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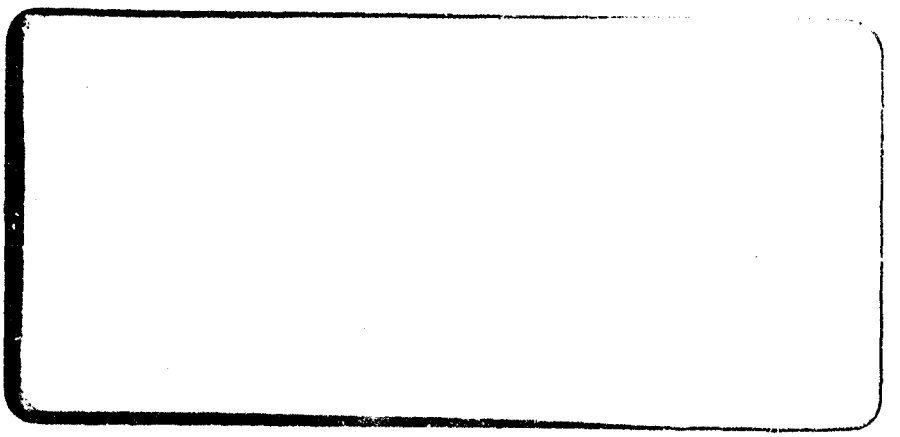

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# Large deformations near a tip of an interface-crack between two Neo-Hookean sheets

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by

J.K.Knowles and Eli Sternberg



Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91125

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# Large deformations near a tip of an interface-crack between two Neo-Hookean sheets*

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## Summary

This paper contains an asymptotic investigation — within the nonlinear theory of elastostatic plane stress — of the deformations and stresses near the tips of a traction-free interface-crack between two dissimilar semi-infinite Neo-Hookean sheets. The results obtained are free of oscillatory singularities of the kind predicted by the linearized theory, which would require the two deformed faces of an interface-crack to overlap in the vicinity of its tips. Instead, the crack is found to open smoothly near its ends, regardless of the specific loading at infinity.

## 1. Introduction

Owing to its importance in fracture mechanics, the plane problem of one or more interface-cracks between two dissimilar elastic slabs has received repeated attention in linearized elastostatics.[1] The earliest such investigation appears to be due to Williams [1] (1959), who examined the local character near a tip of a traction-free interface-crack of elastostatic fields compatible

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[1]A complete bibliography of the extensive literature on this subject is beyond the scope of this paper. Additional references can be found in those listed at the end of the paper.

with the governing field equations, as well as with the appropriate boundary and bond conditions. The analysis in [1] is based on an asymptotic scheme, originated by Knein [2], which Williams [3] had employed previously to explore two-dimensional corner singularities possible in homogeneous elastic slabs. The scheme here alluded to leads from the biharmonic equation satisfied by the generating Airy stress function to an eigenvalue problem for an ordinary fourth-order differential equation. In [1] the prevailing boundary and continuity conditions give rise to a sequence of complex eigenvalues that induces oscillatory singularities at a tip of an interface-crack in the associated sequence of displacement and stress fields.[1]

A global solution to the problem of a uniformly pressurized interface-crack of finite length, between two homogeneous and isotropic (linearly elastic) semi-infinite slabs of possibly distinct material properties, was deduced by England [4] (1965). Let $2\ell$ be the length of the crack and choose rectangular Cartesian coordinates $(x_1, x_2)$ as indicated in Fig.1. Next, let $H_1$ and $H_2$ be the half-planes $x_2 > 0$ and $x_2 < 0$, occupied by the interior of the middle cross-section of the upper and lower slab, respectively, while denoting by $\mu_k$ and $\nu_k$ $(k = 1,2)$ the corresponding shear moduli and Poisson-ratios of the two slabs.[2] If $u_\alpha$ and $\sigma_{\alpha\beta}$ are the components of displacement and stress in the underlying coordinate frame, England's solution conforms to the following formulation of his problem: it satisfies the two-dimensional displacement-stress relations and stress equations of equilibrium (in the absence of body forces),

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[1] Similar oscillatory singularities – likewise traceable to complex eigenvalues – are encountered in [3] in connection with certain mixed boundary conditions.

[2] From here on the "material index" k, as well as all Greek subscripts, are understood to take on the values (1,2); further, Greek subscripts are subject to the usual summation and differentiation conventions.

so that

$$\sigma_{\alpha\beta} = \mu_k\left[\frac{\kappa_k - 3}{1 - \kappa_k}\,\delta_{\alpha\beta}u_{\rho,\rho} + 2u_{(\alpha,\beta)}\right],\ \sigma_{\alpha\beta,\beta} = 0 \quad \text{on} \quad H_k \quad , \tag{1.1}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and

$$\left.\begin{aligned} &\kappa_k = 3 - 4\nu_k \quad \text{for plane strain,} \\ &\kappa_k = \frac{3 - \nu_k}{1 + \nu_k} \quad \text{for generalized plane stress} \quad ; \end{aligned}\right\} \tag{1.2}$$

it obeys the loading conditions

$$\left.\begin{aligned} &\sigma_{2\beta}(x_1,0\pm) = -\delta_{2\beta}\sigma,\ (\sigma > 0),\ (|x_1| < \ell) \quad , \\ &\sigma_{\alpha\beta}(x_1,x_2) = o(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty \quad , \end{aligned}\right\} \tag{1.3}$$

in which $\sigma$ is the constant pressure applied to either crack-face, and it meets the bond conditions

$$u_\alpha(x_1,0+) = u_\alpha(x_1,0-),\ \sigma_{2\beta}(x_1,0+) = \sigma_{2\beta}(x_1,0-) \quad (|x_1| > \ell) \quad . \tag{1.4}$$

In addition, $u_\alpha$ is twice continuously differentiable on $H_k$, the limits $u_\alpha(x_1,0\pm)$ exist and depend continuously on $x_1$ for $-\infty < x_1 < \infty$, whereas the limits $\sigma_{\alpha\beta}(x_1,0\pm)$ exist and are continuous for all $x_1 \neq \pm\ell$.

It should be recalled that the generalized plane-stress solution pertains to "thin" slabs (elastic sheets). In this instance $u_\alpha(x_1,x_2)$, $\sigma_{\alpha\beta}(x_1,x_2)$ at once approximate the thickness-averages and the mid-plane values of the corresponding displacements and stresses.

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[1] Here $u_{(\alpha,\beta)} \equiv (u_{\alpha,\beta} + u_{\beta,\alpha})/2$ are the infinitesimal strains.

We now introduce local polar coordinates $(r,\theta)$ with the origin at $x_1 = \ell$, $x_2 = 0$ (see Fig.1) and describe the asympotic behavior near the right-hand crack-tip of the solution obtained in [4]. If this tip is fixed by superposition of a suitable rigid translation upon the displacement field at hand, one has as $r \to 0$,

$$\left.\begin{aligned} u_\alpha &\sim \sqrt{r}\,\{\varphi_\alpha(\theta)\cos(\gamma \log r) + \psi_\alpha(\theta)\sin(\gamma \log r)\} \quad , \\ \sigma_{\alpha\beta} &\sim \frac{1}{\sqrt{r}}\{\varphi_{\alpha\beta}(\theta)\cos(\gamma \log r) + \psi_{\alpha\beta}(\theta)\sin(\gamma \log r)\} \quad , \end{aligned}\right\} \tag{1.5}$$

where $\varphi_\alpha$, $\psi_\alpha$, $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$ are fully determinate functions of the polar angle $\theta$, and $\gamma$ is the material parameter defined by

$$\gamma = \frac{1}{2\pi}\log\alpha, \quad \alpha = \frac{s + \kappa_1}{1 + s\kappa_2}, \quad s = \frac{\mu_1}{\mu_2} \quad , \tag{1.6}$$

$s$ being the shear stiffness-ratio of the two slabs. Equations (1.5) bring into evidence the unboundedness of $\sigma_{\alpha\beta}$ at the crack-tips, as well as the oscillatory nature of the resulting crack-tip singularities.

With a view toward examining the difficulty that led England to reject his solution as physically inadmissible, we cite from [4] the formula for the relative normal displacement of the two crack-faces:

$$\Delta_2(x_1) \equiv u_2(x_1,0+) - u_2(x_1,0-) =$$

$$\frac{\sigma\sqrt{\alpha}}{2(1+\alpha)}\left[\frac{1+\kappa_1}{\mu_1} + \frac{1+\kappa_2}{\mu_2}\right]\sqrt{\ell^2 - x_1^2}\,\cos\left(\gamma \log\frac{\ell + x_1}{\ell - x_1}\right) \qquad (|x_1| < \ell) \quad . \tag{1.7}$$

When $\gamma \neq 0$, (1.7) reveals an infinity of oscillations in the sign of $\Delta_2(x_1)$ as $x_1 \to \pm\ell$ and thus implies <u>wrinkling</u> of the faces of the crack in the

vicinity of its ends.[1] In order to infer the unacceptable prediction of overlapping crack-faces it is essential to consider also the corresponding relative tangential displacement, which is readily computed with the aid of results given in [4]:

$$\Delta_1(x_1) \equiv u_1(x_1,0+) - u_1(x_1,0-) =$$

$$-\frac{\sigma\sqrt{\alpha}}{2(1+\alpha)}\left[\frac{1+\kappa_1}{\mu_1}+\frac{1+\kappa_2}{\mu_2}\right]\sqrt{\ell^2-x_1^2}\,\sin\left(\gamma\log\frac{\ell+x_1}{\ell-x_1}\right) \quad (|x_1|<\ell) \quad . \quad (1.8)$$

Equations (1.7), (1.8) evidently imply the presence of interpenetration if there is a value of $x_1$ satisfying

$$\Delta_1(x_1) = 0, \ \Delta_2(x_1) < 0, \ |x_1| < \ell \quad . \quad (1.9)$$

If $\gamma \neq 0$, one confirms easily that (1.9) in fact possess a doubly infinite sequence of roots supplied by

$$x_1^{(j)} = \ell \tanh(j\pi/2\gamma) \quad (j = \pm 1, \pm 3, \pm 5, \ldots) \quad . \quad (1.10)$$

From (1.10) follows

$$\delta = \max_j \left[\ell - |x_1^{(j)}|\right] = \ell[1 - \tanh(\pi/2|\gamma|)] \quad (1.11)$$

and one finds that $\delta/\ell < 4 \times 10^{-8}$ for all physically realistic values of the elastic constants.[2] Consequently, the roots of (1.9) are confined to

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[1] Since $\mu_k > 0$, $-1 < \nu_k \leq 1/2$, it is clear from (1.2), (1.6) that the material parameters $\kappa_k$, $s$, and $\alpha$ are positive; $1 \leq \kappa_k < 7$ for plane strain, while $5/3 \leq \kappa_k < \infty$ for generalized plane stress. Thus $\Delta_2(x_1)$ cannot vanish identically.

[2] $\mu_k > 0$, $0 \leq \nu_k \leq 1/2$.

exceedingly small intervals adjacent to the ends of the crack. Nevertheless the presence of any overlap violates the physical requirement that the resulting deformation mapping be one-to-one and thus — as pointed out in [4] — invalidates the solution.

Evidently, the foregoing violation disappears if and only if $\gamma = 0$ or, equivalently,

$$s + \kappa_1 = 1 + s\kappa_2 \quad (s = \mu_1/\mu_2) \quad , \tag{1.12}$$

in which instance the ensuing singularities are no longer oscillatory and

$$\Delta_1(x_1) = 0 \ , \ \Delta_2(x_1) > 0 \quad (|x_1| < \ell) \quad . \tag{1.13}$$

Condition (1.12) is met in particular for plane strain, regardless of the value of $s$, in the limiting case of two incompressible slabs, since $\nu_1 = \nu_2 = 1/2$ here implies $\kappa_1 = \kappa_2 = 1$. In contrast, it is essential for our purposes to observe that (1.12) fails to hold for generalized plane stress of two incompressible sheets, unless the two materials are identical.[1]

We turn next to the analogous problem for a traction-free interface-crack between two semi-infinite slabs that are subjected to loads at infinity. In this connection we confine our attention to the case in which the loading conditions (1.3) give way to

$$\left.\begin{aligned} \sigma_{2\beta}(x_1,0\pm) &= 0 \quad (|x_1| < \ell) \quad , \\ \sigma_{11}(x_1,x_2) &= \overset{\infty}{\sigma}{}^{(k)}_{11} + o(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty, \quad (x_1,x_2) \in H_k \quad , \\ \sigma_{2\beta}(x_1,x_2) &= \delta_{2\beta}\overset{\infty}{\sigma}_{22} + o(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty \quad (\overset{\infty}{\sigma}_{22} > 0) \quad . \end{aligned}\right\} \tag{1.14}$$

[1] Note from the second of (1.2) that $\nu_k = 1/2$ now gives $\kappa_k = 5/3$, so that (1.12) demands $\mu_1 = \mu_2$.

As was made clear by Rice and Sih [5], an arbitrary assignment of the constants $\overset{\infty}{\sigma}{}^{(k)}_{11}$ and $\overset{\infty}{\sigma}_{22}$ $(\overset{\infty}{\sigma}_{22} > 0)$ is in general inconsistent with the bond conditions (1.4), the first of which demands the continuity across the bonded interface of the extensional strain $u_{1,1}$. In view of the displacement-stress relations (1.1) this requirement leads to the loading constraint

$$(1+\kappa_1)\overset{\infty}{\sigma}{}^{(1)}_{11} - s(1+\kappa_2)\overset{\infty}{\sigma}{}^{(2)}_{11} + [\kappa_1 - 3 - s(\kappa_2 - 3)]\overset{\infty}{\sigma}_{22} = 0 \quad . \qquad (1.15)^1$$

Observe that $\overset{\infty}{\sigma}{}^{(k)}_{11} = 0$ is admitted by (1.15) if and only if

$$\kappa_1 - 3 = s(\kappa_2 - 3), \ (s = \mu_1/\mu_2) \quad . \qquad (1.16)$$

Accordingly, the problem of an interface-crack in a uni-axial tension field (at right angles to the interface) cannot possibly possess a solution unless (1.16) holds.[2]

When the given load parameters $\overset{\infty}{\sigma}{}^{(k)}_{11}$, $\overset{\infty}{\sigma}_{22}$ obey (1.15), the solution to the problem governed by (1.1), (1.4), (1.14) is obtained at once by superposition upon England's [4] solution for the uniformly pressurized crack with $\sigma = \overset{\infty}{\sigma}_{22}$ of the piecewise homogeneous deformation with the displacements and stresses:

$$\left.\begin{aligned} u_1 &= \frac{1}{8\mu_k}\left[(\kappa_k+1)\overset{\infty}{\sigma}{}^{(k)}_{11} + (\kappa_k - 3)\overset{\infty}{\sigma}_{22}x_1\right] \quad , \\ u_2 &= \frac{1}{8\mu_k}\left[(\kappa_k+1)\overset{\infty}{\sigma}_{22} + (\kappa_k - 3)\overset{\infty}{\sigma}{}^{(k)}_{11}x_2\right] \quad , \\ \sigma_{11} &= \overset{\infty}{\sigma}{}^{(k)}_{11}, \ \sigma_{22} = \overset{\infty}{\sigma}_{22}, \ \sigma_{12} = \sigma_{21} = 0 \quad \text{on} \quad H_k \quad . \end{aligned}\right\} \qquad (1.17)$$

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[1] This relation is equivalent to Eq.(19) in [5]. No additional constraints on the load parameters arise if $\sigma_{21} \equiv \sigma_{12}$ tends to a constant non-zero value $\overset{\infty}{\sigma}_{12}$ at infinity.

[2] This fact appears to go unnoticed in some papers. Note that (1.16) is satisfied not only in the special case $s = 1$, $\nu_1 = \nu_2$ (identical materials), but also for $\nu_k = 0$ $(\kappa_k = 3)$, regardless of the value of $s$.

Since $u_\alpha$ in (1.17) is continuous across the entire $x_1$-axis, it is clear that (1.7), (1.8) and the discussion following these equations remain valid also for the problem of the traction-free crack under present consideration. Further, the asymptotic results (1.5) evidently continue to hold in the present circumstances. Indeed, the right-hand members in (1.5) are consistent with the displacements and stresses appropriate to the only member of the sequence of elastostatic fields deduced by Williams [1] that has continuous displacements but unbounded stresses at the crack-tip.[1] This field encompasses at the same time the asymptotic structure near a crack-tip of the global solution associated with an all-around uniform shear loading at infinity,[2] which is once again described by (1.5).[3]

The objections raised in [4][4] to violations of the impenetrability requirement apply to a host of papers on various interface-crack problems that have appeared in the literature. These misgivings have prompted a renewed theoretical concern with interface-cracks in recent years. Thus Comninou [7][5] (1977), [8] (1978) sought to remove the inadmissible field oscillations on the basis of the ad-hoc assumption that the crack-faces remain in frictionless contact over two sub-segments (adjacent to the two ends of the crack) of initially undetermined length. Still more recently Achenbach, Keer, Khetan, and Chen [9] (1979) relied on the Dugdale-Barenblatt [10], [11] model of

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[1] It should be recalled that the results in [1] involve certain undetermined amplitude coefficients which elude the local analysis carried out there.

[2] See [5], where such a loading is included.

[3] In this instance $\varphi_\alpha$, $\psi_\alpha$, $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$ are no longer the same functions of the polar angle as before.

[4] The same criticism was voiced independently by Malyshev and Salganik [6].

[5] This publication contains a fairly extensive list of references to earlier work.

inelastic behavior near the tips of a crack in order to eliminate unbounded stress singularities altogether from problems involving interface-cracks.

In the current paper we adhere to strictly elastic behavior but relinquish the hypothesis of infinitesimal deformations, which is in fact violated by solutions exhibiting locally unbounded stresses and hence unbounded displacement gradients. We aim to show that the offensive oscillatory singularities arising in interface-crack problems stem from the linearization of such problems, as conjectured by England [4], rather than from the assumption of perfect elasticity or the particular idealization underlying the formulation of the boundary and bond conditions. With a view toward accomplishing this purpose in an analytically amenable setting we deal asymptotically with the elastostatic field near a tip of a traction-free crack between two otherwise bonded incompressible slabs of Neo-Hookean material. Furthermore, the present study is carried out within the nonlinear theory of elastostatic plane stress since a plane-strain analysis appropriate to incompressible slabs would be pointless.[1] To be sure, the theory of plane stress – in contrast to plane strain – is approximate and presupposes the slab-thickness to be small compared to a characteristic in-plane dimension of its cross-section. Objections based upon the approximate nature of this theory, which might be raised especially in connection with crack problems, however apply equally to linearized and finite elastostatics and do not interfere with our primary purpose.

The method used in pursuing the local issue at hand is an adaptation to the nonlinear theory of the asymptotic scheme underlying [1], [2], [3]. The same approach was employed in a sequence of previous studies of crack-problems

[1] Recall from the discussion following (1.13) that the linear theory does not lead to oscillatory singularities in these circumstances.

in the nonlinear equilibrium theory, under various loading conditions and diverse constitutive assumptions. These papers are referenced and briefly summarized in two recent survey articles [12], [13]. Particularly pertinent to the present study is a related asymptotic exploration [14] of elastostatic singularities induced by certain mixed boundary conditions of the kind arising in the problem of the "rough punch". While the linearized theory in these circumstances predicts oscillatory singularities, their absence is shown in [14] to be consistent with the nonlinear theory of plane strain for compressible materials of the harmonic type.

