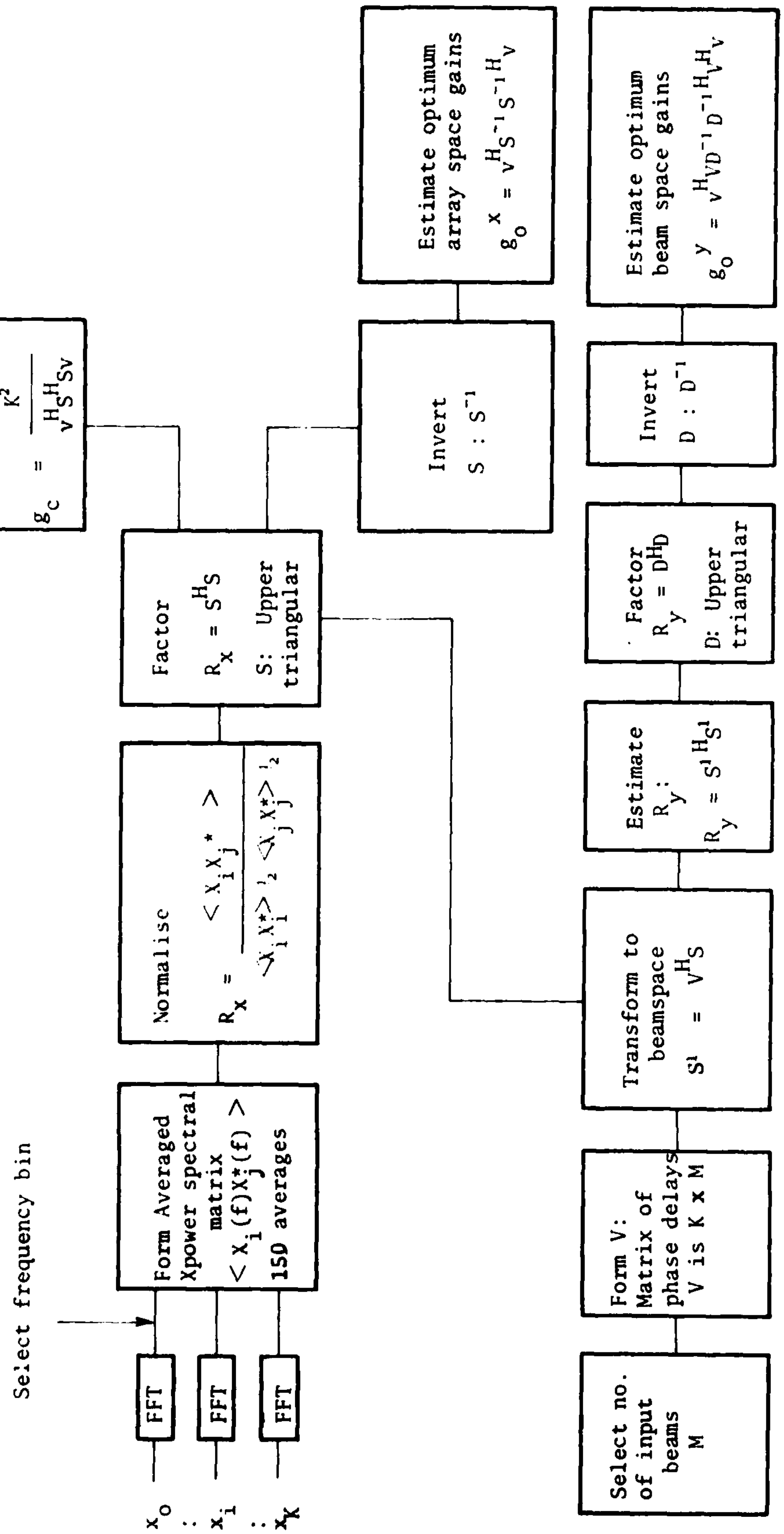

Figure 8. Block diagram for comparison of conventional, array space and beam space array gain estimates

