IDA PAPER P-1395

NATIONWIDE DEFENSE AGAINST NUCLEAR WEAPONS: PROPERTIES OF PRIM-READ DEPLOYMENTS

Alan F. Karr

April 1981

INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

The work reported in this document was conducted under IDA's Central Research Program. Its publication does not imply endorsement by the Department of Defense or any other government agency, nor should the contents be construed as reflecting the official position of any government agency.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A101 702
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) NATIONWIDE DEFENSE AGAINST NUCLEAR WEAPONS PROPERTIES OF PRIM-READ DEPLOYMENTS	5. TYPE OF REPORT & PERIOD COVERED Final
	6. PERFORMING ORG. REPORT NUMBER IDA Paper P-1395
7. AUTHOR(s) Alan F. Karr	8. CONTRACT OR GRANT NUMBER(s) IDA Central Research Program
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Defense Analyses, Program Analysis Division, 400 Army-Navy Drive, Arlington, VA 22202	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE April 1981
	13. NUMBER OF PAGES 209
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

This document is unclassified and suitable for public release.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nationwide defense, Prim-Read deployment, interceptor deployment, defense against strategic attack, ballistic missile, anti-ballistic missile, target price, optimization of defensive resources, target defense principle, min-max problem, mathematical model.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The problem of allocation of defensive resources in nationwide defense against strategic nuclear attack is examined. Distinctive assumptions are that defenses are local, that attacking weapons directed at each target arrive sequentially (requiring that interceptors be allocated without knowledge of how many additional attacking weapons will follow), and that neither side can re-allocate its resources during an attack. Prim-Read deployments are defined and analyzed in detail. Effects

(continued)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) (continued from item 20).

of the "Target Defense Principle" that initially defended targets must remain more attractive than undefended targets (up to the point where destruction is certain) are investigated. Optimality and non-optimality properties of Prim-Read deployments are established for the criteria of target value destroyed and target value destroyed per attacking weapon committed. Variations on the basic model, numerical examples, comparison of Prim-Read and proportional deployments, and discussion of physical implications of the mathematical results are included.

IDA PAPER P-1395

NATIONWIDE DEFENSE AGAINST NUCLEAR WEAPONS: PROPERTIES OF PRIM-READ DEPLOYMENTS

Alan F. Karr

April 1981

RE: IDA Paper P-1395, Classified References, Distribution Unlimited-
No change per Ms. Betty Pringle, IDA

INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION
400 Army-Navy Drive, Arlington, Virginia 22202

IDA Central Research Program

ACKNOWLEDGMENTS

The author is indebted to Lowell Bruce Anderson and Jerome Bracken of IDA for their comments and suggestions throughout the preparation of this paper. Stefan Burr (City University of New York), A. Ross Eckler (Bell Laboratories), Wilbur B. Payne (Department of the Army), Clayton J. Thomas (Department of the Air Force) and Iram Weinstein (System Planning Corporation) provided helpful reviews for which the author thanks them.

CONTENTS

FIGURES

TABLES

APPENDICES

Chapter I

INTRODUCTION

This paper reports the results of a rather lengthy and detailed investigation of some problems that arise in the defense of targets by interceptors when attacking weapons arrive at each target sequentially. For some time there has existed a particular interceptor deployment, generally known as the *Prim-Read deployment*, that equalizes the probabilities that each of a prescribed number ρ of attacking weapons actually destroys the target. (See the discussion of related research in Chapter VIII.) The parameter ρ is chosen by the defending side. The Prim-Read deployment has long been suspected to possess (some sort of) optimality properties for the defending side, but these properties had never been carefully formulated or rigorously established. In this paper we fill both gaps, at least partially. A number of optimality properties are formulated and shown to be satisfied, in many cases uniquely, by Prim-Read deployments. We also present extensive discussions of the physical interpretations and practical implications of our optimality results.

A. THE UNDERLYING PHYSICAL PROBLEM

Let us begin with a discussion of the physical real-world problem that has motivated essentially all work on Prim-Read deployments. That problem is the defense of point targets against attacking ballistic missiles by means of interceptors, which are missiles (ABMs) themselves. On the national scale, one is dealing with the defense of a nation's entire population and industrial capacity. Because of the defending side's limited interceptor resources, it will thus be impossible (in general) to defend all of the targets. The defending side must choose which targets to defend and, for each defended target, how to allocate interceptors assigned there among incoming attacking weapons, all in a manner that attempts to minimize some measure of target value destroyed. The situation is further complicated by the defending side's not knowing how the attacking side will allocate its weapons among the targets and also by the possibility that the attacking side may be able to discern the interceptor deployment and may allocate its weapons on the basis of such knowledge.

Yet another complication arises if attacking weapons arrive sequentially in time at each target. To each attacking weapon there must be assigned some of the interceptors deployed at the target and this assignment must be made without knowledge of how many additional attacking weapons will follow. Such assignments could therefore be prescribed in advance of any attack and the defending side may even wish to allow for the possibility of their being known to the attacking side.

Despite all the difficulties which the structure described above imposes on the defending side, we shall show that there exist reasonable and in some cases even optimal deployments that can be undertaken. Such deployments can (1) minimize the expected target value destroyed, (2) limit the use which the attacking side can make of its knowledge of the deployment, (3) limit the effects of rational actions available to the attacking side, and (4) force the attacking side to choose actions known to the defending side. That Prim-Read deployments possess such properties, and in precisely what form, is established in Chapters III through VII of this paper. Often, it is only Prim-Read deployments that have these properties.

We have not yet considered in detail what the defending side's objective should be. In general, that objective should be to make optimal use of limited interceptor resources according to some measure of target value destroyed. As discussed below in this Chapter and at some length in Chapters IV, V and

VI, we shall deal with two such criteria: target value destroyed and target value destroyed per attacking weapon committed. At the moment, however, we wish to consider the defending side's objective in somewhat more general (albeit vaguer) terms.

Let V_d denote the payoff function, in terms of target value destroyed, to the attacking side when the defending side implements the deployment d (a precise definition of a deployment is unnecessary at this point; cf. Chapter II). For each number i of attacking weapons, $V_d(i)$ is the maximum expected target value destroyed by i attacking weapons, where the maximum is over all allocations of those weapons among the targets, with the deployment d held fixed. We propose, as has been proposed elsewhere, that the deployment d be chosen so that the graph of V_d is of the form shown in Figure 1. That is, d should be chosen such that

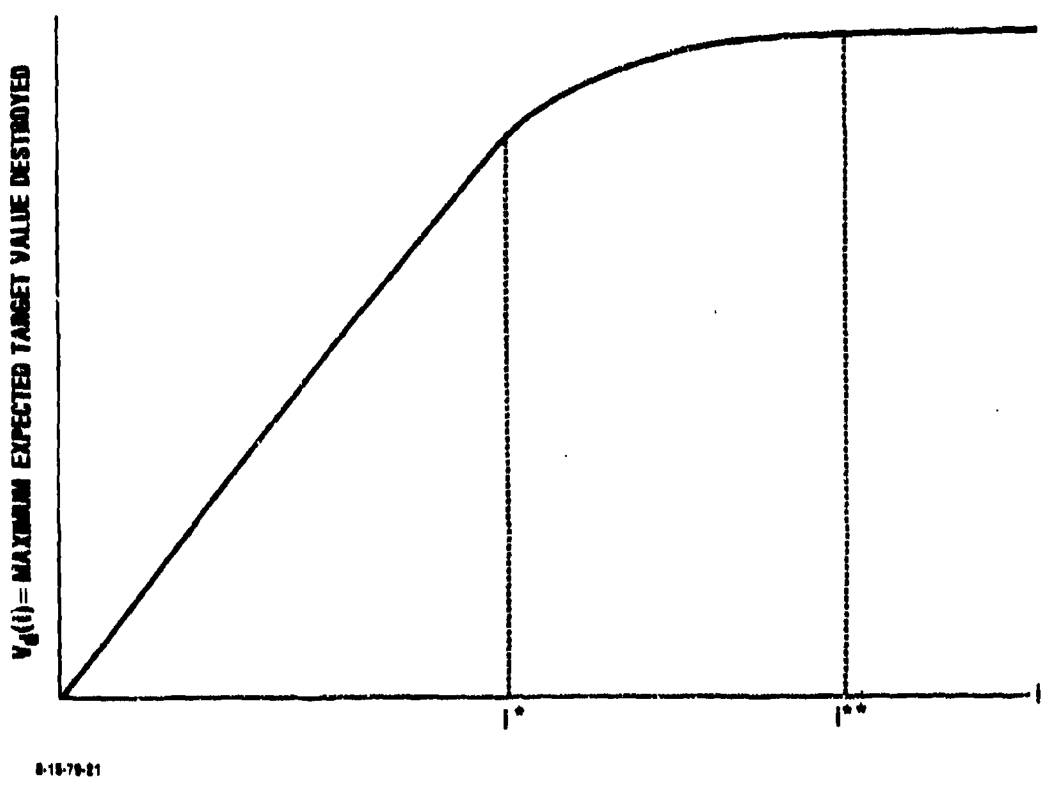


Figure 1. PROPOSED FORM OF PAYOFF FUNCTION

V_d has the following properties:

1) V_d is continuous, concave and increasing (the latter, of course, is unavoidable);

2) There is some i* such that V_d is linear on the interval [0,i*];

3) On the interval (i*,i**], V_d is strictly concave.

Although it is not clear yet, the linear portion of V_d represents attack and destruction of defended targets, while the strictly concave segment corresponds to destruction of undefended targets.

The reasoning underlying this proposed payoff function merits more detailed explication, some of which follows here, and more of which appears in Chapter V and also in Chapter VIII.

In general, the defending side will not possess sufficient interceptor resources to defend all the targets. If there are targets of different values (as there will be in most cases), then the more valuable targets should be defended and the less valuable targets must be left undefended. The reason for this is that the attacking side must never find initially undefended targets more attractive than defended targets, for otherwise the defending side's resources are being wasted. Crucial to this line of argument, and a tenet of many philosophies of defense, is the idea that the purpose of defending some targets is to force the attacking side to expend so much of its resources attacking the defended targets that not all targets can be attacked. Therefore, the deployment of interceptors at defended targets must be such that the attacking side will commit enough attacking weapons to destroy all the defended targets before it attacks any of the undefended targets. This is the *target defense principle* upon which many of our results rest.

In Figure 1, i* is the number of attacking weapons necessary to destroy all the defended targets, and i** is the number of attacking weapons needed to destroy all the targets. To the right of i^{**}, the payoff function is constant. Between i^* and i^{**} the payoff function is strictly concave. However, we have not yet justified linearity of V_d on the interval $[0,i^*]$, which corresponds to destruction of the defended targets.

To make that justification, we begin with two observations.

1) Regardless of precisely which targets are defended, they must be defended in such a manner that the slope, s, of V_d just to the right of i^*, which corresponds to the point at which undefended targets are first attacked, be less than or equal to the slope of V_d everywhere on $[0,i^*]$. This is a consequence of the tenet that defended targets must always remain, even accounting for possible prior destruction, more attractive than undefended targets, up to the point where all defended targets are destroyed. In particular, therefore,

$$V_d(i^*) \geq i^*s.$$

2) The value of i^* determines $V_d(i^*)$, which is simply the total value of the defended targets.

The restrictions engendered by these observations are depicted graphically in Figure 2. Because of the slope requirement, on $[0,i^*]$ the payoff function must lie everywhere above the dotted line, which has slope s, in Figure 2. Moreover, if the graph of V_d ever intersects the dashed line, which extends the graph of V_d backward from i^* to 0 with slope s, it must remain along this line up to the point i^*. The payoff function must then be of general form shown in Figure 3.

There are (at least) three ways to force linearity of V_d on $[0,i^*]$, which we now discuss.

1) If

$$V_d(i^*) = i^*s,$$

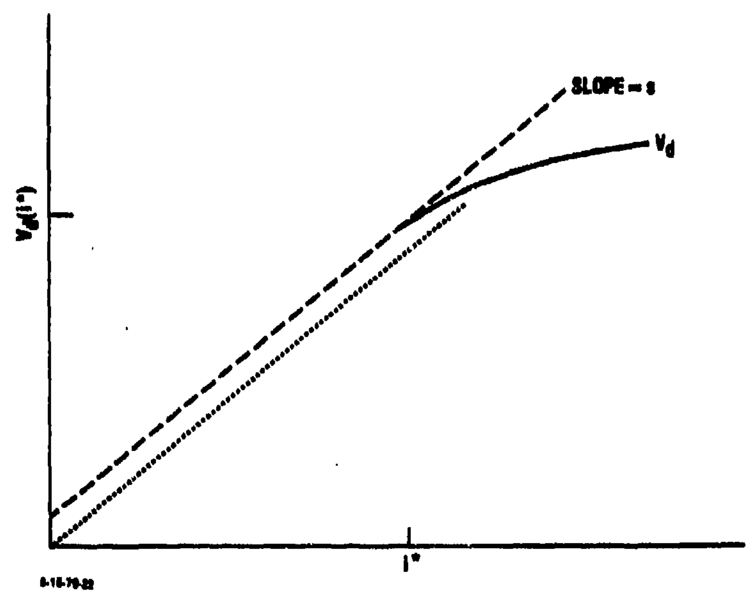


Figure 2. REQUIREMENTS OF PAYOFF FUNCTION

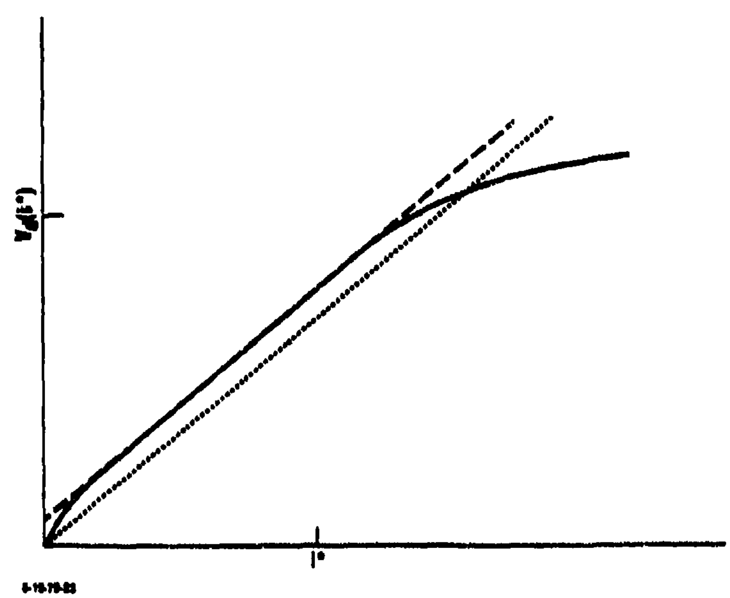


Figure 3. POSSIBLE FORM OF PAYOFF FUNCTION

which would essentially always be the case were it not for discreteness difficulties (targets are discrete in our model and nearly discrete, at least, in reality), then the dotted and dashed lines in Figure 2 coincide, which entails a payoff function of the form shown in Figure 1, whose derivative is continuous at i^*.

2) With i^* and hence s and $V_d(i^*)$ determined, the defending side should seek to minimize the maximum slope of the payoff function V_d on the interval $[0,i^*]$. The rationale for this goal is that by doing so the defending side limits the maximum effectiveness of any single attacking weapon. To realize this goal, the defending side, in view of the limitations represented in Figure 2, should choose the deployment d in such a manner that V_d is linear on $[0,i^*]$. When this is done, the slope of V_d on $[0,i^*]$ will be

$$s^* = \frac{V_d(i^*)}{i^*} .$$

In general, it will then be the case that $s^* > s$, which is consistent with the previously discussed principle that defended targets always be more attractive than undefended targets. Consequently, the derivative of V_d will be discontinuous at i^*, but will remain decreasing, preserving concavity.

3) Also with i^* previously specified, the defended targets should be made equally attractive, attacking weapon by attacking weapon, to the attacking side; this leads directly to linearity of V_d on $[0,i^*]$, with slope there the s^* given above. Arguments supporting this particular assumption are somewhat vague grounds of symmetry and uniformity and also analogies to a large number of decision-making models in which optimal solutions tend to possess an appropriate (and often obvious) uniformity property; cf. [4,6,7].

Any of the three lines of reasoning just described suffices to yield a payoff function V_d of the form shown in Figure 1.

Questions that naturally arise at this point concern possible optimality of deployments with payoff functions of the given form and implications of optimality results for the decision-making processes of both sides and on the underlying problem of interceptor defense of point targets. It is such questions that are addressed in this paper. Deployments with payoff functions of the form shown in Figure 1, when more carefully defined, will be called *Prim-Read deployments*. Our principal mathematical results establish that Prim-Read deployments possess a number of (theoretical) optimality properties; ensuing discussions attempt to clarify and illuminate important practical consequences of the optimality properties. In many cases, we are able to demonstrate that Prim-Read deployments uniquely possess certain optimality properties. Finally, nonoptimality of Prim-Read deployments in some situations of interest is also demonstrated.

In terms of the mathematical model to be presented in the next Section, an absolutely essential feature is our assumption that attacking weapons directed at each target arrive there sequentially in time. For the underlying physical problem introduced at the beginning of this Section, this is a plausible assumption because the attacking side will not wish to reveal its allocation of weapons earlier than necessary; otherwise the defending side could possibly alter its interceptor assignments and decrease the target value destroyed. Consequently, it is nonoptimal for the attacking side not to have attacking missiles allocated to each target arrive there sequentially. Of course, "sequential" is a relative term; the absolute time scale of the attack may still be short.

B. MATHEMATICAL ASSUMPTIONS

We now present the mathematical model to be analyzed in this paper. The prototypical physical situation we wish to consider is that of the defense of a nation's centers of population and production against attack by incoming ballistic missiles, using interceptors that are presumably (but not necessarily) missiles themselves. Other potential applications of the model will be discussed briefly in Section D of this Chapter.

Some essential physical properties of the prototypical process to be studied are the following:

1) Attacking weapons arrive in the vicinity of each target sequentially in time.

2) A certain number of interceptors (possibly zero) must be deployed against each attacking weapon as it arrives in the target vicinity, in order to attempt to destroy the attacking weapon before it reaches the target. This number must be chosen without knowledge of how many more attacking weapons might arrive later; once assigned, interceptors are irrevocably committed and cannot be reassigned even if the target is destroyed before intended engagements can take place.

3) The numbers of interceptors deployed against various attacking weapons need not be equal.

4) Each interceptor can be deployed against only (at most) one attacking weapon. If it fails to engage or destroy the attacking weapon, it is of no further use.

5) When several interceptors are deployed against a single attacking weapon, the resultant engagement consists of independent one-on-one engagements, one for each interceptor. If the attacking weapon is destroyed by one interceptor, the other interceptors are rendered useless and cannot be redeployed.

6) When there are multiple targets, each interceptor must be assigned in advance to defense of some particular target. Interceptors cannot be shifted from target to target as an attack progresses.

The model with which most of this paper deals will incorporate not only these assumptions but also some additional assumptions. In Chapter VII we discuss ways whereby certain of the hypotheses can be weakened. Additional assumptions will incorporate into the model the following effects:

1) There is a central detection system (whether a single system for all targets or separate systems for individual targets is immaterial) that warns of the approach of each attacking weapon, whereupon the interceptors assigned to it are activated against it. All attacking weapons are detected.

2) The defending side is able to discern with certainty the intended target of each attacking weapon. Against each weapon only interceptors assigned to its intended target can be deployed.

3) Neither side has the capability to adaptively reassign its resources during the course of an attack. Attacking weapons allocated to a target that has been destroyed cannot be retargetted, and interceptors assigned to the target cannot be redeployed at some other target.

4) An unintercepted attacking weapon destroys with certainty the target at which it is directed. Only one penetration is required to destroy each target.

We will discuss below the plausibility and physical interpretations of the assumptions, but to give that discussion sufficient focus and specificity, we first state those assumptions, beginning with the single target case.

(1.1) ASSUMPTIONS. a) Attacking weapons arrive at the target sequentially, one at a time, with sufficient time between successive arrivals that interactions involving different attacking weapons do not overlap in time.

b) The defending side decides in advance the number of interceptors that will be deployed against the i^{th} attacking weapon to arrive, $i = 1, 2, \ldots$, provided that such an attacking weapon be in fact committed. Interceptors designated for deployment against attacking weapons that do not arrive cannot (within the framework of the process under immediate consideration) be used for other purposes. (The knowledge that may be available to the defending side when this decision is taken will be discussed below.)

c) Each interceptor can be deployed against (at most) one attacking weapon, which it engages and destroys with probability $1 - q$, where $0 < q < 1$. Different interceptors assigned to a given attacking weapon are (probabilistically) independent.

d) An attacking weapon that is not intercepted destroys the target with probability one.

e) Interactions involving different attacking weapons are mutually independent. □

Consequently, an attacking weapon against which there are deployed d interceptors penetrates the defense and destroys the target with probability q^d;

$$P\{\text{penetration and target destruction} | d \text{ interceptors}\} = q^d ,$$

and is intercepted and itself destroyed--without any harm done to the target-- with the complementary probability $1 - q^d$.

For the multiple target case, the hypotheses are analogous but slightly more complicated.

(1.2) ASSUMPTIONS. a) The targets are separated to the extent that each attacking weapon must be directed at only one target and each interceptor must be assigned in advance to defense of a single target. The attacking side has no shoot-look-shoot capability to redirect attacking weapons during the course of an attack, while the defending side is unable to reassign interceptors from one target to another.

b) Attacking weapons directed at each target arrive sequentially in time with sufficient gaps that interactions involving different attacking weapons do not overlap in time. Whether and how interactions at different targets coincide or overlap in time are immaterial.

c) The defending side decides in advance the number of interceptors to deploy against the i^{th} attacking weapon to arrive at each target, provided that it actually arrive.

d) Each interceptor must be assigned to a single target for deployment against at most one attacking weapon. The probability that an interceptor engages and destroys an attacking weapon against which it is deployed is $1 - q$, where $0 < q < 1$. Different interceptors deployed against a given attacking weapon are independent. The probability q is the same for all targets, interceptors and attacking weapons.

e) An unintercepted attacking weapon destroys with probability one the target at which it is directed. There is no collateral damage.

f) Interactions involving different attacking weapons at each given target are mutually independent.

g) The entire interception/target destruction processes at different targets are mutually independent. []

C. DISCUSSION OF THE ASSUMPTIONS

The assumptions are restrictive in physical terms in order that we obtain a specific and tractable mathematical model. While this paper does not contain detailed consideration of possible applicability of the model, we do treat that question in slightly more detail below. In Chapter VII we discuss ways of relaxing some of the assumptions.

Others of the assumptions, however, are for our purposes immutable. These are the assumptions of sequential arrivals of attacking weapons, and of preassigned targets to defend and attacking weapons to be deployed against for interceptors. The extent to which these hypotheses are plausible depends not only on the time and geography of the physical process under study but also on the knowledge available to the attacking and defending sides. Since this is a fairly important point, especially in the context of the min-max optimality properties discussed in Chapters V and VI (and also elsewhere in this paper), we wish to consider it in somewhat more detail.

In terms of the attacking side, Assumption (1.2a) does not allow representation of shoot-look-shoot processes, i.e., the attacking side cannot during the course of an attack make use of any information it may acquire about which targets have already been destroyed. If such information were usable, the attacking side could adaptively reassign its weapons to undestroyed targets. If the length of the attack is sufficiently short or the targets are sufficiently far away from the attacking weapons or the cost and effort necessary to re-target attacking weapons are sufficiently great, this assumption will not be unreasonable.

Whether the attacking side has knowledge of the defensive deployment to be followed by the defending side is not yet specified. The defending side, however, will often wish to protect itself against this possibility. It will be shown below that Prim-Read deployments have the property that the attacking side can make only minimal use of such information; cf. Theorems (4.7), (4.8) and (4.10) for specific manifestations of this lack of ability of the attacking side to use its knowledge of the defending side's choice of deployment.

Especially if the attacking side knows the interceptor deployment, it is plausible to assume that attacking weapons are pre-targetted. However, the assumption that a pre-attack allocation cannot be changed during the attack still remains both in effect and open to question.

To the author, it is initially more difficult to accept the assumption that the defending side must in advance assign each interceptor not only to a target but also to a specific--but hypothetical--attacking weapon directed at that target. This may lead to interceptors being designated for use against attacking weapons that are never launched against the target and thus, in effect, being wasted. One consequence of this "waste" may be seen in Theorem (6.13). However, the extent to which there really is waste depends on the knowledge available to the defending side; if there is, as we assume, incomplete knowledge *a priori*, the waste may exist only *ex post facto*, and may therefore be unpreventable.

Moreover, sequential arrival of attacking weapons at each target not only can be brought about by the attacking side (by means of its launch procedures) but also is desirable to the attacking side because this procedure maximally delays full revelation of the allocation of attacking weapons to targets. If the defending side believes that the attacking side will so act (or if the defending side at least wishes to protect against the possibility), it is then forced to make advance assignments in the manner assumed above.

In the situation to which we envision the model as applicable, the defending side does indeed possess relatively less information than the attacking side. At most, the defending side knows the total stockpile of available attacking weapons, but never knows the targets to which those attacking weapons are assigned, except to the extent that its own deployment can force an attacking side assumed to use certain specific decision rules to make particular allocations. It is even possible that the defending side does not know the size of the attacking side's stockpile. Many of our results are valid in either case.

That there should exist asymmetry in the information available to the two sides about each other's resource assignments is not unreasonable physically. Interceptors must be located near to the targets they are assigned to defend, while attacking weapons may be directed at specific targets by navigational means and not by the initial position of the attacking weapons.

While it is possibly unrealistic to assume that the attacking side know how many interceptors are assigned to each (potential) incoming weapon at each target, it is extremely interesting that--as we show in Chapter V--there are defensive deployments that minimize target destruction even if this knowledge were available to the attacking side.