The present work is also closely related to a paper by Wong and Shield [15] (1969) that predates the publications mentioned above. In [15] an approximate global plane-stress solution is deduced for the problem occasioned by a finite crack in an all-around infinite Neo-Hookean sheet, subjected to bi-axial tension at infinity. The approximative approach adopted there requires the deformations to be large throughout the sheet. Our local results pertaining to an interface-crack between two distinct Neo-Hookean sheets, upon proper specialization, are found to be in asymptotic agreement with the solution reported in [15].

The chief conclusion reached by us concerns the absence of oscillatory singularities in the present nonlinear setting of the interface-crack problem. Somewhat surprisingly we find that a result arrived at in [15], according to which the crack opens up smoothly under the symmetric loading considered there, continues to hold true for the interface-crack — regardless of the particular loading conditions.[1]

[1] We exclude degenerate loadings that give rise to finite displacement gradients at the crack-tips, such as uni-axial tension parallel to the crack-faces and compatible with (1.15).

Finally, as a by-product, the analysis presented in this paper yields certain results of interest in connection with the Mode II crack problem for a single (homogeneous) Neo-Hookean sheet. The global solution of this problem within the infinitesimal theory predicts that the crack-faces fail to separate in this instance and that a Mode II loading of simple shear at infinity leads to an elastostatic field which is anti-symmetric with respect to the crack-axis. Both of these predictions are found to be in conflict with the local results established in this paper and hence reflect degeneracies stemming from the linearization of the problem. Analogous conclusions regarding the Mode II crack problem were arrived at previously by Stephenson [16] in a nonlinear asymptotic study encompassing a class of incompressible materials under conditions of plane strain.[1] In particular, Stephenson succeeded in proving that the global nonlinear Mode II crack problem for plane strain of a Neo-Hookean material cannot admit a solution anti-symmetric about the crack-axis.

## 2. Preliminaries from the theory of plane stress in finite elastostatics. Neo-Hookean materials.

As prerequisites for the analysis to follow, we assemble in the present, largely expository, section, some basic ingredients of the nonlinear equilibrium theory of plane stress[2] and in this connection confine our attention to incompressible, homogeneous and isotropic, elastic solids.

Let $(x_1,x_2,x_3)$ be rectangular Cartesian material coordinates and consider a body which, in an undeformed configuration, occupies the closed

[1] See also [17], where nonlinear effects bearing on the question as to whether or not a crack opens up in the presence of a Mode II loading are explored further.

[2] An account of this approximate theory, couched in general tensor notation, is contained in a paper by Adkins, Green, and Nicholas [18] (1954), who cite pertinent earlier work. See also [15].

cylindrical region $\mathcal{R}$ described by

$$\mathcal{R} = \{\underset{\sim}{x} \mid (x_1, x_2) \in \Pi,\ -t/2 \leq x_3 \leq t/2\} \quad , \tag{2.1}$$[1]

where $\Pi$ is the cross-section of $\mathcal{R}$ in the mid-plane $x_3 = 0$ and $t$ the constant thickness of the cylindrical slab.

A locally volume preserving deformation of the body is characterized by

$$\underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{for all} \quad \underset{\sim}{x} \in \mathcal{R} \quad , \tag{2.2}$$[2]

with

$$\det \underset{\sim}{F} = 1, \quad \underset{\sim}{F} = \nabla \underset{\sim}{y} \quad \text{on} \quad \mathcal{R} \quad , \tag{2.3}$$

in which $\underset{\sim}{u}$ and $\underset{\sim}{F}$ stand for the displacement vector field and the deformation-gradient tensor field, respectively. We shall temporarily take for granted that the mapping $\underset{\sim}{y}$ is twice continuously differentiable and uniquely invertible on $\mathcal{R}$. If $\underset{\sim}{\sigma}$ is the nominal (Piola) stress field accompanying the deformation, equilibrium — in the absence of body forces — demands that

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on} \quad \mathcal{R} \quad . \tag{2.4}$$

Further, $\underset{\sim}{\sigma}$ is linked to the corresponding actual (Cauchy) stress field $\underset{\sim}{T}$ by means of

$$\underset{\sim}{T} = \underset{\sim}{\sigma}\underset{\sim}{F}^T \quad \text{on} \quad \mathcal{R} \tag{2.5}$$[3]

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[1] For the time being, letters in boldface denote vectors and second-order tensors in three dimensions.

[2] Thus $y_i(\underset{\sim}{x})$ $(i = 1,2,3)$ are the spatial coordinates of the material point $\underset{\sim}{x} = (x_1, x_2, x_3)$ after the deformation.

[3] A superscript $T$ indicates transposition. Note that at present $\underset{\sim}{T}$ is regarded as a function of position on $\mathcal{R}$.

Next, call $\underset{\sim}{G}$ the left Cauchy-Green deformation tensor associated with the deformation (2.2) and let $I_1$, $I_2$, $I_3$ designate the fundamental scalar invariants of $\underset{\sim}{G}$. Thus,

$$\left.\begin{aligned} \underset{\sim}{G} &= \underset{\sim}{F}\underset{\sim}{F}^T \quad \text{on} \quad \mathcal{R} \ , \\ I_1 = \operatorname{tr}\underset{\sim}{G}, \quad I_2 &= \tfrac{1}{2}[(\operatorname{tr}\underset{\sim}{G})^2 - \operatorname{tr}\underset{\sim}{G}^2], \quad I_3 = \det\underset{\sim}{G} = 1 \ . \end{aligned}\right\} \tag{2.6}$$

Suppose now the body at hand possesses an elastic potential $W(I_1,I_2)$, so that $W(I_1,I_2)$ represents the strain-energy density per unit undeformed volume. The appropriate constitutive law then takes the equivalent alternative forms

$$\underset{\sim}{\tau} = A\underset{\sim}{G} - B\underset{\sim}{G}^2 - p\underset{\sim}{1}, \quad \underset{\sim}{\sigma} = A\underset{\sim}{F} - B\underset{\sim}{3}\underset{\sim}{F} - p\underset{\sim}{F}^{-T} \ , \tag{2.7}$$[1]

where

$$A = 2(W_{I_1} + I_1 W_{I_2}), \quad B = 2W_{I_2} \ , \tag{2.8}$$[2]

while $p$ stands for the arbitrary scalar pressure needed to accommodate the kinematical constraint of incompressibility.

Finally, we recall that if $\lambda_i$ $(i = 1,2,3)$ are the local principal stretches associated with the deformation (2.2), their squares are the local eigenvalues of the symmetric positive-definite tensor field $\underset{\sim}{G}$. Accordingly, (2.6) gives

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1 \ . \tag{2.9}$$

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[1] Here $\underset{\sim}{1}$ is the idem tensor and $\underset{\sim}{F}^{-T}$ the transposed inverse of $\underset{\sim}{F}$.

[2] We write $W_{I_1}$ and $W_{I_2}$ for the partial derivatives of $W$ with respect to the corresponding deformation invariants.

At this stage we subject the deformation (2.2) to the following specializing assumptions:

(a) it is symmetric about the mid-plane $x_3 = 0$, so that $y_\alpha(x_1,x_2,x_3)$ is an even, and $y_3(x_1,x_2,x_3)$ an odd function, of $x_3$;

(b) it is consistent with the requirement that the deformed faces of the cylindrical slab be free of tractions, whence

$$\sigma_{i3}(x_1,x_2,\pm t/2) = 0, \ (x_1,x_2) \in \Pi, \ (i = 1,2,3) \quad . \tag{2.10}$$

In addition, we suppose the cylindrical slab to be "thin" in the sense that $t$ is small compared to a characteristic dimension of its mid-section $\Pi$. Motivated by this supposition, as well as by (a) and by (2.10) with $i = 3$, we adjoin to (a) and (b) the approximative assumption

(c) $$\sigma_{33} = 0 \ \text{ on } \ \Pi, \ \sigma_{\alpha\beta,3} = 0 \ \text{ on } \ \mathcal{R} \quad . \tag{2.11}$$

If $f$ is a function defined on $\mathcal{R}$, we shall write $\overset{\circ}{f}$ for the restriction of $f$ to $\Pi$. The relevant theory of plane stress aims at an approximate characterization of $\overset{\circ}{y}_\alpha$, $\overset{\circ}{\sigma}_{\alpha\beta}$, and $\overset{\circ}{\tau}_{\alpha\beta}$ (in the present constitutive setting) on the basis of assumptions (a), (b), (c).

In view of (2.2), one has

$$\overset{\circ}{y}_\alpha(x_1,x_2) = x_\alpha + \overset{\circ}{u}_\alpha(x_1,x_2) \quad \text{on} \quad \Pi \quad . \tag{2.12}$$

Further, the equilibrium equation (2.4) implies

$$\int_{-t/2}^{t/2} [\sigma_{\alpha\beta,\beta}(x_1,x_2,x_3) + \sigma_{\alpha 3,3}(x_1,x_2,x_3)]dx_3 = 0 \quad , \tag{2.13}$$

which, because of (2.10) and the second of (2.11), reduces to

$$\mathring{\sigma}_{\alpha\beta,\beta} = 0 \quad \text{on} \quad \Pi \quad . \tag{2.14}$$

Next, upon putting

$$\lambda(x_1,x_2) = y_{3,3}(x_1,x_2,0) \quad \text{on} \quad \Pi \quad , \tag{2.15}$$

one draws from (2.3), (2.5), (2.6), (2.7), together with (a), that

$$\left.\begin{aligned} &\mathring{F}_{\alpha 3} = \mathring{F}_{3\alpha} = 0,\ \mathring{F}_{33} = \lambda,\ \mathring{G}_{\alpha 3} = \mathring{G}_{3\alpha} = 0,\ \mathring{G}_{33} = \lambda^2 \quad , \\ &\mathring{\sigma}_{3\alpha} = \mathring{\sigma}_{\alpha 3} = 0,\ \mathring{\sigma}_{33} = \tfrac{1}{\lambda}(\mathring{A}\lambda^2 - \mathring{B}\lambda^4 - \mathring{p}) \quad , \\ &\mathring{\tau}_{\alpha 3} = \mathring{\tau}_{3\alpha} = 0,\ \mathring{\tau}_{33} = \lambda\mathring{\sigma}_{33} \quad \text{on} \quad \Pi \quad . \end{aligned}\right\} \tag{2.16}$$

Accordingly, $\lambda^2$ is a principal value of $\underset{\sim}{G}$ at all points of $\Pi$, with the $x_3$-axis as an associated principal axis. This identifies $\lambda$ as the transverse principal stretch on $\Pi$ and entitles us to take

$$\mathring{\lambda}_3 = \lambda \quad \text{on} \quad \Pi \quad . \tag{2.17}$$[1]

Also, since $\mathring{\sigma}_{33} = 0$ by the first of (2.11), one infers from (2.16) that

$$\mathring{p} = \mathring{A}\lambda^2 - \mathring{B}\lambda^4 \quad \text{on} \quad \Pi \quad . \tag{2.18}$$

Bearing in mind (2.6), (2.9), and (2.16), we set

$$I = \mathrm{tr}[\mathring{G}_{\alpha\beta}] = \mathring{\lambda}_1^2 + \mathring{\lambda}_2^2,\ J = \det[\mathring{F}_{\alpha\beta}] = \mathring{\lambda}_1\mathring{\lambda}_2 > 0 \tag{2.19}$$

and note with the aid of (2.3),(2.17), (2.9) that

[1] Note that $\lambda < 1$ represents a thinning, and $\lambda > 1$ a thickening, of the sheet.

$$\lambda = J^{-1}, \quad \overset{\circ}{I}_1 = I + J^{-2}, \quad \overset{\circ}{I}_2 = J^2 + IJ^{-2} \quad . \tag{2.20}$$

We now evaluate $\overset{\circ}{\sigma}_{\alpha\beta}$ by means of (2.7), making use of (2.18), (2.20), and thereafter appeal to (2.19) and the Cayley-Hamilton theorem in two dimensions in order to eliminate $\overset{\circ}{G}_{\alpha\beta}$ from the resulting stress-deformation relation. This computation yields

$$\overset{\circ}{\sigma}_{\alpha\beta} = (\overset{\circ}{A} - \overset{\circ}{B}I)\overset{\circ}{F}_{\alpha\beta} + (\overset{\circ}{B}J^2 - \overset{\circ}{A}J^{-2} + \overset{\circ}{B}J^{-4})\overset{\circ}{F}^{-1}_{\beta\alpha} \quad . \tag{2.21}$$[1]

Finally, (2.5) and (2.16) justify

$$\overset{\circ}{\tau}_{\alpha\beta} = \overset{\circ}{\sigma}_{\alpha\gamma}\overset{\circ}{F}_{\beta\gamma} \quad \text{on} \quad \Pi \quad . \tag{2.22}$$

For convenience we summarize at this point in coordinate-free notation the equations governing the approximate two-dimensional theory of plane stress reviewed above. To this end we shall from here on let $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{u}$ and $\underset{\sim}{F}$, $\underset{\sim}{G}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{I}$ denote the <u>two</u>-dimensional vector and tensor fields with the components $x_\alpha$, $\overset{\circ}{y}_\alpha$, $\overset{\circ}{u}_\alpha$ and $\overset{\circ}{F}_{\alpha\beta}$, $\overset{\circ}{G}_{\alpha\beta}$, $\overset{\circ}{\sigma}_{\alpha\beta}$, $\overset{\circ}{\tau}_{\alpha\beta}$. At the same time we omit henceforth the superscript zero from symbols that denote restrictions to $\Pi$ of functions originally defined on the cylindrical region $\mathcal{R}$.

Thus, (2.12) and (2.14) now become

$$\underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{on} \quad \Pi \quad , \tag{2.23}$$

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on} \quad \Pi \quad . \tag{2.24}$$

Next, keeping (2.19), (2.20) in mind, we introduce the <u>plane-stress elastic potential</u> $U$ through

[1] $\overset{\circ}{F}^{-1}_{\beta\alpha}$ stands for the components of $\overset{\circ}{\underset{\sim}{F}}^{-T}$.

$$U(I,J) = W(I_1,I_2) = W(I + J^{-2}, J^2 + IJ^{-2}) \quad . \tag{2.25}$$

Equation (2.21) then readily leads to

$$\underset{\sim}{\sigma} = 2U_I \underset{\sim}{F} + JU_J \underset{\sim}{F}^{-T} \quad \text{on} \quad \Pi \quad , \tag{2.26}$$

where

$$\left.\begin{aligned} \underset{\sim}{F} = \nabla \underset{\sim}{y} \ , \quad \underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T \ , \\ I = \operatorname{tr} \underset{\sim}{G} = \lambda_1^2 + \lambda_2^2, \quad J = \det \underset{\sim}{F} = \lambda_1\lambda_2 = 1/\lambda \quad \text{on} \quad \Pi \ , \end{aligned}\right\} \tag{2.27}$$

while $U_I$ and $U_J$ designate the corresponding partial derivatives of $U(I,J)$. Further, (2.22) implies

$$\underset{\sim}{T} = \underset{\sim}{\sigma}\underset{\sim}{F}^T \quad \text{on} \quad \Pi \quad . \tag{2.28}$$

Suppose now $\Gamma$ is a regular arc in $\Pi$ and $\Gamma^* = \underset{\sim}{y}(\Gamma)$ its deformation image in $\Pi^* = \underset{\sim}{y}(\Pi)$. Moreover, let $\underset{\sim}{n}$ and $\underset{\sim}{n}^*$ be the orienting unit normal vectors of $\Gamma$ and $\Gamma^*$, respectively. Then (2.28) is easily found to imply

$$\underset{\sim}{T}\underset{\sim}{n}^* = \underset{\sim}{0} \quad \text{on} \quad \Gamma^* \quad \text{if and only if} \quad \underset{\sim}{\sigma}\underset{\sim}{n} = \underset{\sim}{0} \quad \text{on} \quad \Gamma \quad , \tag{2.29}$$

so that the vanishing of the nominal tractions on $\Gamma$ is necessary and sufficient in order that the actual tractions vanish on $\Gamma^*$. Note that the three-dimensional counterpart of this result was used earlier in writing the boundary conditions (2.10).

We also recall parenthetically that (2.26) becomes identical with the analogous two-dimensional constitutive relation for <u>plane strain</u> of a homogeneous and isotropic <u>compressible</u> material, provided $U(I,J)$ is regarded as

the plane-strain elastic potential.

We now deduce an implication of the constitutive relation (2.26) that will play an essential role later on. Let $M$ and $N$ be the invariants of $\underset{\sim}{\sigma}\underset{\sim}{\sigma}^T$ defined by

$$M = \mathrm{tr}(\underset{\sim}{\sigma}\underset{\sim}{\sigma}^T), \quad N = \det \underset{\sim}{\sigma} \quad . \tag{2.30}$$[1]

From (2.26), (2.27) follows

$$\underset{\sim}{\sigma}\underset{\sim}{\sigma}^T = 4U_I^2\underset{\sim}{G} + 4JU_IU_J\underset{\sim}{1} + J^2U_J^2\underset{\sim}{G}^{-1} \quad . \tag{2.31}$$

Now $\underset{\sim}{G}$ satisfies the Cayley-Hamilton equation

$$\underset{\sim}{G}^2 - (\mathrm{tr}\,\underset{\sim}{G})\underset{\sim}{G} + (\det \underset{\sim}{G})\underset{\sim}{1} = \underset{\sim}{0} \quad , \tag{2.32}$$

so that, upon post-multiplying (2.32) by $\underset{\sim}{G}^{-1}$, one arrives at

$$\mathrm{tr}\,\underset{\sim}{G}^{-1} = \frac{1}{\det \underset{\sim}{G}}\mathrm{tr}\,\underset{\sim}{G} = J^{-2}\mathrm{tr}\,\underset{\sim}{G} = IJ^{-2} \quad , \tag{2.33}$$

whence (2.30), (2.31) yield

$$M = (4U_I^2 + U_J^2)I + 8U_IU_JJ \quad . \tag{2.34}$$

On the other hand, post-multiplication of (2.26) by $\underset{\sim}{F}^T$ at once gives

$$J \det \underset{\sim}{\sigma} = \det(2U_I\underset{\sim}{G} + JU_J\underset{\sim}{1}) \quad . \tag{2.35}$$

With the aid of the characteristic polynomial of the tensor $2U_I\underset{\sim}{G}$, (2.35) is seen to furnish

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[1] Thus $M$ and $N$ are analogous to the scalar invariants $I$ and $J$ of the deformation tensor $\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T$

$$J \det \underset{\sim}{g} = J^2 U_J^2 + 2JU_I U_J \operatorname{tr} \underset{\sim}{G} + 4U_I^2 \det \underset{\sim}{G} \quad , \tag{2.36}$$

so that (2.27) and the second of (2.30) enable one to write

$$N = 2U_I U_J I + (4U_I^2 + U_J^2)J \quad . \tag{2.37}$$

Elimination of the explicit dependence on I between (2.34) and (2.37) results in the identity

$$2U_I U_J \operatorname{tr} (\underset{\sim}{g}\underset{\sim}{g}^T) - (4U_I^2 + U_J^2) \det \underset{\sim}{g} + (4U_I^2 - U_J^2)^2 J = 0 \quad , \tag{2.38}$$

which is the desired consequence of (2.26).