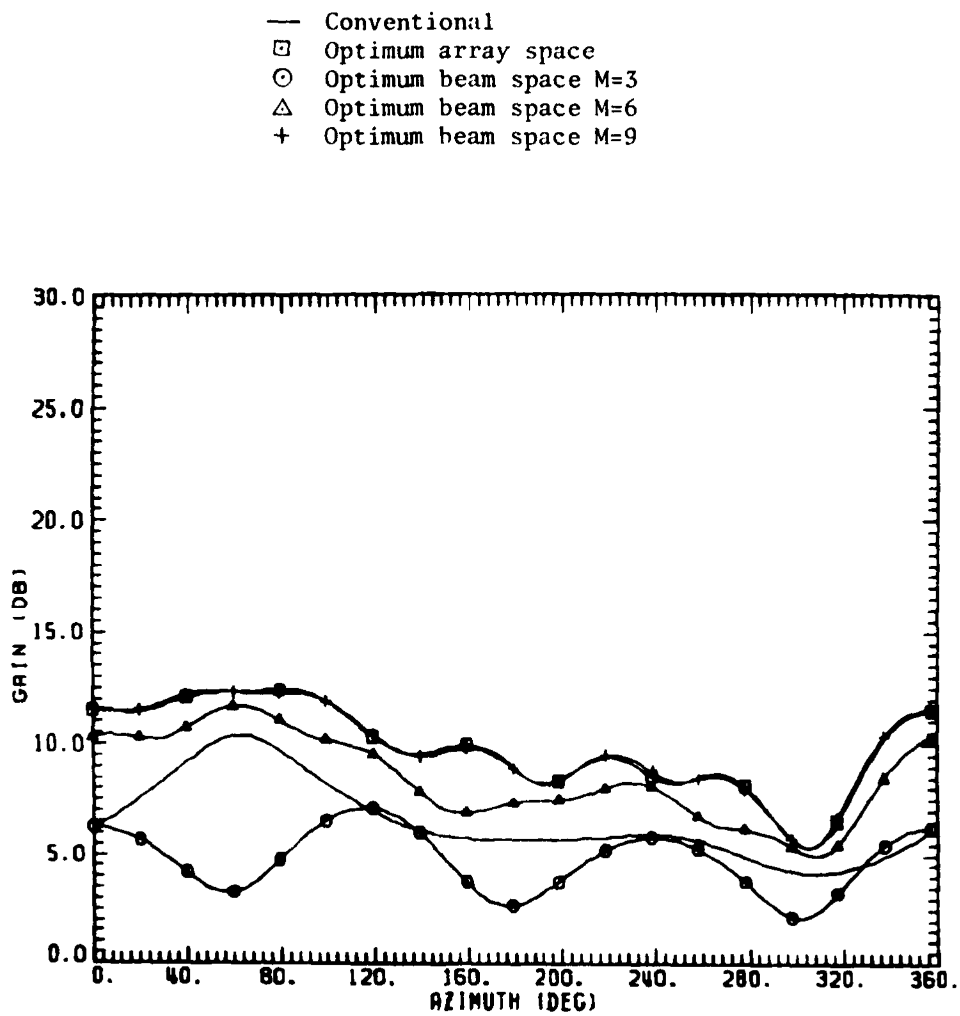

Figure 9. Array gain estimates for r/λ = 1/8

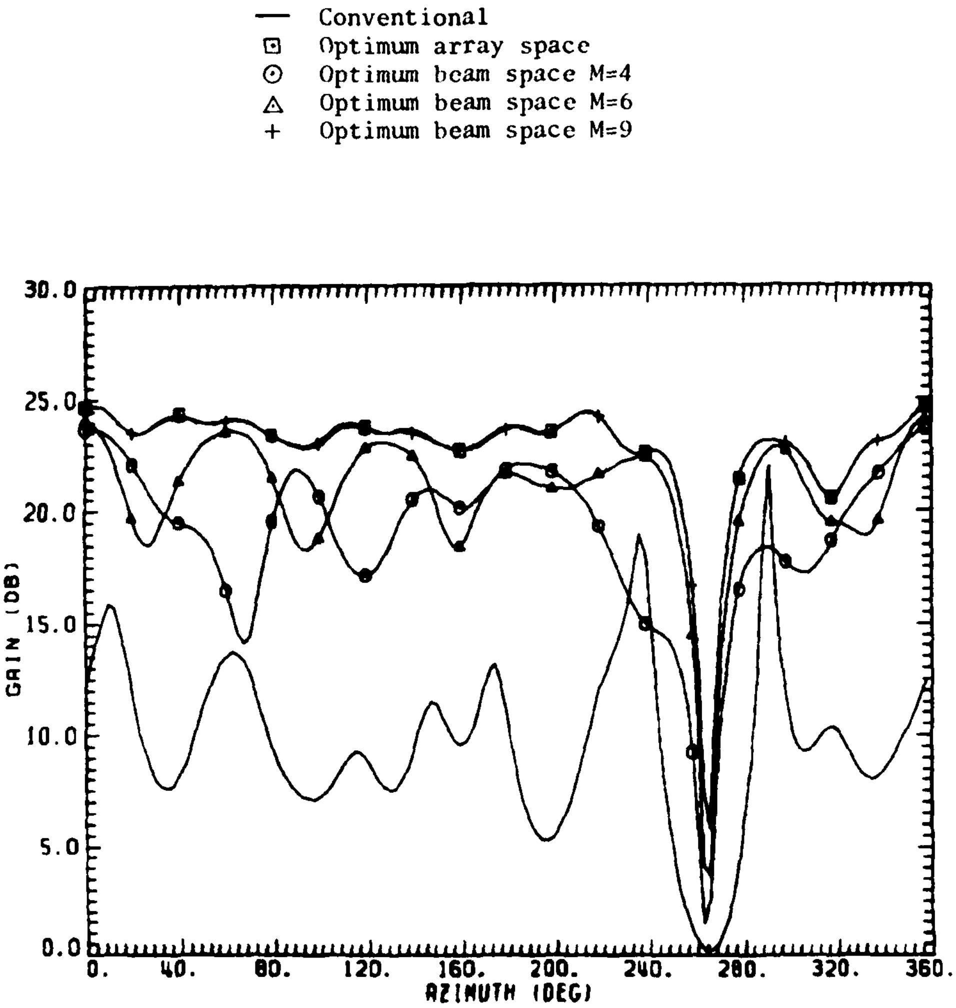

Figure 10. Array gain estimates for r/λ = 0.42

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The maximum signal-to-noise ratio array processor is formulated in beam space. Expressions for the optimum narrowband weight vectors and array gain are derived. Some general properties of the beamspace formulation are derived and conditions for the equivalence of the array and beam space formulations are proved.

Examples using both simulated and sonar data are given to compare the beam and array space formulations.

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