Most of the other assumptions are fairly straightforward. Independence of interactions involving different targets or different attacking weapons seems quite plausible. The simplistic attrition structure, which decomposes an attacking weapon/interceptors interaction into independent one-on-one interactions, may be less plausible in some circumstances. However, this is one assumption that we are able to relax; cf. Chapter VII. The assumption that an unintercepted attacking weapon destroys the target at which it is directed can also be weakened, but this requires a re-interpretation of the idea of the "price" imposed by a deployment, as defined in Chapter II. In view of the implications of the assumption of preassigned interceptors in regard to target separation, the assumption of no collateral damage is entirely natural.

When there is more than one target, the targets must be assigned values in order that the defending side be able to choose which of them to defend, and how to deploy interceptors at defended targets. The mathematical results we derive below involve (expected) target value destroyed when the attacking side optimizes its allocation of weapons against a given defensive deployment. To do so in practice, the attacking side would need to know the values of the various targets, which may be fairly clear in some cases (e.g., centers of population) but less clear in others (e.g., industrial centers of different kinds). We assume that either the attacking side does possess fairly accurate estimates of target values or that the defending side wishes to guard against this possibility. In regard to Prim-Read deployments this will be seen not to be restrictive, since a Prim-Read deployment, in effect, reveals the (relative) values of at least the defended targets.

As explained and explored in more detail in the Chapters below, the defending side's goal is taken to be to make optimal use--according to a criterion that must be specified--of its limited interceptor resources. Two principal optimality criteria will be considered in this paper, each of which the defending side will attempt to minimize:

1) Expected target value destroyed by an allocation of attacking weapons that is optimized against the chosen deployment.

2) Expected target value destroyed per attacking weapon committed, with the allocation of attacking weapons optimized against the chosen deployment.

The physical interpretation is that the defending side chooses its deployment (which assigns both interceptors to targets and interceptors at each target to sequentially arriving attacking weapons) and the attacking side then allocates its weapons, with knowledge of the deployment, so as to maximize either target value destroyed or target value destroyed per attacking weapon committed. The defending side seeks to minimize this maximum.

The principal results of this paper show that Prim-Read deployments are essentially always optimal for the second criterion and are optimal for the first criterion provided that one assume that the target defense principle stated above be satisfied and that the attack size be less than is required to destroy all the defended targets. Recall that the target defense principle states that targets should be defended in order of decreasing value in such a manner that no defended target is ever less attractive than an undefended target, even when one accounts for the possibility that previous attacking weapons may have destroyed the defended target.

It will further be seen that Prim-Read deployments achieve certain goals that are of secondary importance, but nice to achieve nonetheless. Among these are specifying the commitment of attacking weapons required to destroy the defended targets, limiting the number of actions that might be undertaken by the attacking side and in general limiting effects of the uncertainty and lack of information imposed on the defending side by the fundamental asymmetry of the choice problem we treat.

Whichever criterion is established by the defending side, it will be necessary in general to leave each target defenseless after a certain number of attacking weapons have been directed at it. Possibly not all targets can be defended. The defending side must therefore choose, for each target,

1) the number of attacking weapons against which the target will be defended (possibly zero);

2) the number of interceptors to be assigned to each attacking weapon against which the target is defended.

This is the choice problem with which this paper is concerned.

Under the target attrition model defined by (1.1c) or (1.2d), commitment of one more attacking weapon than the number of attacking weapons defended against assures destruction of the target (since $q < 1$, no smaller commitment assures destruction). The effect is to specify a "price" for each target: the minimum commitment of attacking weapons by which destruction of the target is certain. By deploying interceptors in such a manner that the attacking side--under the assumption that it use certain forms of decision rules--will pay the full price of each defended target, the defending side thereby gains control (despite its having to choose first) of the attacking side's resource allocation.

D. POTENTIAL APPLICABILITY OF THE MODEL

Although this paper contains no detailed analyses of possible applicability of the model to specific physical combat processes, we do wish to discuss briefly and in a fairly general manner the kind of situations it might be used to represent. As previously noted, this paper is motivated by and directed primarily toward defense of national population and production resources against a strategic nuclear attack. We have discussed several of the assumptions in the context of this particular application and concluded that the assumptions were at least plausible and in many cases rather reasonable. We believe, therefore, that our results are applicable and are of some importance to the problem that motivated this research.

However, it is possible, that the model may be applicable to other, and perhaps smaller scale, situations. Those situations, it seems to the author, must, like the prototypical missile/ABM problem, involve defense of point targets by interceptors. That defense must in some sense be a barrier through which attacking weapons attempt to penetrate; destruction of the barrier is not important and not necessarily attempted. However, the attacking side may attempt to "exhaust" the barrier by committing enough weapons to force deployment of all the interceptors. The crucial aspects are that each interceptor be deployed against exactly one (or, more accurately, not more than one) attacking weapon and, of course, that attacking weapons arrive sequentially at the barrier.

An alternative model of barrier penetration processes appears in [1], to which the reader is referred for comparison.

To consider a concrete situation, let us examine defense of an aircraft carrier task force by aircraft stationed on the carrier. The carrier is the target. Against an attack by enemy aircraft, the interceptors might be deployed on patrol. If we assume that there is also present an efficient central detection system, such as AWACS, then detection of all penetrators seems reasonable. Not so reasonable is the assumption that penetrators (enemy aircraft) arrive sequentially in time. Presumably the enemy might attempt to saturate and confuse the defense by attempting many penetrations simultaneously. Hence if the model were used in this situation, it should be with the explicit understanding that its hypotheses are only imperfectly satisfied. If penetrations are attempted over a sufficiently short period of time, it is plausible that each interceptor can be involved in only one attempted engagement. Destruction of the target by a successful penetrator is not certain, but we indicate in Chapter VII how to weaken this assumption.

In the same context, the model might be applied to close-in defense of the carrier by surface-to-air missiles launched from the associated escort ships. As in the preceding example, the greatest difficulty is with the assumption that penetrators arrive sequentially in time; the other assumptions seem relatively plausible.

Much the same analysis might apply to defense of an air base against an attack by enemy aircraft, using either aircraft or surface-to-air missiles. Once again, the troublesome assumption is that of sequential arrivals of attacking weapons. The assumption concerning "unreusability" of interceptors seems better satisfied when interceptors are SAMs than when interceptors are aircraft.

It is difficult to justify applicability of the model to any process in ground combat; the author (though his knowledge is limited) is unable to find a ground combat process for which the assumptions of the model are less than patently untrue. Nonetheless, the previously noted tenet that one aspect of a defense is to force a known--and also possibly unacceptably large--commitment of the attacking side's resources if the attack is undertaken at all seems very important to us. Its consequences in other combat situations need to be explored.

E. STRUCTURE OF THE PAPER

In order to improve accessibility of the paper to readers without advanced mathematical training or without interest in detailed derivations of our results, we have attempted to provide sufficient summary information so that the content and interpretations of the principal results are understandable without a line-by-line reading of the paper. To this end, we have provided a summary at the end of each Chapter that describes the important results of that Chapter in both mathematical and physical terms, but not in full detail. Therefore, a substantial understanding of the paper could be obtained by reading:

1) Chapter I;
2) The summaries of Chapters II, III, and IV;
3) The general discussion in Chapter V, Section A and the summary of Chapter V;
4) The summaries of Chapters VI and VII;
5) Chapter VIII.

Both the Chapter summaries and Chapter VIII list by number the principal mathematical results so that they may be located easily for further details.

We emphasize, however, that the "substantial understanding" noted above is only a minimal substantial understanding. Many examples, illustrations and discussions of practical consequence appear in the body of the paper and would be missed in a partial reading. We urge the reader who has background and interest to read the entire paper, or at least Chapter V.

As for specific content, Chapter II contains derivations of Prim-Read deployments from uniformity hypotheses on expected target values destroyed by various attacking weapons, and also introduces a number of important concepts, the most important being that of the target prices imposed by a deployment. In Chapter III we present optimality properties of Prim-Read deployments in the single target case. Although this case is perhaps of relatively little intrinsic interest, the results and derivations presented there serve as useful motivations for the more complicated results appearing later in the paper. Chapter IV develops optimality properties in the multiple target case, but under the assumption that either all targets are defended or only targets of unit value are left undefended. The mathematical and practical heart of the paper, we believe, is Chapter V, in which we treat optimality properties of Prim-Read developments when many targets must be left undefended and when the target defense principle is assumed to hold. In these circumstances, the Prim-Read deployment is shown to be the unique solution to several optimization problems, each of which is of clear practical importance. We also show that there are important problems to which Prim-Read deployments are *not* optimal solutions. Chapter V also contains a discussion, extending that of Chapter I, Section A, of desirable properties of payoff functions and ways of attaining them. Some additional optimality results complementary to those in Chapters IV and V are given in Chapter VI. Finally, in Chapter VII we obtain the form of Prim-Read deployments under hypotheses weaker than those given in (1.1) and (1.2); we allow inclusion of target-dependent intercept probabilities, unreliable attacking weapons, and alternative attrition structures. The entire paper is summarized in Chapter VIII, where we also discuss related literature and some aspects of the problem (which do exist) that are not treated here. An Index of Notation is given following the references. Appendix A contains numerical examples and Appendix B contains some additional results comparing Prim-Read and proportional deployments.

We believe that this paper presents a clear and comprehensive analysis of the strengths and weaknesses of Prim-Read deployments, and believe that it contributes to understanding of the underlying defense problem.

CHAPTER II

DERIVATION OF THE PRIM-READ DEPLOYMENTS

In this Chapter we derive the interceptor allocation and requirement for Prim-Read defensive deployments, first for a single target and then for multiple targets. The reader will observe that the Prim-Read deployments equalize the expected target values destroyed by various attacking weapons and might then conjecture (by analogy with various other game-theoretic allocation problems) that this "equal risk" deployment possesses certain optimality properties. Those properties constitute the subject of Chapters III, IV, V, and VI.

A. SINGLE TARGET CASE

Let us first consider the single target case. An important idea in the Prim-Read defense strategy is that of the "price" of a target, which is a function of the chosen deployment. The price of the target is the minimum number of weapons that must be expended by the attacking side to be *certain* that the target is destroyed. After introducing some notation, we shall give the precise definition.

A *deployment* is specified by a vector d, where

$d(i)$ = number of interceptors allocated to attempt to destroy the i^{th} attacking weapon.

A deployment for the defending side consists of choices $d(1)$, $d(2)$,... that define an interceptor allocation. We explicitly permit the $d(i)$ to have non-integer values.

(2.1) DEFINITION. The *price* imposed on the attacking side by a deployment $d = (d(i))$ is

$$\rho(d) = \min\{i : d(i) = 0\} .$$

In view of the Assumptions (1.1), the attacking side can be certain of destroying the target by launching against it an attack of size equal to the price, and would, provided it knew the price, never commit more weapons. To impose a price ρ the defending side need only deploy $(\rho-1)$ interceptors in the deployment

$$\text{(2.2)} \qquad d(i) = 1, \quad 1 \le i \le \rho-1, \qquad = 0, \quad i \ge \rho .$$

This is true provided we require that $d(i) \ge 1$ whenever $d(i) > 0$, a property not satisfied by Prim-Read deployments.

Given a deployment d, let

$$\text{(2.3)} \quad p(d,i) = P\{\text{target is destroyed by attacking weapon } i\} = \left[\prod_{\ell=1}^{i-1} \left(1-q^{d(\ell)}\right)\right] q^{d(i)} .$$

Evidently $p(d,i) = 0$ for $i > \rho(d)$; i.e., $\sum_{i=1}^{\rho(d)} p(d,i) = 1$.

EXAMPLE. Let d be the deployment given by (2.2). Then

$$p(d,i) = (1-q)^{i-1} q$$

for $i = 1, \ldots, \rho - 1$, while

$$p(d,\rho) = 1 - q \sum_{i=1}^{\rho-1} (1-q)^{i-1} = (1-q)^{\rho-1} .$$

Much of our attention in this paper focusses on deployments d that equalize the probabilities $p(d,1),\ldots,p(d,\rho(d))$; these will be called Prim-Read deployments.

(2.4) DEFINITION. A deployment d^* is a *Prim-Read deployment* if

$$p(d^*,i) = \frac{1}{\rho(d^*)} \tag{2.5}$$

for $i = 1, \ldots, \rho(d^*)$.

The following Theorem verifies that Prim-Read deployments exist and also provides their explicit form.

(2.6) THEOREM. For each integer $\rho \geq 1$ there exists a Prim-Read deployment d^* such that

$$\rho(d^*) = \rho ;$$

the deployment d^* is given by

$$d^*(i) = -\frac{\log(\rho-i+1)}{\log q} \tag{2.7}$$

for $i = 1, \ldots, \rho$.

PROOF. In order to satisfy

$$p(d^*,i) = \frac{1}{\rho(d^*)} = \frac{1}{\rho}$$

for each i, we must have, first of all,

$$\frac{1}{\rho} = p(d^*,1) = q^{d^*(1)} ,$$

which is equivalent to

$$\begin{aligned} d^*(1) \log q &= \log \rho^{-1} \\ &= -\log \rho , \end{aligned}$$

or

$$d^*(1) = \frac{-\log \rho}{\log q} ,$$

which is expression (2.7) for $i = 1$. Suppose now that (2.7) holds for $i = 1, \ldots, k-1$; then

$$\frac{1}{\rho} = p(d^*,k)$$

$$= \left[\prod_{\ell=1}^{k-1} \left(1-q^{d^*(\ell)}\right)\right] q^{d^*(k)}$$

$$= \left[\prod_{\ell=1}^{k-1} \left(1 - \frac{1}{\rho-\ell+1}\right)\right] q^{d^*(k)}$$

(by the induction hypothesis)

$$= \left[\prod_{\ell=1}^{k-1} \left(\frac{\rho-\ell}{\rho-\ell+1}\right)\right] q^{d^*(k)}$$

$$= \frac{\rho - k + 1}{\rho} q^{d^*(k)} .$$

Therefore,

$$q^{d^*(k)} = \frac{1}{\rho - k + 1} ,$$

which is the same as

$$d^*(k) = \frac{-\log(\rho - k + 1)}{\log q},$$

verifying (2.7) for $i = k$. ∎

(2.8) COROLLARY. The Prim-Read interceptor requirement is given by

$$I(d^*) = -\log \rho!/\log q .$$

In order to aid the reader in understanding this requirement we present the following Table of Prim-Read interceptor requirements for the case $q = .50$.

Table 1. PRIM-READ INTERCEPTOR REQUIREMENTS

$\rho(d^*)$	$I(d^*)$
1	0
2	1.00
3	2.58
4	4.58
5	6.91
6	9.49
$\rho \to \infty$	$\frac{(\rho+1/2)\log \rho - \rho}{\log q}$

The asymptotic expression for $\rho \to \infty$ is obtained using Stirling's approximation for $\rho!$ [16, p.194] and drops a constant term of $\frac{1}{2} \log 2\pi$.

Because, as evidenced by Table 1, the Prim-Read deployment need not require integral numbers of interceptors, the results there may be slightly misleading as well as impossible to implement physically. The following example illustrates.

(2.9) EXAMPLE. Let $q = .50$. For $\rho = 5$ we then have

$$d^*(1) = 2.32$$
$$d^*(2) = 2.00$$
$$d^*(3) = 1.58$$
$$d^*(4) = 1.00$$
$$d^*(5) = 0 .$$

The integral interceptor requirement is therefore

$$I_1(d^*) = 8 .$$

In most of our discussion below, we simply ignore such "discreteness" problems, despite their obvious physical importance. For values of q (the penetration probability) and ρ that are moderately large, the errors are probably not substantial.

The rationale for choice of a deployment satisfying (2.5) is not entirely clear, although the results presented in Chapters III through VI provide some clarification. One can argue heuristically that (2.5) might be desirable because it imposes an ineffectiveness of choice on the attacking side. That is, if the Prim-Read deployment is implemented and the attacking side is to choose the number of weapons with which to attack the target, no one weapon has greater marginal effect than any other. The precise result of this "ineffectiveness of choice" is described by Theorem (4.10).

For one target, other than choosing the deployment and thereby defining the target price, the defending side has no choices to make. If there are many targets to be defended, more choices are required, but before considering that situation, we consider one additional aspect of the single target case.

In the previous discussion it was assumed that the defending side first chose a deployment, based on which it computed an interceptor requirement. In reality, however, the available number of interceptors may be fixed in advance, say at I_0. The defending side could then implement a Prim-Read deployment d^* as follows: Let ρ be the largest integer μ for which

$$(2.10) \qquad - \log \mu!/\log q \leq I_0 .$$

Then implement the Prim-Read deployment with price ρ. If equality does not hold for $\mu = \rho$ in (2.10), then the above strategy does not utilize all available resources. The remaining interceptors, however, can clearly be distributed in such a manner that (2.5) remains nearly satisfied. In light of

Theorem (2.6) the defending side could, for example, make a choice such that

$$p(d^*,i) \leq \frac{1}{\rho} , \qquad 1 \leq i < \rho,$$

but

$$p(d^*,\rho) \geq \frac{1}{\rho} .$$

In this case, if the target were attacked at all and if the resources were available, the attacking side would expend the full price necessary to destroy the target in order to obtain the marginal return $p(d^*,\rho)$ of the final weapon, which exceeds those of the earlier weapons. See also Theorem (3.1), which further considers the effect of increasing marginal returns.

B. MULTIPLE TARGET CASE

We now consider the more important, multiple target case. Suppose that there are targets numbered 1, ..., T with integer values v(1), ..., v(T), respectively, the latter according to some scalar measure of target value. An interceptor deployment is then specified by a matrix $d = (d(j,i))$, where

$d(j,i)$ = number of interceptors deployed at target j against the i^{th} attacking weapon (if there is one) directed at that target.

The *price vector* for a deployment d is the vector $\rho(d) = (\rho(d,1),...,\rho(d,T))$ defined by

$$\rho(d,j) = \min\{i:d(j,i) = 0\} .$$

Observe that while the deployment d defines the price vector $\rho(d)$, it nonetheless makes sense to speak of a deployment d for which $\rho(d)$ equals a prescribed vector ρ.

The uniformity property (2.5) used in the single target case can be extended to the multiple target case, up to a scaling parameter, in a manner which we now describe. Given a deployment d, let

$$p(d,j,i) = P\{\text{target } j \text{ destroyed by } i^{th} \text{ attacking weapon directed at it}\}$$

$$= \left[\prod_{\ell=1}^{i-1} (1-q^{d(j,\ell)})\right] q^{d(j,i)} ;$$

recall that the attrition structure is given by (1.2).

By analogy with Definition (2.4) we introduce the following terminology.

(2.11) DEFINITION. Let k be a positive integer. A deployment d* is said to be a *Prim-Read deployment with scaling factor* k provided that

(2.12) $$v(j)p(d^*,j,i) = \frac{1}{k}$$

for all $j = 1, \ldots, T$ and $i = 1, \ldots, \rho(d^*,j)$.

The content of (2.12) is that the expected target value destroyed by each attacking weapon is the same (namely, is equal to 1/k) up to the point at which all targets are destroyed. While the analogy of (2.12) to (2.5) is rather strong to begin with, it is strengthened by the following result.

(2.13) THEOREM. For each integer k there exists a unique Prim-Read deployment d* with scaling factor k. The deployment d* is given by

(2.14) $$d^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q}$$

for $j = 1, \ldots, T$ and $i = 1, \ldots, kv(j)$.

Before proving the Theorem, we take note of the following consequences of it.

(2.15) COROLLARY. If d^* is the Prim-Read deployment with scaling factor k then

$$\rho(d^*,j) = kv(j) \tag{2.16}$$

for $j = 1, \ldots, T$.

(2.17) COROLLARY. The Prim-Read interceptor requirement is

$$I(d^*) = -\sum_{j=1}^{T} \frac{\log([kv(j)]!)}{\log q} .$$

The expression (2.16) states that for a Prim-Read deployment d^*, target prices are proportional to target values, and this--as we shall see in more detail below--leads the attacking side to distribute its weapons among the targets in proportion to their respective values. However, as (2.14) shows, the defending side *does not* distribute its interceptor resources among the targets in direct proportion to their values (except in the special case when all targets have the same value).

We now prove Theorem (2.13).

PROOF of THEOREM (2.13). Consider some target j. In order that

$$v(j)p(d^*,j,1) = \frac{1}{k} ,$$

we must have

$$q^{d^*(j,1)} = \frac{1}{kv(j)} ,$$

which is equivalent to

$$d^*(j,1) = -\frac{\log kv(j)}{\log q} .$$

Proceeding as in the proof of Theorem (2.6), suppose that

$$d^*(j,\ell) = -\frac{\log(kv(j)-\ell+1)}{\log q}$$

for $\ell = 1, \ldots, i-1$. Then (2.12) implies that

$$\begin{aligned}\frac{1}{k} &= v(j)p(d^*,j,i)\\ &= v(j)\left[\prod_{\ell=1}^{i-1} (1-q^{d^*(j,\ell)})\right]q^{d^*(j,i)}\\ &= v(j)\,\frac{kv(j)-i+1}{kv(j)}q^{d^*(j,i)} ,\end{aligned}$$

which in turn implies that

$$q^{d^*(j,i)} = \frac{1}{kv(j)-i+1}$$

and hence that

$$d^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q} .$$

Since j was arbitrary, the proof is complete. ∥

REMARKS. There are two distinct ways in which one can envision the defending side choosing the scaling factor k, given that it has chosen to have a deployment of Prim-Read form. First, its interceptor resources may be fixed at some level, say I_0. The maximum parameter k such that

$$I(d^*) \leq I_0$$

would be chosen and the remaining interceptors either diverted to other uses or allocated among the targets according to some (more or less arbitrary) scheme. See also Chapter V.

Second, it might be the case that the defending side knows the maximum number of attacking weapons that could ever be launched against the targets in question. For example, the number of attacking weapons might be fixed by negotiated agreement between the two sides, as is done with strategic nuclear weapons. The defending side might then wish to construct a deployment sufficiently strong that, in order to destroy all the targets, the attacking side would have to commit its entire stock of weapons. This would lead to the defending side's choosing the minimal value of k for which the total of the target prices exceeds the number of weapons available to the attacking side. An interceptor requirement could then be determined.

We observe once more that (2.12) above suffices to imply that target prices be proportional to target values. When $k = 1$, targets of unit value are left undefended, but in reality the defending side will be forced to leave many targets--usually of differing values and some of values greater than one--undefended. In Chapter V we consider in detail this important problem of undefended targets. Chapters III and IV are devoted to exploring mathematical optimality properties of the Prim-Read deployments and to understanding the practical implications of such properties.

C. CHAPTER SUMMARY

In this Chapter we have defined Prim-Read deployments as satisfying certain uniformity properties, namely (2.5) in the single target case and (2.12) in the multiple target case. The former equalizes the probabilities of target destruction for all attacking weapons against which there is a defense with that of the first against which there is no defense. The latter equalizes expected target values destroyed for all attacking weapons that are defended against and those of the first attacking weapon at each target that is not defended against. The important concepts of the price of a single target defense and the price vector of a multiple target defense are also introduced in this Chapter.

The main results of this Chapter are the following:

1) Theorem (2.6), which gives the form of a Prim-Read deployment d^* in the single target case:

$$d^*(i) = -\frac{\log(\rho(d^*)-i+1)}{\log q}, \qquad i=1,\ldots,\rho(d^*).$$

The associated interceptor requirement is calculated in Corollary (2.8) and found to be

$$I(d^*) = -\frac{\log(\rho(d^*)!)}{\log q}.$$

2) Theorem (2.13), which gives the form of Prim-Read deployments for the multiple target case. If k is the scaling factor, so that the equalization condition is

$$v(j)p(d^*,j,i) = \frac{1}{k}, \qquad j=1,\ldots,T; \quad i=1,\ldots,\rho(d^*,j),$$

then

$$d^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q},$$

also for $j = 1, \ldots, T$ and $i = 1, \ldots, \rho(d^*,j) = kv(j)$. The interceptor requirement is

$$I(d^*) = -\frac{1}{\log q}\sum_{j=1}^{T} \log[(kv(j))!].$$

These results are Corollaries (2.15) and (2.17), respectively.

Chapter III

OPTIMALITY PROPERTIES: SINGLE TARGET CASE

A. INTRODUCTION

This Chapter considers some ways in which the Prim-Read deployment represents an optimal use of the interceptor resources of the defending side. To motivate the more complicated results concerning the multiple target case (presented in the next two Chapters) we first present here some results concerning the single target case. In both cases (as the reader will observe throughout this Chapter and the next two) one must formulate optimality questions with care and in the correct context. The results of this Chapter are of relatively less intrinsic interest than those of Chapter IV and, especially, Chapter V, and might perhaps best be viewed as a means of motivation and introduction to the ideas and mathematics used in the sequel.

Let the defending side be protecting a single target. Throughout this Chapter we use the following notation: if d is a deployment,

a) $\rho(d) = \min\{i : d(i) = 0\}$ is the price imposed by d. Without loss of generality (because attacking weapons are perfect) we assume that $d(\ell) = 0$ for all $\ell \geq \rho(d)$.

b) $p(d,i)$ is the probability that the target is destroyed by the i^{th} attacking weapon directed at it; cf. (2.3).

c) $P(d,i) = \sum_{k=1}^{i} p(d,k)$ is the probability that the target is destroyed by one of the first i attacking weapons directed at it.

d) $I(d) = \sum_{i=1}^{\rho(d)-1} d(i)$ is the interceptor requirement.

We use d^* to denote Prim-Read deployments.