We conclude this section by considering as a special case the Neo-Hookean material, with which we shall be concerned hereafter. In this instance the three-dimensional elastic potential is given by

$$W(I_1, I_2) = \tfrac{\mu}{2}(I_1 - 3) \quad , \tag{2.39}$$

where $\mu > 0$ is the material's shear modulus. In view of (2.25), the corresponding plane-stress elastic potential takes the form

$$U(I,J) = \tfrac{\mu}{2}(I + J^{-2} - 3) \tag{2.40}$$

and (2.26), in component form, thus reduces to

$$\sigma_{\alpha\beta} = \mu(F_{\alpha\beta} - \lambda^2 F^{-1}_{\beta\alpha}), \; F_{\alpha\beta} = y_{\alpha,\beta} \; , \; \lambda = J^{-1} \quad . \tag{2.41}$$

But

$$JF^{-1}_{\beta\alpha} = \varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}F_{\mu\nu}, \; J = \det[F_{\alpha\beta}] > 0 \quad , \tag{2.42}$$

if $\varepsilon_{\alpha\mu}$ is the two-dimensional alternator ($\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$). Hence (2.41) yield

$$\sigma_{\alpha\beta} = \mu(y_{\alpha,\beta} - \lambda^3 \varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}y_{\mu,\nu}) \quad , \tag{2.43}$$

while according to (2.28), (2.41), the actual stresses obey

$$\tau_{\alpha\beta} = \mu(y_{\alpha,\gamma}y_{\beta,\gamma} - \lambda^2\delta_{\alpha\beta}) \quad . \tag{2.44}$$

Substituting from (2.43) into the component version of (2.24) and remembering the first of (2.27), one arrives at the equilibrium equations in terms of the spatial coordinates:

$$\nabla^2 y_\alpha = (\lambda^3)_{,\beta}\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}y_{\mu,\nu} \tag{2.45}$$

or

$$\left.\begin{aligned} \nabla^2 y_1 &= (\lambda^3)_{,1}y_{2,2} - (\lambda^3)_{,2}y_{2,1} \quad , \\ \nabla^2 y_2 &= (\lambda^3)_{,2}y_{1,1} - (\lambda^3)_{,1}y_{1,2} \quad . \end{aligned}\right\} \tag{2.46}$$

Further, in view of (2.40) and since $J = 1/\lambda$, the relation (2.38) here becomes

$$\lambda^{12} - \frac{1}{\mu^2}(\lambda^7 + \lambda)N - 2\lambda^6 - \frac{1}{\mu^2}\lambda^4 M + 1 = 0 \quad , \tag{2.47}$$

while (2.30), (2.43) lead to

$$\left.\begin{aligned} N &= \det[\sigma_{\alpha\beta}] = \mu^2(\lambda^5 + \lambda^{-1} - \lambda^3 y_{\alpha,\beta}y_{\alpha,\beta}) \quad , \\ M &= \sigma_{\alpha\beta}\sigma_{\alpha\beta} = \mu^2[(1 + \lambda^6)y_{\alpha,\beta}y_{\alpha,\beta} - 4\lambda^2] \quad . \end{aligned}\right\} \tag{2.48}$$

The foregoing equations agree with the analogous equations in the paper by Wong and Shield [15].[1] In particular, (2.47) is in agreement with Eq.(5.4) of [15], which is deduced there in a different manner.

Finally, it will be helpful to recall the response of a Neo-Hookean material to a three-dimensional homogeneous deformation corresponding to simple shear or uni-axial stress. For this purpose we first note with the aid of (2.3), (2.6), (2.8), and(2.39) that the first of the constitutive relations (2.7) at present reduces to

$$\underset{\sim}{T} = \mu \underset{\sim}{F}\underset{\sim}{F}^T - p\underset{\sim}{1}, \quad \underset{\sim}{F} = \nabla \underset{\sim}{y} \quad . \tag{2.49}$$

A simple shear parallel to the plane $x_3 = 0$ is represented by the mapping

$$y_1 = x_1 + kx_2, \; y_2 = x_2, \; y_3 = x_3 \quad , \tag{2.50}$$

where $k$ is the amount of shear (shearing strain). According to (2.49) this deformation induces the true stresses

$$\tau_{12} = \mu k, \; \tau_{23} = \tau_{31} = 0, \; \tau_{11} = \mu k^2, \; \tau_{22} = \tau_{33} = 0 \quad , \tag{2.51}$$

provided the arbitrary hydrostatic pressure $p$ is adjusted so as to enforce the vanishing of the transverse normal stress $\tau_{33}$.

The deformation appropriate to uni-axial stress parallel to the $x_2$-axis, in turn, takes the form

$$y_1 = x_1/\sqrt{\lambda}, \; y_2 = x_2, \; y_3 = x_3/\sqrt{\lambda} \quad (\lambda > 0) \quad , \tag{2.52}$$

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[1] It should be noted, however, that the plane-stress elastic potential employed in [15] is $tU(I,J)$, if $U$ is the potential defined in (2.40). Further, in [15] the equations are cast in terms of thickness stress-resultants (forces per unit length), rather than in terms of stresses. Also $t$ is not restricted to be constant in [15].

so that the associated principal stretches obey

$$\lambda_2 = \lambda, \ \lambda_1 = \lambda_3 = 1/\sqrt{\lambda} \quad . \tag{2.53}$$

Moreover, p in (2.49) must now be chosen so as to insure that $\tau_{11} = \tau_{33} = 0$. In this manner one arrives at the true stresses

$$\left.\begin{aligned} \tau_{22} \equiv \tau(\lambda) = \mu(\lambda^2 - \lambda^{-1}), \ \tau_{11} = \tau_{33} = 0 \quad , \\ \tau_{12} = \tau_{23} = \tau_{31} = 0 \quad . \end{aligned}\right\} \tag{2.54}$$

A graph of the first of (2.54), depicting the axial stress $\tau$ as a function of the axial stretch $\lambda$, is given in Fig.2.

3. <u>The global nonlinear interface-crack problem for two Neo-Hookean sheets. Local analysis</u>.

Here we first formulate, within the finite theory of plane stress for Neo-Hookean materials, the global interface-crack problem whose local analysis constitutes our objective. Figure 1 is henceforth taken to refer to two semi-infinite Neo-Hookean sheets, the interiors of which occupy the open half-planes $H_1$ and $H_2$ in an undeformed configuration, while $(x_1,x_2)$ now represent material coordinates in this configuration. Further, $\mu_1$ and $\mu_2$ at present denote the respective shear moduli of the two sheets. The latter are bonded along the $x_1$-axis but for the crack of length $2\ell$, which is assumed to be traction-free, each sheet being subjected to a homogeneous deformation at infinity.

One is thus required to find a deformation $y_\alpha(x_1,x_2)$ that satisfies the equilibrium equations (2.45) on $H_k$ and — through (2.43) — generates a nominal stress field

$$\left.\begin{aligned} \sigma_{\alpha\beta} &= \mu_k(y_{\alpha,\beta} - \lambda^3 \varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}y_{\mu,\nu}) \quad \text{on} \quad H_k \quad , \\ \lambda &= J^{-1}, \quad J = \det[y_{\alpha,\beta}] \quad , \end{aligned}\right\} \tag{3.1}$$

obeying the boundary conditions

$$\sigma_{\alpha 2}(x_1, 0\pm) = 0 \quad (|x_1| < \ell) \quad . \tag{3.2}^1$$

In addition, the desired deformation is to conform to the kinematic loading conditions

$$y_\alpha(x_1, x_2) = \overset{\infty}{F}{}^{(k)}_{\alpha\beta} x_\beta + O(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty, \quad (x_1, x_2) \in H_k \quad , \tag{3.3}$$

in which $\overset{\infty}{F}{}^{(k)}_{\alpha\beta}$ are pre-assigned constants, and $y_\alpha(x_1, x_2)$ must meet the bond conditions

$$y_\alpha(x_1, 0+) = y_\alpha(x_1, 0-), \; \sigma_{\alpha 2}(x_1, 0+) = \sigma_{\alpha 2}(x_1, 0-) \quad (|x_1| > \ell) \quad . \tag{3.4}^2$$

Finally, $y_\alpha$ and $\sigma_{\alpha\beta}$ are to have the same smoothness as $u_\alpha$ and $\sigma_{\alpha\beta}$ in England's problem.[3]

The gradient components $\overset{\infty}{F}{}^{(k)}_{\alpha\beta}$ of the homogeneous deformation at infinity cannot be assigned at will since (3.3) must be compatible with the bond conditions (3.4). With the aid of (3.1) one is thus led to the four conditions

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[1] See the remark following (2.29), according to which (3.2) assure that the deformed crack-faces are free of actual tractions.

[2] The second of (3.4) is easily seen to imply the continuity of the actual tractions across the deformed bonded portions of the interface.

[3] See the regularity assertions following (1.4).

$$\left.\begin{aligned} \overset{\infty}{F}{}^{(1)}_{\alpha 1} &= \overset{\infty}{F}{}^{(2)}_{\alpha 1} \quad , \\ s\{\overset{\infty}{F}{}^{(1)}_{\alpha 2} + [\overset{\infty}{J}{}^{(1)}]^{-3}\varepsilon_{\alpha\mu}\overset{\infty}{F}{}^{(1)}_{\mu 1}\} &= \overset{\infty}{F}{}^{(2)}_{\alpha 2} + [\overset{\infty}{J}{}^{(2)}]^{-3}\varepsilon_{\alpha\mu}\overset{\infty}{F}{}^{(2)}_{\mu 1} \quad , \end{aligned}\right\} \tag{3.5}$$

where

$$s = \frac{\mu_1}{\mu_2}, \quad \overset{\infty}{J}{}^{(k)} = \det[\overset{\infty}{F}{}^{(k)}_{\alpha\beta}] > 0 \quad . \tag{3.6}$$[1]

The relations (3.5) are a counterpart in the present nonlinear setting of the loading constraint (1.15).

If $\mu_1 = \mu_2$ $(s = 1)$, so that one is dealing with a crack in a single all-aroung homogeneous Neo-Hookean sheet, (3.5) are automatically satisfied, provided

$$\overset{\infty}{F}{}^{(1)}_{\alpha\beta} = \overset{\infty}{F}{}^{(2)}_{\alpha\beta} \equiv F_{\alpha\beta} \quad . \tag{3.7}$$

Moreover, the particular assignment

$$[\overset{\infty}{F}_{\alpha\beta}] = \begin{bmatrix} \overset{\infty}{\lambda}_1 & 0 \\ 0 & \overset{\infty}{\lambda}_2 \end{bmatrix}, \quad \overset{\infty}{\lambda}_2 > 1 \tag{3.8}$$

corresponds to a Mode I loading with the principal stretches $\overset{\infty}{\lambda}_1$, $\overset{\infty}{\lambda}_2$ at infinity, while

$$[\overset{\infty}{F}_{\alpha\beta}] = \begin{bmatrix} 1 & \overset{\infty}{k} \\ 0 & 1 \end{bmatrix} \tag{3.9}$$

constitutes a Mode II loading, $\overset{\infty}{k}$ being the amount of shear applied at

[1] The inequality in (3.6) follows from the inequality in (2.41).

infinity.

With a view toward an asymptotic analysis of the elastostatic field near the right-hand crack-tip, it is expedient to regard the spatial coordinates $y_\alpha$ from here on as functions of the local (material) polar coordinates $(r,\theta)$ (see Fig.1) and to introduce the complex spatial coordinate

$$y(r,\theta) = y_1(r,\theta) + iy_2(r,\theta) \quad (r > 0,\ -\pi \le \theta \le \pi) \quad . \tag{3.10}$$

One then has

$$J = \det[y_{\alpha,\beta}] = r^{-1}\,\mathrm{Im}\left\{\overline{\frac{\partial y}{\partial r}}\,\frac{\partial y}{\partial \theta}\right\}, \quad \lambda = J^{-1} \quad , \tag{3.11}$$

provided $\overline{\partial y/\partial r}$ stands for the complex conjugate of $\partial y/\partial r$. The equilibrium equations (2.45) are equivalent to

$$\left.\begin{aligned} \nabla^2 y &= \frac{i}{r}\left\{\frac{\partial y}{\partial r}\,\frac{\partial \lambda^3}{\partial \theta} - \frac{\partial y}{\partial \theta}\,\frac{\partial \lambda^3}{\partial r}\right\} \quad , \\ \nabla^2 y &\equiv \frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2} \quad , \end{aligned}\right\} \tag{3.12}$$

while the constitutive relations (2.43) lead to

$$\left.\begin{aligned} \sigma_{11} + i\sigma_{21} &= \mu\left[\frac{\partial y}{\partial r}(\cos\theta + i\lambda^3\sin\theta) - \frac{1}{r}\frac{\partial y}{\partial \theta}(\sin\theta - i\lambda^3\cos\theta)\right] \quad , \\ \sigma_{12} + i\sigma_{22} &= \mu\left[\frac{\partial y}{\partial r}(\sin\theta - i\lambda^3\cos\theta) + \frac{1}{r}\frac{\partial y}{\partial \theta}(\cos\theta + i\lambda^3\sin\theta)\right] \quad . \end{aligned}\right\} \tag{3.13}$$

From (3.13) one deduces formulas for the polar components of nominal stress:

$$\left.\begin{aligned} \sigma_{rr} + i\sigma_{\theta r} &= \mu e^{-i\theta}\left\{\frac{\partial y}{\partial r} + i\lambda^3\ \frac{1}{r}\frac{\partial y}{\partial \theta}\right\} \quad , \\ \sigma_{r\theta} + i\sigma_{\theta\theta} &= \mu e^{-i\theta}\left\{\frac{1}{r}\frac{\partial y}{\partial \theta} - i\lambda^3\ \frac{\partial y}{\partial r}\right\} \quad . \end{aligned}\right\} \tag{3.14}$$[1]

Further, (2.48) at once yield

$$\left.\begin{aligned} N = \det[\sigma_{\alpha\beta}] &= \mu^2\left\{\lambda^3 + \lambda^{-1} - \lambda^3\left(\left|\frac{\partial y}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial y}{\partial \theta}\right|^2\right)\right\} \quad , \\ M = \sigma_{\alpha\beta}\sigma_{\alpha\beta} &= \mu^2\left\{(1+\lambda^6)\left(\left|\frac{\partial y}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial y}{\partial \theta}\right|^2\right) - 4\lambda^2\right\} \quad . \end{aligned}\right\} \tag{3.15}$$

Equations (3.12) hold for $0 < \theta < \pi$, as well as for $-\pi < \theta < 0$. On the other hand, (3.13), (3.14), and (3.15) apply to $0 \le \theta \le \pi$ with $\mu = \mu_1$ and to $-\pi \le \theta \le 0$ with $\mu = \mu_2$. Moreover, because of (3.2), (3.13), one has the boundary conditions

$$\left[\frac{1}{r}\frac{\partial y}{\partial \theta} - i\lambda^3\ \frac{\partial y}{\partial r}\right]_{(r,\pm\pi)} = 0 \quad (0 < r < 2\ell) \quad , \tag{3.16}$$

whereas (3.4), (3.13) furnish the bond conditions

$$\left.\begin{aligned} y(r,0+) &= y(r,0-) \quad , \\ s\left[\frac{1}{r}\frac{\partial y}{\partial \theta} - i\lambda^3\frac{\partial y}{\partial r}\right]_{(r,0+)} &= \left[\frac{1}{r}\frac{\partial y}{\partial \theta} - i\lambda^3\frac{\partial y}{\partial r}\right]_{(r,0-)} \quad (r > 0) \quad , \end{aligned}\right\} \tag{3.17}$$

and analogous bond conditions at $\theta = \pm\pi$.

Taking for granted the existence of a solution to the global interface-

[1] The subscripts in the left members of (3.14) are exempted from the range and summation conventions adopted earlier for Greek indices.

crack problem stated at the beginning of this section, we now assume that this solution admits an asymptotic representation of the form

$$\left.\begin{aligned} y(r,\theta) &\sim \ell + r^{m} v(\theta) \quad \text{as} \quad r \to 0 \quad (0 \le \theta \le \pi) \ , \\ y(r,\theta) &\sim \ell + r^{m'} v(\theta) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le 0) \ , \end{aligned}\right\} \tag{3.18}^{1}$$

$$m = \alpha + i\beta, \ m' = \alpha' + i\beta' \quad (\alpha,\alpha' \text{ and } \beta,\beta' \text{ real}) \ , \tag{3.19}$$

so that the exponents $m$ and $m'$ are permitted to be complex numbers, while $v$ is a complex-valued function, twice continuously differentiable on $[0,\pi]$ and on $[-\pi,0]$, that fails to vanish identically on either of these two intervals. Further, we stipulate that

$$\alpha = \operatorname{Re} m > 0, \ \alpha' = \operatorname{Re} m' > 0 \ ; \tag{3.20}$$

accordingly the displacements are required to be continuous as $r \to 0$ and the elastostatic field at hand is taken to have been normalized by means of a suitable rigid translation so as to keep the crack-tip at $r = 0$ fixed.

Note that (3.18), (3.19), (3.10) furnish

$$\left.\begin{aligned} y_1(r,\theta) &\sim \ell + r^{\alpha}\{v_1(\theta)\cos(\beta \log r) - v_2(\theta)\sin(\beta \log r)\} \ , \\ y_2(r,\theta) &\sim r^{\alpha}\{v_1(\theta)\sin(\beta \log r) + v_2(\theta)\cos(\beta \log r)\} \end{aligned}\right\} \tag{3.21}^{2}$$

when $0 \le \theta \le \pi$; if $-\pi \le \theta \le 0$, $\alpha$ and $\beta$ in (3.21) are to be replaced by $\alpha'$

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[1] Here and in the sequel the asymptotic equality symbol "$\sim$" is used in the following connotation: the first of (3.18) means that for $0 \le \theta \le \pi$,

$$y(r,\theta) = \ell + r^{m} v(\theta) + o(r^{m}) \quad \text{as} \quad r \to 0 \ ,$$

where $o(r^{m}) = o(|r^{m}|) = o(r^{\alpha})$, if $\alpha = \operatorname{Re} m$.

[2] All asymptotic equalities and order-of-magnitude estimates are henceforth understood to refer to the limit as $r \to 0$.

and $\beta'$, respectively. Equations (3.21) exhibit the structure of the oscillatory behavior admitted by the Ansatz (3.18). Clearly, the right-hand members in (3.21) are non-oscillatory if and only if $m$ and $m'$ are both real ($\beta = \beta' = 0$). Observe also that (3.18) encompass the possibility that $y_1(r,\theta) - \ell$ and $y_2(r,\theta)$ are of different dominant orders of mangitude as $r \to 0$, for a fixed value of $\theta$. For example, according to (3.21), $\beta = 0$ together with $v_2 = 0$ on $[0,\pi]$ gives $y_1(r,\theta) - \ell \sim r^{\alpha} v_1(\theta)$, but $y_2(r,\theta)$ in this instance vanishes to a higher order as $r \to 0$.

