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LONG UNIMODAL SUBSEQUENCES: A PROBLEM OF F.R.K. CHUNG.(U)
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By

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J. Michael Steele

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A PROBLEM OF F.R.K. CHUNG

By

J. Michael Steele

I. Introduction.

Let p denote a permutation of $\{1,2,\ldots,n\}$ and call $\{a_1 < a_2 < \ldots < a_t\}$ a unimodal subsequence provided there is a j such that

$$p(a_1) < p(a_2) < \ldots < p(a_j) > p(a_{j+1}) > \ldots > p(a_t)$$

or

$$p(a_1) > p(a_2) > \ldots > p(a_j) < p(a_{j+1}) < \ldots < p(a_t) .$$

Let $\ell(n)$ denote the expected length of the longest unimodal subsequence of a randomly permuted subsequence i.e. $\ell(n) = \sum_p \rho(p)/n!$, where $\rho(p)$ denotes the length of the longest unimodal subsequence of the permutation p.

F.R.K. Chung [1] conjectured that

$$\lim_{n \to \infty} \ell(n)/\sqrt{n} = C \text{ exists} .$$

The point of this note is to prove Chung's conjecture and show $C = 2\sqrt{2}$. Actually, Chung's conjecture is slightly more general than this introductory version, and this more general conjecture is obtained by the same proof.

II. Proof of F.R.K. Chung's Conjecture.

Suppose (X_i, Y_i), $1 \leq i < \infty$ are independent and uniformly distributed in $[0,1]^2$. For any $A \subset [0,1]$ let

$$I_n(A) = \max\{k: Y_{i_1} < Y_{i_2} < \ldots < Y_{i_k} \text{ with } X_{i_1} < X_{i_2} < \ldots < X_{i_k},\ X_{i_j} \varepsilon A \text{ and } i_j \varepsilon [1,\ldots,n]\}$$

and

$$D_n(A) = \max\{k: Y_{i_1} > Y_{i_2} > \ldots > Y_{i_k} \text{ with } X_{i_1} < X_{i_2} < \ldots < X_{i_k},\ X_{i_j} \varepsilon A \text{ and } i_j \varepsilon [1,2,\ldots,n]\}.$$

Next set

$$U_n = \max_{0 \leq t \leq 1} \{\max(I_n([0,t]) + D_n([t,1]),\ D_n([0,t]) + I_n([t,1]))\} .$$

The desired proof will be obtained by applying known results to the random variable U_n. To begin it is easy to check that

$$EU_n = \ell(n) .$$

Next we note that by the work of Hammersley [2] and Kesten [3] that almost surely and in L^1 we have the limits

$$(2.2)\qquad \lim_{n\to\infty} I_n(A)/\sqrt{n} = C\sqrt{\lambda(A)} \quad\text{and}\quad \lim_{n\to\infty} D_n(A)/\sqrt{n} = C\sqrt{\lambda(A)}$$

where $\lambda(A)$ is the Lebesgue measure of $A \subset [0,1]$, and C is a universal constant. The work of Logan and Shepp [9] and Vershik and Kerov [5] established that $C = 2$.

For any N and $1 \le k \le N$ we define

$$U_n^N(k) = \max[I_n(0,k/n) + D_n((k-1)/N,1),\ D_n(0,k/N) + I_n((k-1))/N,1)]$$

and

$$U_n^N = \max_{1\le k\le N} U_n^N(k) .$$

Clearly, for all N, $U_n \le U_n^N$ and by the above mentioned limit results we have

$$\lim_{n\to\infty} U_n^N/\sqrt{n} = 2 \max_{1\le k\le N} (\sqrt{k/N} + \sqrt{(N-k+1)/N}) ,$$

where the limit is almost sure and in L^1. The arbitrariness of N then shows $\limsup_{n\to\infty} U_n/\sqrt{n} \le 2 \max_{0<t<1} (\sqrt{t} + \sqrt{1-t}) = 2\sqrt{2}$ a.s., so by Fatou's lemma we get $\limsup_{n\to\infty} \ell(n)/\sqrt{n} \le 2\sqrt{2}$.

For the opposite direction note the trivial bound

$$U_n \ge I_n([0,\tfrac{1}{2}]) + D_n([\tfrac{1}{2},1])$$

so

$$\liminf_{n\to\infty} \ell(n)/\sqrt{n} \ge \liminf_{n\to\infty} E(I_n[0,\tfrac{1}{2}] + D_n[\tfrac{1}{2},1]) = 2\sqrt{2}$$

which completes the proof.

III. The Generalization.

Instead of allowing the subsequence to make "one turn" as in the unimodal case, one can consider subsequences which make k turns. Explicitly, let $\ell_k(n)$ be the expected length of the longest subsequence S of a random permutation with the following property:

S can be decomposed into $k+1$ segments which are monotone and which alternate between increasing and decreasing.

The method of the preceeding section can be used easily to show

$$\lim_{n \to \infty} \ell_k(n)/\sqrt{n} = 2\sqrt{k+1} ;$$

all one has to do is define the proper analogue $U_n(k)$ of U_n and argue as before. One should also note that the preceding bounds also prove the almost sure and L^1 convergence of $U_n(k)/\sqrt{n}$ to $2\sqrt{k+1}$.

References

[1] Chung, F.R.K., On Unimodal Subsequences, Bell Laboratories Technical Report (1979).

[2] Hammersley, J.M., A Few Seedlings of Research, Proc. Sixth Berk. Symp. Math. Stat. and Prob., 1 (1972), 345-394.

[3] Kesten, H. Comments on J.F.C. Kingman's Subadditive Ergodic Theory, Ann. Prob. 1, (1973), 883-889.

[4] Logan, B.F. and Shepp, L.A., A Variational Problem for Young Tableau, Adv. in Math., 26, (1977), 206-222.

[5] Vershik, A.M. and Kerov, C.V., Asymptotics of the Plancherel Measure of the Symmetric Group and a Limiting Form for Young Tableau, Dokl. Akad. Nauk U.S.S.R., 233, (1977), 1024-1027.

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Let $\ell(n)$ be the expected length of the longest unimodal subsequence of a random permutation. It is proved here that $\ell(n)/\sqrt{n}$ converges to $2\sqrt{2}$. This settles a conjecture of F.R.K. Chung.

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