For ρ a positive integer, $D(\rho)$ denotes the set of deployments d for which $\rho(d) = \rho$ and if, in addition, α is a positive number, $D(\rho,\alpha)$ denotes the set of those $d \in D(\rho)$ for which $I(d) \leq \alpha$. Given ρ we denote by $\alpha^*(\rho)$ the interceptor requirement of the Prim-Read deployment $d^* \in D(\rho)$, namely

$$\alpha^*(\rho) = - \log \rho! / \log q \ .$$

Finally, let $\tilde{D}(\rho) = D(\rho,\alpha^*(\rho))$. Deployments $d \in \tilde{D}(\rho)$ require at most as many interceptors as the Prim-Read deployment d^*.

B. RESULTS

We now proceed to our first optimality result.

(3.1) THEOREM. If $d \in \tilde{D}(\rho)$ and $P(d,\cdot)$ is convex on $[0,\rho]$, then $d = d^*$.

REMARK. Convexity of P means that for each $i = 2,\ldots,\rho - 1$,

$$P(d,i) \leq \frac{1}{2} [P(d,i-1) + P(d,i+1)] \ ,$$

which is equivalent to

(3.2) $$P(d,i) - P(d,i-1) \leq P(d,i+1) - P(d,i) \ .$$

It follows at once from (2.5) that the Prim-Read deployment d^* satisfies (3.2), in fact with equality for each i.

Implications of (3.2) for the attacking and defending sides, and its desirability for the defending side, will be discussed following the proof of the Theorem.

PROOF of THEOREM (3.1). For a given deployment $d \in \tilde{D}(\rho)$ we have

(3.3) $$P(d,1) = q^{d(1)} ;$$

for $i = 2, \ldots, \rho$ we have

(3.4) $$\begin{aligned} P(d,i) - P(d,i-1) &= p(d,i) \\ &= \left[\prod_{\ell=1}^{i-1} \left(1-q^{d(\ell)}\right)\right] q^{d(i)} , \end{aligned}$$

The constraint

$$\sum_{i=1}^{\rho-1} d(i) \le I(d^*)$$

is equivalent to

(3.5) $$\begin{aligned} \prod_{i=1}^{\rho-1} q^{d(i)} &\ge q^{I(d^*)} \\ &= q^{-\log \rho!/\log q} \\ &= \frac{1}{\rho!} . \end{aligned}$$

Suppose that $d \in \tilde{D}(\rho)$ and that $P(d,\cdot)$ is convex. For $i = \rho - 1$, the inequality (3.2) becomes

$$P(d,\rho) - P(d,\rho-1) \ge P(d,\rho-1) - P(d,\rho-2) ,$$

which by (3.4) is equivalent to

(3.6) $$\prod_{i=1}^{\rho-1} (1-q^{d(i)}) \ge \left[\prod_{i=1}^{\rho-2} \left(1-q^{d(i)}\right)\right] q^{d(\rho-1)} ;$$

Under the assumption that $d(i) > 0$ for $i < \rho$, (3.6) reduces to

$$(1-q^{d(\rho-1)}) \ge q^{d(\rho-1)} ,$$

or

(3.7) $$q^{d(\rho-1)} \le \frac{1}{2} .$$

Similarly, the requirement

$$P(d,\rho-1) - P(d,\rho-2) \geq P(d,\rho-2) - P(d,\rho-3)$$

is equivalent to

$$\left[\prod_{i=1}^{\rho-2}\left(1-q^{d(i)}\right)\right] q^{d(\rho-1)} \geq \left[\prod_{i=1}^{\rho-3}\left(1-q^{d(i)}\right)\right] q^{d(\rho-2)},$$

and hence to

$$(1-q^{d(\rho-2)})q^{d(\rho-1)} \geq q^{d(\rho-2)} .$$

By (3.7), this becomes

$$q^{d(\rho-2)} \leq \frac{1}{2}(1-q^{d(\rho-2)}) ,$$

which is the same as

$$q^{d(\rho-2)} \leq \frac{1}{3} .$$

Proceeding inductively, we infer that

$$q^{d(i)} \leq (\rho-i+1)^{-1} \tag{3.8}$$

for all i whenever $P(d,\cdot)$ is convex. The requirements (3.5) and (3.8) together imply that

$$q^{d(i)} = \frac{1}{\rho - i + 1}$$

for $i = 1, \ldots, \rho - 1$, which completes the proof. ▯

REMARKS. 1) The convexity property (3.2) is that of "increasing returns to scale." It ensures that a rational attacking side with at least ρ weapons available will--provided it attacks the target at all--expend the full price ρ necessary to destroy the target. This is because each attacking weapon, when (3.2) holds, is more valuable to the attacking side, in terms of expected target value destroyed, than the previous one. Therefore, if an i^{th} attacking weapon is launched, it is irrational not to launch an $(i+1)^{st}$. Theorem (3.1) shows that no deployment other than the Prim-Read can use the same number of interceptors and still produce a convex payoff function on $[0,\rho]$. When the payoff function is convex, and if the attacking side seeks to maximize target value destroyed per attacking weapon committed, it is forced to expend the full resources needed to destroy the target.

2) By using the Prim-Read deployment the defending side reduces to two the number of actions that a rational attacking side would choose, namely, "do not attack the target at all" and "attack the target and pay the full price," and destroy it. Thus the defending side, even though it must choose first, is able to limit the choices available to the attacking side. The defending side is thereby able to exert control over the attacking side's expenditure of resources. If the attacking side seeks to maximize either expected target value destroyed or expected target value destroyed per attacking weapon committed, it will choose to destroy the target.

3) In the preceding discussion the price ρ was taken to be exogenously determined. If a limited number of interceptors is available, a suitable price may be calculated using (2.10).

Another manifestation of the uniformity of the Prim-Read deployment is given in our next Theorem, which deals with a situation in which the defending side wishes to minimize the number of interceptors required while controlling the maximum cumulative damage per attacking weapon. Recall that $D(\rho)$ is the set of all deployments with price ρ.

(3.9) THEOREM. Let ρ be an integer. Then the Prim-Read deployment d^* uniquely solves the optimization problem

(3.10) $$\text{minimize } I(d)$$

$$\text{s.t.} \quad d \in D(\rho)$$

$$\max_{1 \le r \le \rho} \frac{P(d,r)}{r} \le \frac{1}{\rho} .$$

PROOF. Since (2.7) implies that $P(d^{*},r) = r/\rho$ for $r = 1, \ldots, \rho$ the Prim-Read deployment d^{*} is a feasible solution to (3.10). Optimality is suggested, although not proved of course, by the fact that all constraints are satisfied as equalities.

Suppose now that $\tilde{d}$ is an optimal solution to (3.10). Then first of all we must have

$$q^{\tilde{d}(1)} = P(\tilde{d},1) \le \frac{1}{\rho} ,$$

which we write in the form

$$\rho q^{\tilde{d}(1)} \le 1 .$$

Using this inequality, the fact that $q^{d^{*}(1)} = 1/\rho$ and the constraint

$$1 - (1-q^{\tilde{d}(1)})(1-q^{\tilde{d}(2)}) = P(\tilde{d},2) \le 2/\rho$$

we see that

$$q^{\tilde{d}(2)} \le \frac{2/\rho - q^{\tilde{d}(1)}}{(1-q^{\tilde{d}(1)})} \le \frac{2/\rho - q^{\tilde{d}(1)}}{(1-q^{d^{*}(1)})} = \frac{2 - \rho q^{\tilde{d}(1)}}{\rho - 1} ,$$

which we write as

$$(\rho-1)q^{\tilde{d}(2)} + \rho q^{\tilde{d}(1)} \leq 2 \ .$$

By an inductive continuation of the procedure above we can conclude that for each $i = 1, \ldots, \rho - 1$ the inequality

$$\sum_{j=1}^{i} (\rho-j+1)q^{\tilde{d}(j)} \leq i$$

is valid. Note that so far optimality of $\tilde{d}$ has not been used.

If $\tilde{d}$ is optimal, then in addition to the preceding inequalities, $I(\tilde{d}) \leq I(d^*)$, which is the same as

$$\prod_{i=1}^{\rho-1} q^{\tilde{d}(i)} \geq \prod_{i=1}^{\rho-1} q^{d^*(i)} \ .$$

Together, the previous $\rho - 1$ linear inequalities and this one nonlinear inequality imply that $\tilde{d} = d^*$. For an illustration, see the proof of Theorem (3.11) below. ▯

REMARKS. 1) Theorems (3.1) and (3.9) are evidently rather closely related to one another; they are nearly, but not quite, dual optimization problems. See also Theorem (3.11) below.

2) If one supposes that the attacking side, once the defensive deployment is chosen and implemented, is able to discern that deployment and then choose a number of attacking weapons that maximizes

$$\frac{P(d,r)}{r}$$

over $\{1,\ldots,\rho\}$, then Theorem (3.9) states that by employing the Prim-Read deployment, if there is a constraint $1/\rho$ on the per-attacking weapon payoff for the optimal choice of the attacking side, the defending side minimizes the requirement for interceptors.

3) Since

$$P(d^*,r)/r = 1/\rho$$

for all r, the Prim-Read deployment does not, according to the attacking side's criterion of maximizing $P(d,r)/r$, limit the number of actions that are then feasible (indeed, optimal) for the attacking side; it does, however, ensure that all actions have the same effect. This is yet a different way in which the Prim-Read deployment limits the decision-making ability of the attacking side: in this situation no alternatives are precluded, but all actions lead to the same outcome.

4) The content of Theorem (3.9) is that--for a given target price--the Prim-Read deployment minimizes the defending side's interceptor requirement subject to a constraint on the maximum (cumulative) damage per attacking weapon. Theorem (3.11) below states that, in effect, the constraint and the objective function in (3.9) can be interchanged. That is, the Prim-Read deployment uniquely minimizes the maximum cumulative damage per attacking weapon, subject to an upper bound on the number of available interceptors. This result should be compared with the various results in Chapters IV, V and VI, which treat min-max optimality properties in the multiple target case.

(3.11) THEOREM. The Prim Read deployment d^* is the unique solution to the optimization problem

(3.12)
$$\text{minimize} \max_{1 \leq r \leq \rho} \frac{P(d,r)}{r}$$
$$\text{s.t.} \quad d \in \tilde{D}(\rho) .$$

PROOF. Feasibility of d^* is obvious. If $d \in \tilde{D}(\rho)$ then

$$\frac{P(d,\rho)}{\rho} = \frac{1}{\rho} ,$$

which obviously implies that

$$\max_{1 \le r \le \rho} \frac{P(d,r)}{r} \ge \frac{1}{\rho}$$

$$= \max_{1 \le r \le \rho} \frac{P(d^*,r)}{r} .$$

This shows that d^* is an optimal solution to (3.12) and that if $\tilde{d}$ is any other optimal solution, then

$$\max_{1 \le r \le \rho} \frac{P(\tilde{d},r)}{r} = \frac{1}{\rho} . \tag{3.13}$$

To show that d^* is in fact the only optimal solution to (3.12), let $\tilde{d}$ be any optimal solution and define

$$x_i = q^{\tilde{d}(i)} , \qquad i = 1, \ldots, \rho - 1 .$$

The equality (3.13) implies the inequality

$$\frac{1}{\rho} \ge P(\tilde{d},1)$$

$$= q^{\tilde{d}(1)} = x_1 ,$$

which we rewrite as

$$\rho x_1 \le 1. \tag{3.14}$$

The inequality

$$\frac{1}{\rho} \ge \frac{P(\tilde{d},2)}{2}$$

$$= \frac{1}{2}\left[q^{\tilde{d}(1)} + (1-q^{\tilde{d}(1)})q^{\tilde{d}(2)}\right]$$

$$= \frac{1}{2}\left[x_1 + (1-x_1)x_2\right] ,$$

which also results from (3.13), can be rewritten--with the aid of (3.14)-- as

$$\frac{2}{\rho} \geq x_1 + \left(\frac{\rho-1}{\rho}\right)x_2 ,$$

and hence as

$$\rho x_1 + (\rho-1)x_2 \leq 2 .$$

By continuing this procedure we obtain the following set of (linear) inequalities (one of which must hold as equality):

$$\begin{aligned} \rho x_1 &\leq 1 \\ \rho x_1 + (\rho-1)x_2 &\leq 2 \\ \rho x_1 + (\rho-1)x_2 + (\rho-2)x_3 &\leq 3 \\ &\vdots \\ \rho x_1 + (\rho-1)x_2 + \ldots + 2x_{\rho-1} &\leq \rho - 1 . \end{aligned}$$

Together with the nonlinear inequality

$$\prod_{i=1}^{\rho-1} x_i \geq 1/\rho! ,$$

which is but a rewritten form of the constraint $I(\tilde{d}) \leq I(d^*)$, the preceding inequalities and optimality of $\tilde{d}$ imply that

$$x_i = \frac{1}{\rho-i+1}$$

for each i, i.e., that $\tilde{d} = d^*$.

To illustrate (rather than overwhelm the reader with technical details) let us consider the case $\rho = 3$. The relevant inequalities are then

$$3x_1 \leq 1$$

$$3x_1 + 2x_2 \leq 2$$

and

$$x_1 x_2 \geq \frac{1}{6} ;$$

the only point (x_1, x_2) satisfying these three inequalities is $(1/3, 1/2)$, which does indeed correspond to $\tilde{d} = d^*$. The higher dimensional cases are notationally more complicated but conceptually the same. □

It is important to realize that the Prim-Read deployment is not a universal solution to every optimization problem involved with interceptor defense of point targets. The following discussion shows, however, that for some problems to which it is not an optimal solution, it is still a reasonably good choice.

(3.15) EXAMPLE. The optimization problem we wish to treat in this case is the following: fix the attacking side's resources at some integer level ρ and the defending side's interceptor resources at

$$I = -\log \rho!/\log q.$$

These are the levels of resource expenditure for the two sides when the defending side implements a Prim-Read deployment with price ρ. The optimization problem to be considered is

$$\text{(3.16)} \qquad \text{minimize} \max_a \left[1 - \prod_{i=1}^{a} (1 - q^{d(i)})\right] ,$$

where the maximum is over $a \leq \rho$ and the minimum is over all deployments d for which $I(d) \leq I$. The objective function

$$P(d,a) = 1 - \prod_{i=1}^{a} (1 - q^{d(i)})$$

is the probability that the target is destroyed (and is hence directly proportional to the expected target value destroyed) when the defensive deployment is d and the attacking side commits a attacking weapons. It is assumed in (3.16) that ρ and I are known to both sides.

This optimization problem is certainly a natural problem that one might pose in a target defense problem: the defending side wants to minimize, and the attacking side to maximize, the expected target value destroyed. Formulation of (3.16) as a min-max problem rather than a (simultaneous move) game represents our assumption that the attacking side can discern the defending side's deployment before the attack occurs. The principal difference between the problem (3.15) and those discussed above in this Chapter (especially in Theorems (3.9) and (3.11) is that the latter account for the level of expenditure of attacking weapons by replacing P(d,a) in (3.16) by P(d,a)/a, the expected target value destroyed per attacking weapon expended.

What we will do is to compare the solution to (3.16) with the Prim-Read deployment d^* with price ρ which solves (3.12) above.

The solution to (3.16) may be obtained by the methods of calculus once some preliminary analysis is used to simplify the problem. For any d,

$$\max_{a \leq \rho} P(d,a) = P(d,\rho) ,$$

so (3.16) reduces to the minimization problem

$$\text{(3.17)} \qquad \text{minimize } 1 - \prod_{i=1}^{\rho} (1-q^{d(i)})$$

$$\text{s.t.} \quad I(d) \leq I .$$

To solve (3.17) we may restrict attention to those d for which $d(i) = 0$ for $i > \rho$; there is no point in deploying interceptors against nonexistent attacking weapons. Note that since $d^*(\rho) = 0$ we have

$$P(d^*,\rho) = 1 ,$$

whereas if d is any feasible point for (3.17) such that $d(i) > 0$ for $i = 1, \ldots, \rho$, then

$$P(d,\rho) < 1 .$$

Consequently, the Prim-Read deployment is *not* an optimal solution to (3.17).

To find the optimal solution to (3.17) one may use the classical methods of calculus (either Lagrange multipliers or substitution of $I - \sum_{i=1}^{\rho-1} d(i)$ for $d(\rho)$). The resultant solution is the uniform allocation $\tilde{d}$ given by

$$\text{(3.18)} \qquad \tilde{d}(i) = \frac{I}{\rho} , \qquad i = 1, \ldots, \rho .$$

We observe that at the optimum, the objective function value for the problem (3.17) is

$$\text{(3.19)} \qquad P(\tilde{d},\rho) = 1 - \left[1 - \left(\frac{1}{\rho!}\right)^{1/\rho}\right]^{\rho} .$$

We now wish to compare $P(d^*,\rho)$ with $P(\tilde{d},\rho)$ in order to understand to what extent the Prim-Read deployment d^* fails to be an optimal solution to (3.16). Since, as previously noted, $P(d^*,\rho) = 1$, to investigate the ratio $P(\tilde{d},\rho)/P(d^*,\rho)$ it suffices to consider only the behavior of $P(\tilde{d},\rho)$ as given in (3.19). The behavior we investigate is that of $P(\tilde{d},\rho)$ as $\rho \to \infty$. Using Stirling's approximation

$$\rho! \sim \sqrt{2\pi}\, \rho^{\rho+1/2} e^{-\rho}$$

and the fact that

$$\lim_{\rho\to\infty} \rho^{1/\rho} = 1 \; ,$$

cf. [16, p.57], we conclude that

$$\text{(3.20)} \qquad \lim_{\rho\to\infty} P(\tilde{d},\rho) = \lim_{\rho\to\infty} 1 - (1 - \frac{e}{\rho})^{\rho}$$

$$= 1 - e^{-e}$$

$$\simeq .9340 \; .$$

The practical significance of (3.20) is that for large values of ρ, the Prim-Read deployment is 93 percent optimal in (3.16) and optimal in (3.12). In fact, the convergence in (3.20) is extremely fast; $P(\tilde{d},\rho) \geq .92$ once $\rho \geq 20$.

On the other hand, let us consider how the solution $\tilde{d}$ to (3.16) fares as a possible candidate in the problem, (3.12), to which the Prim-Read deployment d^* is the optimal solution. For $\tilde{d}$ as given by (3.18), the cumulative payoff function $P(\tilde{d},\cdot)$ to the attacking side is easily seen to be strictly concave on $[0,\rho]$, i.e., is of the form of the payoff function given in Figure 4a (Chapter V). Consequently

$$\max_{r\leq\rho} \frac{P(\tilde{d},r)}{r} = P(\tilde{d},1)$$

$$= 1 - \left[1-\left(\frac{1}{\rho!}\right)^{1/\rho}\right]$$

$$= \left(\frac{1}{\rho!}\right)^{1/\rho} \; .$$

Once again using Stirling's approximation, one can see that

$$\max_{r \le \rho} \frac{P(\tilde{d},r)}{r} \sim \frac{e}{\rho} . \tag{3.21}$$

Since

$$\max_{r \le \rho} \frac{P(d^*,r)}{r} = \frac{1}{\rho} ,$$

the inference to be drawn from (3.21) is that the uniform deployment $\tilde{d}$ is nonoptimal for the problem by a factor of $1 - \frac{1}{e} \simeq .63$. This should be compared with the factor $e^{-e} \simeq .07$, by which the Prim-Read deployment fails to be optimal for the problem (3.16).

C. CHAPTER SUMMARY

This Chapter is devoted to development of optimality properties of Prim-Read deployments in the single target case. As previously explained, this case is of less interest in its own right than as a motivation for and simplification of the multiple target case. The results derived here are, for the most part, qualitatively similar to the results appearing in Chapter IV, but are mathematically less complicated and imposing. A reader who wishes to understand the flavor of the derivations given in this paper could read those in this Chapter and omit those in the following Chapters.

The main mathematical results presented in Chapter III are the following:

1) Theorem (3.1), which states that if $d \in \tilde{D}(\rho)$ and the payoff function $P(d,\cdot)$ is convex on $[0,\rho]$, then d is the Prim-Read deployment d^*.

2) Theorem (3.9), which asserts that for each ρ the Prim-Read deployment d^* is the unique solution to the optimization problem

$$\begin{aligned} &\text{minimize } I(d) \\ &\text{s.t. } d \in D(\rho) \\ &\max_{r \leq \rho} \frac{P(d,r)}{r} \leq \frac{1}{\rho} . \end{aligned}$$

That is, subject to the upper bound $1/\rho$ on the cumulative probability of target destruction per attacking weapon committed, the Prim-Read deployment d^* uniquely requires the fewest interceptors.

3) Theorem (3.11), which shows that for each ρ the Prim-Read deployment d^* is the unique solution to the optimization problem

$$\begin{aligned} &\text{minimize } \max_{r \leq \rho} \frac{P(d,r)}{r} \\ &\text{s.t. } d \in \tilde{D}(\rho) . \end{aligned}$$

That is, among deployments d with price ρ and interceptor requirement at most that of the Prim-Read deployment d^*, the deployment d^* uniquely minimizes the maximum probability of target destruction per attacking weapon committed.

4) Example (3.15), which contains several conclusions of interest. First, it is shown that the Prim-Read deployment is *not* a solution to the optimization problem

$$\begin{aligned} &\text{minimize } P(d,\rho) \\ &\text{s.t. } I(d) \leq I(d^*) \\ &\quad\;\; \rho(d) \leq \rho + 1 \end{aligned}$$

and that the solution to this problem is the uniform deployment $\tilde{d}$ defined in expression (3.18). However, it is shown that d^* is 93 percent optimal for this latter problem--in which the objective function is the probability of target destruction rather than probability of target destruction per attacking weapon committed -- whereas the uniform deployment $\tilde{d}$ is only 37 percent optimal for the optimization problem of Theorem (3.11) and result 3) presented above. This is a useful robustness property of Prim-Read deployments.

Chapter IV

OPTIMALITY PROPERTIES: MULTIPLE TARGET CASE

A. INTRODUCTION

The objective of this Chapter is to present analogues, for the multiple target case, of the optimality results derived in Chapter III. In this Chapter we concentrate on the situation where all targets, except possibly those of unit value, are defended. The more important situation in which many targets of differing values must be left undefended is treated in detail in Chapter V.

We first derive analogues of Theorems (3.1), (3.9) and (3.11) for the multiple target case. As in Chapter II we assume that there are targets 1, ..., T of positive, integral values v(1), ..., v(T). An interceptor deployment is a matrix $d = (d(j,i))$, where $d(j,i)$ is the number of interceptors deployed at target j to attempt to intercept the i^{th} attacking weapon directed there. The deployment d specifies a price vector $\rho(d,\cdot)$ defined by

$$\rho(d,j) = \min\{i: d(j,i) = 0\} \quad ;$$

without loss of generality we assume that $d(j,\ell) = 0$ for each j and all $\ell \geq \rho(d,j)$.

For each integer k we denote by d_k^* the Prim-Read deployment with scaling factor k, given by Theorem (2.13) as

$$d_k^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q}, \qquad j = 1, \ldots, T; \quad i = 1, \ldots, kv(j) .$$

The associated price vector and interceptor requirement are

$$\rho(d_k^*, j) = kv(j)$$

and

$$I(d_k^*) = -\frac{1}{\log q} \sum_{j=1}^{T} \log[(kv(j))!] ,$$

respectively, as given by Corollaries (2.15) and (2.17).

Throughout what follows, a caret over a vector denotes the sum of its components; for example,

$$\hat{v} = \sum_{j=1}^{T} v(j)$$

is the total value of the targets.

Against a deployment d, the attacking side can be certain of destroying all of the targets by committing

$$\hat{\rho}(d) = \sum_{j=1}^{T} \rho(d, j)$$

properly allocated weapons. In what follows, therefore, it seems more reasonable that one should fix the total of target prices (i.e., the commitment of offensive resources required to destroy all of the targets) rather than fix the individual target prices (which, in effect, was done in Chapter III). Consequently, for each k, let $\hat{D}_k$ denote the set of deployments d for which

$$\hat{\rho}(d) = \hat{\rho}(d_k^*)$$

and

$$I(d) \leq I(d_k^*) .$$

Deployments in $\hat{D}_k$ impose the same commitment of attacking weapons to destroy all the targets as does the Prim-Read deployment d_k^* and require no more interceptors.

Given a deployment d, define

$$(4.1) \qquad V(d,i) = \max_a \sum_{j=1}^{T} v(j) P\{\text{target } j \text{ destroyed}\} ,$$

where the maximum is taken over all allocations $a = (a(1),\ldots, a(T))$, of attacking weapons among the targets such that

$$\hat{a} = \sum_{j=1}^{T} a(j) \le i .$$

This is the payoff function discussed in Chapter I,A and Chapter V,A. More specifically,

$$(4.2) \qquad V(d,i) = \max_a \sum_{j=1}^{T} v(j) \left[1 - \prod_{\ell=1}^{a(j)} \left(1-q^{d(j,\ell)}\right)\right] ,$$

and is the maximum expected target value that can be destroyed by i attacking weapons. When we treat optimization problems whose objective functions involve V, the attacking side is assumed to optimize its allocation of weapons among targets, based on knowledge of the deployment d. As noted in Chapter I, we suppose that either the attacking side is in fact able to so optimize or the defending side is guarding against this particular possibility.

B. OPTIMALITY RESULTS

The first result of this Chapter is the following, which is analogous to Theorem (3.1). We emphasize that we fix only the total of the target prices and not, which could be less reasonable, the individual target prices.

(4.3) THEOREM. Let $k \geq 1$ be fixed. If $d \in \hat{D}_k$ and $V(d,\cdot)$ is convex on $[0,\hat{\rho}(d_k^*)]$, then $d = d_k^*$.

PROOF. From Theorem (2.13) we have that

$$V(d_k^*,i) = i \frac{\sum_{j=1}^{T} v(j)}{\sum_{j=1}^{T} \rho(d_k^*,j)}$$

$$= i \frac{\hat{v}}{\hat{\rho}(d_k^*)}$$

$$= \frac{i}{k}$$

for $1 \leq i \leq \hat{\rho}(d_k^*)$; that is, $V(d_k^*,\cdot)$ is linear and, in particular, convex. Here $\hat{v} = \sum v(j)$ is the total of the target values.