To (3.18) we adjoin an additional asymptotic assumption concerning the transverse stretch $\lambda$ by requiring that

$$\lambda(r,\theta) = 1/J(r,\theta) = O(r^q) \quad \text{as} \quad r \to 0 \quad (-\pi \leq \theta \leq \pi) \quad \text{for some} \quad q > 0 \quad . \tag{3.22}$$

At the same time we demand the validity of the asymptotic equalities resulting from two consecutive formal differentiations of (3.18) with respect to $r$ or $\theta$, as well as the "differentiability" of (3.22).[1] Equation (3.22) implies that $J(r,\theta)$ becomes unbounded as $r \to 0$ and hence anticipates that not all deformation-gradient components remain bounded in this limit, so that $\alpha < 1$, $\alpha' < 1$. Since $\lambda(r,\theta)$ tends to zero as $r \to 0$, in view of (3.22), the deformation prevailing in the vicinity of the crack-tip entails an extreme thinning of each sheet. Evidently (3.22) excludes all regular deformations — in particular homogeneous deformations — from our present considerations.

Aiming at the lowest-order asymptotic near-field behavior, we now seek to determine $v(r,\theta)$, as well as the exponents $m$, $m'$ with the smallest real parts, consistent with (3.18), (3.20), (3.22), the governing equilibrium equation (3.12) and the accompanying boundary and bond conditions (3.16),

[1] Thus $\nabla\lambda(r,\theta) = O(r^{q-1})$ as $r \to 0$ $(0 < |\theta| < \pi)$.

(3.17). On account of (3.18) and (3.22), the nonlinear $\lambda^3$-terms in (3.12), (3.16), (3.17) do not come into play in the lowest-order analysis. Upon entering (3.12), (3.16) with (3.18), (3.22), one readily arrives at

$$\ddot{v} + m^2 v = 0 \quad \text{on} \quad (0,\pi), \quad \ddot{v} + (m')^2 v = 0 \quad \text{on} \quad (-\pi,0) \quad , \tag{3.23}$$[1]

$$\dot{v}(\pi) = \dot{v}(-\pi) = 0 \quad . \tag{3.24}$$

From (3.23) follows

$$\left.\begin{aligned} v(\theta) &= a \sin m\theta + b \cos m\theta \quad (0 < \theta \leq \pi) \quad , \\ v(\theta) &= a' \sin m'\theta + b' \cos m'\theta \quad (-\pi \leq \theta < 0) \quad , \end{aligned}\right\} \tag{3.25}$$

in which $a$, $b$ and $a'$, $b'$ are complex constants. On the other hand, (3.18), (3.22) in conjunction with the bond conditions (3.17) and (3.25) give

$$\left.\begin{aligned} b\, r^m + o(r^\alpha) &= b'\, r^{m'} + o(r^{\alpha'}) \quad , \\ s\, m\, a\, r^{m-1} + o(r^{\alpha-1}) &= m' a' r^{m'-1} + o(r^{\alpha'-1}) \quad . \end{aligned}\right\} \tag{3.26}$$

If $\alpha \neq \alpha'$, (3.26), (3.20) imply that $a$, $a'$, $b$, $b'$ vanish simultaneously, which is inadmissible since in this instance $v \equiv 0$ on $[-\pi,\pi]$. Therefore,

$$\alpha = \mathrm{Re}\, m = \alpha' = \mathrm{Re}\, m' \quad . \tag{3.27}$$

Further, (3.26), (3.27) necessitate that

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[1] Here and in what follows superior dots connote differentiation with respect to the polar angle $\theta$.

$$\left.\begin{aligned} b + o(1) &= b' r^{i(\beta-\beta')} \quad , \\ s\,m\,a + o(1) &= m' a' r^{i(\beta-\beta')} \quad , \end{aligned}\right\} \tag{3.28}$$

and the left members in (3.28) tend to $b$ and $s\,m\,a$, respectively, as $r \to 0$, whereas the right members fail to possess limits unless $\beta = \beta'$. Consequently, one draws from (3.27) and (3.28) that

$$m = m', \; s\,a = a', \; b = b' \quad . \tag{3.29}$$

Combining (3.29) with (3.25), invoking (3.24), and recalling that $m = 0$ is precluded by (3.20), one has

$$\left.\begin{aligned} a \cos m\pi - b \sin m\pi &= 0 \quad , \\ s\,a \cos m\pi + b \sin m\pi &= 0 \quad . \end{aligned}\right\} \tag{3.30}$$

Thus, since $s > 0$ and $a = b = 0$ is ruled out by $v \not\equiv 0$ on $[0,\pi]$,

$$\sin 2m\pi = 0 \quad , \tag{3.31}$$

whence $m$ is <u>real</u>. The smallest positive root of (3.31) is

$$m = \alpha = \frac{1}{2} \quad , \tag{3.32}$$

and (3.30), (3.32) lead to $b = 0$, while the constant $a$ remains arbitrary. Also, (3.32), (3.29), (3.25), and (3.18) yield

$$\left.\begin{aligned} y(r,\theta) &\sim \ell + r^{1/2} v(\theta) \quad (-\pi \le \theta \le \pi) \quad , \\ v(\theta) &= a h(\theta) \sin\frac{\theta}{2} \quad (-\pi \le \theta \le \pi) \quad , \end{aligned}\right\} \tag{3.33}$$

in which $a \neq 0$ is an arbitrary complex constant and $h$ is the step-function defined by

$$h(\theta) = 1 \ (0 \leq \theta \leq \pi), \ h(\theta) = s \ (-\pi \leq \theta < 0), \ s = \mu_1/\mu_2 \ . \tag{3.34}$$

On setting $a = a_1 + ia_2$ and referring to (3.10), one finally concludes that

$$\left.\begin{aligned} y_1(r,\theta) &\sim \ell + a_1 r^{1/2} h(\theta) \sin\tfrac{\theta}{2} \ , \\ y_2(r,\theta) &\sim a_2 r^{1/2} h(\theta) \sin\tfrac{\theta}{2} \quad (-\pi \leq \theta \leq \pi) \ , \end{aligned}\right\} \tag{3.35}$$

where $a_1$, $a_2$ are arbitrary real constants with $a_1^2 + a_2^2 > 0$.

One confirms with the aid of (3.11) that the Jacobian determinant of the right members in (3.35) vanishes identically. Therefore the lowest-order approximation to the local deformation (near the crack-tip) obtained above does not constitute a mapping that is one-to-one. Furthermore, since $\lambda = 1/J$, the one-term asymptotic solution (3.35) fails to supply an estimate for the transverse stretch. These inadequacies make it necessary to seek at least a <u>two</u>-term approximation to the elastostatic field in the vicinity of the crack-tip.

In preparation for this task we show first that (3.33), <u>together with</u> (2.47) and (3.22), in fact enable one to deduce an estimate for $\lambda(r,\pm\pi)$ as $r \to 0$, i.e. for the transverse stretch along the faces of the crack.[1] Indeed, the boundary conditions (3.2), referred to the polar coordinates $(r,\theta)$, give

$$N(r,\pm\pi) = \det[\sigma_{\alpha\beta}(r,\pm\pi)] = 0 \quad (0 < r < 2\ell) \ , \tag{3.36}$$

[1] This appeal to the identity (2.47) was suggested by Wong and Shield [15], who use (2.47) in a similar manner in their treatment of a Mode I problem for a crack in an all-around homogeneous Neo-Hookean sheet.

so that (2.47) supplies

$$\frac{M\lambda^4}{\mu_1^2} = 1 + \lambda^{12} - 2\lambda^6 \quad \text{at} \quad \theta = \pi \quad (0 < r < 2\ell) \quad . \tag{3.37}$$

Next, (3.33), (3.22), and the second of (3.15) are found to imply

$$M = \sigma_{\alpha\beta}\sigma_{\alpha\beta} \sim \frac{\mu_1^2|a|^2}{4r} \quad (-\pi \le \theta \le \pi), \quad a \neq 0 \quad . \tag{3.38}$$

Keeping in mind that $\lambda(r,\theta) = o(1)$ for $-\pi \le \theta \le \pi$ and combining (3.38) with (3.37), as well as with its companion for $\theta = -\pi$, one readily confirms that

$$\lambda(r,\pi) \sim \sqrt{\frac{2}{|a|}}\, r^{1/4}, \quad \lambda(r,-\pi) \sim \sqrt{\frac{2}{s|a|}}\, r^{1/4} \quad \text{as} \quad r \to 0 \quad , \tag{3.39}$$

which is consistent with (3.22) and necessitates $q \le 1/4$. Guided by (3.39), we now refine the original a priori assumption (3.22) by anticipating that

$$\left.\begin{aligned} \lambda(r,\theta) &= O(r^{1/4}) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad , \\ \nabla\lambda(r,\theta) &= O(r^{-1/4}) \quad \text{as} \quad r \to 0 \quad (0 < |\theta| < \pi) \quad . \end{aligned}\right\} \tag{3.40}$$

It is helpful to observe that (3.40), (3.33) imply

$$\lambda^3 \frac{\partial y}{\partial r} = O(r^{1/4}) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad , \tag{3.41}$$

$$\frac{i}{r}\left(\frac{\partial y}{\partial r}\frac{\partial \lambda^3}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial \lambda^3}{\partial r}\right) = O(r^{-3/4}) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad . \tag{3.42}$$

Proceeding to the second-order asymptotic analysis, we set

$$\left.\begin{aligned} y(r,\theta) &\sim \ell + r^{1/2}v(\theta) + r^{n}w(\theta) \quad (0 \le \theta \le \pi) \quad , \\ y(r,\theta) &\sim \ell + r^{1/2}v(\theta) + r^{n'}w(\theta) \quad (-\pi \le \theta \le 0) \quad . \end{aligned}\right\} \tag{3.43}$$

Here $v$ is known from (3.33), $n$ and $n'$ are as yet unknown complex exponents obeying

$$\gamma = \mathrm{Re}\, n > \frac{1}{2}, \quad \gamma' = \mathrm{Re}\, n' > \frac{1}{2} \quad , \tag{3.44}$$

and $w$ is an initially undetermined complex-valued function that is not permitted to vanish identically on $[0,\pi]$ or $[-\pi,0]$. Further, $w$ and (3.43) are subject to the same differentiability hypotheses made previously in connection with $v$ and (3.18).

Our current objective is to find $w$ on $[-\pi,\pi]$ and $n$, $n'$, with the smallest real parts admitted by (3.44), consistent with (3.12), (3.16), (3.17), and (3.40). In view of (3.41), (3.42), (3.43), and because $v$ satisfies (3.23), (3.24), the equilibrium equation (3.12) and the boundary conditions (3.16) require

$$\left.\begin{aligned} r^{n-2}[\ddot{w}(\theta) + n^2 w(\theta)] + o(r^{n-2}) &= O(r^{-3/4}) \quad (0 < \theta < \pi) \quad , \\ r^{n'-2}[\ddot{w}(\theta) + (n')^2 w(\theta)] + o(r^{n'-2}) &= O(r^{-3/4}) \quad (-\pi < \theta < 0) \quad , \end{aligned}\right\} \tag{3.45}$$

$$\left.\begin{aligned} r^{n-1}\dot{w}(\pi) + o(r^{n-1}) &= O(r^{1/4}) \quad , \\ r^{n'-1}\dot{w}(-\pi) + o(r^{n'-1}) &= O(r^{1/4}) \quad . \end{aligned}\right\} \tag{3.46}$$

If

$$\mathrm{Re}\, n < \frac{5}{4}, \quad \mathrm{Re}\, n' < \frac{5}{4} \quad , \tag{3.47}$$

as we shall temporarily take for granted, (3.45), (3.46) evidently imply (3.23), (3.24) – and hence (3.25) – with $v$, $m$, $m'$ replaced by $w$, $n$, $n'$, respectively. Also, (3.47) insure that the nonlinear $\lambda^3$-terms in the bond conditions (3.17) likewise fail to enter the second-order analysis explicitly.

An argument strictly parallel to that following (3.25) easily completes the second-order asymptotic solution. In particular one finds in this manner that

$$n = n', \ \sin 2n\pi = 0 \quad , \tag{3.48}$$

whence $n$ and $n'$ are <u>real</u>. The smallest root of (3.48) conforming to (3.44) is

$$n = n' = \gamma = \gamma' = 1 \quad , \tag{3.49}$$

and since (3.49) is consistent with (3.47), there is no need to explore the complementary ranges $\gamma \geq 5/4$, $\gamma' \geq 5/4$. The second-order approximation thus emerging is given by

$$\left.\begin{aligned} y(r,\theta) &\sim \ell + r^{1/2} v(\theta) + r w(\theta) \quad (-\pi \leq \theta \leq \pi) \quad , \\ v(\theta) &= a\, h(\theta) \sin\tfrac{\theta}{2} \ , \ w(\theta) = b \cos\theta \quad (-\pi \leq \theta \leq \pi) \quad , \end{aligned}\right\} \tag{3.50}$$

in which $a \neq 0$ and $b \neq 0$ are otherwise arbitrary complex constants, while $h$ again stands for the step-function introduced in (3.34). From (3.50), (3.10) finally follows:

$$\left.\begin{aligned} y_1(r,\theta) &\sim \ell + a_1 r^{1/2} h(\theta) \sin\tfrac{\theta}{2} + b_1 r \cos\theta \quad , \\ y_2(r,\theta) &\sim a_2 r^{1/2} h(\theta) \sin\tfrac{\theta}{2} + b_2 r \cos\theta \quad (-\pi \leq \theta \leq \pi) \quad , \end{aligned}\right\} \tag{3.51}$$

with

$$a_1^2 + a_2^2 > 0, \; b_1^2 + b_2^2 > 0 \quad , \tag{3.52}$$

$a_1$, $a_2$ and $b_1$, $b_2$ being otherwise arbitrary real constants.

Upon using (3.11) to compute the Jacobian determinant on the basis of the two-term asymptotic solution (3.50), one arrives at the estimate

$$J(r,\theta) \sim A r^{-1/2} h(\theta) \cos\frac{\theta}{2} \quad (-\pi \le \theta \le \pi) \quad , \tag{3.53}$$

in which

$$A = \frac{1}{2}(a_2 b_1 - a_1 b_2) \ge 0 \quad , \tag{3.54}$$

the inequality being a consequence of the requirement $J > 0$. The approximation (3.53) gives merely $J(r,\theta) = o(r^{-1/2})$ $(-\pi \le \theta \le \pi)$ unless $A > 0$, as we shall assume until further notice.[1] From (3.53) one draws the asymptotic characterization of the transverse stretch $\lambda = 1/J$:

$$\lambda(r,\theta) \sim \frac{r^{1/2}}{Ah(\theta)\cos(\theta/2)} \quad (-\pi < \theta < \pi) \quad \text{if} \quad A > 0 \quad , \tag{3.55}$$

which is consistent with the a priori assumption (3.40) employed in the derivation of (3.50). Equations (3.53), (3.55), (3.34) reflect the discontinuity in $J(r,\theta)$ and $\lambda(r,\theta)$ at $\theta = 0$ if $s \neq 1$ $(\mu_1 \neq \mu_2)$.

Although (3.55) and (3.39) together furnish estimates of $\lambda(r,\theta)$, as $r \to 0$, for each fixed $\theta$ in the complete range $[-\pi,\pi]$, the approximation given by (3.55) evidently deteriorates severely near the crack-faces $\theta = \pm\pi$. In order to obtain a crack-tip estimate for $\lambda$ that is free of this deficiency

[1] The special case $A = 0$ will be treated in the next section.

and that at the same time reveals the nature of the transition from (3.55) to (3.39) as $\theta \to \pm\pi$, one may once again appeal to the identity (2.47) used earlier to deduce (3.39). Since (3.22) implies $\lambda(r,\theta) = o(1)$ as $r \to 0$, (2.47) near $r = 0$ is approximated by

$$M(r,\theta)\lambda^4 + N(r,\theta)\lambda - \mu_1^2 = 0 \quad (0 \le \theta \le \pi) \quad . \tag{3.56}$$[1]

This suggests that

$$\lambda(r,\theta) \sim \lambda_0(r,\theta) \quad \text{as} \quad r \to 0 \quad (0 \le \theta \le \pi) \quad , \tag{3.57}$$

where $\lambda_0$ is the unique positive root for $\lambda$ of (3.56), as is not difficult to confirm rigorously. On setting

$$\chi = \mu_1^{-3/2} M^{-1/4} N, \quad \eta = \mu_1^{-1/2} M^{1/4} \lambda \quad , \tag{3.58}$$

(3.56) becomes

$$\eta^4 + \chi\eta - 1 = 0 \quad . \tag{3.59}$$

Let $\eta = \varphi(\chi)$ be the unique positive root of (3.59)[2], so that (3.58), (3.57) give

$$\lambda(r,\theta) \sim \mu_1^{1/2} M^{-1/4} \varphi(\mu_1^{-3/2} M^{-1/4} N) \quad (0 \le \theta \le \pi) \quad . \tag{3.60}$$

The desired estimate for $\lambda$ is obtained by inserting small-r approximations for $M(r,\theta)$ and $N(r,\theta)$ into (3.60). Such an approximation to

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[1] We sketch in detail only the analysis pertaining to the upper sheet $(0 \le \theta \le \pi)$. Recall from (2.48) that $M = \sigma_{\alpha\beta}\sigma_{\alpha\beta} > 0$, $N = \det[\sigma_{\alpha\beta}]$.

[2] An explicit elementary representation for $\varphi(\chi)$ is readily established, but serves no particular purpose here.

$M(r,\theta)$ is available from (3.38). On the other hand, an analogous estimate for $N(r,\theta) = \det[\sigma_{\alpha\beta}]$ is computable, without recourse to (3.55), from (3.50) and (3.13) by relying merely on (3.40). In this manner one obtains

$$N(r,\theta) \sim \mu_1^2 A r^{-1/2} \cos\frac{\theta}{2} \quad (0 \le \theta < \pi),\ N(r,\pi) = o(r^{-1/2}) \quad . \tag{3.61}$$[1]

Substitution from (3.38), (3.61) into (3.60), along with a parallel argument applied to the lower sheet, results in

$$\left.\begin{aligned} &\lambda(r,\theta) \sim r^{1/4} g(\theta) \varphi(r^{-1/4} A g(\theta) \cos(\theta/2)) \quad , \\ &g(\theta) = \sqrt{2/|a|h(\theta)} \quad (-\pi \le \theta \le \pi),\ A > 0 \quad . \end{aligned}\right\} \tag{3.62}$$

Under suitable assumptions concerning the uniformity in $\theta$ of the Ansatz (3.43), one can show that the estimate (3.62), the validity of which is contingent upon our current supposition that $A > 0$, holds <u>uniformly</u> for $-\pi \le \theta \le \pi$.

Finally, we note that one recovers from (3.62) through appropriate specialization, the previous estimates (3.39) and (3.55), which apply to $\theta = \pm\pi$ and $-\pi < \theta < \pi$ separately. To see this we infer from (3.59) that

$$\varphi(0) = 1,\ \varphi'(\chi) < 0 \quad (-\infty < \chi < \infty),\ \varphi(\chi) \sim \frac{1}{\chi} \quad \text{as} \quad \chi \to \infty \quad . \tag{3.63}$$

The first of (3.63) reveals that (3.62) reduces to (3.39) for $\theta = \pm\pi$; the last of (3.63) leads immediately from (3.62) to (3.55).