Suppose now that $d \in \hat{D}_k$ and that $V(d,\cdot)$ is convex. We shall first show that

(4.4) $$\rho(d,j) = \rho(d_k^*,j)$$

for each $j = 1, \ldots, T$. If (4.4) fails, then since $d \in \hat{D}_k$, we must have $\rho(d,j_0) < \rho(d_k^*,j_0)$ for some j_0, which would imply that

$$V(d,\rho(d,j_0)) \geq v(j_0)$$

$$= \frac{\rho(d_k^*,j_0)}{k} .$$

On the other hand, convexity of $V(d,\cdot)$, together with the fact that $V(d,\hat{\rho}(d)) = \hat{v}$, implies that

$$V(d,\rho(d,j_0)) \leq \frac{\rho(d,j_0)}{k}$$

$$< \frac{\rho(d_k^*,j_0)}{k} .$$

Since the two preceding expressions are contradictory, (4.4) must hold.

Convexity of $V(d,\cdot)$ further implies that

$$\max_j v(j)q^{d(j,1)} = V(d,1) \leq \frac{1}{k}$$

for all j, which is the same as

$$(4.5) \qquad d(j,1) \geq d_k^*(j,1) .$$

However, strict inequality in (4.5) for some j would lead to a violation of the constraint that $d \in \hat{D}_k$ and therefore

$$d(j,1) = d_k^*(j,1)$$

for $j = 1, \ldots, T$.

The remainder of the proof follows by induction. $\square$.

A crucial property of the Prim-Read deployment in the multiple target case is that

$$(4.6) \qquad \rho(d_k^*,j)/v(j) = k$$

for all j, i.e., the price of a target is proportional to its value. As Theorem (2.13) shows, the defending side does not then distribute its interceptor resources among the targets in proportion to target values. What the defending side does is to distribute its resources among the targets in a manner that forces a rational attacking side to distribute its attacking weapons proportionally to target values. The point is an important one: being forced to choose first, the defending side chooses so as to limit the alternatives available to the attacking side when the latter behaves rationally. For the multiple target case, (4.6) is an important effect of the defending side's choice.

By analogy to Theorem (3.9) we obtain the following result, which states that in the multiple target case as well as the single target case, Prim-Read deployments (uniquely!) minimize the number of interceptors required by the defending side, subject to a constraint on the maximum average target destruction per attacking weapon committed.

(4.7) THEOREM. Let k be a positive integer and let $\hat{v}$ be the total value of all targets. Then the Prim-Read deployment d_k^* uniquely solves the optimization problem

$$\begin{aligned} &\text{minimize } I(d) \\ &\text{s.t.} \quad \hat{\rho}(d) = \hat{\rho}(d_k^*) \\ &\max_{1 \leq r \leq \hat{\rho}(d)} \frac{V(d,r)}{r} \leq \frac{1}{k} \, . \end{aligned}$$

PROOF. Recall also that $\hat{\rho}(d_k^*) = k\hat{v}$. It follows from earlier results that d_k^* is a feasible solution to this problem.

Consider now an optimal solution d'. Taking r = 1 in the second constraint gives the requirement

$$\max_{j=1,\ldots,T} v(j)q^{d'(j,1)} \leq \frac{1}{k} \, .$$

Hence for each j we must have

$$\begin{aligned} d'(j,1) &\geq -\frac{\log kv(j)}{\log q} \\ &= d_k^*(j,1) \, . \end{aligned}$$

Since the Prim-Read deployment is feasible, we infer that

$$d'(j,1) = d_k^*(j,1)$$

for j = 1, ..., T. The remainder of the proof follows by induction in the manner of the proof of Theorem (3.9). ∎

In the same way that Theorem (3.11) is related to Theorem (3.9), the next Theorem is a complementary version of Theorem (4.7).

(4.8) THEOREM. The Prim-Read deployment d_k^* is the unique solution to the optimization problem

$$\text{(4.9)} \qquad \text{minimize} \max_{1 \leq r \leq \hat{\rho}(d)} \frac{V(d,r)}{r}$$

$$\text{s.t.} \quad d \in \hat{D}_k .$$

We omit the proof.

A further property of Prim-Read deployments, which has no analogue in the single target case, is that Prim-Read deployments uniquely prevent the attacking side from being able to benefit from its being permitted to optimize its allocation of attacking weapons among targets based on knowledge of the deployment chosen by the defending side.

(4.10) THEOREM. Let k be a positive integer and for an integer-component vector $a = (a(1),...,a(T))$ let

$$\text{(4.11)} \qquad \tilde{V}(d,a) = \sum_{j=1}^{T} v(j)\left[1- \prod_{\ell=1}^{a(j)} \left(1-q^{d(j,\ell)}\right)\right] ,$$

which is the expected target value destroyed by the allocation a of attacking weapons. Then of all deployments $d \in \hat{D}_k$, the Prim-Read deployment d_k^* is uniquely characterized by the property that $\tilde{V}(d_k^*,\cdot)$ depends on the allocation a only through $\hat{a}$, provided that $a(j) \leq \rho(d,j)$ for each j.

PROOF. From (2.12) we have

$$\tilde{V}(d_k^*,a) = \frac{\hat{a}}{k}$$

for all a. Suppose now that $d \in \hat{D}_k$ satisfies the stated property. By considering $(1,0,\ldots,0)$, $(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ as successive choices of a, we conclude that

$$v(j)q^{d(j,1)} = v(j')q^{d(j',1)} \tag{4.12}$$

for all j and j'. Consider next two targets j and j', and the two allocations

$$a_1 = (0,\ldots,\underset{\uparrow j}{1},\ldots,\underset{\uparrow j'}{1},\ldots,0)$$

and

$$a_2 = (0,\ldots,\underset{\uparrow j}{2},\ldots,\underset{\uparrow j'}{0},\ldots,0)\ .$$

We must then have

$$v(j)q^{d(j,1)} + v(j')q^{d(j',1)} = v(j)\left[1-\prod_{\ell=1}^{2}(1-q^{d(j,\ell)})\right]. \tag{4.13}$$

Using (4.12) we may transform (4.13) to

$$q^{d(j,2)} = \frac{q^{d(j,1)}}{1-q^{d(j,1)}}\ ; \tag{4.14}$$

of course, a similar expression holds for j'.

We may continue this process of simplification, which becomes harder notationally but not conceptually, to conclude that

$$q^{d(j,i)} = \frac{q^{d(j,i-1)}}{1-q^{d(j,i-1)}} \tag{4.15}$$

for all j and i. Finally, (4.15), together with the constraint that $d \in \hat{D}_k$, implies that $d = d_k^*$. □

The interpretation of Theorem (4.10) is not that the Prim-Read deployment restricts the attacking side in its choice of actions, given a fixed stockpile of attacking weapons, but rather that the Prim-Read deployment limits the range of effects of those actions. In particular, the attacking side, when confronted by a Prim-Read deployment, cannot gain from being able to allocate its weapons among the targets. According to Theorem (4.10), in this case the target value destroyed depends only on how many attacking weapons are committed and not on how those weapons are distributed among the targets, provided only that no weapons are wasted by being directed at targets that are certain to have been destroyed.

C. ADDITIONAL PROPERTIES

As is also true for the single target case, in the multiple target case the Prim-Read deployment does not solve every optimization problem arising from the basic target defense model. The following Example parallels Example (3.15), to which the reader is referred for background and comparison.

(4.16) EXAMPLE. In this Example we consider an optimization problem that stands in the same relation to the problem (4.9) solved in Theorem (4.8) as does (3.16) to the problem (3.12) solved in Theorem (3.11). Specifically, that problem is the following: Let k be a fixed positive integer, fix the attacking side's resources at

$$A = k\hat{v} = \hat{\rho}(d_k^*) ,$$

and fix the defending side's interceptor resources as

$$I = I(d_k^*) = -\frac{1}{\log q} \sum_{j=1}^{T} \log[(kv(j))!] .$$

These are the resources expended by the attacking side (to destroy all of the targets) and required by the defending side, when the latter uses the Prim-Read deployment d_k^*. Then, consider the optimization problem

$$(4.17)\quad \text{minimize} \; \max_a \sum_{j=1}^{T} v(j)\left[1-\prod_{i=1}^{a(j)}\left(1-q^{d(j,i)}\right)\right]$$

$$\text{s.t} \quad \sum_{j=1}^{T} a(j) \le A$$

$$\sum_{i=1}^{\rho(d,j)} d(j,i) \le -\frac{\log[(kv(j))!]}{\log q}, \quad j = 1, \ldots, T .$$

In the problem (4.17) the defending side seeks a deployment d that:

1) At each target deploys at most as many interceptors as does the Prim-Read deployment d_k^*.

2) Minimizes the maximum expected target value destroyed when the attacking side is able to optimize its allocation $a = (a(1), \ldots, a(T))$ of attacking weapons among the targets, given knowledge of the defending side's deployment.

We observe that, in keeping with Assumptions (1.2a) and (1.2d), interceptors cannot be reassigned to different targets.

We have not been able to solve (4.17). However, consider by analogy with Example (3.15) the uniform deployment $\tilde{d}$ given for each j by

$$(4.18)\quad \tilde{d}(j,i) = \frac{I_j}{kv(j)}$$

for $i = 1, \ldots, kv(j)$, where

$$(4.19)\quad I_j = -\frac{\log[(kv(j))!]}{\log q}$$

is the total number of interceptors deployed at target j, as an alternative to the Prim-Read deployment d*. Against the deployment $\tilde{d}$, the optimal allocation of attacking weapons remains the same as it was against the Prim-Read deployment d*, namely

$$(4.20) \qquad \tilde{a}(j) = kv(j) , \qquad j = 1, \ldots, T.$$

The optimal value of the associated objective function is

$$(4.21) \qquad V(\tilde{d},\tilde{a}) = \sum_{j=1}^{T} v(j)\left(1 - \left[1 - \left(\frac{1}{(kv(j))!}\right)^{\frac{1}{kv(j)}}\right]^{kv(j)}\right) .$$

By comparison with (3.19) and the subsequent analysis we can see that as $k \to \infty$

$$(4.22) \qquad V(\tilde{d},\tilde{a}) \simeq (1-e^{-e})\hat{v}$$

$$\simeq .93\hat{v} .$$

Therefore, for large values of k we have

$$(4.23) \qquad \frac{V(\tilde{d},\tilde{a})}{V(d_k^*,\tilde{a})} \simeq .93 ,$$

so that for the problem (4.17), the Prim-Read deployment is demonstrably 93 percent as effective as the uniform deployment $\tilde{d}$, as it was in the single target case.

On the other hand, let us consider the problem (4.9) to which d_k^* is the optimal solution. The optimal objective function value is

$$\max_{i \leq A} \frac{V(d^*,\tilde{a},i)}{i} = \frac{1}{k} .$$

For the problem (4.9) the uniform deployment $\tilde{d}$ defined by (4.18) is far from optimal. In fact,

$$\max_{i \le A} \frac{V(\tilde{d},\tilde{a},i)}{i} = \max_j v(j) \left[\frac{1}{(kv(j))!}\right]^{1/kv(j)}$$

$$\sim \frac{e}{k}$$

as $k \to \infty$. Moreover, the maximum value is attained for $i = T$ with one attacking weapon directed at each target.

One may draw the same robustness conclusion as was valid in the single target case; cf. Example (3.15). ∎

REMARK. A more sensible alternative to the problem (4.17) is the problem

$$\text{(4.24)} \qquad \underset{d}{\text{minimize}} \ \max_a \sum_{j=1}^{T} v(j) \left[1 - \prod_{i=1}^{a(j)} \left(1 - q^{d(j,i)}\right)\right]$$

$$\text{s.t.} \quad \sum_{j=1}^{T} a(j) \le A$$

$$I(d) \le I \ .$$

In this problem, the defending side is constrained to the number of interceptors required for the Prim-Read deployment, but not on a target-by-target basis, as was the case in (4.17). Since the defense constructs the deployment from scratch, interceptors can initially be assigned to any target, even though they may be difficult to move from target to target thereafter. One can view (4.17), therefore, as being restricted to possible improvement of an existing Prim-Read deployment. Unfortunately, the problem (4.24) has been intractable.

We do wish to observe, however, that it is a consequence of Example (4.16) that $d_k^{\#}$ is *not* an optimal solution to the problem (4.24). As would be expected, $d_k^{\#}$ is relatively less optimal for the latter problem than for the problem (4.17), since the only difference between the two is that (4.24) has a larger constraint set. However, we have not been able to obtain lower bounds for this case such as obtained in Example (3.15).

D. CHAPTER SUMMARY

This Chapter contains a development of mathematical optimality properties of Prim-Read deployments in the multiple target case, but under the assumption that either all targets are defended or only targets of unit value are left undefended. To read the summary below, recall that for each integer $k \geq 1$, d_k^* is the Prim-Read deployment with scaling factor k (as given in (2.14)), $\hat{D}_k$ is the set of deployments with the same total of target prices as d_k^* and interceptor requirement not greater than that of d_k^*, and $V(d,i)$ is the target value destroyed by an attack consisting of i attacking weapons that is optimized against the deployment d. The principal results of this Chapter are then the following.

1) Theorem (4.3), which asserts that if k is fixed, if $d \in \hat{D}_k$, and if $V(d,\cdot)$ is convex on the interval $[0,\hat{\rho}(d_k^*)]$, then $d = d_k^*$. That is, of deployments with a prescribed total of target prices (i.e., a prescribed commitment of attacking weapons required to destroy all of the targets) and needing no more interceptors, only the Prim-Read deployment entails a convex payoff function, with the associated increasing returns to scale, for the attacking side. Faced with such a payoff function the attacking side would choose to expend its resources in order to destroy the defended targets and not divert those resources to other uses.

2) Theorem (4.7), which demonstrates that for each k the Prim-Read deployment d_k^* is the unique solution to the optimization problem

$$\text{minimize} \quad I(d)$$

$$\text{s.t.} \quad \hat{\rho}(d) = \hat{\rho}(d_k^*)$$

$$\max_{1 \le r \le \hat{\rho}(d)} \frac{V(d,r)}{r} \le \frac{1}{k} .$$

In physical terms, this result means that subject to a fixed total of target prices and an upper bound on target value destroyed per attacking weapon committed, the Prim-Read deployment uniquely requires the fewest interceptors.

3) Theorem (4.8), in which is stated that for each k the Prim-Read deployment d_k^* is the unique solution to the optimization problem

$$\text{minimize} \quad \max_{1 \le r \le \hat{\rho}(d)} \frac{V(d,r)}{r}$$

$$\text{s.t.} \quad d \in \hat{D}_k .$$

The interpretation here is that with the total of target prices and the supply of available interceptors fixed, the Prim-Read deployment uniquely minimizes the maximum (over all attack sizes) target value destroyed per attacking weapon committed. This property is essentially argument 2) used in Chapter I, A to justify linearity of the payoff function V_d.

4) Theorem (4.10), which states that if for an allocation a of attacking weapons we define

$$\tilde{V}(d,a) = \sum_{j=1}^{T} v(j)\left[1 - \prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right] ,$$

which is the expected target value destroyed by the allocation a, then of all deployments in $\hat{D}_k$, only the Prim-Read deployment makes $\tilde{V}(d,a)$ dependent on the allocation a only through its total size $\hat{a} = \sum_{j=1}^{T} a(j)$. That is, the Prim-Read deployment d_k^* is characterized by its imposing on the attacking side a "uniformity of effect," in the sense that the target value destroyed is a function only of the number of attacking weapons committed and not of the way those weapons are allocated among the targets.

5) The nonoptimality statements gathered in Example (4.16). Specifically, it is shown there that with the two sides' resources fixed at appropriate Prim-Read levels, the Prim-Read deployment d_k^* is *not* a solution to the optimization problem

$$\text{minimize} \quad V(d,A)$$

$$\text{s.t.} \quad A = k\hat{v}$$

$$\sum_{i=1}^{\rho(d,j)} d(j,i) \leq \sum_{i=1}^{\rho(d_k^*,j)} d_k^*(j,i)\ , \qquad j = 1, \ldots, T.$$

That is, with no reassignment of interceptors from target to target permitted, the Prim-Read deployment does not minimize the expected target value destroyed by an optimized attack. However, the Prim-Read deployment is robust in the sense that it is nearly as effective as the uniform deployment; by comparison the uniform deployment is far from optimal for the problems, cf. 2) and 3) just above, to which the Prim-Read deployment is the optimal solution.

As a consequence of the results noted in 5), the Prim-Read deployment is not an optimal solution to the corresponding problem in which interceptors may be reassigned from one target to another.

We emphasize, in concluding this Chapter, that in the discussion herein, essentially all targets have been defended. In reality, however, the defending side will possess insufficient interceptor resources to defend all (or probably even most) of the targets. It must, therefore, leave many targets, generally not all of the same value, undefended. The choice problem faced by the defending side thus becomes yet more complicated. The target defense principle discussed in Chapter I, A provides a basis for choosing which targets to defend and a partial basis for deciding how to defend them. The next Chapter is devoted to exploration of the consequences and implications of this very important principle.

Chapter V

CONSEQUENCES OF THE TARGET DEFENSE PRINCIPLE

As observed in Chapter I, the important practical problem in nationwide target defense concerns the case when many targets, almost always of differing values, must be left undefended. The discussion in Chapter IV did not include this case; there we assumed that either all targets were defended or only targets of unit value were left undefended. In this Chapter we discuss in detail properties of Prim-Read deployments when there are many undefended targets and when the target defense principle is assumed to be satisfied. We obtain not only results analogous to those of Chapter IV but also some additional results.

A. GENERAL DISCUSSION

Before presenting the mathematical results of this Chapter, we will discuss the target defense problem in fairly general terms, but more mathematically than in Chapter I, A. At this point we refer the reader to Figure 4, which depicts several payoff functions corresponding to different defensive deployments. Recall that

$$V(d,i) = \max_{a:\hat{a}=i} \sum_{j=1}^{T} v(j)\left[1- \prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right],$$

which is the maximum expected target value that can be destroyed by commitment of i attacking weapons. To obtain the payoff V(d,i), the attacking side optimizes its allocation of weapons among targets given knowledge of the deployment d.

In Figure 4a we show the payoff function $V(0,\cdot)$ corresponding to the case when no targets are defended ("0" denotes

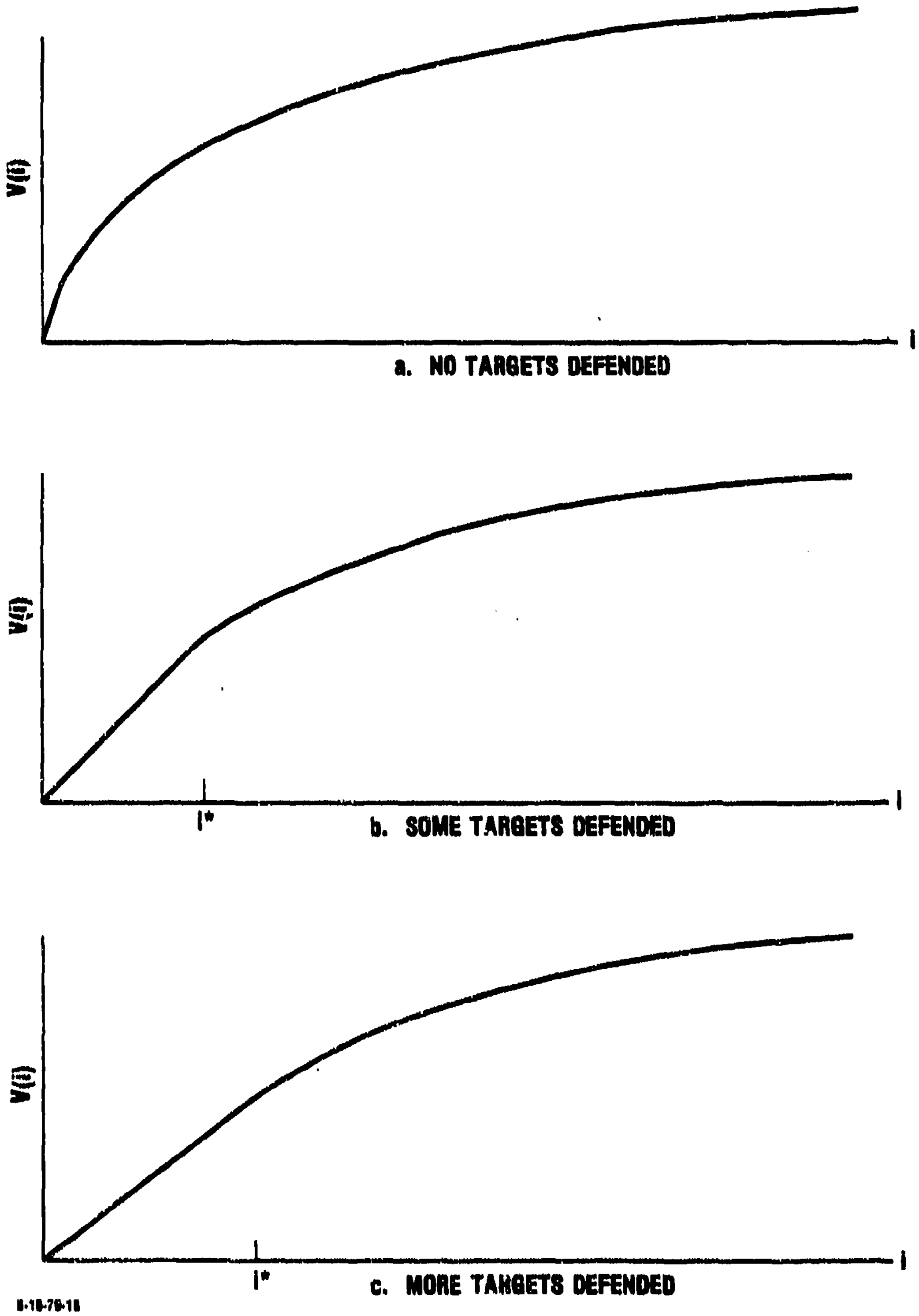


Figure 4. POSSIBLE PAYOFF FUNCTIONS

the deployment which deploys no interceptors). This payoff function is strictly concave on $(0,\infty)$, since when there is no defense (and attacking weapons are assumed to be reliable) the attacking side's optimal response is to destroy the targets in order of decreasing value. To maximize target value destroyed, the attacking side commits as many weapons as possible, whereas to maximize target value destroyed per attacking weapon committed, the attacking side should commit only one weapon (directed at the most valuable target, of course).

The payoff functions shown in Figures 4b and 4c arise when some targets, but not all, are defended using a Prim-Read deployment constructed in the manner described below in this Chapter. In Figure 4c, more targets are defended than in Figure 4b. For both Figures i^* is the number of attacking weapons necessary to destroy all of the *initially defended* targets. Details of the construction of Prim-Read payoff functions will be given presently.

Underlying the defending side's choice of which targets to defend, and how to deploy interceptors at each defended target, is the target defense principle put forth in Chapter I, which we restate here.

TARGET DEFENSE PRINCIPLE. a) Targets must be defended in order of decreasing value.

b) If a target is initially defended, then up to the point at which it is destroyed with certainty, the expected target value destroyed by an attacking weapon directed at it must be greater than or equal to the value of every initially undefended target.

The rationale for the target defense principle is that if defensive resources are to be expended, it must be in such a manner that the attacking side be forced to commit its weapons to defended targets rather than undefended targets, at least up to the point that all defended targets are destroyed.

In more mathematical terms, both parts of the target defense principle may be combined into the following statement: a deployment d satisfies the target defense principle if whenever $\rho(d,j) > 1$ (i,e., target j is initially defended) and $\rho(d,j') = 1$ (target j' is not defended), then

$$v(j') \leq v(j)p(d,j,i)$$
$$= v(j)\left[\prod_{\ell=1}^{i-1}\left(1-q^{d(j,\ell)}\right)\right]q^{d(j,i)}$$

for $i = 1, \ldots, \rho(d,j)$. Recall that $p(d,j,i)$ is the probability that target j is destroyed by the i^{th} attacking weapon directed at it. In particular, $v(j') < v(j)$, which is the first part of the target defense principle.

For further discussion of the target defense principle we refer to Chapters I and VIII and to references [15,19].

Figures 4b and 4c represent the payoff functions of Prim-Read deployments that satisfy the target defense principle; these payoff functions are linear up to the point i^* at which all of the defended targets are destroyed and thereafter coincident with a translated portion of the payoff function from the undefended case. To respond optimally to a Prim-Read deployment, the attacking side should destroy the defended targets in an arbitrary order and then destroy the undefended targets in order of decreasing value.

The more targets are defended, the lower the slope of the linear segment of the payoff function, which is illustrated by comparing Figures 4b and 4c. In fact, there are three ways in which the defending side's choice of a Prim-Read payoff function can be interpreted:

(1) choice of which targets to defend;
(2) specification of the slope of the linear segment of the payoff function;

(3) specification of the number of attacking weapons required to destroy all of the defended targets.

We will describe how to construct a Prim-Read payoff function based on each of these three methods of choice.

For the first method of construction, based on choosing which targets to defend, we refer the reader to Figure 5. If $V(0,\cdot)$ denotes the payoff function for the undefended case, then the procedure is the following:

1) According to the target defense principle, the defended targets must be the i_1 most valuable targets for some $i_1 < T$ (T is the total number of targets).