Although we have not determined in detail any further terms in the expansion (3.50), we have ascertained that additional terms of orders $r^{5/4}$,

[1] Here $A$ is again the auxiliary amplitude parameter defined in (3.54).

$r^{3/2}$, and $r^{7/4}$ can be obtained by arguments parallel to those used in this and the preceding section. If one attempts to continue the expansion beyond the term of order $r^{7/4}$, however, one encounters difficulties since at that stage the $\lambda^3$-terms in the equilibrium equation (3.12) begin to enter explicitly into the asymptotic analysis. Indeed, it appears from the structure of the uniform approximation (3.62) for $\lambda$ that such higher-order terms can no longer be of the "factored" form $r^k f(\theta)$.

4. Rotation of the deformed composite sheet. The degenerate case $A = 0$.

The complex version (3.50) of the two-term asymptotic approximation deduced in Section 3 involves the complex amplitude parameters $a$ and $b$ ($a \neq 0$, $b \neq 0$), the determination of which eludes the local analysis carried out there. The values of these two parameters are bound to depend on the material parameters $\mu_1$ and $\mu_2$, the crack-length $2\ell$, as well as on the particular loading at infinity, which has not entered the derivation of (3.50). Indeed, these local results are not confined to semi-infinite sheets and encompass essentially arbitrary loadings applied to the outer boundary. We show presently that (3.50) may, without loss of generality, be simplified by subjecting the deformed composite sheet to a rigid rotation about the crack-tip at $r = 0$. Moreover, this transformation — as will become evident — renders the intrinsic content of the asymptotic solution (3.50) more transparent.

If $\hat{y}(r,\theta)$ is the complex spatial coordinate (in the original fixed Cartesian frame) of a material point $(r,\theta)$ after such a rotation through an angle $\vartheta$, one has

$$\hat{y}(r,\theta) = \ell + e^{i\vartheta}\{y(r,\theta) - \ell\} \tag{4.1}$$

we now take

$$\vartheta = \frac{\pi}{2} - \varepsilon \quad , \tag{4.2}$$

where $\varepsilon$ is the argument of the complex parameter $a$, whence

$$a = a_1 + ia_2 = |a|e^{i\varepsilon} \quad . \tag{4.3}$$

For this choice of $\vartheta$, equations (4.1), (3.50) imply

$$\hat{y}(r,\theta) \sim \ell + \hat{a}r^{1/2}h(\theta)\sin\frac{\theta}{2} + \hat{b}r\cos\theta \quad (-\pi \le \theta \le \pi) \quad , \tag{4.4}$$

$$\hat{a} = i|a|, \ \hat{b} = i\bar{a}b/|a| \quad (a \neq 0, \ b \neq 0) \quad . \tag{4.5}$$

From (4.5) follows

$$\left.\begin{aligned} &\hat{a} = \hat{a}_1 + i\hat{a}_2, \ \hat{a}_1 = 0, \ \hat{a}_2 = |a| > 0 \quad , \\ &\hat{b} = \hat{b}_1 + i\hat{b}_2, \ \hat{b}_1 = 2A/|a| \geq 0, \ \hat{b}_2 = 2B/|a| \quad , \end{aligned}\right\} \tag{4.6}$$

where

$$A = \frac{1}{2}(a_2b_1 - a_1b_2) \geq 0, \quad B = \frac{1}{2}(a_1a_2 + b_1b_2) \quad , \tag{4.7}$$

so that $A$ is the non-negative auxiliary amplitude parameter originally introduced in (3.54). Also, (4.6) give

$$\left.\begin{aligned} &|\hat{a}| = |a|, \ \hat{A} = \frac{1}{2}(\hat{a}_2\hat{b}_1 - \hat{a}_1\hat{b}_2) = A \quad , \\ &|\hat{b}| = |b|, \ \hat{B} = \frac{1}{2}(\hat{a}_1\hat{b}_1 + \hat{a}_2\hat{b}_2) = B \quad , \end{aligned}\right\} \tag{4.8}$$

which assert the invariance of $|a|$, $|b|$, $A$, and $B$ under the rotation (4.1).

To avoid cumbersome notation we henceforth write $y$, $a$, $b$ in place of

$\hat{y}$, $\hat{a}$, $\hat{b}$ and thus note that (4.4), on account of (4.6), leads to

$$\left.\begin{aligned} y_1(r,\theta) &\sim \ell + b_1 r\cos\theta \quad , \\ y_2(r,\theta) &\sim a_2 r^{1/2} h(\theta)\sin\tfrac{\theta}{2} + b_2 r\cos\theta \quad (-\pi \le \theta \le \pi) \quad , \end{aligned}\right\} \tag{4.9}$$

where

$$a_2 > 0,\ b_1 \ge 0 \quad (b_1 = 2A/a_2) \quad , \tag{4.10}$$

and $a_2$, $b_1$, $b_2$ are otherwise arbitrary real constants. As is now apparent, no generality is lost in the original two-term asymptotic approximation (3.51) by setting $a_1 = 0$, while subjecting $a_2$ and $b_1$ to the inequalities in (4.10)

In view of (4.10), the exceptional case $A = 0$, which was excluded in establishing the estimates (3.55) and (3.62) for the transverse stretch $\lambda$, corresponds to $b_1 = 0$ in (4.9). On the other hand, when $b_1 = 0$, the approximating deformation supplied by (4.9) is no longer one-to-one[1] and the current approximation to $y_1(r,\theta)$ degenerates into the weak estimate

$$y_1(r,\theta) = \ell + o(r) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad , \tag{4.11}$$

which is inadequate. The case $A = 0$ $(b_1 = 0)$ thus entails an essential deterioration of the asymptotic solution (4.9) and necessitates higher-order considerations.

Motivated by the foregoing observations and bearing in mind (4.9), (4.10) with $b_1 = 0$, we now adopt the <u>three</u>-term asymptotic Ansatz:

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[1] Since $J$ is invariant under the rotation (4.1), the Jacobian determinant of this mapping is given by the right member in (3.53), which vanishes for $A = 0$.

$$\left.\begin{aligned} y(r,\theta) &\sim \ell + ia_2 r^{1/2}\sin\tfrac{\theta}{2} + ib_2 r\cos\theta + r^{p}z(\theta) \quad (0 \le \theta \le \pi) \ , \\ y(r,\theta) &\sim \ell + isa_2 r^{1/2}\sin\tfrac{\theta}{2} + ib_2 r\cos\theta + r^{p'}z(\theta) \quad (-\pi \le \theta \le 0) \ , \end{aligned}\right\} \tag{4.12}$$

where

$$a_2 > 0, \ \mathrm{Re}\, p > 1, \ \mathrm{Re}\, p' > 1 \ , \tag{4.13}$$

while the as yet unknown complex-valued function $z = z_1 + iz_2$ must not vanish identically on $[0,\pi]$ or $[-\pi,0]$ and is required to be twice continuously differentiable on each of these two intervals.[1]

From (4.12) one gathers

$$\left.\begin{aligned} \nabla^2 y &\sim r^{p-2}\{\ddot{z} + p^2 z\} \quad (0 < \theta < \pi) \ , \\ \nabla^2 y &\sim r^{p'-2}\{\ddot{z} + (p')^2 z\} \quad (-\pi < \theta < 0) \ . \end{aligned}\right\} \tag{4.14}$$

Consequently, the equilibrium equation (3.12) together with the estimate (3.42) enables one to conclude that

$$\ddot{z} + p^2 z = 0 \quad (0 < \theta < \pi) \quad \text{if} \quad 1 < \mathrm{Re}\, p < 5/4 \ , \tag{4.15}$$

$$\ddot{z} + (p')^2 z = 0 \quad (-\pi < \theta < 0) \quad \text{if} \quad 1 < \mathrm{Re}\, p' < 5/4 \ . \tag{4.16}$$

Further, on applying the boundary conditions (3.16) to (4.12) and taking account of the estimate (3.41), one has

$$\dot{z}(\pi) = 0 \quad \text{if} \quad 1 < \mathrm{Re}\, p < 5/4 \ , \tag{4.17}$$

[1] We take for granted the "differentiability" of the asymptotic identities (4.12).

$$\dot{z}(-\pi) = 0 \quad \text{if} \quad 1 < \operatorname{Re} p' < 5/4 \quad . \tag{4.18}$$

Next, imposing the transition conditions (3.17) on (4.12) and appealing once again to (3.41), we see that

$$z(0+) = \dot{z}(0+) = 0 \quad \text{if} \quad 1 < \operatorname{Re} p < 5/4, \quad \operatorname{Re} p < \operatorname{Re} p' \quad , \tag{4.19}$$

$$z(0-) = \dot{z}(0-) = 0 \quad \text{if} \quad 1 < \operatorname{Re} p' < 5/4, \quad \operatorname{Re} p' < \operatorname{Re} p \quad . \tag{4.20}$$

But, according to (4.15), (4.17), (4.19), the inequalities in (4.19) demand that $z = 0$ on $[0,\pi]$, which is inadmissible; similarly, the inequalities in (4.20), in view of (4.16), (4.18), (4.20), necessitate $z = 0$ on $[-\pi,0]$, which is also unacceptable. On the other hand, the transition conditions (3.17) reduce to

$$z(0+) = z(0-), \; s\dot{z}(0+) = \dot{z}(0-) \quad \text{if} \quad 1 < \operatorname{Re} p = \operatorname{Re} p' < 5/4 \; , \tag{4.21}$$

and in this event (4.15) to (4.21) require that $z = 0$ on $[-\pi,\pi]$.

The preceding conclusions entitle us to claim that $\operatorname{Re} p \geq 5/4$, $\operatorname{Re} p' \geq 5/4$. Moreover, since our present objective is an asymptotically consistent three-term approximation of the form (4.12) in which the exponents $p$ and $p'$ have the <u>smallest</u> real parts admitted by (4.13), there is no need to consider $\operatorname{Re} p > 5/4$, $\operatorname{Re} p' > 5/4$ if we succeed in determining $z(\theta)$ on the supposition that $\operatorname{Re} p = \operatorname{Re} p' = 5/4$. We are thus led to set

$$p = \frac{5}{4} + i\delta, \; p' = \frac{5}{4} + i\delta' \quad (\delta,\delta' \text{ real}) \quad . \tag{4.22}$$

Calculating the Jacobian determinant $J(r,\theta)$ from (4.12) by means of (3.11) and using (4.22), one arrives at

$$J(r,\theta) \sim \frac{a_2}{2} r^{-1/4} \omega(r,\theta) \quad (0 < \theta \leq \pi) \quad , \tag{4.23}$$

provided

$$\omega(r,\theta) = \mathrm{Re}\{(\tfrac{5}{4} - i\delta) r^{-i\delta} \bar{z}(\theta) \cos\tfrac{\theta}{2} - r^{i\delta} \dot{\bar{z}}(\theta) \sin\tfrac{\theta}{2}\} \quad . \tag{4.24}$$

If $\delta \neq 0$, Eqs.(4.23), (4.24) are easily found to yield two alternative implications: <u>either</u> $J(r,\theta)$ changes sign infinitely often as $r \to 0$ for each $\theta$ in $(0,\pi]$ <u>or</u> $z(\theta)$ is such that $\omega(r,\theta)$ vanishes identically on this interval. The first of these eventualities is precluded by the requirement $J(r,\theta) > 0$; the second alternative leads to $r^{1/4} J(r,\theta) \to 0$ as $r \to 0$ and hence gives $r^{-1/4} \lambda(r,\theta) \to \infty$ as $r \to 0$, for each $\theta$ in $(0,\pi]$, which is incompatible with (3.40). Therefore $\delta = 0$ and $p$ must be real. One shows analogously that $p'$ has to be real as well, so that (4.22) now reduce to

$$p = p' = 5/4, \quad \delta = \delta' = 0 \quad . \tag{4.25}$$

With (4.25) in force one infers from (4.23), (4.24) and their counterpart for $-\pi \leq \theta < 0$ that

$$J(r,\theta) \sim r^{-1/4} \omega(\theta) \quad (-\pi \leq \theta \leq \pi) \quad , \tag{4.26}$$

$$\omega(\theta) = \tfrac{1}{2} a_2 h(\theta) \{\tfrac{5}{4} z_1(\theta) \cos\tfrac{\theta}{2} - \dot{z}_1(\theta) \sin\tfrac{\theta}{2}\} \quad , \tag{4.27}$$[1]

in which $h(\theta)$ is the step-function defined in (3.34) and $z_1(\theta)$ is the real part of $z(\theta)$. Since $\lambda(r,\theta) = 1/J(r,\theta)$, a zero of $\omega(\theta)$ anywhere on $[-\pi,\pi]$ would contradict (3.40), so that

[1] As $\omega(r,\theta)$ is now independent of $r$, we write $\omega(\theta)$ in place of $\omega(r,\theta)$.

$$\lambda(r,\theta) \sim r^{1/4}/\omega(\theta),\ \omega(\theta) > 0 \quad (-\pi \le \theta \le \pi) \quad . \tag{4.28}$$

We are now in a position to use the equilibrium equation (3.12) in conjunction with (4.14), (4.25), (4.27) to obtain the differential equation governing $z(\theta)$. In contrast to the lower-order analyses carried out in Section 3, the right and left sides of (3.12) now balance asymptotically, and one arrives at

$$\ddot{z} + \frac{25}{16} z = \frac{3a_2}{2\omega^4} h(\theta)\{\frac{\omega}{4}\cos\frac{\theta}{2} - \dot{\omega}\sin\frac{\theta}{2}\} \quad (0 < |\theta| < \pi) \quad , \tag{4.29}$$

where $\omega$ is related to $z$ through (4.27). The boundary conditions (3.16), in turn, lead from (4.12), by virtue of (4.25) and (4.28), to the requirements

$$\dot{z}(-\pi) = \frac{1}{2} s a_2 \omega^{-3}(-\pi),\ \dot{z}(\pi) = -\frac{1}{2} a_2 \omega^{-3}(\pi) \quad , \tag{4.30}$$

while the transition conditions (3.17) are found to yield

$$z(0+) = z(0-),\ s\dot{z}(0+) = \dot{z}(0-) \quad . \tag{4.31}$$

Since the right members in (4.29) and (4.30) are real-valued functions, it follows with the aid of (4.31) that $z_2 \equiv \mathrm{Im}\, z$ must satisfy

$$\ddot{z}_2 + \frac{25}{16} z_2 = 0 \quad (0 < |\theta| < \pi) \quad , \tag{4.32}$$

subject to the boundary and transition conditions

$$\dot{z}_2(-\pi) = \dot{z}_2(\pi) = 0,\ z_2(0+) = z_2(0-),\ s\dot{z}_2(0+) = \dot{z}_2(0-) \quad . \tag{4.33}$$

This homogereous liner boundary-value problem admits only the trivial solution

$$z_2(\theta) = 0 \quad (-\pi \le \theta \le \pi) \quad . \tag{4.34}$$

The boundary-value problem governing $z_1 \equiv \mathrm{Re}\, z$ is obtained by taking real parts in (4.29), (4.30), (4.31). The resulting problem can be cast into a more convenient form by means of a suitable rescaling of $z_1$. To this end we set

$$z_1(\theta) = 2^{3/2}3^{1/4}a_2^{-1/2}\zeta(\theta) \quad (-\pi \le \theta \le \pi) \tag{4.35}$$

and note first that (4.27) now becomes

$$\omega(\theta) = 2^{-3/2}3^{1/4}a_2^{1/2}h(\theta)\psi(\theta) \quad (-\pi \le \theta \le \pi) \quad , \tag{4.36}$$

provided

$$\psi(\theta) = 5\zeta(\theta)\cos\tfrac{\theta}{2} - 4\dot{\zeta}(\theta)\sin\tfrac{\theta}{2} \quad (-\pi \le \theta \le \pi) \quad . \tag{4.37}$$

Further, in view of (4.36) and the positivity of $\omega(\theta)$, one has to stipulate that

$$\psi(\theta) > 0 \quad (-\pi \le \theta \le \pi) \quad . \tag{4.38}$$

On equating real parts in (4.29) and making use of (4.35), (4.36), (4.37), one obtains the differential equation to be satisfied by $\zeta$:

$$[8(1 - \cos\theta) + h^2(\theta)\psi^4(\theta)]\ddot{\zeta} - 4\sin\theta\dot{\zeta}$$

$$+ [\tfrac{5}{2}(1 - 3\cos\theta) + \tfrac{25}{16}h^2(\theta)\psi^4(\theta)]\zeta = 0 \quad (0 < |\theta| < \pi) \quad . \tag{4.39}$$

The boundary and transition conditions accompanying (4.39) follow from (4.33) by recourse to (4.35) to (4.38). In this manner one deduces

$$\dot{\zeta}(-\pi) = \tfrac{1}{2}3^{-1/4}s^{-1/2}, \quad \dot{\zeta}(\pi) = -\tfrac{1}{2}3^{-1/4} \quad , \tag{4.40}$$[1]

$$\zeta(0+) = \zeta(0-), \quad s\dot{\zeta}(0+) = \dot{\zeta}(0-) \quad . \tag{4.41}$$

We thus seek a solution $\zeta(\theta)$ of the nonlinear boundary-value problem consisting of (4.39) subject to (4.40), (4.41), such that $\psi(\theta)$ – defined by (4.37) – is strictly positive on $[-\pi,\pi]$.

As far as the numerical determination of $\zeta(\theta)$ is concerned, the problem at hand is awkward because (4.40) prescribe the boundary values of the derivative of the unknown function, rather than $\zeta(\pm\pi)$. For this reason we now convert the foregoing problem for $\zeta(\theta)$ to one for $\psi(\theta)$ that circumvents this difficulty.

With this aim in mind we note first that (4.37), (4.40) give

$$\psi(-\pi) = 2\times 3^{-1/4}s^{-1/2}, \quad \psi(\pi) = 2\times 3^{-1/4} \quad . \tag{4.42}$$

We observe parenthetically that the above boundary values of $\psi(\theta)$ are consistent with the asymptotic formulas for $\lambda(\pm\pi)$ arrived at in (3.39), as is seen, on setting $|a| = a_2$ in $(3.39)^2$, by comparing this estimate with (4.28) and by referring to (4.36). Next, we differentiate (4.37) with respect to $\theta$ and eliminate $\ddot{\zeta}$ from the ensuing identity by means of (4.39). The equation thus obtained relates $\zeta$, $\dot{\zeta}$ to $\psi$, $\dot{\psi}$, is linear in the former two variables, and may be employed along with (4.37) to express $\zeta$, $\dot{\zeta}$ in terms of $\psi$, $\dot{\psi}$ and trigonometric functions of $\theta$. In this manner one finds that

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[1] ...erve that (4.37), (4.38) necessitate $\dot{\zeta}(-\pi) > 0$ and $\dot{\zeta}(\pi) < 0$.