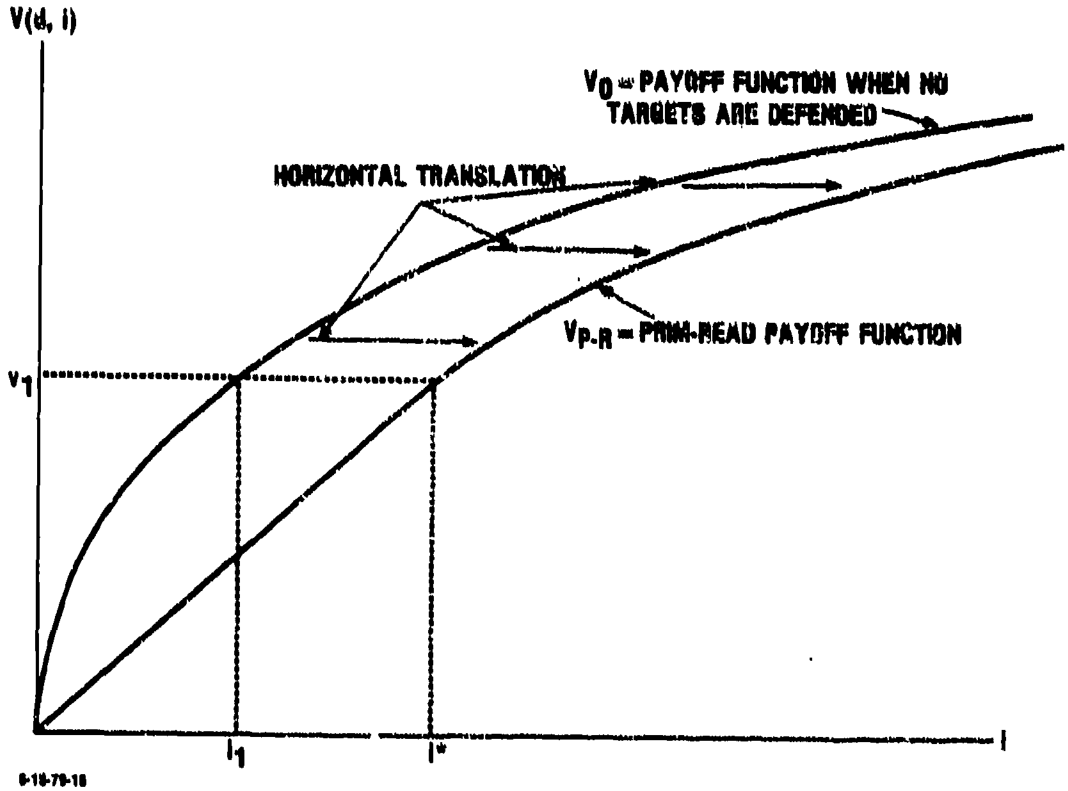


Figure 5. CONSTRUCTION OF PRIM-READ PAYOFF FUNCTION BY SPECIFYING DEFENDED TARGETS

2) Let

$$v_1 = V(0,i_1) ,$$

which is the total value of the targets to be defended.

3) The Prim-Read payoff function is to be linear on the interval $[0,i^*]$, where i^* is the (yet to be determined) number of attacking weapons needed to destroy the defended targets. By the target defense principle, the slope of the linear segment, namely

$$s = \frac{v_1}{i^*} ,$$

must be greater than or equal to the slope of $V(0,\cdot)$ at i_1. This is because the payoff function $V(d^*,\cdot)$ for the Prim-Read deployment will be a translation of $V(0,\cdot)$ once all the defended targets are destroyed. Hence we must have

$$s \geq V'(0,i_1) ,$$

where the prime denotes differentiation. The defending side will now choose i^* as large as possible but such that

$$\frac{v_1}{i^*} \geq V'(0,i_1) .$$

Ignoring possible discreteness difficulties, it will be possible for the defending side to take

$$i^* = \frac{v_1}{V'(0,i_1)} ,$$

which completes determination of the Prim-Read payoff function.

Specifically, the Prim-Read payoff function is then given by

$$V(d^*,i) = iV'(0,i_1) \qquad \text{on } [0,i^*]$$
$$= V(0,i-(i^*-i_1)) \qquad \text{on } (i^*,\infty) ,$$

where $i^* = v_1/V'(0,i_1)$. One must then derive a deployment having this payoff function, which we do in Section B below.

The second construction of Prim-Read deployments requires first specifying the slope of the linear segment; this is illustrated in Figure 6. The procedure to be used is the following:

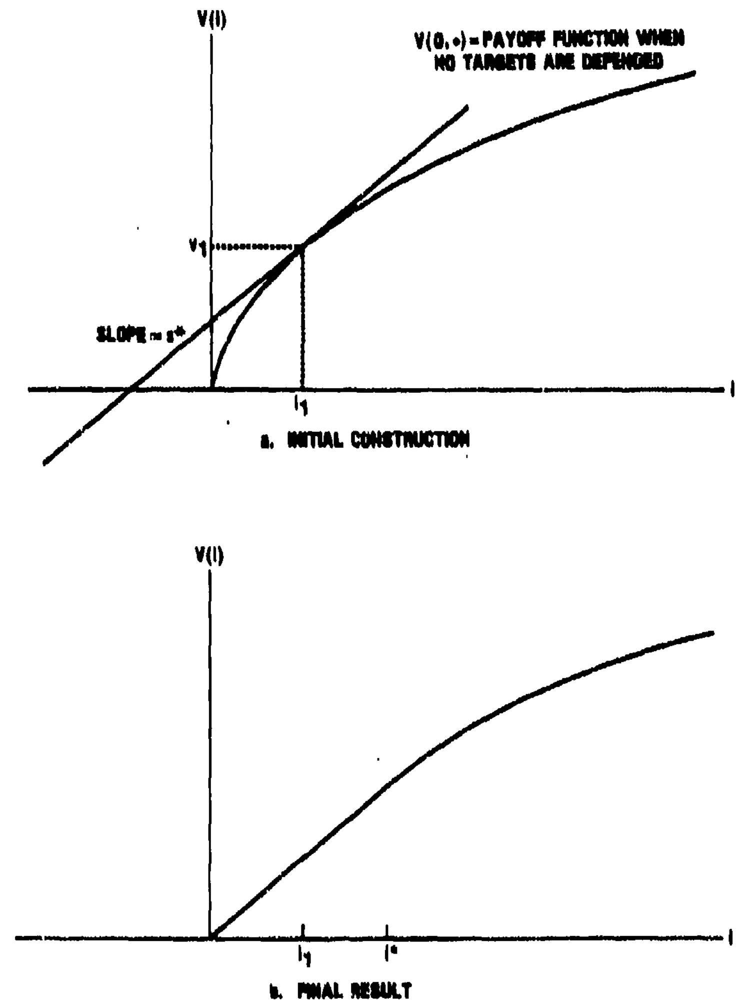


Figure 6. CONSTRUCTION OF PRIM-READ DEPLOYMENT BY SPECIFYING SLOPE OF THE LINEAR SEGMENT

1) Let s^* be the prescribed slope of the linear segment. By strict concavity of $V(0,\cdot)$ there is at most one point i_1 such that

$$V'(0,i_1) = s^* .$$

If there is no such point, it is impossible to construct a Prim-Read deployment for which the linear segment has slope s^*.

2) If there exists a unique i_1 such that $V'(0,i_1) = s^*$, then the i_1 most valuable targets must be defended and the remainder of the construction follows the preceding case. Alternatively, as shown in Figure 6, one takes the linear portion to the left of i_1 and $V(0,\cdot)$ to the right of i_1 and translates this graph to the right until the linear segment passes through the origin. The flatter the linear segment the more translation and the greater the commitment of attacking weapons required to destroy the defended targets.

The final method for constructing a Prim-Read deployment involves specification of the commitment i^* of attacking weapons necessary to destroy the defended targets, and is carried out as follows:

1) By concavity of $V(0,\cdot)$ there is at most one point i_1 such that

$$i^* = \frac{V(0,i_1)}{V'(0,i_1)} .$$

If there is no such point, it is impossible to construct a Prim-Read deployment with the prescribed value of i^*.

2) If there is such a point i_1, proceed as in the two previous constructions, defending the i_1 most valuable targets.

The reader will observe that all three procedures yield payoff functions of the same form; they differ based on what is prescribed at the start by the defending side. Some other relevant points are:

1) The amount by which target value destroyed is decreased relative to the undefended case, i.e., the difference $V(0,i) - V(d^*,i)$, initially increases in i until it attains its maximum at $i = i_1$ and thereafter decreases to zero as $i \to \infty$.

2) The more targets the defending side has the resources to defend (or, equivalently, the flatter the linear segment of the payoff function, or the greater the commitment of attacking weapons needed to destroy all of the defended targets), the greater the decrease relative to the undefended case.

3) For Prim-Read deployments the maximum value of $V(d^*,i)/i$ is attained for all $i \in [0,i^*]$. The attacking side, if it seeks to maximize target value destroyed per attacking weapon committed and if destroying more targets is preferred to destroying fewer, will then expend the resources required to destroy all of the defended targets, but will not attack any of the undefended targets.

As a final point, we observe that when there are undefended targets, there is no arbitrariness in the choice of target prices, as there was in the unspecified scaling parameter k of Definition (2.11). The prices of the defended targets are made as large as possible without violating the target defense principle.

B. MATHEMATICAL RESULTS

Before presenting our principal mathematical results, we motivate them, and also illustrate the target defense principle, by means of an example.

(5.1) EXAMPLE. Let there be four targets with values $v(1) = 8$, $v(2) = 4$, $v(3) = 2$, $v(4) = 1$ and suppose that $q = 0.8$. We will determine Prim-Read deployments for various choices of defended targets and also the associated required numbers of interceptors. In accordance with the target defense principle, the targets must be defended in order of decreasing value, i.e., in the order 1,2,3,4.

If only target 1 is defended using a Prim-Read deployment d*, then in order to satisfy the second part of the target defense principle, its price, $\rho(d^*,1)$, must be (less than or) equal to 2. Each of the first two attacking weapons directed at it then yields an expected target value destroyed equal to 4, which is the value of the most valuable undefended target. To implement the resultant Prim-Read deployment requires 3.11 interceptors.

Suppose now that (only) targets 1 and 2 are to be defended. To satisfy the target defense principle, d* must now satisfy

$$\rho(d^*,2) = \frac{v(2)}{v(3)} = 2 \;;$$

to maintain parity between targets 1 and 2, both of which are now defended, we must also have

$$\begin{aligned}\rho(d^*,1) &= \frac{v(1)}{v(2)}\,\rho(d^*,2)\\ &= \frac{v(1)}{v(3)}\\ &= 4 \;.\end{aligned}$$

The interceptor requirement becomes 17.35.

If targets 1, 2 and 3 are to be defended, which corresponds to k = 1 in the notation of Definition (2.11), then the prices must be

$$\rho(d^*,3) = \frac{v(3)}{v(4)} = 2 \;,$$

$$\rho(d^*,2) = \frac{v(2)}{v(4)} = 4$$

and

$$\rho(d^*,1) = \frac{v(1)}{v(4)} = 8 \;;$$

the interceptor requirement is 64.87.

If more than 65 interceptors are to be used in a Prim-Read deployment, then target 4 must be defended before additional interceptors can be deployed at the more valuable targets. For k = 2, the interceptor requirement is 202.33, while those for k = 3 and k = 4 are 372.60 and 564.71, respectively.

Figure 7 illustrates four of the choices described above, along with the payoff function when no targets are defended. In that Figure, a "discrete" graph has been linearly interpolated for clarity. Each Prim-Read payoff function is linear up to and including the destruction of the most valuable undefended target.

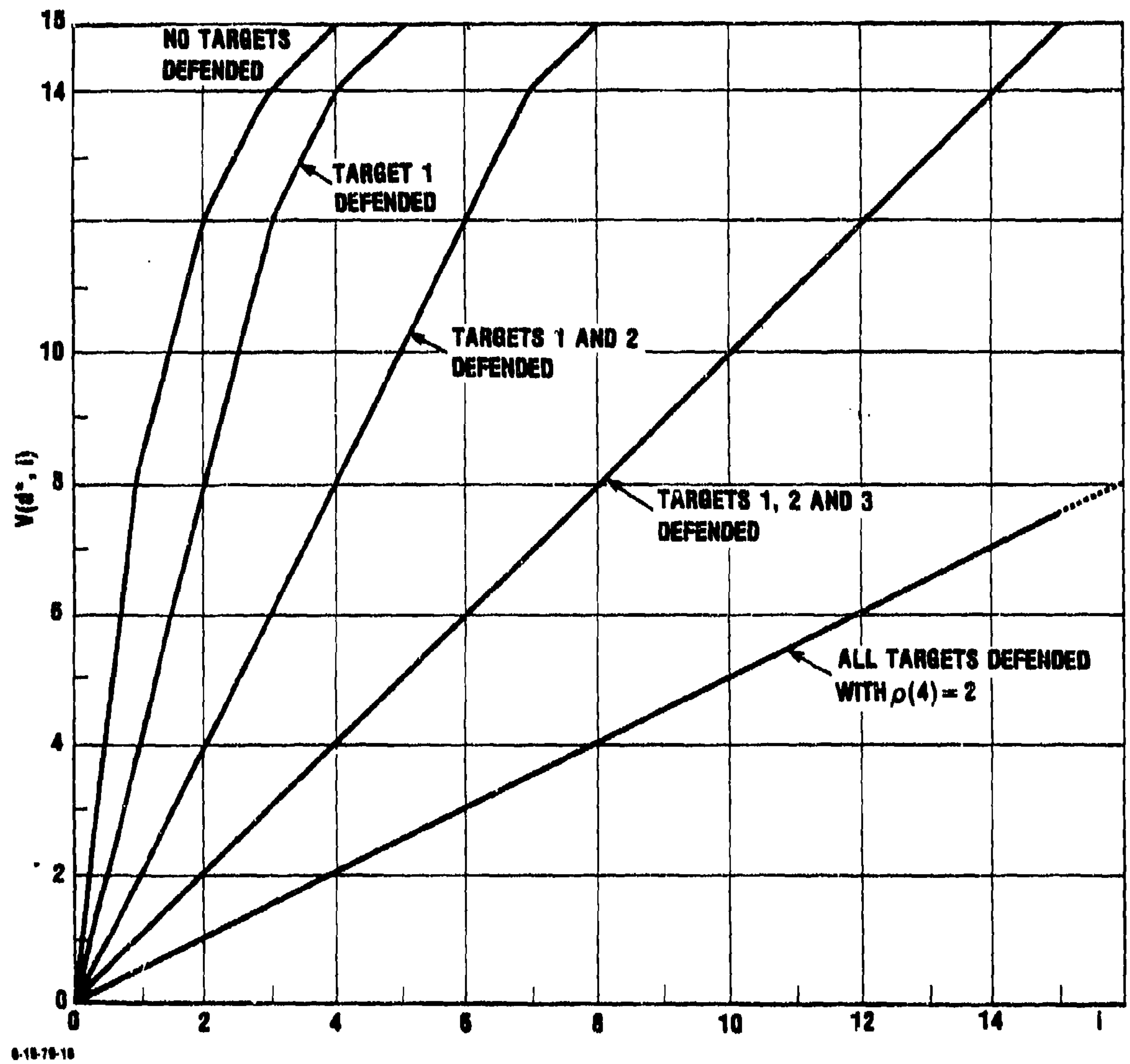


Figure 7. PRIM-READ PAYOFF FUNCTIONS FOR EXAMPLE (5.1)

We now proceed to a precise definition of Prim-Read deployments. Suppose that there are targets 1, ..., T of respective integer values v(1), ..., v(T), so that

(5.2) $$v(1) > v(2) > \ldots > v(T-1) > v(T) \geq 1 ,$$

and that

(5.3) $$m_i = \frac{v(i)}{v(i+1)}$$

is a positive integer for $i = 1, \ldots, T - 1$. While the assumption (5.2) is not restrictive, the assumption (5.3) is; however, we have been unable to avoid it in some form. Its usefulness will become apparent momentarily and ways of minimizing its undesirable effects will be discussed below.

By analogy with Definitions (2.4) and (2.11) we propose the following.

(5.4) DEFINITION. Suppose that $j_0 < T$. A deployment d^* is a *Prim-Read deployment* defending targets 1, ..., j_0 provided

a) $\rho(d^*,j) \geq 2$ for $j = 1, \ldots, j_0$;

b) $\rho(d^*,j) = 1$ for $j = j_0 + 1, \ldots, T$;

c) If $j \leq j_0$, then

(5.5) $$v(j)p(d^*,j,i) = \frac{\sum_{\ell=1}^{j_0} v(\ell)}{\sum_{\ell=1}^{j_0} \rho(d^*,\ell)}$$

for $i = 1, \ldots, \rho(d^*,j)$;

d) If $j \leq j_0$ and $i \leq \rho(d^*,j)$, then

(5.6) $$v(j)p(d^*,j,i) \geq v(j_0+1) .$$

Some comments are in order before we explore the consequences of Definition (5.4).

1) A Prim-Read deployment d* satisfies the target defense principle. By (5.2) the more valuable targets are defended and the less valuable targets left undefended. The second part of the target defense principle is expressed by (5.6): the expected yield from an attacking weapon directed at a defended target, up to the point where destruction of the target is certain, exceeds the value of the most valuable undefended target.

2) Although (5.6) is stated as an inequality, (5.3) will allow us to construct deployments for which it holds as an equality.

3) The condition (5.5) is the Prim-Read equalization criterion appearing in operationally different, but conceptually identical, form in (2.5) and (2.12) above and in (7.1), (7.10) and (7.15) below. Given that targets 1, ..., j_0 are defended by a deployment d,

$$i^* = \sum_{\ell=1}^{j_0} \rho(d,\ell)$$

is the number of attacking weapons needed to destroy them, while

$$\hat{v}_1 = \sum_{\ell=1}^{j_0} v(\ell)$$

is the total of their values. Therefore the ratio $\hat{v}_1/i^*$ is the target value destroyed per attacking weapon committed to the defended targets, and (5.5) stipulates that each such attacking weapon have this as its yield.

4) Against a Prim-Read deployment, the attacking side's optimal response is to first destroy the defended targets and then destroy the undefended targets in order of decreasing

value. Consequently, the payoff function $V(d^*,\cdot)$ is given by

$$V(d^*,i) = i\,\frac{\hat{v}_1}{i^*} \qquad \text{for } 1 \le i \le i^*$$

$$= \sum_{\ell=1}^{j_0+i-i^*} v(\ell) \qquad \text{for } i \ge i^*.$$

That is, the payoff function is a linear segment followed by a translated portion of the undefended case payoff function and is of the form discussed in Chapters I,A and V,A above.

The following analogue of Theorems (2.6) and (2.13) can be proved using our earlier methods, so we omit a derivation.

(5.7) THEOREM. For each $j_0 < T$ there exists a Prim-Read deployment d^* defending targets 1, ..., T, which is given by

$$d^*(j,i) = -\frac{\log(n(j)-i+1)}{\log q}, \qquad j=1,\ldots,j_0; \quad i=1,\ldots,n(j), \tag{5.8}$$

where $n(j) = v(j)/v(j_0+1)$, which is an integer by (5.3). For this deployment, (5.6) holds as an equality for each j and i.

In particular, we have $\rho(d^*,j) = n(j)$ for $j = 1, \ldots, j_0$, so that target prices for defended targets are proportional to target values. Also, the interceptor requirement is

$$I(d^*) = -\frac{1}{\log q}\sum_{j=1}^{j_0} \log[n(j)!] .$$

As noted in the statement of (5.7), the deployment d^* given by (5.8) satisfies (5.6) as an equality, which makes it the maximally strong Prim-Read deployment defending targets 1, ..., j_0 and satisfying the target defense principle.

Our first optimality result is analogous to Theorem (4.8).

(5.9) THEOREM. Let $j_0 < T$ be fixed and let d^* be the Prim-Read deployment given by (5.8). Then d^* is the unique solution to the optimization problem

$$\text{(5.10)} \qquad \text{minimize} \max_{1 \le r \le \hat{\rho}(d^*)} \frac{V(d,r)}{r}$$

$$\text{s.t.} \quad I(d) \le I(d^*) \ .$$

Before proving Theorem (5.9) we point out that the constraint does not include a requirement that d defend precisely the targets 1, ..., j_0; however, we will show that any optimal solution to (5.10) has this property.

PROOF. We observe first that, from (5.6),

$$\max \frac{V(d^*,r)}{r} = v(j_0+1).$$

It will be shown that if

$$\text{(5.11)} \qquad \max \frac{V(d,r)}{r} \le v(j_0+1),$$

then $d = d^*$, which suffices to establish unique optimality of d^* for the problem (5.10).

We begin by showing that if d satisfies (5.11), then $\rho(d,j) = \rho(d^*,j)$ for all $j = 1, \ldots, T$. First of all, if $\rho(d,j) < \rho(d^*,j)$ for some $j \le j_0$ then

$$\frac{V(d,\rho(d,j))}{\rho(d,j)} \ge \frac{v(j)}{\rho(d,j)}$$

$$> \frac{v(j)}{\rho(d^*,j)} = v(j_0+1) \ ,$$

which contradicts (5.11). Therefore $\rho(d,j) \ge \rho(d^*,j)$ for all j; in particular, $\rho(d,j) > 1$ for $j \le j_0$.

On the other hand, if $\rho(d,j) > \rho(d^*,j)$ for some j, then

$$d(j,\rho(d^*,j)) > 0 = d^*(j,\rho(d^*,j)),$$

so that in order that the constraint in (5.10) not be violated there must be h with $1 \leq h \leq j_0$ such that

$$\sum_{i=1}^{\rho(d^*,h)-1} d(h,i) < \sum_{i=1}^{\rho(d^*,h)-1} d^*(h,i) \; . \tag{5.12}$$

However, (5.12) implies that

$$\frac{V(d,\rho(d^*,h)-1)}{\rho(d^*,h)-1} > \frac{V(d^*,\rho(d^*,h)-1)}{\rho(d^*,h)-1} = v(j_0+1) \; ,$$

which again contradicts (5.11). Consequently, $\rho(d,j) = \rho(d^*,j)$ for all j.

It remains to show that

$$d(j,i) = d^*(j,i) \tag{5.13}$$

for $1 \leq j \leq j_0$ and $1 \leq i \leq \rho(d^*,j)$. Since $I(d) \leq I(d^*)$ either

$$\sum_{i=1}^{\rho(d^*,j)-1} d(j,i) = \sum_{i=1}^{\rho(d^*,j)-1} d^*(j,i) \tag{5.14a}$$

for all $j \leq j_0$ or there exists some h for which

$$\sum_{i=1}^{\rho(d^*,h)-1} d(h,i) < \sum_{i=1}^{\rho(d^*,h)-1} d^*(h,i) \; . \tag{5.14b}$$

If (5.14b) were true, the argument of the preceding paragraph could be applied to yield a contradiction to (5.11). Therefore, (5.14a) holds for each j and Theorem (3.11) may now be invoked to conclude that (5.13) is satisfied. ▯

Consequently, when (5.3) holds, in order to minimize the maximum target value destroyed per attacking weapon committed, the defending side should implement a Prim-Read deployment.

If I is the supply of available interceptors, the defending side would then choose

$$j_0 = \max\{j: I(d^*) \leq I\} . \tag{5.15}$$

If $j_0 < T$, targets 1, ..., j_0 would be defended using the deployment d^* given by (5.8). If $j_0 = T$, then all targets can be defended, which can be done using a Prim-Read deployment as described in Chapters II and IV.

To circumvent the nondiscreteness difficulties that are excluded by (5.3) but clearly do arise in reality, one could do the following:

1) Determine j_0 by (5.15).

2) Take

$$\rho(d^*,j_0) = \max\{k: v(j_0) \geq kv(j_0+1)\}$$

as the price of target j_0.

3) For $j < j_0$ take either

$$\rho(d^*,j) = \max\{\ell: v(j) \geq \ell v(j_0+1)\}$$

or (by backward recursion)

$$\rho(d^*,j) = \max\{\ell: v(j) \geq \ell v(j+1)\}\rho(d^*,j+1) .$$

The first of the two methods in step 3) places greater emphasis on (5.6), the target defense principle, and the second, greater emphasis on (5.5), the Prim-Read equalization criterion. Although we have not investigated the question in detail, we suspect that the Theorems of this Chapter remain approximately true for such deployments.

In order to choose a Prim-Read deployment that forces expenditure of a known stockpile A_0 of attacking weapons, the defending side would proceed in the following manner (assuming that (5.2) and (5.3) are satisfied):

1) If there exists $j_0 < T$ such that

$$A_0 \leq \sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)} , \tag{5.16a}$$

then implement the Prim-Read deployment d^* that defends targets $1, \ldots, j_0$, which is given by (5.8).

2) If

$$A_0 = \sum_{j=1}^{T-1} \frac{v(j)}{v(T)} + 1 , \tag{5.16b}$$

then implement the Prim-Read deployment that defends targets $1, \ldots, T-1$; the last attacking weapon will be used on the undefended target T.

3) If

$$A_0 > \sum_{j=1}^{T} \frac{v(j)}{v(T)} , \tag{5.16c}$$

then all targets must be defended, and a Prim-Read deployment may be chosen as described in Chapter II.

It was demonstrated in Theorem (4.8) and Example (4.16) that when all targets are defended, Prim-Read deployments do not solve optimization problems in which the objective function is target value destroyed, whereas they do solve problems in which the objective function is target value destroyed per attacking weapon committed. Specifically, for problems of the forms (4.17) and (4.24) Prim-Read deployments are not optimal. Therefore, it is of interest to examine the same sorts of questions for the situation to which this Chapter is devoted.

It will be shown, in Theorems (5.17) and (5.32), that when the target defense principle is assumed to hold, the following properties are valid:

a) Prim-Read deployments minimize, in most cases uniquely, expected target value destroyed by an attack that is not sufficiently large that it exhaust the Prim-Read defenses.

b) There exists a deployment with the same interceptor requirement as the Prim-Read deployment that is uniformly better (i.e., has a smaller payoff function) for all attack sizes between the Prim-Read exhaustion point and the point (against the Prim-Read deployment) at which all the targets are destroyed, and which also satisfies the target defense principle.

The following result is one of the most important results in this paper; together with Theorem (5.32) it conclusively and unambiguously delineates optimality properties of Prim-Read deployments as a function of the number of attacking weapons committed.

(5.17) THEOREM. Let $j_0 < T$ be fixed and let d^* be the Prim-Read deployment given by (5.8). Assume that for each $j \leq j_0$ there is a positive integer $n(j)$ such that $v(j) = n(j)v(j_0+1)$. Then for each A such that

$$A \leq \sum_{j=1}^{j_0} n(j) = \sum_{j=1}^{j_0} \rho(d^*,j) ,$$

the Prim-Read deployment d^* is a solution to the optimization problem

$$\text{(5.18)} \quad \text{minimize} \quad \max_{a:\hat{a}\leq A} \sum_{j=1}^{T} v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right]$$

$$\text{s.t.} \quad I(d)\leq I(d^*)$$

$$\text{(5.19)} \qquad v(j)p(d,j,i) \geq v(j_0+1) , \qquad j=1,\ldots,j_0 ; \quad i=1,\ldots,\rho(d,j) .$$

If $A \geq n(1) - 1$, then the Prim-Read deployment d^* is the *unique* solution to (5.18).

As indicated symbolically in (5.18), the maximization there is over all allocations $a = (a(1),\ldots,a(T))$ of attacking weapons among the targets for which

$$\hat{a} = \sum_{j=1}^{T} a(j) \leq A ;$$

of course, $\hat{a} = A$ for the maximizing allocation a.

The interpretation of Theorem (5.17) is that, given a choice of which targets are to be defended and if the target defense principle must be satisfied, then the Prim-Read deployment minimizes the expected target value destroyed, provided that the number of attacking weapons not be more than that required to destroy all of the defended targets. If the attack size is large enough to exhaust the Prim-Read defense at the most valuable target (and hence at any defended target), the Prim-Read deployment is uniquely optimal.

We now give the proof of Theorem (5.17).

PROOF of THEOREM (5.17). To establish optimality of d^* for the problem (5.18), the following computation suffices: if d satisfies the constraints there, then

$$\max_{\hat{a} \leq A} \sum_{j=1}^{T} v(j) \left[1 - \prod_{\ell=1}^{a(j)} \left(1 - q^{d(j,\ell)}\right)\right] = \max \sum_{j=1}^{T} v(j) \sum_{\ell=1}^{a(j)} p(d,j,\ell)$$

$$\geq \max \sum_{j=1}^{T} \sum_{\ell=1}^{a(j)} v(j_0+1)$$

[by (5.19)]

$$= v(j_0+1)A$$

$$= \max_{a} \sum_{j=1}^{T} v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d^*(j,\ell)}\right)\right].$$

We now prove uniqueness under the assumption that

$$(5.20) \qquad A \geq n(1) - 1 = (\max_{j \leq j_0} n(j)) - 1 .$$

To this end, observe first that the proof of Theorem (5.9) shows that if d satisfies the constraints of (5.18) then

$$\rho(d,j) \leq n(j) , \qquad j=1,\ldots,j_0 ,$$

and

$$q^{d(j,\ell)} \geq \frac{1}{n(j)-\ell+1} , \qquad j=1,\ldots,j_0 , \quad \ell=1,\ldots,\rho(d,j) .$$

For each target j, either (5.19) holds with equality for $i = 1, \ldots, n(j)$, in which case

$$\rho(d,j) = n(j)$$

and

$$d(j,\ell) = d^*(j,\ell)$$

for $\ell = 1, \ldots, n(j)$, or else there is i_0 (depending on j) such that

$$(5.21) \qquad v(j)p(d,j,i_0) > v(j_0+1)$$

and such that no smaller value of i satisfies (5.21). If we put

$$P(d,j,\ell) = 1 - \prod_{r=1}^{\ell} (1-q^{d(j,r)}) ,$$

which is the probability of destruction of target j by one of the first ℓ attacking weapons directed at it, then we may infer from (5.21) that

$$v(j)P(d,j,\ell) = \ell v(j_0+1)$$

for $\ell = 1, \ldots, i_0 - 1$, while

$$v(j)P(d,j,i_0) > i_0 v(j_0+1) .$$

In order that the target defense principle (5.19) remain satisfied, we must then have

$$v(j)P(d,j,\ell) > \ell v(j_0+1)$$

for all $\ell \geq i_0$, which is possible only if

$$\rho(d,j) < n(j) .$$

Suppose now that $d \neq d^*$ satisfies the constraints of (5.18) and that (5.20) holds. It follows from the preceding paragraph that there is some $j \leq j_0$ such that

$$\rho(d,j) \leq n(j) - 1 .$$

Consequently, with A weapons the attacking side can destroy strictly more target value than against the Prim-Read deployment by destroying target j with (at most) $n(j) - 1$ shots and then destroying one of the undefended targets. Therefore,

$$\max_a \sum_{j=1}^{T} v(j)P(d,j,a(j)) > Av(j_0+1) ,$$

which completes the proof of the uniqueness assertion. ▯

REMARK. In response to the question whether the assumption (5.20) is essential for uniqueness, it is easy to answer in the affirmative. For example, if $T = 3$ with $v(1) = 8$, $v(2) = 4$ and $v(3) = 1$ and if $j_0 = 2$ then we have $n(1) = 8$ and $n(2) = 4$. The deployment d given by

$$d(1,i) = -\frac{\log(9-i)}{\log q}, \qquad i=1,\ldots,8,$$

$$d(2,1) = -\frac{\log 4}{\log q},$$

and

$$d(2,2) = 0$$

satisfies the target defense principle and also has the property that

$$\max_{\hat{a}=1} \sum_j v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right] = 1$$

$$= \max_{\hat{a}=1} \sum_j v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d^*(j,\ell)}\right)\right],$$

and therefore d^* is not the unique solution to (5.18) when $A = 1 < \max_{j \le j_0} \{n(j)-1\}$.

Yet another optimization problem of interest is uniquely solved by the Prim-Read deployment when the target defense principle is assumed to be satisfied: Prim-Read deployments uniquely maximize the number of attacking weapons that must be committed in order to destroy the defended targets. We state this property next, but omit the proof, which is rather lengthy and tedious, albeit straightforward.

(5.22) THEOREM. Let $j_0 < T$ be fixed and assume that for each $j \le j_0$ there is a positive integer $n(j)$ such that $v(j) = n(j)v(j_0+1)$. Then the Prim-Read deployment d^* given by (5.8) is the unique solution to the optimization problem

(5.23) $$\text{maximize} \sum_{j=1}^{j_0} \rho(d,j)$$

$$(5.24) \qquad \text{s.t.} \quad v(j)p(d,j,i) \geq v(j_0+1), \qquad j=1,\ldots,j_0; \; i=1,\ldots,\rho(d,j).$$

Here the target defense principle is embodied in the constraint (5.24).

REMARKS. 1) One motivation for Theorems (5.17) and (5.22) was an attempt to discover to what extent the target defense principle alone determines the form of a deployment d. The constraint (5.24), i.e., the second part of the target defense principle, implies that

$$(5.25) \qquad \rho(d,j) \leq n(j)$$

and

$$(5.26) \qquad d(j,\ell) \leq -\frac{\log(n(j)-\ell+1)}{\log q}$$

for $j \leq j_0$ and $\ell \leq \rho(d,j)$. (Incidentally, the first part of the target defense principle is embodied in our labeling the targets in order of decreasing value.)

It is easy to show that there exist deployments satisfying (5.25) and (5.26) that are not Prim-Read deployments. To consider a specific example, let T = 3 with v(1) = 8, v(2) = 4, v(3) = 1 and let j_0 = 2. Then n(1) = 8 and n(2) = 4. One deployment satisfying (5.25) and (5.26), and not of Prim-Read form, is given by

$$q^{d(1,1)} = \frac{1}{2} \qquad q^{d}(2,1) = \frac{1}{3}$$

$$q^{d(1,2)} = \frac{1}{2} \qquad q^{d(2,2)} = \frac{4}{9}$$

$$q^{d(1,3)} = \frac{1}{2} \qquad q^{d(2,3)} = 1 .$$

$$q^{d(1,4)} = 1$$

Verification of (5.25) and (5.26) is easy; the associated interceptor requirement is $I(d) = 17.876$. Against this deployment the attacking side should commit its weapons against the targets in the order 1,1,2,2,2,1,1,3.

The deployment d given by

$$q^{d(1,1)} = \frac{1}{2} \qquad q^{d(2,1)} = \frac{3}{4}$$

$$q^{d(1,2)} = \frac{1}{2} \qquad q^{d(2,2)} = 1$$

$$q^{d(1,3)} = \frac{1}{2}$$

$$q^{d(1,4)} = 1$$

also satisfies (5.25) and (5.26), and has interceptor requirement $I(d) = 10.608$. The Prim-Read deployment d^* with $\rho(d^*,1) = 4$ and $\rho(d^*,2) = 2$ has interceptor requirement $I(d^*) = 17.348$.

2) In general, uniform deployments such as that defined by (4.18) do not satisfy the target defense principle. Suppose, for example, that $T = 4$ with $v(1) = 8$, $v(2) = 4$, $v(3) = 2$ and $v(4) = 1$. If targets 1 and 2 were to be defended, the Prim-Read deployment d^* chosen in accordance with Definition (5.4) would have $\rho(d^*,1) = 4$ and $\rho(d^*,2) = 2$, with interceptor requirement $I(d^*) = 17.348$. A uniform defense for target 1 for 4 shots (i.e., equal numbers of interceptors deployed against each of the first four attacking weapons, along the lines of (4.18)), and of target 2 for 2 shots, would lead the optimal order of allocation of attacking weapons among the targets to be 1,2,3,1,4,2,1,2,1,1. It is then immediate that the target defense principle is violated.

3) In the strictest sense, the assumption under which Theorems (5.17) and (5.22) are proved, namely that $v(j)/v(j_0+1)$ be an integer for each $j \leq j_0$, is weaker than the assumption (5.3). There do exist choices of target values and of j_0 for which the weaker assumption on only the ratios $v(j)/v(j_0+1)$ holds,

but (5.3) fails. However, validity of the weaker assumption for all j_0 implies that of (5.3).

C. ADDITIONAL PROPERTIES

In Theorem (5.17) it was shown that provided that

$$(5.27) \qquad A \leq \sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)} ,$$

the Prim-Read deployment d* minimizes expected target value destroyed by an optimized allocation of A attacking weapons, assuming that the target defense principle is satisfied. The question of whether d* is similarly optimal when (5.27) fails is resolved in this Section: not only is d* not optimal but there even exists a deployment d such that

$$V(d,i) < V(d^*,i)$$

for all i such that

$$(5.28) \qquad \sum_{j=1}^{j_0} n(j) < i \leq \sum_{j=1}^{j_0} n(j) + (T-j_0) ,$$

which uses no more interceptors than the Prim-Read deployment d*, and which satisfies the target defense principle. Of course, d will defend more targets than does the Prim-Read deployment d*. For a discussion and interpretation of the condition (5.28) the reader is referred to the discussion that follows the proof of Theorem (5.32).

The optimization problem we consider in this Section is, therefore,

$$(5.29) \qquad \text{minimize } V(d,A)$$

$$\text{s.t. } I(d) \leq I(d^*)$$

$$(5.30) \qquad v(j)p(d,j,i) \geq v(j') ,$$

where the constraint (5.30) is to hold for $i = 1, \ldots, \rho(d,j)$ whenever $\rho(d,j) \geq 2$ and $\rho(d,j') = 1$. Here (5.30) represents the target defense principle: the expected yield from an attacking weapon directed at a defended target that is not yet destroyed with certainty exceeds the value of every undefended target. It is further assumed that j_0 is fixed and that A satisfies (5.28).

The following Example illustrates nonoptimality of the Prim-Read deployment d* for the problem (5.29) in a specific case. Thereafter, nonoptimality in general will be demonstrated.

(5.31) EXAMPLE. Assume that $T = 4$ and that $v(1) = 8$, $v(2) = 4$, $v(3) = 2$, $v(4) = 1$. For $j_0 = 2$, the total of the Prim-Read prices is $n(1) + n(2) = 6$. For $A = 7$ we have

$$V(d^*,7) = 14,$$

where d* is the Prim-Read deployment. The deployment d defined by

$$q^{d(1,1)} = \frac{1}{3} \qquad q^{d(2,1)} = \frac{1}{2} \qquad q^{d(3,1)} = \frac{1}{2}$$

$$q^{d(1,2)} = \frac{1}{2} \qquad q^{d(2,2)} = \frac{1}{2} \qquad q^{d(3,2)} = 1 ,$$

$$q^{d(1,3)} = 1 \qquad q^{d(2,3)} = 1$$

then satisfies

$$\prod_{j,\ell} q^{d(j,\ell)} = \frac{1}{48}$$

$$= \prod_{j,\ell} q^{d^*(j,\ell)} ,$$

from which we infer that $I(d) = I(d^*)$. Further, the target defense principle is satisfied by d since

$$v(1)p(d,1,j) = \frac{8}{3} > 1 = v(4) , \qquad j=1,2,3 ,$$

and

$$v(2)p(d,2,1) = 2 > v(4) ,$$

and also

$$\begin{aligned} v(2)p(d,2,2) &= v(2)p(d,2,3) \\ &= v(3)p(d,3,1) \\ &= v(3)p(d,3,2) \\ &= 1 \\ &= v(4) . \end{aligned}$$

Finally, it is immediate that $V(d,7) = 13 < V(d^*,7)$ and therefore d^* is not an optimal solution to (5.29) for $A = 7$. []

The following result shows that in general the Prim-Read deployment d^* is not an optimal solution to (5.29) when (5.28) holds, and actually shows more: we explicitly construct a deployment d such that $V(d,A) < V(d^*,A)$ for all A satisfying (5.28).

(5.32) THEOREM. Let j_0 be fixed and let d^* be the Prim-Read deployment given by (5.8); assume that (5.3) holds. Then there exists a deployment d such that:

a) d satisfies the target defense principle in the form (5.30);

b) $I(d) = I(d^*)$;

c) $V(d,A) < V(d^*,A)$ for all A such that

$$\sum_{j=1}^{j_0} n(j) < A \leq \left[\sum_{j=1}^{j_0} n(j)\right] + [T-j_0] . \qquad (5.33)$$

PROOF. Recall that

$$n(j) = v(j)/v(j_0+1)$$

and, from the assumption (5.3), that

$$v(j_0+1) = m(j_0+1)v(j_0+2) .$$

Note also that the requirement $I(d) = I(d^*)$ is equivalent to

$$(5.34) \qquad \prod_{j,\ell} q^{d(j,\ell)} = \prod_{j,\ell} q^{d^*(j,\ell)} = \prod_{j=1}^{j_0} (1/n(j)!) .$$

The deployment d will defend targets 1, ..., $j_0 + 1$ and is defined by means of the following equalities:

$$q^{d(j_0+1,1)} = \frac{m(j_0+1) - 1}{m(j_0+1)} ,$$

$$q^{d(j_0+1,2)} = 1 ,$$

$$q^{d(j,\ell)} = q^{d^*(j,\ell)} , \qquad j=2,\ldots,j_0; \ \ell=1,\ldots,n(j),$$

$$q^{d(1,1)} = \frac{1}{n(1)} \frac{m(j_0+1)}{m(j_0+1) - 1}$$

and

$$(5.35) \qquad q^{d(1,\ell)} = q^{d^*(1,\ell)} , \qquad \ell=2,\ldots,n(1).$$

It is immediate that (5.34) is satisfied. Provided that the target defense principle be satisfied, it will then be true that

$$V\left(d,\left(\sum_{j=1}^{j_0} n(j)\right)+1\right) = \sum_{j=1}^{j_0+1} v(j) - 1$$

$$= V\left(d^*,\left(\sum_{j=1}^{j_0} n(j)\right)+1\right) - 1 ,$$

which verifies that

(5.36) $$V(d,A) = V(d^*,A) - 1$$

for $A = \left(\sum_{j=1}^{j_0} n(j)\right) + 1$. The forms of the deployments d* and d then imply that (5.36) holds for all A satisfying (5.33). Therefore, to complete the proof of the Theorem, it suffices to show that d satisfies the target defense principle (5.30).

For target $j_0 + 1$ the definition of d implies that

$$v(j_0+1)p(d,j_0+1,1) = v(j_0+1) - v(j_0+2)$$

and

$$v(j_0+1)p(d,j_0+1,2) = v(j_0+2) .$$

Keeping in mind that target $j_0 + 2$ is now the most valuable undefended target and that, by virtue of (5.2) and (5.3), $v(j_0+1) = m(j_0+1)v(j_0+2)$ with $m(j_0+1) \geq 2$, we infer from the two preceding expressions that (5.30) holds for $j = j_0 + 1$.

If $2 \leq j \leq j_0$ then the definitions of d and of the Prim-Read deployment d* yield

$$v(j)p(d,j,\ell) = v(j_0+1) > v(j_0+2)$$

for $\ell = 1, \ldots, n(j)$, which verifies the target defense principle for these values of j.

It is slightly less straightforward to verify that the target defense principle holds for $j = 1$. To begin on this, for $\ell = 1$ we have

$$v(1)p(d,1,1) = v(1)q^{d(1,1)}$$

$$= \frac{v(1)}{n(1)} \frac{m(j_0+1)}{m(j_0+1) - 1}$$

$$= v(j_0+1) \frac{m(j_0+1)}{m(j_0+1) - 1}$$

$$> v(j_0+1)$$

$$> v(j_0+2) ,$$

which gives (5.30) for $j = \ell = 1$. The expected target value that survives the first attacking weapon directed at target 1 is

$$v(1)(1-q^{d(1,1)}) = v(1) \frac{n(1)[m(j_0+1)-1] - m(j_0+1)}{n(1)[m(j_0+1)-1]} .$$

Together with (5.35) this expression gives

$$v(1)p(d,1,\ell) = \frac{1}{n(1)-1} v(1) \frac{n(1)[m(j_0+1)-1] - m(j_0+1)}{n(1)[m(j_0+1)-1]}$$

$$= \left\{ m(j_0+1) \frac{n(1)[m(j_0+1)-1]-m(j_0+1)}{[n(1)-1][m(j_0+1)-1]} \right\} v(j_0+2) ,$$

which is valid for $\ell = 2, \ldots, n(1)$. Therefore, it now suffices to prove that

(5.37)
$$1 \leq m(j_0+1) \frac{n(1)[m(j_0+1)-1] - m(j_0+1)}{[n(1)-1][m(j_0+1)-1]}$$

$$= \frac{m(j_0+1)}{m(j_0+1)-1} \left[m(j_0+1) - \frac{n(1)}{n(1)-1} \right] .$$

The assumptions of the Theorem imply that $n(1) > 2$, which gives

$$\frac{n(1)}{n(1)-1} \leq \frac{3}{2} .$$

If $m(j_0+1) \geq 3$, it is then clear that the final expression in (5.37) exceeds one; if $m(j_0+1) = 2$ this expression becomes

$$\frac{2}{2-1} \left[2 - \frac{n(1)}{n(1)-1} \right] \geq 1 .$$

This completes the proof of Theorem (5.32). ∎

In the condition (5.33), the leftmost term is the total of the Prim-Read prices of the defended targets, while the rightmost term is the total of the Prim-Read prices of all of the targets. We have already demonstrated that the Prim-Read deployment d^* is optimal for the problem (5.29) if

$A \leq \sum_{j=1}^{j_0} n(j)$; for $A > \sum_{j=1}^{j_0} n(j) + (T-j_0)$ the difficulties in obtaining a deployment better than the Prim-Read deployment are associated with having to satisfy the target defense principle. The content of Theorem (5.32) is that by defending one additional target against (exactly) one attacking weapon, one can construct a deployment d that is uniformly better than the Prim-Read deployment for values of A satisfying (5.33). Furthermore, this deployment uses the same number of interceptors and also satisfies the target defense principle. Here is a numerical illustration.

(5.38) EXAMPLE. Assume that $T = 4$, $v(1) = 8$, $v(2) = 4$, $v(3) = 2$, $v(4) = 1$ and $j_0 = 2$. The construction given in Theorem (5.32) then yields the deployment d defined by

$$q^{d(1,1)} = \frac{1}{2} \qquad q^{d(2,1)} = \frac{1}{2} \qquad q^{d(3,1)} = \frac{1}{2}$$

$$q^{d(1,2)} = \frac{1}{3} \qquad q^{d(2,2)} = 1 \qquad q^{d(3,2)} = 1 .$$

$$q^{d(1,3)} = \frac{1}{2}$$

$$q^{d(1,4)} = 1$$

For this example $\sum_{j=1}^{j_0} n(j) = 6$ and $\left[\sum_{j=1}^{j_0} n(j)\right] + (T-j_0) = 8$ and from the easily checked facts that

$$v(1)p(d,1,1) = 4 \,,$$

$$v(1)p(d,1,\ell) = \frac{4}{3} \,, \qquad \ell=2,3,4,$$

$$v(2)p(d,2,\ell) = 2 \,, \qquad \ell=1,2,$$

and

$$v(3)p(d,3,\ell) = 1 \,, \qquad \ell=1,2,$$

it is seen that $V(d,A) = V(d^*,A) - 1$ for $A = 7$ and $A = 8$, as predicted by Theorem (5.32). In Figure 8 we present a graphical comparison of the payoff functions $V(d^*,\cdot)$ and $V(d,\cdot)$.

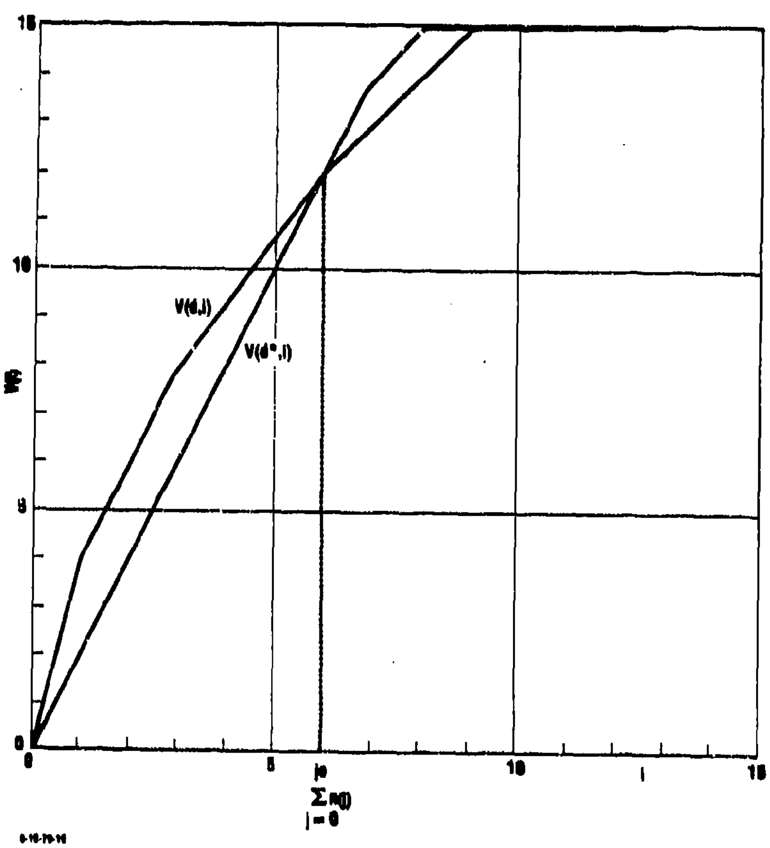


Figure 8. COMPARISON OF PAYOFF FUNCTIONS FOR EXAMPLE (5.38)

D. CHAPTER SUMMARY

In this Chapter we discuss in detail the consequences of the following two-part *target defense principle*:

1) Targets must be defended in order of decreasing value.

2) The expected target value destroyed by an attacking weapon directed at an initially defended target, even allowing for possible destruction of the target by another (earlier) attacking weapon, must always be at least as great as the value of every initially undefended target.

We discuss optimality properties of Prim-Read deployments under the target defense principle and also some important optimization problems for which Prim-Read deployments are not optimal solutions.

The beginning of the Section presents a general discussion of the target defense problem and describes three ways whereby Prim-Read deployments may be constructed. This discussion parallels and extends that in Chapter I,A; the reader is referred to both discussions for further details concerning the heuristic justification for Prim-Read deployments.

In the principal mathematical results of the Chapter several optimality properties and one important nonoptimality property of Prim-Read deployments are demonstrated. Specifically, those main results are the following.

1) Theorem (5.7), which states that, subject to the restriction represented by (5.3), if targets 1, ..., j_0 are defended, the appropriate Prim-Read deployment d* is given by

$$d^*(j,\ell) = -\frac{\log(n(j)-\ell+1)}{\log q}$$

for each j and $\ell = 1, \ldots, n(j)$, where

$$n(j) = \frac{v(j)}{v(j_0+1)} .$$

Recall that targets are labeled in order of decreasing value. The associated interceptor requirement is

$$I(d^*) = -\frac{1}{\log q} \sum_{j=1}^{j_0} \log \left[\left(\frac{v(j)}{v(j_0+1)}\right)!\right] .$$

2) Theorem (5.9), which asserts that if targets $j, \ldots, j_0$ are defended, then the Prim-Read deployment d^* is the unique solution to the optimization problem

$$\text{minimize} \max_{1 \leq r \leq \hat{\rho}(d^*)} \frac{V(d,r)}{r}$$

$$\text{s.t.} \quad I(d) \leq I(d^*) .$$

That is, of deployments requiring no more interceptors than the Prim-Read deployment d^*, the latter uniquely minimizes the maximum target value destroyed per attacking weapon committed, even when the allocation of attacking weapons is optimized against the chosen defensive deployment.