...l that $a_1 = 0$, $a_2 > 0$. See (4.10) and the remarks following (4.10).

$$\zeta = \frac{1}{15h^2(\theta)\psi^4}\{[3h^2(\theta)\psi^4 - 8\cos\theta + 8]\psi\cos\frac{\theta}{2}$$

$$+ 4[h^2(\theta)\psi^4 - 8\cos\theta + 8]\dot{\psi}\sin\frac{\theta}{2}\} \qquad (-\pi \le \theta \le \pi) \quad , \tag{4.43}$$

$$\dot{\zeta} = \frac{1}{12h^2(\theta)\psi^4}\{[8 + 8\cos\theta - 3h^2(\theta)\psi^4]\psi\sin\frac{\theta}{2}$$

$$+ 4[h^2(\theta)\psi^4 - 8\cos\theta + 8]\dot{\psi}\cos\frac{\theta}{2}\} \qquad (-\pi \le \theta \le \pi) \quad . \tag{4.44}$$

Differentiating (4.43) and equating the result to $\dot{\zeta}$ given by (4.44), one is led to the differential equation for $\psi$:

$$\{h^2(\theta)\psi^4 + 8(1 - \cos\theta)\}\ddot{\psi} - 32(1 - \cos\theta)\psi^{-1}\dot{\psi}^2$$

$$- 4\sin\theta\dot{\psi} - \frac{1}{2}(3 - \cos\theta)\psi + \frac{9}{16}h^2(\theta)\psi^5 = 0 \qquad (0 < |\theta| < \pi) \quad . \tag{4.45}$$

Finally, (4.41) together with (4.43), (4.44) furnish the transition conditions appropriate to $\psi$:

$$\psi(0+) = \psi(0-), \; s\dot{\psi}(0+) = \dot{\psi}(0-) \quad . \tag{4.46}$$

It is readily shown that (4.45), accompanied by conditions (4.42), (4.46) and supplemented by (4.43), in turn implies (4.39), (4.40), (4.41), as well as (4.37).

The completion of the three-term asymptotic representation (4.12) of $y(r,\theta)$ pertaining to the special case $b_1 = 0$ has thus been reduced to the task of finding a solution $\psi(\theta)$ of the nonlinear second-order differential equation (4.45) that is positive for $-\pi \le \theta \le \pi$ and meets the boundary and transition conditions (4.42), (4.46). Assuming for the moment that $\psi(\theta)$

has been so determined, $\zeta(\theta)$ is then supplied by (4.43), $z_1(\theta)$ follows from (4.35), while $z_2(\theta)$ vanishes identically according to (4.34). Further, the exponents $p$ and $p'$ are given by (4.25). Taking real and imaginary parts in (4.12) one therefore has the following asymptotic results for the degenerate case $b_1 = 0$ $(A = 0)$:

$$\left.\begin{aligned} y_1(r,\theta) &\sim \ell + cr^{5/4}\zeta(\theta) \quad , \\ y_2(r,\theta) &\sim a_2 r^{1/2} h(\theta)\sin\tfrac{\theta}{2} + b_2 r\cos\theta \quad (-\pi \le \theta \le \pi) \quad , \end{aligned}\right\} \tag{4.47}$$

with

$$c = 2^{3/2}3^{1/4}a_2^{-1/2}, \quad a_2 > 0 \quad . \tag{4.48}$$

The first of (4.47) supplants the weak estimate (4.11) for $y_1(r,\theta)$ furnished by (4.9) when $b_1 = 0$. Note from (4.41) that $\zeta(\theta)$ is continuous at $\theta = 0$, although its derivative suffers a finite jump-discontinuity there. On account of (4.26), (4.28), (4.36), and (4.48), $J(r,\theta)$ and $\lambda(r,\theta)$ at present conform to the estimates

$$J(r,\theta) \sim \frac{1}{c} r^{-1/4} h(\theta)\psi(\theta) \quad (-\pi \le \theta \le \pi) \quad , \tag{4.49}$$

$$\lambda(r,\theta) \sim cr^{1/4}[h(\theta)\psi(\theta)]^{-1} \quad (-\pi \le \theta \le \pi) \quad , \tag{4.50}$$

whereas (3.53) gives merely the inadequate estimate $J(r,\theta) = o(r^{-1/2})$ $(-\pi \le \theta \le \pi)$ when $A = 0$. The result (4.50) reduces to (3.39) for $\theta = \pm\pi$ and is consistent with the a priori assumption (3.40). Since the right member in (4.49) is the Jacobian determinant of the right members in (4.47), the positivity of $\psi$ on $[-\pi,\pi]$ assures that (4.47) furnish a one-to-one approximation to the local deformation map.

Our attempts to deal analytically with the highly nonlinear two-point boundary-value problem (4.45), (4.42), (4.46) for $\psi(\theta)$ have remained unsuccessful. In contrast, no particular difficulties were encountered in its numerical solution. The results thus obtained leave no doubt as to the existence of a solution; although its uniqueness cannot be taken for granted, there is no indication of non-uniqueness.

Figure 3 and Figure 4 display illustrative graphs of $\psi(\theta)$ and $\zeta(\theta)$ for $-\pi \le \theta \le \pi$, appropriate to two choices of the stiffness-ratio $s = \mu_1/\mu_2$. In Figure 3, $s = 1/2$, whereas Figure 4 pertains to $s = 1$, that is to an all-around homogeneous Neo-Hookean sheet. In the latter instance both $\psi(\theta)$ and $\zeta(\theta)$ are even functions and there is no discontinuity in the slope of the corresponding curves at $\theta = 0$.

## 5. Discussion of the elastostatic field near the tip of the interface-crack. The special case $s = 1$.

In this section we first examine the structure of the deformation field near the tip of an interface-crack as predicted by the nonlinear theory of plane stress for Neo-Hookean sheets. Thereafter we shall discuss the associated near-field of stress.

For the purpose at hand it is convenient to put the asymptotic results (4.9) and (4.47) into a non-dimensional form by means of the scaling

$$\left.\begin{aligned} \xi_1 = (x_1 - \ell)/a_2^2,\ \xi_2 = x_2/a_2^2,\ \rho = r/a_2^2 \\ \eta_1 = (y_1 - \ell)/a_2^2,\ \eta_2 = y_2/a_2^2 \ . \end{aligned}\right\} \qquad (5.1)$$

Then (4.9), (4.10) yield

$$\left.\begin{aligned} \eta_1(\rho,\theta) &\sim b_1\rho\cos\theta \quad (b_1 > 0) \quad , \\ \eta_2(\rho,\theta) &\sim \rho^{1/2}h(\theta)\sin\tfrac{\theta}{2} + b_2\rho\cos\theta \quad (-\pi \le \theta \le \pi) \quad , \end{aligned}\right\} \tag{5.2}$$

while, in the event that $b_1 = 0$, (4.47), (4.48) give

$$\left.\begin{aligned} \eta_1(\rho,\theta) &\sim 2^{3/2}3^{1/4}\rho^{5/4}\zeta(\theta) \quad , \\ \eta_2(\rho,\theta) &\sim \rho^{1/2}h(\theta)\sin\tfrac{\theta}{2} + b_2\rho\cos\theta \quad (-\pi \le \theta \le \pi) \quad . \end{aligned}\right\} \tag{5.3}$$

We observe to begin with that the foregoing asymptotic results are entirely free of oscillations of the kind arising in the treatment of the linearized problem and discussed in Section 1. Mathematically, the absence of this oscillatory singular behavior can be traced to the fact that the initially undetermined exponents $m$, $m'$, $n$, $n'$, and $p$, $p'$ introduced in (3.18), (3.43), and (4.12) were all found to be real-valued.

Of particular interest is the shape of the deformed upper and lower crack-face in the vicinity of the (fixed) end of the crack. From (5.2) and (3.34) one has

$$\left.\begin{aligned} &\eta_1(\rho,\pi) \sim b_1\rho, \ \eta_2(\rho,\pi) \sim \rho^{1/2} \quad , \\ &\eta_1(\rho,-\pi) \sim -b_1\rho, \ \eta_2(\rho,-\pi) \sim -s\rho^{1/2} \quad , \\ &b_1 \ge 0, \ s = \mu_1/\mu_2 \quad . \end{aligned}\right\} \tag{5.4}$$

Elimination of $\rho$ between the first two and the second two of (5.4) leads to the subsequent approximate description of the curves into which the crack-faces at $\theta = \pi$ and $\theta = -\pi$ are deformed:

$$\left.\begin{aligned} \eta_1 &= -b_1\eta_2^2,\ \eta_2 \geq 0 \quad \text{for} \quad \theta = \pi \quad , \\ \eta_1 &= -\frac{b_1}{s^2}\eta_2^2,\ \eta_2 \leq 0 \quad \text{for} \quad \theta = -\pi \quad . \end{aligned}\right\} \tag{5.5}$$

In the non-degenerate case $b_1 > 0$ this description is adequate and asserts that each crack-face, after deformation, is locally approximated by an arc of a parabola, as indicated in Fig.5. The two parabolic arcs determined by (5.5) join up with a common tangent at the crack-tip, this tangent being perpendicular to the crack-axis following the rigid rotation that led from (3.51) to (4.9). The deformed upper and lower crack-faces, when $s \neq 1$, are seen to lie on two distinct parabolas, both of which are concave toward the undeformed crack.

In the degenerate case $b_1 = 0$, one draws from (5.3) and (3.34) that

$$\left.\begin{aligned} \eta_1(\rho,\pi) &\sim 2^{3/2}3^{1/4}\rho^{5/4}\zeta(\pi),\ \eta_2(\rho,\pi) \sim \rho^{1/2} \quad , \\ \eta_1(\rho,-\pi) &\sim 2^{3/2}3^{1/4}\rho^{5/4}\zeta(-\pi),\ \eta_2(\rho,-\pi) \sim -s\rho^{1/2} \quad , \end{aligned}\right\} \tag{5.6}$$

whence in first approximation

$$\left.\begin{aligned} \eta_1 &= 2^{3/2}3^{1/4}\zeta(\pi)\eta_2^{5/2},\ \eta_2 \geq 0 \quad \text{for} \quad \theta = \pi \quad , \\ \eta_1 &= 2^{3/2}3^{1/4}\zeta(-\pi)s^{-5/2}(-\eta_2)^{5/2},\ \eta_2 \leq 0 \quad \text{for} \quad \theta = -\pi \quad . \end{aligned}\right\} \tag{5.7}$$

Thus once again the deformation images of the upper and lower crack-faces have a common tangent at the tip of the crack. Figure 6 illustrates the shape of the deformed crack-faces when $b_1 = 0$, for the two stiffness-ratios $s = 1/2$ and $s = 1$ (homogeneous sheet). As is apparent from the graphs in

Fig.3 and Fig.4,

$$\left.\begin{aligned} &\zeta(\pi) < 0,\ \zeta(-\pi) > 0 \quad \text{for} \quad s = \mu_1/\mu_2 = 1/2 \quad , \\ &\zeta(\pi) = \zeta(-\pi) > 0 \quad \text{for} \quad s = 1 \quad . \end{aligned}\right\} \tag{5.8}$$

These inequalities, in conjunction with (5.7), imply that the boundary of the sheet near the tip of the crack, when $b_1 = 0$, is deformed into an S-shaped curve for $s = 1/2$, whereas for $s = 1$ the region occupied by the deformed sheet is locally convex. Accordingly, the special case $b_1 = 0$ marks the transition from circumstances in which the deformed sheet is wholly concave sufficiently close to the crack-tip to conditions under which this is no longer true. As will become clear later on, $b_1 > 0$ in the particular crack-problem for a homogeneous sheet $(s = 1)$ treated by Wong and Shield [15]. It is not obvious, however, whether or not there exist global loading and sheet geometries that induce the degeneracy $b_1 = 0$.

In view of the primary purpose of this study, the most important conclusion emerging from the preceding results is that – at least for an interface-crack between Neo-Hookean sheets – the finite theory of plane stress does not give rise to interpenetration of the deformed crack-faces in the vicinity of the crack-tips. The prediction of such an unacceptable overlap is thus seen to stem from the linearization of this singular problem. Indeed, somewhat surprisingly, the crack is found to open smoothly even if $s \neq 1$.

As is apparent from (5.2), (5.3), the leading term in the corresponding approximations for $\eta_2$ vanishes when $\theta = 0$. Consequently, a more detailed discussion of the deformations in the vicinity of the crack-tip at $\rho = 0$ necessitates that one take account of the second term, whose coefficient $b_2$ in particular enters into the lowest-order approximation to the

deformation-image of the bonded interface. Confining this discussion to the non-degenerate case $b_1 > 0$, we gather from (5.2) and (5.1) that

$$\eta_1 \sim b_1\xi_1, \quad \eta_2 \sim H(\xi_2)\{\sqrt{\xi_1^2+\xi_2^2} - \xi_1\}^{1/2} + b_2\xi_1 \quad , \tag{5.9}$$

provided $H$ is the step-function defined by

$$H(\xi_2) = 1/\sqrt{2} \quad \text{if} \quad \xi_2 > 0, \quad H(\xi_2) = -s/\sqrt{2} \quad \text{if} \quad \xi_2 < 0 \quad . \tag{5.10}$$

Equations (5.9) reveal that the material coordinate lines $\xi_1 = \text{constant}$ are — in first approximation — carried into the straight lines $\eta_1 = b_1\xi_1$, while the coordinate lines $\xi_2 = \text{constant}$ are deformed into the family of curves approximated by

$$\eta_2 = H(\xi_2)\{\sqrt{(\eta_1/b_1)^2+\xi_2^2} - \eta_1/b_1\}^{1/2} + b_2\eta_1/b_1 \quad . \tag{5.11}$$

Within this approximation the deformation-image of the interface $\xi_2 = 0$, $\xi_1 > 0$ is rectilinear and furnished by

$$\eta_2 = b_2\eta_1/b_1, \quad \eta_1 > 0 \quad . \tag{5.12}$$

Figure 7 displays qualitative sketches, based on (5.9) to (5.12), of the deformed material lines $\xi_1 = \text{constant}$ and $\xi_2 = \text{constant}$, appropriate to $b_1 > 0$, for $b_2 > 0$, $b_2 = 0$, and $b_2 < 0$, depending on whether the stiffness-ratio $s < 1$ or $s = 1$.

We conclude the discussion of the near-field deformations with some observations concerning the transverse stretch-ratio $\lambda$, which reflects the local thinning of the two sheets. The asymptotic behavior of $\lambda(r,\theta)$, as $r \to 0$, is supplied by (3.62) if $b_1 > 0$ $(A > 0)$ and follows from (4.28),

(4.36)[1] when $b_1 = 0$. In either instance $\lambda(r,\theta)$, at any fixed value of $\theta$ in $[-\pi,\pi]$, is an increasing function of $r$ near $r = 0$, so that the thinning becomes more severe as the crack-tip is approached. Barring the degenerate case $b_1 = 0$, we note that $\lambda(r,\theta)$, for any fixed (sufficiently small) value of $r$, is smallest at $\theta = 0$ and increases steadily with $|\theta|$ as $|\theta| \to \pi$. Consequently the thinning is most pronounced at the interface and less prominent at the crack-faces. If $s < 1$ $(\mu_1 < \mu_2)$, the local thinning is more prevalent in the upper than in the lower sheet, as is to be anticipated.

We turn now to the determination of the stresses near the crack-tip. Asymptotic results for the cartesian components of nominal stress are readily obtainable from (4.9) with the aid of (3.13) and (3.40). Thus,

$$\left.\begin{aligned} &\frac{\sigma_{11}}{\mu_1} \sim \frac{b_1}{h(\theta)}\,, \quad \frac{\sigma_{22}}{\mu_1} = \frac{1}{2}\rho^{-1/2}\cos\frac{\theta}{2} + o(1)\,, \\ &\frac{\sigma_{12}}{\mu_1} = o(1), \quad \frac{\sigma_{21}}{\mu_1} \sim -\frac{1}{2}\rho^{-1/2}\sin\frac{\theta}{2} + \frac{b_2}{h(\theta)} \quad (-\pi \le \theta \le \pi)\,, \end{aligned}\right\} \tag{5.13}$$

where $\rho$ is the dimensionless radial coordinate introduced in (5.1). The associated components of actual stress are deducible from (4.9) by means of (2.44) and (3.40). This computation yields:

$$\left.\begin{aligned} &\frac{\tau_{11}}{\mu_1} = o(\rho^{-1/2}), \quad \frac{\tau_{22}}{\mu_1} \sim \frac{1}{4}\rho^{-1}h(\theta) - b_2\rho^{-1/2}\sin\frac{\theta}{2}\,, \\ &\frac{\tau_{12}}{\mu_1} = \frac{\tau_{21}}{\mu_1} \sim -\frac{b_1}{2}\rho^{-1/2}\sin\frac{\theta}{2} \quad (-\pi \le \theta \le \pi)\,. \end{aligned}\right\} \tag{5.14}$$

[1] See also Fig.3 and Fig.4, where $\psi(\theta)$ is plotted for $s = 1/2$ and $s = 1$.

Evidently, the expansion (4.9) is insufficient to produce a dominant estimate for $\sigma_{12}$ and $\tau_{11}$; also, when $b_1 = 0$, one merely infers from (5.13), (5.14) that $\sigma_{11}$ and $\tau_{12}$ tend to zero as $\rho \to 0$.

Equations (5.14) lead to the following asymptotic results for the actual principal stresses:

$$\frac{\tau_1}{\mu_1} = o(\rho^{-1/2}), \quad \frac{\tau_2}{\mu_1} \sim \frac{1}{4}\rho^{-1}h(\theta) - b_2\rho^{-1/2}\sin\frac{\theta}{2} \quad (-\pi \leq \theta \leq \pi) \quad . \tag{5.15}$$

The above weak estimate for $\tau_1$ can easily be improved if $b_1 > 0$. To this end we first draw from (2.28) that

$$\det \underset{\sim}{\tau} = \tau_1\tau_2 = \det \underset{\sim}{F} \det \underset{\sim}{\sigma} = J \det \underset{\sim}{\sigma} \tag{5.16}$$

and then invoke (4.10), (3.53), as well as (5.13), to see that

$$\tau_1\tau_2 \sim \frac{1}{4}\mu_1^2 b_1^2 \rho^{-1} h^2(\theta)\cos^2\frac{\theta}{2} \quad . \tag{5.17}$$

On the other hand, (5.17) and the second of (5.15) justify

$$\frac{\tau_1}{\mu_1} \sim b_1^2 h(\theta)\cos^2(\theta/2) \quad (-\pi \leq \theta \leq \pi) \quad . \tag{5.18}$$

Since the deformed crack-faces are free of tractions, the fact that $\tau_1$ vanishes at $\theta = \pm\pi$ is not accidental.

It is of interest to examine the dependence — to dominant order — of the primary actual normal stress $\tau_{22}$ upon the radial distance from the crack-tip after deformation, i.e. upon the spatial radial coordinate. We do so merely for an approach to the tip along the interface $\theta = 0$ and for this purpose gather from (5.2) that

$$\rho_* \equiv |\eta(\rho,0)| \sim |b|\rho, \quad \eta = \eta_1 + i\eta_2 \quad . \tag{5.19}$$

Hence (5.14) now give

$$\tau_{22}(\rho,0\pm) \sim \frac{\mu_1}{4} h(0\pm)\rho^{-1} \sim \frac{\mu_1}{4|b|} h(0\pm)\rho_*^{-1} \quad , \tag{5.20}$$

so that $\tau_{22}(\rho,0)$ is both $O(\rho^{-1})$ and $O(\rho_*^{-1})$ as $\rho \to 0$. The conclusion that this singularity is stronger than the analogous square-root singularity arising in the linearized theory is consistent with earlier findings concerning crack-tip singularities in hardening materials.[1]

We record next the asymptotic behavior of the strain-energy density. By (2.40), (2.41),

$$U = \frac{\mu}{2}(I + \lambda^2 - 3) \quad , \tag{5.21}$$

with $\mu = \mu_1$ and $\mu = \mu_2$ in the upper and lower half-plane, respectively. Further, from (2.27), (3.10), (4.9), and (3.40), one has

$$I = y_{\alpha,\beta} y_{\alpha,\beta} = |\nabla y|^2 \sim \frac{1}{4}\rho^{-1} h^2(\theta), \quad \lambda = O(\rho^{-1/4}) \quad , \tag{5.22}$$

whence

$$U \sim \frac{\mu_1}{8}\rho^{-1} h(\theta) \quad , \tag{5.23}$$

which reflects the jump discontinuity in $U$ at $\theta = 0$.