3) Theorem (5.17), which is one of the most important results in the paper, and states that if targets $1, \ldots, j_0$ are defended and if

$$A \leq \sum_{j=1}^{j_0} [v(j)/v(j_0+1)] ,$$

then the corresponding Prim-Read deployment d^* (given by (5.8)) is a solution to the optimization problem

$$\text{minimize } V(d,A)$$

$$\text{s.t.} \quad I(d) \leq I(d^*)$$

$$v(j)p(d,j,i) \geq v(j_0+1), \qquad 1 \leq j \leq j_0 ; \; 1 \leq i \leq \rho(d,j).$$

Furthermore, provided that $A \geq n(1) - 1 = [v(1)/v(j_0+1)] - 1$, the Prim-Read deployment d* is the unique solution to the stated optimization problem. That is, if the number of attacking weapons is at most that required to destroy all of the defended targets, then among all deployments satisfying the target defense principle, the Prim-Read deployment minimizes the expected target value destroyed by an allocation of attacking weapons that is optimized against the chosen defensive deployment. If the number of attacking weapons is sufficient to exhaust the defenses at every individual target (although not necessarily sufficient to destroy the then undefended target), the Prim-Read deployment uniquely minimizes the expected target value destroyed.

This is the one instance where it is shown that Prim-Read deployments minimize target value destroyed as well as target value destroyed per attacking weapon committed , although only rélative to deployments satisfying the target defense principle.

4) Theorem (5.22), which states that if targets 1, ..., j_0 are to be defended, then the corresponding Prim-Read deployment d* is the unique solution to the optimization problem

$$\text{maximize} \sum_{j=1}^{j_0} \rho(d,j)$$

$$\text{s.t.} \quad v(j)p(d,j,i) \geq v(j_0+1) , \quad 1 \leq j \leq j_0; 1 \leq i \leq \rho(d,j).$$

The interpretation here is that of all deployments that defend targets 1, ..., j_0 in a manner satisfying the target defense principle, the Prim-Read deployment d* uniquely maximizes the number of attacking weapons that must be committed in order to destroy all of the defended targets.

5) Theorem (5.32), another very important result, which demonstrates that if d* is the Prim-Read deployment corresponding to defense of targets $1,\ldots,j_0$, then there exists a deployment d, which is computed explicitly, such that

a) d satisfies the target defense principle;

b) $I(d) = I(d^*)$

c) $V(d,A) < V(d^*,A)$ for all values of A satisfying

$$\sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)} < A \leq \left[\sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)}\right] + [T-j_0] .$$

The content of this Theorem is that if A satisfies the restrictions above, then the Prim-Read deployment d* is not a solution to the optimization problem

$$\begin{aligned} &\text{minimize } V(d,A) \\ &\text{s.t. } I(d) \leq I(d^*) \\ &\qquad v(j)p(d,j,i) \geq v(j') \qquad \text{if } \rho(d,j)>1 \text{ and } \rho(d,j')=1. \end{aligned}$$

The Prim-Read deployment, even among deployments satisfying the target defense principle and not requiring more interceptors, does not minimize expected target value destroyed by an optimized attack when the attacking side commits more weapons than are necessary to destroy all of the defended targets.

Although the results listed above were all derived subject to the hypothesis (5.3) that $v(j)/v(j+1)$ be an integer greater than or equal to 2 for each j, they all remain approximately valid without this restriction.

Perhaps the most important conclusion to be drawn from the entire paper is obtained by combining the implications of Theorems (5.17) and (5.32). Suppose that, given its interceptor resources and the targets that are candidates to be defended, the defending side elects to implement a Prim-Read deployment. Based on the supply of available interceptors, it will do so by, for some j_0, defending targets 1, ..., j_0 using the Prim-Read deployment d* given by (5.8); this choice leads to the payoff function $V(d^*,\cdot)$ illustrated in Figure 9.

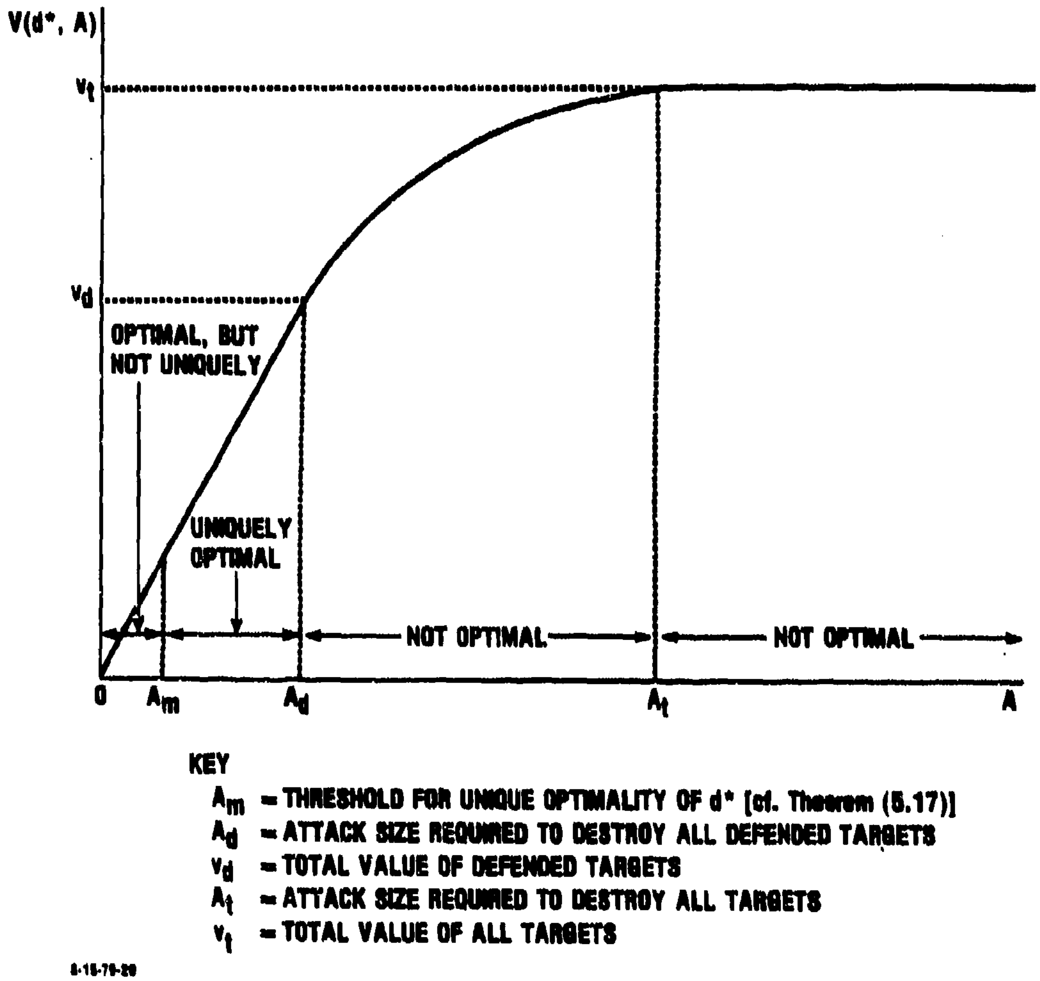


Figure 9. PRIM-READ PAYOFF FUNCTION AND OPTIMALITY PROPERTIES

Also as shown in Figure 9, if A is the number of weapons committed by the attacking side, then depending on the value of A, the Prim-Read deployment may or may not, among deployments that require the same number of interceptors and satisfy the target defense principle, minimize expected target value destroyed. Specifically,

1) If $0 \leq A < A_m$, where $A_m = [v(1)/v(j_0+1)] - 1$, which is maximum number of weapons against which any target is defended, then the Prim-Read deployment minimizes target value destroyed by an attack of size A optimized against the chosen defensive deployment, but not uniquely.

2) If $A_m \leq A \leq A_d$, where

$$A_d = \sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)} = \sum_{j=1}^{j_0} \rho(d^*,j) ,$$

which is the number of attacking weapons required to destroy all of the defended targets, then the Prim-Read deployment uniquely minimizes expected target value destroyed by an optimized allocation of A attacking weapons.

3) If $A_d < A \leq A_t$, where

$$A_t = \left[\sum_{j=1}^{j_0} \frac{v(j)}{v(j_0+1)}\right] + [T-j_0] = \sum_{j=1}^{T} \rho(d^*,j)$$

is the total of the Prim-Read prices of all targets, then the Prim-Read deployment does not minimize expected target value destroyed by an optimized allocation of attacking weapons. Indeed, there exists a deployment d using the same number of interceptors and satisfying the target defense principle such that $V(d,A) < V(d^*,A)$ for all A in the interval $(A_d,A_t]$.

4) If $A > A_t$, then the Prim-Read deployment does not minimize expected target value destroyed by A optimally allocated attacking weapons, but there is no other deployment that is uniformly better than the Prim-Read deployment for all $A > A_t$.

Chapter VI

ADDITIONAL MIN-MAX OPTIMALITY PROPERTIES

A. INTRODUCTION

Chapters IV and V developed optimality and nonoptimality properties of Prim-Read deployments for optimization problems of the general form

$$(6.1) \qquad \text{minimize} \max_{a} \sum_{j=1}^{T} v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right],$$

where the deployment d, and also the allocation a of attacking weapons, satisfy certain constraints, one of which, for example, is the target defense principle underlying all of Chapter V. In this Chapter we treat one additional class of problems of the form (6.1), in which we assume that target prices remain fixed at their Prim-Read levels, and that

$$\hat{a} = \sum_{j=1}^{T} a(j) = i \leq \sum_{j=1}^{T} v(j) ,$$

where i is treated as a parameter of the problem. We further assume that we are in the situation of Chapter IV, rather than that of Chapter V, and that k = 1; therefore, all targets except those of unit value are defended. Consequently the results of this Chapter apply, at most, indirectly to the nationwide defense problem treated in Chapter V, but may apply to other situations. Regardless of direct applicability, of course, the results are of interest as further properties of Prim-Read deployments.

Since we do not assume the target defense principle to hold, the results of this Chapter are complementary to those in Chapter V. Our main result here asserts that, of deployments with the same interceptor requirement, the Prim-Read deployment d* minimizes target value destroyed--barring a small number of known exceptional cases--if and only if there is a set of targets whose prices sum to i. An empirical observation, for which we have indirect but also inconclusive theoretical support, is that in those cases in which d* is not optimal, the deviation from optimality is nearly negligible.

The specific class of optimization problems to be considered is

$$(6.2) \qquad \begin{aligned} &\text{minimize } V(d,i) \\ &\text{s.t. } \; I(d) \le I(d^*) \\ &\qquad \rho(d,j) \le v(j) , \qquad j=1,\dots,T , \end{aligned}$$

where $i \le \sum v(j)$ is fixed, and where d* is the Prim-Read deployment with scaling parameter k = 1, as given by (2.14). That is, for each j

$$d^*(j,i) = -\frac{\log(v(j)-i+1)}{\log q} , \qquad i=1,\dots,v(j).$$

Recall also that

$$V(d,i) = \max_{a:\hat{a}=i} \sum_{j=1}^{T} v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right] .$$

A generalization of the main result of this Chapter (which is Theorem (6.13)) asserts that the Prim-Read deployment d* is an optimal solution to (6.2) if and only if there is $J_0 \subset \{1,\dots,T\}$ such that

$$i = \sum_{j \in J_0} v(j) = \sum_{j \in J_0} \rho(d^*,j) .$$

In words, d* is an optimal solution to (6.2) for a given value of i if and only if there is a subset of targets that can be exactly destroyed by i attacking weapons.

Of the problems treated in earlier Chapters, the problem (4.24) is most closely related to (6.2). The difference between them is that in (6.2) target prices are fixed at Prim-Read levels, whereas in (4.24) they are not. The most important difference between (6.2) and the similar problems (5.18) and (5.29) treated in Chapter V is that in the latter the target defense principle is assumed to be satisfied. When the target defense principle holds, then by Theorem (5.17) the Prim-Read deployment minimizes $V(d,i)$ for all $i \leq \sum v(j)$.

The problem (6.2) is of interest whether one envisions the defending side's choosing a Prim-Read deployment in order to make efficient use of a supply of available interceptors or in order to force exhaustion of a particular stockpile of attacking weapons in order to destroy the targets. In either case (because of faulty decision-making processes or incorrect estimates by the defending side), the number of attacking weapons committed might not be large enough to destroy all of the defended targets, so it is desirable to understand properties of Prim-Read deployments in such contingencies.

To avoid overburdening (the author and) the reader with mathematics at the expense of concepts and their implications, we restrict our attention for the remainder of this Chapter to the case of two targets, which is sufficiently general to illustrate the range of complications when $T \geq 3$, but sufficiently simple to be comprehensible and computationally tractable. Therefore,

$$V(d,i) = \max_{a(1)+a(2)=i} \sum_{j=1}^{2} v(j)\left[1-\prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right].$$

B. EXAMPLES

Before stating the main result of this Chapter we give some illustrative examples.

(6.3) EXAMPLE. Suppose that $v(1) = 3$ and $v(2) = 1$. If we put $\alpha = q^{d(1,1)}$, $\beta = q^{d(1,2)}$, then the constraint $I(d) = I(d^*)$ becomes

$$\alpha\beta = 1/6.$$

By direct calculation, $V(d,1) = \max\{3\alpha,1\} \geq 1$; therefore the Prim-Read deployment d^* does solve (6.2) for $i = 1$.

Suppose now that $i = 2$. Then

$$V(d,2) = \max\{3[1-(1-\alpha)(1-\beta)],3\alpha+1\} .$$

Using elementary algebra one can show that if

$$V(d,2) \leq V(d^*,2) = 2 ,$$

then $\alpha = 1/3$, $\beta = 1/2$, which implies that $d = d^*$ and hence that d^* solves (6.2) for $i = 2$. This turns out to be an exception to the general case, since 2 weapons cannot exactly destroy either one of the targets or both targets.

That the Prim-Read deployment d^* solves (6.2) for $i = 3$ and $i = 4$ follows by the same reasoning used when $i = 1$. []

The next Example gives an illustration of a case when the Prim-Read deployment d^* does not solve (6.2).

(6.4) EXAMPLE. Suppose that $v(1) = 4$ and $v(2) = 2$ and observe that the constraint $I(d) = I(d^*)$ is equivalent to

$$q^{d(1,1)}q^{d(1,2)}q^{(1,3)}q^{d(2,1)} = 1/48 . \tag{6.5}$$

Since

$$V(d,1) = \max\{4q^{d(1,1)}, 2q^{d(2,1)}\} ,$$

it is evident that by taking $q^{d(1,1)}$ greater than but nearly equal to 1/8, $q^{d(2,1)}$ greater than but nearly equal to 1/6, and $q^{d(1,2)}$, $q^{d(1,3)}$ less than but nearly equal to 1, all in such a manner that (6.5) remain satisfied, one can force $V(d,1) < 1 = V(d^*,1)$, so that the Prim-Read deployment does not solve (6.2) for $i = 1$.

That the Prim-Read deployment d^* satisfies (6.2) for $i = 2$, as well as for $i = 4$ and $i = 6$, is obvious.

Consider now the case $i = 3$, in which we have

$$\begin{aligned} V(d,3) = \max\Big\{ & 4[1-(1-q^{d(1,1)})(1-q^{d(1,2)})(1-q^{d(1,3)})] , \\ & 4[1-(1-q^{d(1,1)})(1-q^{d(1,2)})] + 2q^{d(2,1)} , \\ & 4q^{d(1,1)} + 2\Big\} , \end{aligned}$$

since the attacking side can allocate 3, 2 or 1 weapons to the more valuable target 1. To find d such that $V(d,3) < V(d^*,3)$ we must satisfy the three inequalities

$$4\left[1 - \prod_{\ell=1}^{3}\left(1-q^{d(1,\ell)}\right)\right] < 3, \tag{6.6}$$

$$4\left[1 - \prod_{\ell=1}^{2}\left(1-q^{d(1,\ell)}\right)\right] + 2q^{d(2,1)} < 3 , \tag{6.7}$$

and

$$4q^{d(1,1)} + 2 < 3 . \tag{6.8}$$

We were able to find such values of the $d(j,\ell)$ by a perturbation argument that also forms the basis for the omitted proof of Theorem (6.13).

The underlying reasoning is the following. The Prim-Read deployment d* satisfies (6.6) - (6.8) as equalities. Consider a perturbed deployment d that can be written in the form

$$q^{d(1,1)} = \alpha q^{d^*(1,1)} = \alpha/4$$

$$q^{d(1,2)} = \beta q^{d^*(1,2)} = \beta/3$$

$$q^{d(1,3)} = \gamma q^{d^*(1,3)} = \gamma/2$$

$$q^{d(2,1)} = \delta q^{d^*(2,1)} = \delta/2 ,$$

where in order to satisfy (6.5) we assume that $\alpha\beta\gamma\delta = 1$, which we may use to eliminate γ from the equations (6.6) - (6.8).

Now make a first-order Taylor series expansion (in α, β and δ) of the left-hand sides of the equations (6.6) - (6.8). The appropriate first partial derivatives evaluated at the Prim-Read point $\alpha = \beta = \delta = 1$ are

$$\frac{\partial(6.6)}{\partial\alpha} = -\frac{2}{3} \qquad \frac{\partial(6.6)}{\partial\beta} = -\frac{1}{2} \qquad \frac{\partial(6.6)}{\partial\delta} = -1$$

$$\frac{\partial(6.7)}{\partial\alpha} = \frac{2}{3} \qquad \frac{\partial(6.7)}{\partial\beta} = 1 \qquad \frac{\partial(6.7)}{\partial\delta} = 1$$

$$\frac{\partial(6.8)}{\partial\alpha} = 1 \qquad \frac{\partial(6.8)}{\partial\beta} = 0 \qquad \frac{\partial(6.8)}{\partial\delta} = 0 .$$

By Taylor's theorem, cf. [16], we can satisfy (6.6) - (6.8) by choosing perturbations $\Delta\alpha$, $\Delta\beta$, $\Delta\delta$ sufficiently close to zero and such that

(6.9) $$-\frac{2}{3}\Delta\alpha - \frac{1}{2}\Delta\beta - \Delta\delta < 0$$

(6.10) $$\frac{2}{3}\Delta\alpha + \Delta\beta + \Delta\delta < 0$$

(6.11) $$\Delta\alpha < 0 ;$$

we then take $\alpha = 1 + \Delta\alpha$, $\beta = 1 + \Delta\beta$, $\delta = 1 + \Delta\delta$.

By experiment, the choices $\Delta\alpha = -.005$, $\Delta\beta = -.00125$, and $\Delta\delta = .004167$ satisfy (6.9) - (6.11) and lead to the values

$$q^{d(1,1)} = .24875$$
$$q^{d(1,2)} = .332917$$
$$q^{d(1,3)} = .50105$$
$$q^{d(2,1)} = .5020835 ,$$

which do satisfy (6.6) - (6.8). Consequently, the Prim-Read deployment d^* does not solve (6.2) when $i = 3$. One possible reason is that the alternative deployment d is able to leave the largest single payoff to the (irrelevant when $i = 3$) fourth attacking weapon.

For $i = 5$ we obtain

$$V(d,5) = \max\left\{4+2q^{d(2,1)},\ 4\left[1 - \prod_{\ell=1}^{3}\left(1-q^{d(1,\ell)}\right)\right] + 2\right\} .$$

Because $V(d,5)$ depends symmetrically on $q^{d(1,1)}$, $q^{d(1,2)}$ and $q^{d(1,3)}$, if we take

$$q^{d(1,1)} = q^{d(1,2)} = q^{d(1,3)} = .34902$$
$$q^{d(2,1)} = .49 ,$$

then we have $V(d,5) < 5 = V(d^*,5)$.

To summarize this Example, the Prim-Read deployment d^* is an optimal solution to (6.2) if and only if $i = 2$, 4 or 6, i.e., if and only if some subset of targets can be exactly destroyed by i attacking weapons. ▯

Similar effects arise in the case of two targets with equal values.

(6.12) EXAMPLE. Suppose that $v(1) = v(2) = 3$. By symmetry we may restrict attention to deployments d for which

$$q^{d(1,1)} = q^{d(2,1)} \equiv \alpha$$

$$q^{d(1,2)} = q^{d(2,2)} \equiv \beta ,$$

where $\alpha\beta = q^{d^*(1,1)}q^{d^*(1,2)} = q^{d^*(2,1)}q^{d^*(2,2)} = 1/6$.

Since $V(d,1) = 3\alpha$, it is possible to have $V(d,1) < 1$, so the Prim-Read deployment d^* is not an optimal solution to (6.2) when $i = 1$.

When $i = 2$ another exceptional case arises: even though two attacking weapons cannot exactly destroy either target, the Prim-Read deployment is an optimal solution to (6.2) when $i = 2$. To see this, note that

$$V(d,2) = \max\{6\alpha, 3\alpha+3(1-\alpha)\beta\} .$$

If $V(d,2) \leq 2$ then on the one hand $\alpha \leq 1/3$, while on the other hand the inequality

$$3\alpha + 3(1-\alpha)\beta \leq 2$$

implies that $1/3 \leq \alpha \leq 1/2$. Therefore $\alpha = 1/3$, $\beta = 1/2$ and $d = d^*$.

For $i = 3$ and $i = 6$ it is evident that the Prim-Read deployment is an optimal solution to (6.2). The case $i = 4$ is another exceptional case: the Prim-Read deployment can be shown to be the unique optimal solution to (6.2), despite the impossibility of exactly destroying a subset of the targets with i attacking weapons. Finally, for $i = 5$, a symmetry argument analogous to that used in Example (6.4) for the case $i = 5$ can be used to show that the Prim-Read deployment is not an optimal solution to (6.2) □

C. MAIN RESULT

We now present the main mathematical result of this Chapter, which states that--except for the anomalous cases noted in Examples (6.3) and (6.12) above--the Prim-Read deployment d^* is an optimal solution to (6.2), for a given value of i, if and only if one or the other or both targets can be exactly destroyed by i attacking weapons. By the latter phrase we mean that $i = v(1)$, $i = v(2)$ or $i = v(1) + v(2)$; i.e., i is the sum of the prices of a subset of the targets.

(6.13) THEOREM. Suppose that $T = 2$ and that $v(1) \leq v(2)$. Then the Prim-Read deployment d^* is an optimal solution to (6.2), that is,

$$V(d^*,i) = \min\{V(d,i): I(d) = I(d^*), \rho(d) = \rho(d^*)\},$$

if and only if one of the following conditions is satisfied:

a) $v(2) \leq 2$;

b) $v(1) = v(2) = 3$ and $i = 2$ or $i = 4$;

c) $v(1) = 1$, $v(2) = 3$ and $i = 2$;

d) $i = v(1)$;

e) $i = v(2)$;

f) $i = v(1) + v(2)$.

The first three conditions are the exceptional cases noted in Examples (6.3) and (6.12), while the latter three conditions state that i attacking weapons can exactly destroy some subset of the set of targets.

The proof of Theorem (6.13) is lengthy, largely technical, and rather unenlightening, so we have omitted it from this paper. Its essential argument is a generalization of the perturbation method used in the analysis of Example (6.4) above. The technical difficulties arise in having to prove that the perturbed deployment d satisfies $V(d,i) < i = V(d^*,i)$, rather than being able to verify the inequality numerically, as was permissible in the Example. An analogous result holds for $T \geq 3$ but it also is omitted.

As was done in Examples (3.15) and (4.16) in earlier Chapters, it is of interest to study the extent by which the Prim-Read deployment d^* fails to be optimal for the problem (6.2) when d^* is in fact not optimal. Because we have been unable to solve (6.2) in closed form when the Prim-Read deployment is not the optimal solution, we do not have rigorously derived bounds on the possible deviation from optimality. However, the techniques used in Example (6.4) and in the proof of Theorem (6.13)--and here the reader is asked to accept on faith the fact that more simple-minded, straightforward approaches were unsuccessful--strongly suggest that the deviation is minimal. For all of the examples we have analyzed, including several that are not included here, if $\tilde{d}$ is found such that

$$V(\tilde{d},i) < V(d^*,i) ,$$

then both the absolute deviation

$$V(d^*,i) - V(\tilde{d},i)$$

and the (more appropriate) relative deviation

$$\frac{V(d^*,i) - V(\tilde{d},i)}{V(d^*,i)}$$

are quite small. A theoretical investigation of these deviations might be pursued by examination of second derivatives corresponding to the first derivatives that were calculated in Example (6.4). In view of the fact that (6.2) is not the most important optimization problem treated in this paper, we have not performed such an investigation.

D. CHAPTER SUMMARY

This Chapter describes optimality and nonoptimality properties of Prim-Read deployments, in the multiple target case, for a particular class of optimization problems. In these problems, the objective function is expected target value destroyed by an optimized allocation of attacking weapons and both the defending side's resources and the individual target prices (not just the total of all target prices) are fixed at their Prim-Read levels. The attacking side's resources are treated parametrically and are at most the number of weapons needed to destroy all of the targets. It is assumed that only targets of unit value are left undefended. The target defense principle is *not* assumed to be satisfied.

There is only one mathematical result in this Chapter, Theorem (6.13), which when generalized, states that the Prim-Read deployment d^* (with scaling parameter $k = 1$) is a solution to the optimization problem

$$\begin{aligned} &\text{minimize } V(d,i) \\ &\text{s.t. } I(d) = I(d^*) \\ &\qquad \rho(d,j) = \rho(d^*,j) = v(j) , \qquad j=1,\ldots,T, \end{aligned}$$

if and only if one of a small number of known exceptional cases holds (which can happen only for small values of i) or there is a set $J_0 \subset \{1,\ldots,T\}$ such that

$$i = \sum_{j \in J_0} \rho(d^*,j) = \sum_{j \in J_0} v(j) .$$

That is, save in the exceptional cases, the Prim-Read deployment, among all deployments with the same target prices and interceptor requirement, minimizes expected target value destroyed by an attack consisting of i optimally allocated attacking weapons if and only if there is a subset of targets whose prices sum to i. When there are many targets with relatively low individual values, it becomes fairly likely that the latter conditions will be satisfied.