The foregoing asymptotic results involve the three real amplitude

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[1] See [12], [13]. The Neo-Hookean material is "hardening" in the sense that the slope of the response curve in Fig.2 is steadily increasing for $\lambda > 1$.

parameters $a_2$, $b_1$, $b_2$, the first of which – in view of (5.1) – governs the length-scale of the near-field approximations under discussion. As has been pointed out earlier,[1] these parameters elude the local analysis and, for an interface-crack between two semi-infinite sheets, are bound to depend upon the stiffness-ratio $s = \mu_1/\mu_2$, the crack-length $2\ell$, as well as on the specific loading at infinity.

The primary amplitude parameter $a_2$ can, in the usual manner, be expressed in terms of an appropriate path-independent "$\mathcal{J}$-integral" by recourse to a familiar conservation law of finite elastostatics. Let $C_1$ and $C_2$ be the two simple closed curves shown in Fig.8, which lie in the respective half-planes $H_1$ and $H_2$. The conservation law alluded to above then assures that

$$\mathcal{J}^{(k)} \equiv \int_{C_k} (Un_1 - \sigma_{\alpha\beta} n_\beta y_{\alpha,1}) d\mathcal{A} = 0 \qquad (k = 1,2) \quad , \tag{5.24}$$

where $\underset{\sim}{n}$ is the unit outward normal vector of $C_k$. Next, let $L_k(\varepsilon)$ and $\Gamma_k(\varepsilon)$ designate the rectilinear and circular portions of $C_k$ labeled in Fig.8. Noting that $n_1 = 0$ along $L_k(\varepsilon)$ and recalling the boundary conditions (3.2) together with the bond conditions (3.4), one infers that the contributions to $\mathcal{J}^{(1)} + \mathcal{J}^{(2)}$ stemming from $L_1(\varepsilon)$ and $L_2(\varepsilon)$ vanish, whence

$$\mathcal{J} \equiv \int_{C} (Un_1 - \sigma_{\alpha\beta} n_\beta y_{\alpha,1}) d\mathcal{A} = \int_{\Gamma(\varepsilon)} (Un_1 - \sigma_{\alpha\beta} n_\beta y_{\alpha,1}) d\mathcal{A} \quad , \tag{5.25}$$

provided $C$ and $\Gamma(\varepsilon)$ are the two curves (see Fig.8) defined by

$$C = C_1 + C_2 - L_1(\varepsilon) - L_2(\varepsilon) - \Gamma(\varepsilon), \quad \Gamma(\varepsilon) = \Gamma_1(\varepsilon) + \Gamma_2(\varepsilon) \quad . \tag{5.26}$$

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[1] See the introductory remarks in Section 4.

Passing to the limit as $\varepsilon \to 0$ in (5.25) and making use of (5.1), (5.2), (5.13), and (5.23), one readily arrives at

$$a_2^2 = -\frac{8}{\mu_1(1+s)}\,\mathcal{J}\,,\quad \mathcal{J} = \int_C (Un_1 - \sigma_{\alpha\beta}n_\beta u_{\alpha,1})\,d\mathcal{A}\,, \tag{5.27}$$

in which $C$ is <u>any</u> simple curve issuing from an interior point of the crack, terminating at such a point, and surrounding the crack-tip situated at $x_2 = \ell$.

The conservation law underlying (5.25), and hence also this identity, is equally valid in the infinitesimal theory of (generalized) plane stress. One may therefore calculate from (5.25), by letting $\varepsilon$ tend to zero, the value of $\mathcal{J}$ appropriate to an available global solution of the <u>linearized</u> interface-crack problem for particular loading conditions. Moreover, this value of $\mathcal{J}$ in conjunction with the first of (5.27) would yield a small-load estimate for $a_2$ if one could take for granted that the integrand in the second of (5.27) is approximated <u>uniformly</u> on $C$ — at small loads — by its counterpart in the solution of the corresponding linearized problem. While such an assumption is plausible in case $\mu_1 = \mu_2$, it is no longer tenable when $\mu_1 \neq \mu_2$ for curves $C$ that come sufficiently close to the crack-tip since the elastostatic field predicted by the linear theory is oscillatory in any small enough neighborhood of the tip, whereas no such oscillations occur at least at suitably small distances from the tip according to the finite theory. For this reason, no matter how small the loads, the linear theory cannot even supply a <u>pointwise</u> approximation consistent with the finite theory at <u>all</u> material points.

It is conceivable, however, that despite these circumstances, linearized elastostatics furnishes a valid approximation to $\mathcal{J}$ at small loads.

Thus, the global solution based on the infinitesimal theory might uniformly approximate the elastostatic field emerging in the nonlinear theory on every material point set that is sufficiently remote from the tips of the crack. The precise approximative status of solutions to linearized problems involving interface-cracks remains an intriguing issue.

Although the special case $s = 1$ $(\mu_1 = \mu_2)$, encompassed by the asymptotic analysis carried out in this paper, has been included in the preceding discussion of the results obtained, certain aspects of the crack problem for an all-around homogeneous Neo-Hookean sheet merit additional attention. If $s = 1$, equations (3.34), (5.1), (5.2) give

$$\left.\begin{aligned} \eta_1(\rho,\theta) &\sim b_1\rho\cos\theta \quad , \\ \eta_2(\rho,\theta) &\sim \rho^{1/2}\sin\frac{\theta}{2} + b_2\rho\cos\theta \quad (-\pi \le \theta \le \pi) \quad , \end{aligned}\right\} \tag{5.28}$$

where

$$\rho = r/a_2^2, \ \eta_1 = (y_1 - \ell)/a_2^2, \ \eta_2 = y_2/a_2^2 \quad . \tag{5.29}$$

Consider now, in particular, the Mode I problem governed by (3.8), so that the loading at infinity is one corresponding to a pure homogeneous deformation with the $x_\alpha$-axes as principal axes and $\overset{\infty}{\lambda}_\alpha$ as principal stretch-ratios. In this instance one would anticipate the global solution to be symmetric about the $x_1$-axis. Hence $b_2 = 0$ and the near-field of deformation predicted by (5.28) becomes

$$\eta_1(\rho,\theta) \sim b_1\rho\cos\theta, \ \eta_2(\rho,\theta) \sim \rho^{1/2}\sin\frac{\theta}{2} \quad (-\pi \le \theta \le \pi) \quad . \tag{5.30}$$

This conclusion is found to be in asymptotic agreement with the lowest-order

approximation to the global solution deduced by Wong and Shield [15],[1] provided their results are adjusted to keep the crack-tip at $x_2 = \ell$ fixed, and provided one takes

$$b_1 = \overset{\infty}{\lambda}_1, \quad a_2 = \overset{\infty}{\lambda}_2\sqrt{2\ell} \quad . \tag{5.31}$$

In the analogous Mode II problem (3.9) is in force, the loading at infinity being one appropriate to a homogeneous deformation of pure shear of amount $\overset{\infty}{k}$, parallel to the $x_1$-axis. The well-known solution to the linearized version of this problem is anti-symmetric about the crack-axis, i.e. its displacement field

$$u_\alpha(x_1,x_2) = y_\alpha(x_1,x_2) - x_\alpha \tag{5.32}$$

obeys the parity relations

$$u_1(x_1,x_2) = -u_1(x_1,-x_2), \quad u_2(x_1,x_2) = u_2(x_1,-x_2) \quad . \tag{5.33}$$

Since the amplitude parameters in (5.28) cannot be chosen so as to render this local expansion compatible with (5.33), the global solution to the nonlinear Mode II problem at hand apparently cannot possess the anti-symmetry exhibited by its counterpart in the linear theory. This inference is also supported by the observation that the nonlinear equilibrium equations (2.46), when cast in terms of displacements, fail to be invariant under the parity transformation (5.33). Further, whereas the crack-faces fail to separate in the Mode II problem according to infinitesimal elastostatics, the present nonlinear asymptotic analysis predicts that the crack does open

[1] Recall that the iterative scheme employed in [15] presupposes large deformations <u>throughout</u> the sheet.

at least in a neighborhood of its ends.

The foregoing conclusions regarding the Mode II problem for a Neo-Hookean sheet are strictly parallel to results arrived at earlier by Stephenson [16] in an asymptotic study pertaining to finite plane strain for a class of incompressible elastic materials. Moreover, Stephenson succeeded in proving the non-existence of an anti-symmetric global solution to this problem for a Neo-Hookean material.

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We are greatly indebted to Albert F.Chang for his valuable help in connection with the numerical solution of the nonlinear boundary-value problem encountered in Section 4 of this paper.

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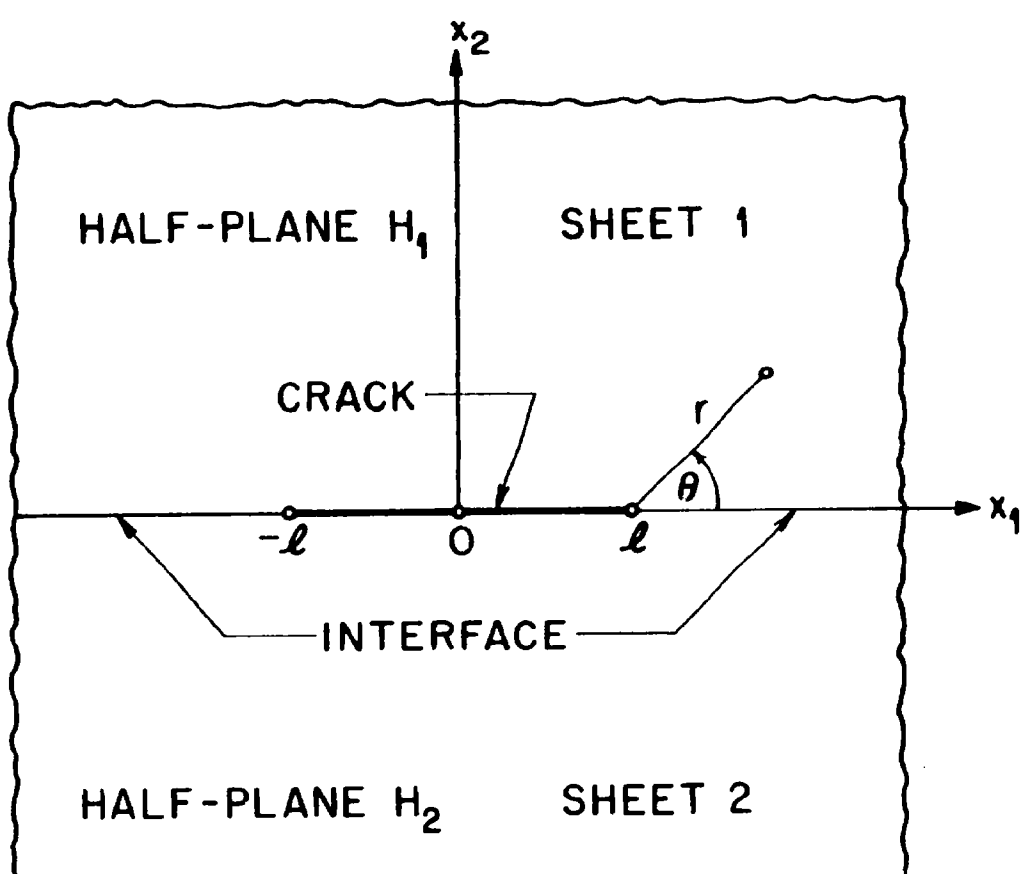


FIGURE 1. INTERFACE-CRACK BETWEEN TWO SEMI-INFINITE ELASTIC SHEETS AND COORDINATES.

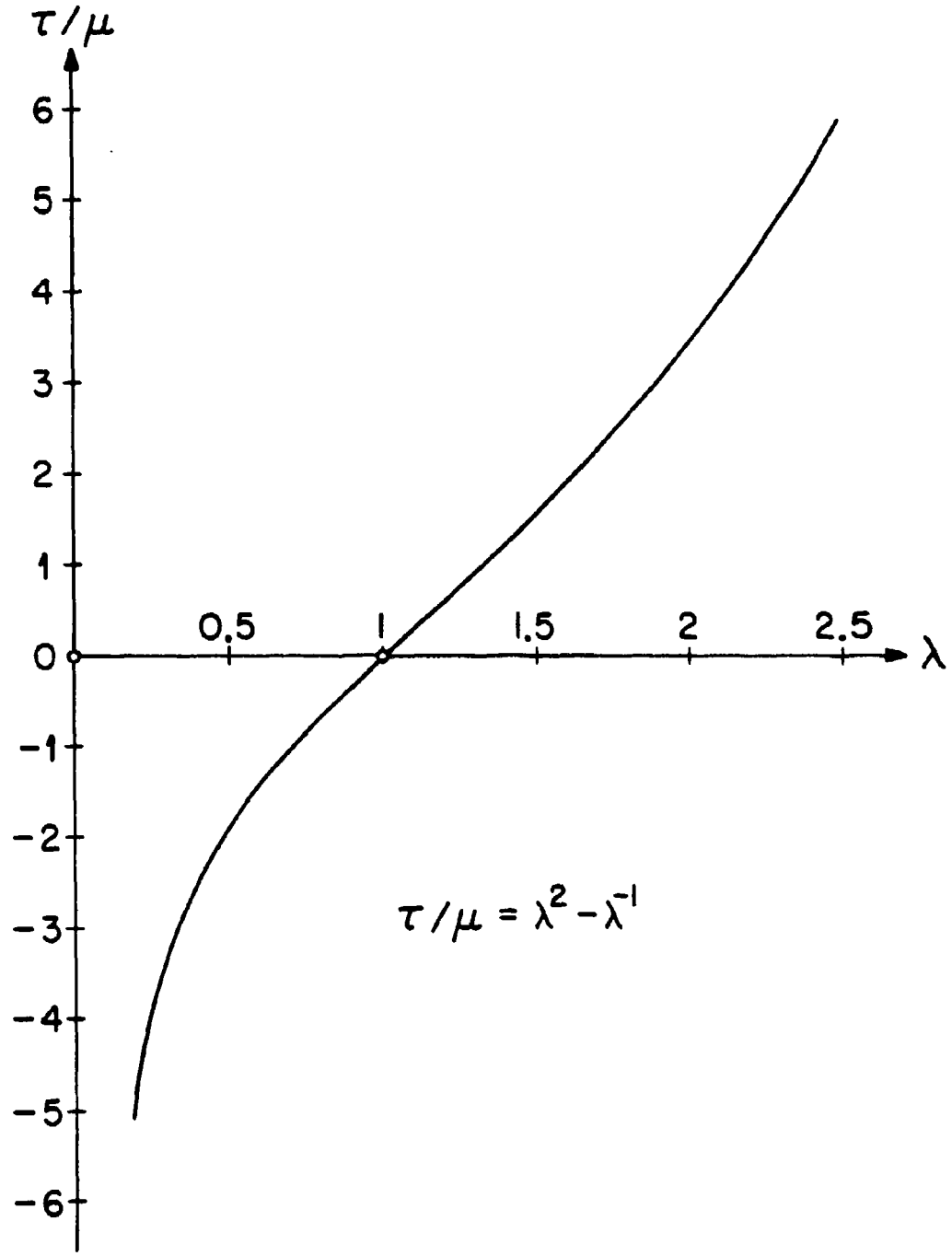


FIGURE 2. RESPONSE OF A NEO-HOOKEAN MATERIAL UNDER UNI-AXIAL STRESS

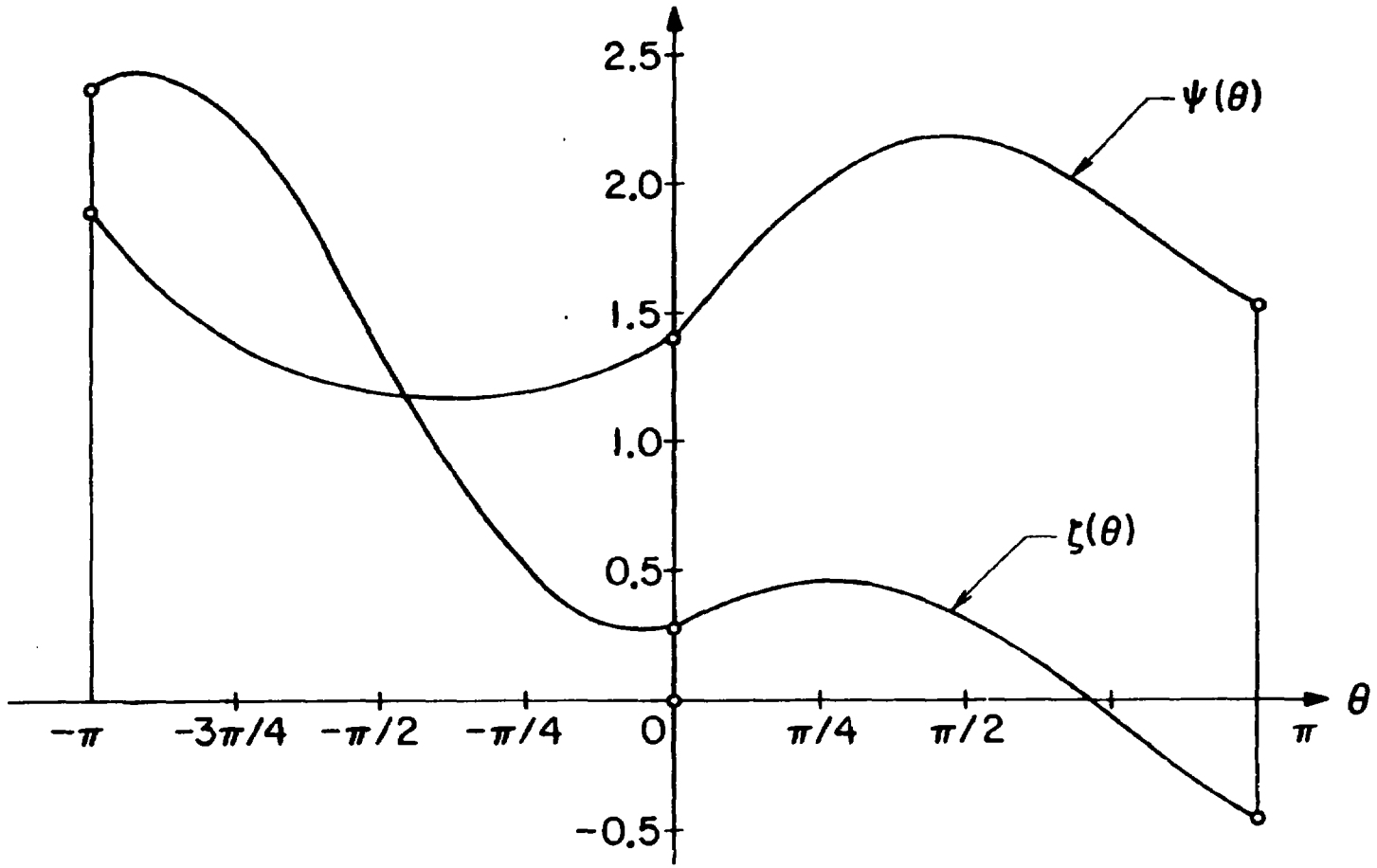


FIGURE 3. GRAPHS OF $\psi(\theta)$ AND $\zeta(\theta)$ FOR $s = \mu_1/\mu_2 = 1/2$

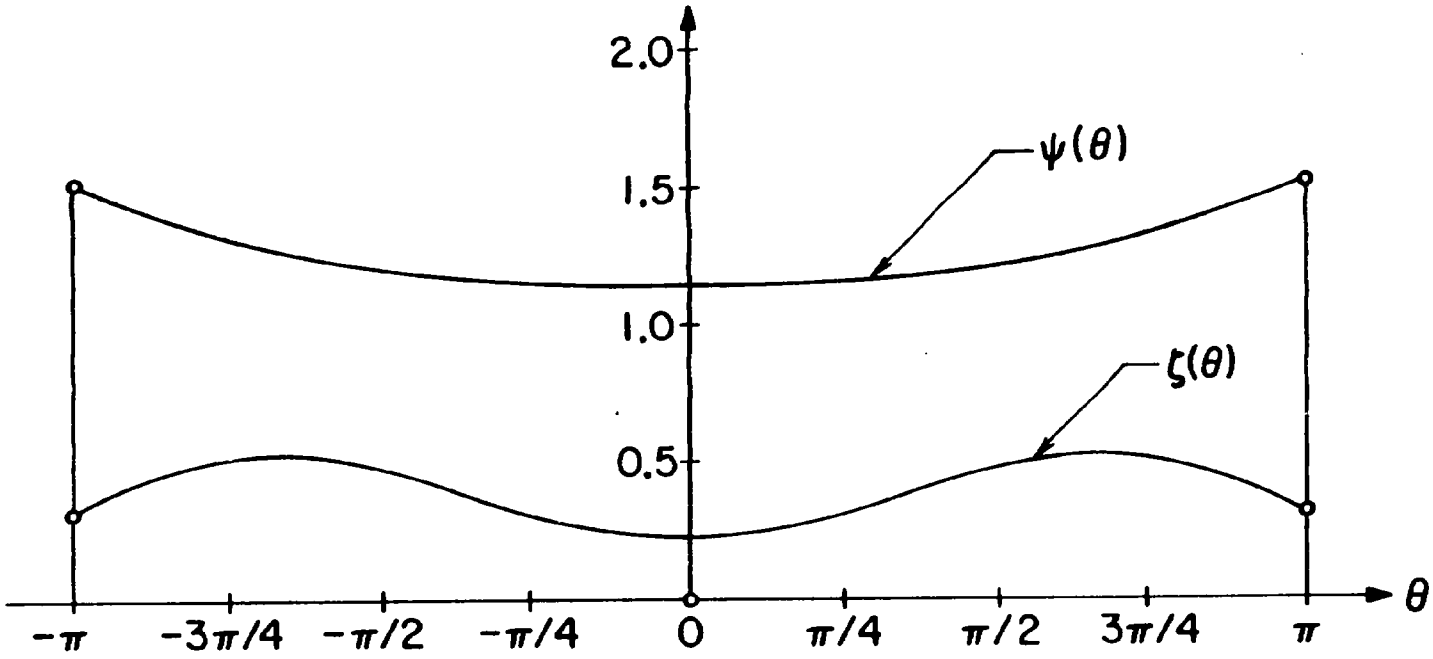


FIGURE 4. GRAPHS OF $\psi(\theta)$ AND $\zeta(\theta)$ FOR $s = \mu_1/\mu_2 = 1$.

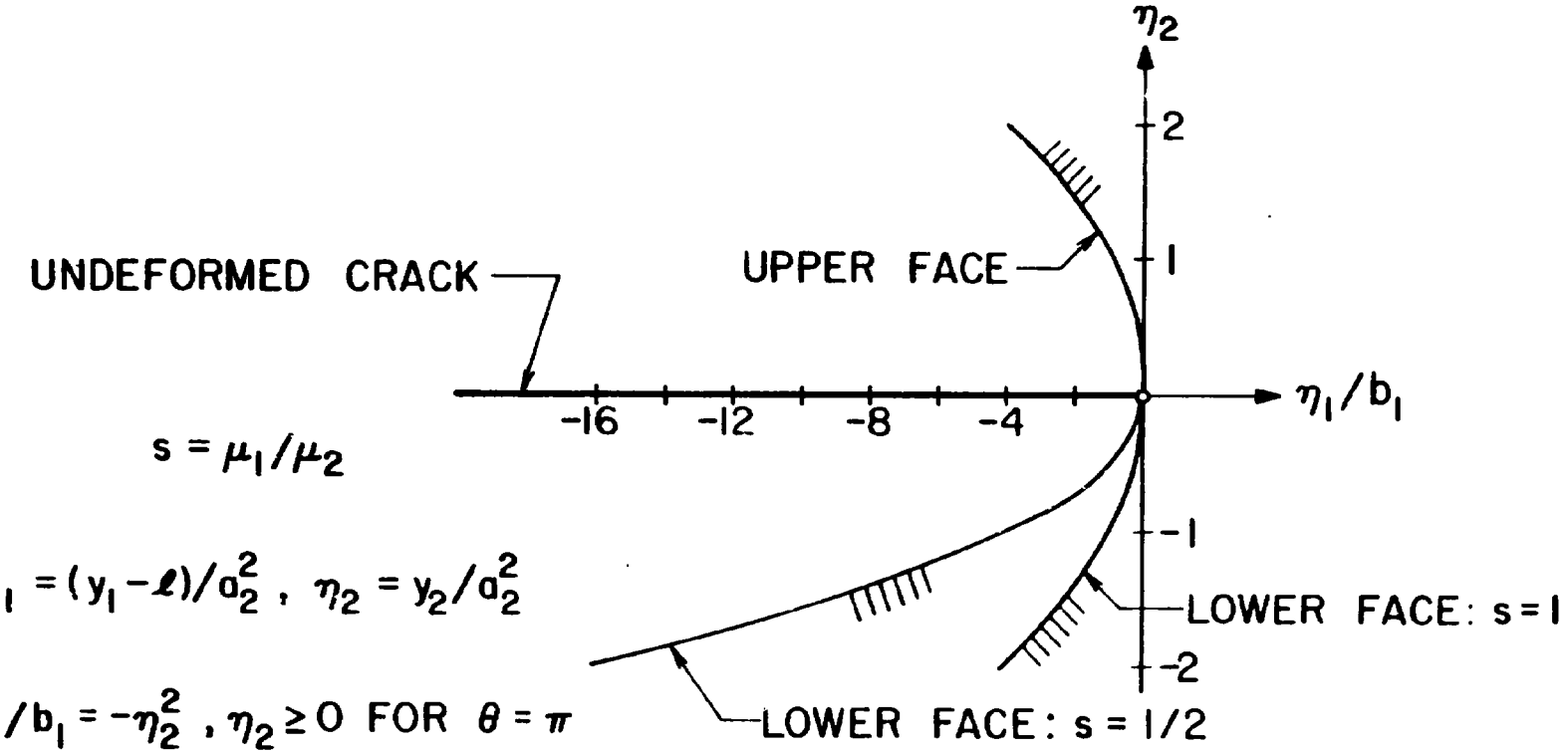


FIGURE 5. DEFORMED CRACK-FACES FOR $b_1 > 0$

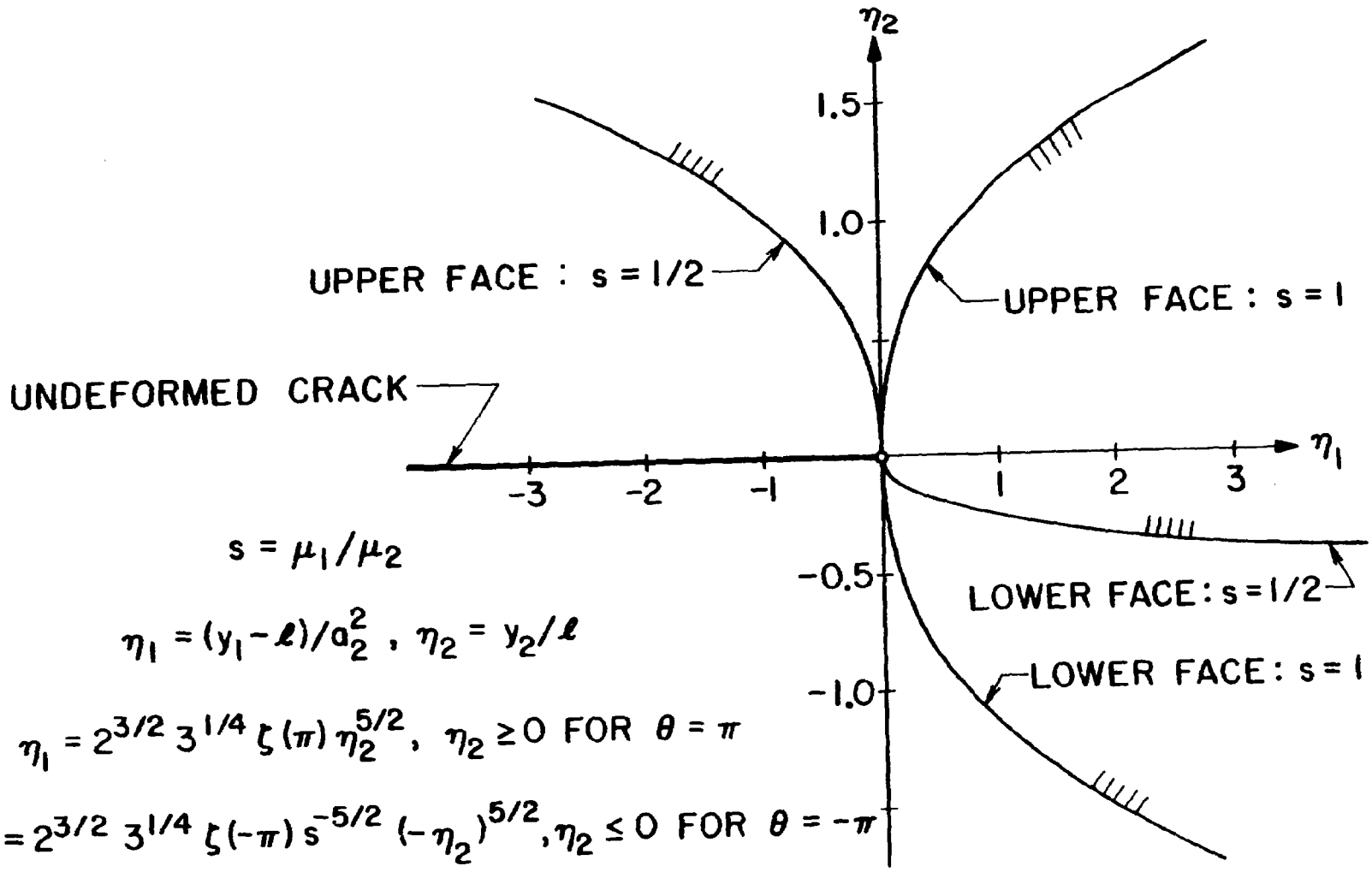


FIGURE 6. DEFORMED CRACK-FACES FOR $b_1 = 0$

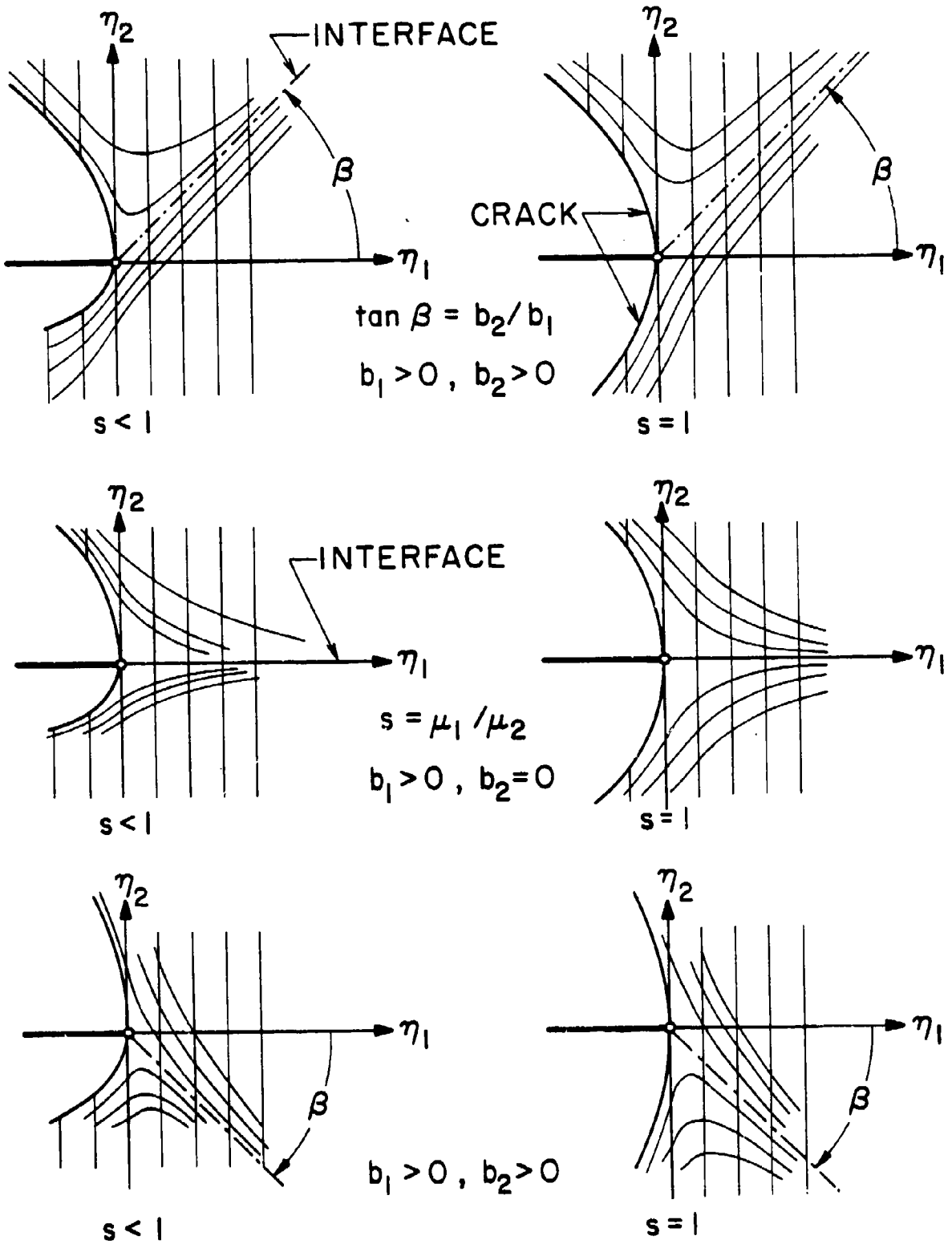


FIGURE 7. DEFORMED MATERIAL COORDINATE LINES FOR $b_1>0$.

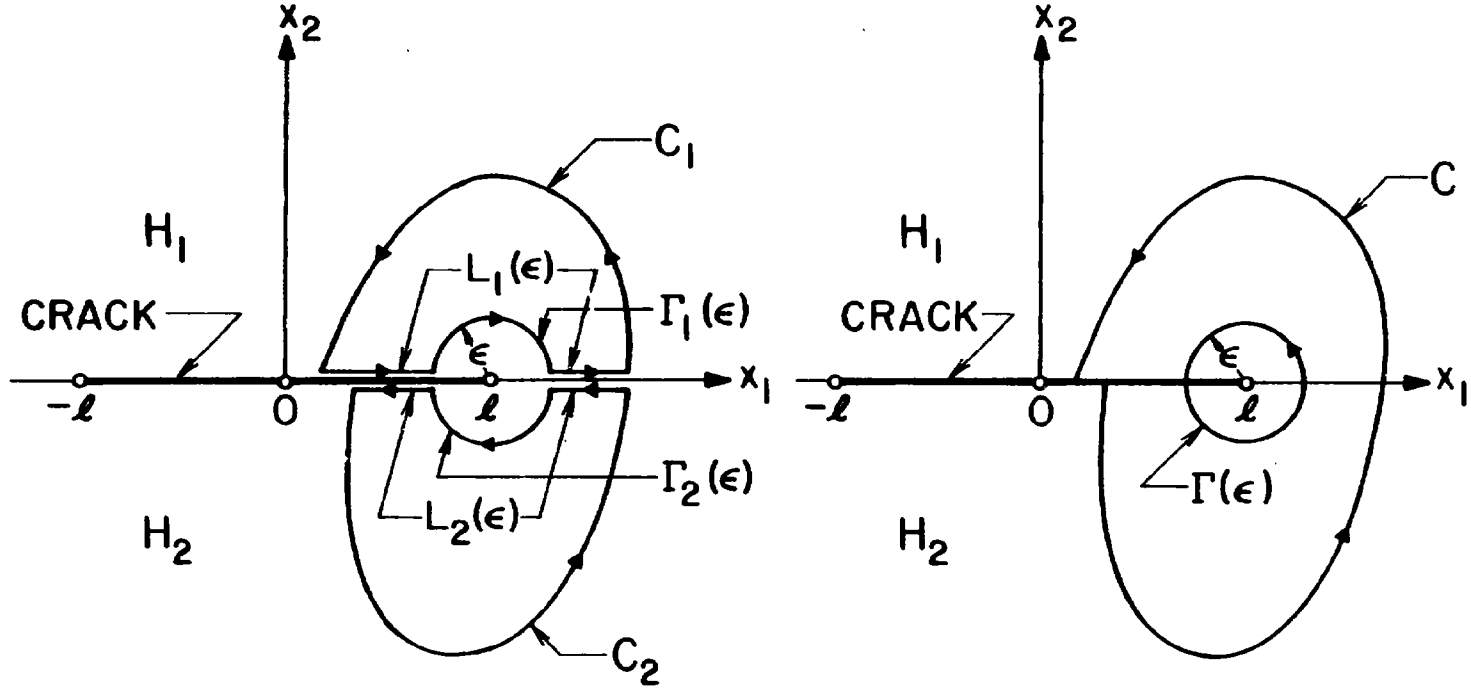


FIGURE 8. INTEGRATION PATHS USED FOR CONSERVATION LAW.

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This paper contains an asymptotic investigation – within the nonlinear theory of elastostatic plane stress – of the deformations and stresses near the tips of a traction-free interface-crack between two dissimilar semi-infinite Neo-Hookean sheets. The results obtained are free of oscillatory singularities of the kind predicted by the linearized theory, which would require the two deformed faces of an interface-crack to overlap in the vicinity of its tips. Instead, the crack is found to open smoothly near its ends, regardless of the specific loading at infinity.

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