Chapter VII

VARIATIONS

In this Chapter we describe how some of the initial assumptions imposed in (1.1) and (1.2) can be weakened in order to represent certain phenomena of physical interest and importance. Specifically, we show how to extend the basic model specified by (1.1) and (1.2) to permit target dependent intercept probabilities, initially unreliable attacking weapons that may malfunction before reaching the interceptor defense, terminally unreliable attacking weapons that may fail to damage a target despite having successfully penetrated the interceptor defense, and alternative attrition structures. The latter, for example, may not represent an engagement of an attacking weapon by n interceptors simply as n independent, one-on-one engagements.

To keep the size of this paper finite, we have only derived the forms of appropriately defined Prim-Read deployments and presented interpretive remarks concerning such deployments. Many of the optimality and nonoptimality results of Chapters IV, V, and VI extend with essentially no difficulty to the cases of target dependent intercept probabilities and unreliable attacking weapons (of both kinds). Validity of these results for alternative attrition structures, however, is open to doubt and cannot be resolved without further research (which might have to be done on a case-by-case basis).

We wish to emphasize that the principal structural assumptions put forth in Chapter I remain in effect. Attacking weapons directed at each target arrive sequentially in time, interactions involving different attacking weapons are probabilistically independent, and neither attacking weapons nor interceptors can be adaptively reassigned during the course of an attack. We have not examined possible ways of weakening these major assumptions.

A. TARGET DEPENDENT INTERCEPT PROBABILITIES

It is a simple matter to modify the basic model of (1.2) to allow target dependent intercept probabilities. This is desirable in physical terms since different targets might be defended by different kinds of interceptors, or the same kind of interceptors may have differing effectiveness at different targets because, for example, of differences in warning time or local environmental conditions. Introducing target dependent intercept probabilities also allows representation of the case where different types of attacking weapons are directed at different targets, provided that all weapons directed at a given target be of the same type. Similarly, there may be different types of interceptors at different targets, but all interceptors at a given target must be of the same type.

We assume that the Assumptions (1.2) are satisfied, except that the one-on-one penetration probability at target j is now some $q_j \in (0,1)$. By analogy with Definition (2.11), a deployment d^* is said to be a Prim-Read deployment with scaling factor k, where k is a positive integer, provided that

(7.1) $$v(j)p(d^*,j,i) = \frac{1}{k}, \qquad j=1,\ldots,T; \; i=1,\ldots,\rho(d^*,j).$$

Note that (7.1) and (2.12) are formally identical, but that now

$$p(d,j,i) = \left[\prod_{\ell=1}^{i-1}\left(1-q_j^{d(j,\ell)}\right)\right]q_j^{d(j,i)} .$$

The counterpart of Theorem (2.13) is the following result.

(7.2) THEOREM. For each $k \geq 1$ there exists a unique Prim-Read deployment d^* with scaling factor k, given by

$$(7.3) \qquad d^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q_j}\,, \qquad j=1,\ldots,T;\; i=1,\ldots,kv(j)\,.$$

The close resemblance between (7.3) and (2.14) is, of course, not coincidental. From Theorem (7.2) the following consequences are immediate.

COROLLARY. If d^* is the Prim-Read deployment with scaling factor k, then

$$\rho(d^*,j) = kv(j)\,, \qquad j=1,\ldots,T,$$

and

$$I(d^*) = -\sum_{j=1}^{T} \frac{\log[(kv(j))!]}{\log q_j}\,.$$

Since the proofs of the Theorem and its Corollary are virtually the same as those of Theorem (2.13) and its Corollaries (2.15) and (2.17), we have omitted them.

As previously remarked, many of the results of Chapters IV, V and VI extend--with virtually only notational changes--to the case of target dependent intercept probabilities.

B. POSSIBLY UNRELIABLE ATTACKING WEAPONS

In this Chapter we show how to represent two forms of possible unreliability of attacking weapons, which we call initial and terminal unreliability, respectively. Initial unreliability accounts for the possibility that some attacking weapons may be launched but never arrive at the interceptor defense of their intended targets, so that no interceptors need to be deployed against them. To be slightly more precise, it may develop during the course of an actual attack that no interceptors have to be deployed against initially unreliable attacking weapons, but the defending side does not know in advance which attacking weapons, if any, will be unreliable and must contend with the possibility that none will be. Initial unreliability might result, for example, from an unsuccessful launch, an extreme navigational error, or a distant defense.

Terminal unreliability is meant to represent the possibility that an attacking weapon may penetrate the defense of interceptors deployed against it and yet fail to destroy the target. For example, there may be a local (but critical) navigational error, a warhead may fail to detonate, or the target may have a single, close-in defense.

We will deal first with initial unreliability, which is the simpler of the two forms. For simplicity we consider mainly the single target case, which nonetheless rather clearly illuminates the problem. To represent initial unreliability we introduce the following hypothesis.

(7.4) ASSUMPTION. Each attacking weapon fails to arrive at the interceptor screen of the target at which it is directed with probability $1-p$, where $0 < p \leq 1$. Initial failures of different weapons are mutually independent and independent of the entire attrition/penetration process at all targets. No interceptors are deployed against initially unreliable attacking weapons.

Except for this modification, the original Assumptions (1.1), (1.2) remain in force.

In what follows, although we continue to call

$$\rho(d) = \min\{i : d(i) = 0\}$$

the price imposed by a deployment d, we wish the reader to be aware that commitment (i.e., launching) of $\rho(d)$ attacking weapons no longer ensures destruction of the target (unless $p = 1$). From the defending side's point of view, however, the interpretation remains nearly unchanged: $\rho(d) - 1$ is the number of attacking weapons reaching the target vicinity against which interceptors are actually deployed.

We now say that a deployment d^* is a Prim-Read deployment with price ρ if $\rho(d^*) = \rho$ and

$$p(d^*,1) = p(d^*,2) = \ldots = p(d^*,\rho(d^*)) .$$

The difference between this condition and (2.5) is that while the marginal contributions $p(d^*,1), \ldots, p(d^*,\rho(d^*))$ remain equal to one another, they are no longer assumed to be equal to $1/\rho$ and their sum will now be less than one. In fact, as Theorem (7.5) below demonstrates, each is equal to p/ρ. Let us emphasize, at this point, that $p(d^*,i)$ is the probability that the target is destroyed by the i^{th} attacking weapon *launched* at it.

Theorem (7.5) also settles another question of interest. Recall that a Prim-Read deployment can be thought of as being either as strong as possible subject to a limited supply of available interceptors or as strong as necessary to force commitment of a prescribed number A_0 of attacking weapons in order to destroy the target. Theorem (7.5) below shows that, even when there is possible initial unreliability, the Prim-Read deployment is the same as when there is no initial unreliability (i.e., is the same deployment given by (2.7)). There then arises the question of the appropriate choice of the price ρ. For the case of a deployment constrained by interceptor resources it is clear that existence of initial unreliability should not lead the defending side to change the price from that when $p = 1$. What should be done by the defending side that attempts to force full commitment of the stockpile of A_0 attacking weapons is less clear, but two principal possibilities emerge:

1) Choose $\rho = A_0$, the same choice as if p were equal to one.

2) Since pA_0 is the expected number of attacking weapons that are initially reliable (and against which interceptors must be deployed), choose a Prim-Read deployment with $\rho = pA_0$.

The following result indicates rather decisively that the defending side should elect the first of the two alternatives.

(7.5) THEOREM. For each integer $\rho \geq 1$, there exists a Prim-Read deployment d^* with $\rho(d^*) = \rho$, which is given by

$$d^*(i) = -\frac{\log(\rho-i+1)}{\log q}, \qquad i=1,\ldots,\rho.$$

The associated payoff function is given by

$$(7.6) \qquad p(d^*,i) = \frac{p}{\rho}, \qquad i=1,\ldots,\rho,$$

$$= \frac{p}{\rho}\sum_{\ell=0}^{\rho-1}\binom{i-1}{\ell}p^{\ell}(1-p)^{i-1-\ell}, \qquad i>\rho.$$

Recall that $p(d^*,i)$ is the probability that the target is destroyed by the i^{th} attacking weapon *committed*. This weapon may fail in three ways to destroy the target: the target may have already been destroyed, the weapon may be initially unreliable, or the weapon may be initially reliable but be destroyed by the interceptors. Observe that when $p = 1$, (7.6) reduces to (2.5). However, if $p < 1$ then $p(d^*,i) > 0$ for all i and no attack with finitely many weapons is certain to destroy the target.

To understand the implications of (7.6) for the defending side, we note that for $i > \rho(d^*) = \rho$ (and provided that $p < 1$) we have

$$0 < \frac{p}{\rho}\sum_{\ell=0}^{\rho-1}\binom{i-1}{\ell}p^{\ell}(1-p)^{i-1-\ell} < \frac{p}{\rho}$$

and that the middle term (i.e., $p(d^*,i)$) decreases to zero as $i \to \infty$. If the attacking side is attempting to maximize target value destroyed it would, as before, expend as many weapons as possible. But if the attacking side is attempting, in the manner of Theorem (3.11), to maximize target value destroyed per attacking weapon committed, it will expend $\rho(d^*)$ attacking weapons. Therefore, of the choices $\rho(d^*) = A_0$ and $\rho(d^*) = pA_0$, the former is definitely better for the defending side.

To summarize, when there is initial unreliability, the defending side should not modify the choice of target price for a Prim-Read deployment.

In the case, as well, of multiple targets and initial unreliability, no changes should be made by the defending side. A Prim-Read deployment d* is defined by its satisfying an equalization condition of the form

$$\begin{aligned} v(j)p(d^*,j,1) &= v(j')p(d^*,j',1) \\ &= v(j')p(d^*,j'',2) \\ &\vdots \\ &= v(j')p(d^*,j',\rho(d^*,j')) \end{aligned}$$

for all j and j'. It is easily seen that--no matter what the value of p--for each k the Prim-Read deployment d* with prices $\rho(d^*,j) = kv(j)$, as given by (2.14), satisfies this condition. The preceding discussion for the single target case therefore applies also to the multiple target case.

The results above show that, regardless of the value of the initial reliability p, the previously chosen Prim-Read deployment should not be discarded in favor of the Prim-Read deployment with price pA_0. However, this does not preclude existence of some other deployment that is superior to the original Prim-Read deployment.

The proof of Theorem (7.5), which is based on a straightforward conditioning argument (the i^{th} attacking weapon committed destroys the target if and only if some number ℓ of the previous $i - 1$ attacking weapons were initially reliable but none of these destroyed the target, the i^{th} attacking weapon is initially reliable and it penetrates the interceptor defense), is omitted.

We proceed now to consider terminal unreliability, which is introduced into the axiom structure of (1.1) and (1.2) by the following hypothesis.

(7.7) ASSUMPTION. An unintercepted attacking weapon destroys its intended target with probability $\tilde{p} \in [0,1]$. Terminal unreliabilities of different attacking weapons are independent of one another and of all other probabilistic aspects of the interception processes.

When $\tilde{p} = 1$ we are in the case treated in Chapters II through VI; assume now that there is no initial unreliability. The single target case will be treated first.

As for the case of initial unreliability, one must correctly interpret the price imposed by a deployment d. Destruction of the target is not ensured by commitment of $\rho(d)$ attacking weapons or even necessarily by penetration of the interceptor defense by $\rho(d)$ attacking weapons, but only by penetration of $\rho(d)$ terminally reliable attacking weapons. While one might introduce an alternative terminology such as "defense level" (although, strictly speaking, the defense level should $\rho(d)-1$), we continue to use "price" with the same mathematical definition as above:

$$\rho(d) = \min\{i : d(i) = 0\} .$$

In Definition (2.4), a Prim-Read deployment d* is defined by the equalization condition

$$p(d^*, i) = \frac{1}{\rho(d^*)} , \qquad i = 1, \ldots, \rho(d^*),$$

which, in particular, implies that

(7.8) $$\sum_{i=1}^{\rho(d^*)} p(d^*,i) = 1 .$$

To obtain an analogous condition for the case of terminal unreliability we equate the probabilities $p(d^*,1),\ldots,p(d^*,\rho(d^*))$ to one another but not to $1/\rho(d^*)$, as a consequence of which (7.8) will no longer hold.

We remind that $p(d,i)$ is the probability that the target is destroyed by the i^{th} attacking weapon launched at it.

(7.9) DEFINITION. A deployment d^* is a *Prim-Read deployment* if

(7.10) $$p(d^*,1) = p(d^*,2) = \ldots = p(d^*,\rho(d^*)-1) = p(d^*,\rho(d^*)) .$$

The effect of (7.10) is to equalize the target value destroyed by each attacking weapon against which there is a defense with that by the first attacking weapon against which there is no defense. It follows that

$$0 < p(d^*,i) < p(d^*,\rho(d^*))$$

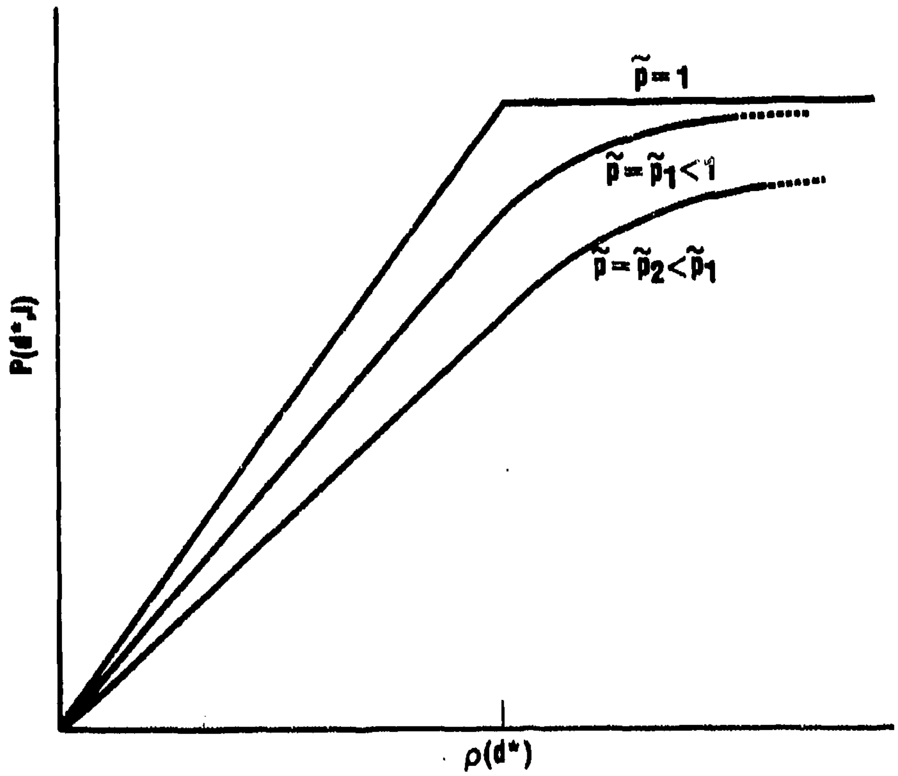
for all $i > \rho(d^*)$, so that the cumulative payoff function P defined by

$$P(d^*,i) = \sum_{\ell=1}^{i} p(d^*,\ell)$$

is of the form shown in Figure 10. For purposes of comparison we have shown the payoff functions for $\tilde{p} = \tilde{p}_1$, $\tilde{p} = \tilde{p}_2 < \tilde{p}_1$ and $\tilde{p} = 1$ (the latter, of course, corresponds to no terminal unreliability).

The following result is analogous to Theorem (2.6) and may be proved by similar methods.

(7.11) THEOREM. For each integer $\rho \geq 1$ there exists a Prim-Read deployment d^* with $\rho(d^*) = \rho$, which is given by



6-15-79-14

Figure 10. PRIM-READ PAYOFF FUNCTIONS IN THE CASE OF TERMINAL UNRELIABILITY

$$d^*(i) = -\frac{\log(1+(\rho-i)\tilde{p})}{\log q}, \qquad i=1,\ldots,\rho. \tag{7.12}$$

It follows from (7.12) that the interceptor requirement is

$$I(d^*) = -\frac{1}{\log q}\sum_{i=1}^{\rho} \log(1+(\rho-i)\tilde{p}), \tag{7.13}$$

and that

$$P(d^*,\rho) = \frac{\rho\tilde{p}}{1+(\rho-1)\tilde{p}}, \tag{7.14}$$

which is the probability of the target's being destroyed by one of the attacking weapons against which there is a defense or the first attacking weapon against which there is no defense. The reader will note that if $\tilde{p} = 1$, (7.12), (7.13) and (7.14) reduce to previously derived results. As one would expect, it is also true that $P(d^*, \rho(d^*))$ increases as $\tilde{p}$ does.

We turn now to terminal unreliability in the multiple target case. As in Chapters II and IV there will be a scaling parameter that is a choice variable for the defending side. However, there now arise complications even with existence of Prim-Read deployments, which are surmounted with a technical assumption similar to (5.3). Another alternative, which we have not pursued, but which has a lengthy history of success in two-sided optimization problems, is randomization; cf. [4]. An additional novelty is that--at least under the assumption we have chosen--target prices are not quite directly proportional to target values. Roughly speaking, the reason for this is that (7.14) implies that the ratio $P(d^*, \rho(d^*))/\rho(d^*)$ is not independent of $\rho(d^*)$, which it would be if there were no terminal unreliability, and this necessitates an adjustment of target prices.

By analogy with earlier notation, let

$$p(d,j,i) = \left[\prod_{\ell=1}^{i-1} \left(1-\tilde{p}q^{d(j,\ell)}\right)\right] \tilde{p}q^{d(j,i)} ,$$

which is the probability that the target j is destroyed by the i^{th} attacking weapon *launched* at it. We then have the following analogues of Definition (7.9) and Theorem (7.11).

(7.15) DEFINITION. A deployment d^* is a *Prim-Read deployment* provided that for each two targets j and j'

$$\begin{aligned} v(j')p(d^*,j',1) &= v(j)p(d^*,j,1) \\ &= v(j)p(d^*,j,2) \\ &\vdots \\ &= v(j)p(d^*,j,\rho(d^*,j)). \end{aligned}$$

The meaning of this condition is that the expected target value destroyed by each attacking weapon against which there are interceptors deployed is the same and is equal to the target value destroyed at each target by the first attacking weapon against which no interceptors are deployed.

Unfortunately, the discreteness difficulties now become severe and a Prim-Read deployment may not exist. Nonetheless we do have the following result.

(7.16) THEOREM. Let $v_0 = \min\{v(1), \ldots, v(T)\}$ be the minimum target value. If $k \geq 1$ is an integer such that for each j the quantity

$$\frac{v(j)}{\tilde{p}v_0}[1 + (k-1)\tilde{p}] - \frac{1}{\tilde{p}}$$

is an integer, then there exists a Prim-Read deployment d^* such that $\rho(d^*,j) = k$ for all targets j for which $v(j) = v_0$. Specifically, for each j

$$\rho(d^*,j) = \frac{v(j)}{\tilde{p}v_0}[1 + (k-1)\tilde{p}] + 1 - \frac{1}{\tilde{p}} \tag{7.17a}$$

and

$$d^*(j,i) = -\frac{\log(1+(\rho(d^*,j)-i)\tilde{p})}{\log q} \tag{7.17b}$$

for $i = 1, \ldots, \rho(d^*,j)$.

We omit the proof of Theorem (7.16).

REMARK. Although we have not checked the details fully, we believe that Theorem (7.16) remains valid if everywhere--including in (7.17)--we replace the terminal reliability $\tilde{p}$ by target-dependent terminal reliabilities $\tilde{p}_j$. This modification is of interest as a surrogate for targets that require more than one hit, or differing numbers of hits, in order to be destroyed.

C. ALTERNATIVE ATTRITION STRUCTURES

The attrition structure specified by Assumptions (1.1c) and (1.2d) may be too simplistic for some potential applications of our model, although it does seem fairly plausible for the nationwide defense problem in which interceptors are anti-ballistic missiles. However, even in this situation interceptor missiles might be under the control of a single tracking system and might not be probabilistically independent. The independence assumption is also questionable if the interceptors are aircraft since interceptor tactics would presumably involve various forms of cooperation. Our purpose in this Section is to indicate how to extend our basic model to include other forms of attrition equations. We restrict our attention to the single target case since the assumption of independence of intercept/attrition processes at different targets in the multiple target case seems reasonable in virtually all potential applications of the model. Also, we continue to assume that interactions involving different attacking weapons are independent.

Let $r(\ell)$ be the probability that an attacking weapon penetrates to the target when ℓ interceptors are deployed against it; we previously assumed that

$$r(\ell) = q^{\ell} . \tag{7.18}$$

Note that the function r appearing in (7.18) can be defined and makes sense for nonintegral values of ℓ, that r is continuous and strictly decreasing as a function of $\ell \in [0,\infty)$, and that $r(0) = 1$ and $\lim_{x \to \infty} r(x) = 0$. The latter properties entail existence of an inverse function $r^{-1}:(0,1) \to [0,\infty)$ such that rr^{-1} and $r^{-1}r$ are the identity functions on $(0,1]$ and $[0,\infty)$, respectively.

The next result shows that invertibility of the attrition function suffices to permit demonstration of existence and calculation of a Prim-Read deployment.

(7.19) THEOREM. Let $r(\ell)$, the probability that an attacking weapon penetrates when ℓ interceptors are deployed against it, be given by a function $r:[0,\infty) \to (0,1]$ such that

a) $r(0) = 1$;
b) r is continuous and strictly decreasing;
c) $\lim_{x\to\infty} r(x) = 0$.

Then for each integer $\rho \geq 1$ there exists a unique deployment d^* such that $\rho(d^*) = \rho$ and

$$p(d^*,i) = \frac{1}{\rho}, \qquad i=1,\ldots,\rho. \tag{7.20}$$

The deployment d^* is given by

$$d^*(i) = r^{-1}\left(\frac{1}{\rho-i+1}\right)$$

for $i = 1, \ldots, \rho(d^*)$.

COROLLARY. The associated interceptor requirement is

$$I(d^*) = \sum_{k=2}^{\rho} r^{-1}\left(\frac{1}{k}\right) .$$

The proof of Theorem (7.19) is an entirely straightforward modification of that of Theorem (2.6) and is therefore omitted.

The optimality results obtained in Chapters III, IV, and V remain valid for more general attrition functions; a careful inspection of their various proofs reveals that the arguments used do not depend on the specific form of the attrition function. The nonoptimality results given in Examples (3.15) and (4.16) seem to depend essentially only on the strict convexity of the attrition function given by (7.18) and may remain valid if strict convexity is imposed as an assumption. It is likely (but we have not worked out the details) that the results of Chapter VI also remain valid provided that the attrition function be strictly convex.

D. CHAPTER SUMMARY

In this Chapter we derive the forms of Prim-Read deployments under several variations of the basic Assumptions (1.1) and (1.2); these variations permit representation of phenomena of physical interest that were excluded (for reasons of economy and simplicity) by our original assumptions. Those phenomena treated in this Chapter are target dependent intercept probabilities, initially unreliable attacking weapons (which may be launched but never arrive at the interceptor defense), terminally unreliable attacking weapons (which, even having penetrated the interceptor defense, may fail to destroy the target), and alternative attrition structures for representing the engagement of a single attacking weapon by one or more interceptors.

The principal mathematical results of this Chapter are the following.

1) Theorem (7.2), which states that if the penetration probability at target j is $q_j \in (0,1)$ and if a Prim-Read deployment d* is defined by the condition that

$$v(j)p(d^*,j,i) = \frac{1}{k}$$

for all j and i, where k is a positive, integral scaling parameter, then

$$d^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q_j}$$

for $j = 1, \ldots, T$ and $i = 1, \ldots, \rho(d^*,j) = kv(j)$. Target prices remain directly proportional to target values.

2) Theorem (7.5), which gives the form and payoff function of a Prim-Read deployment for the case of a single target and initially unreliable attacking weapons. Let p be the probability that each attacking weapon is initially reliable (i.e., reaches the vicinity of the target and requires a reaction by the

defending side). A deployment d* is by definition a Prim-Read deployment if

$$p(d^*,1) = p(d^*,2) = \dots = p(d^*,\rho(d^*)) .$$

It is shown that for each ρ this condition is satisfied by the ordinary Prim-Read deployment d* with $\rho(d^*) = \rho$, namely, the deployment d* given by

$$d^*(i) = -\frac{\log(\rho-i+1)}{\log q} , \qquad i=1,\dots,\rho.$$

Furthermore, the associated payoff function is given by

$$p(d^*,i) = \frac{p}{\rho} , \qquad i=1,\dots,\rho,$$

$$= \frac{p}{\rho} \sum_{\ell=0}^{\rho-1} \binom{i-1}{\ell} p^{\ell}(1-p)^{i-1-\ell} , \qquad i>\rho .$$

In terms of the defending side's choice of a Prim-Read deployment, no change from the case $p = 1$ of perfectly reliable attacking weapons is called for; the defending side should choose as if p were equal to 1 and not as if the attacking side's resources were reduced by a factor of $1 - p$. This conclusion holds also in the multiple target case.

3) Theorem (7.11), which gives the form of a Prim-Read deployment when there is but one target and when attacking weapons may be terminally unreliable. Let $\tilde{p}$ be the terminal reliability of each attacking weapon, so that $1 - \tilde{p}$ is the probability that an attacking weapon fails to destroy the target given that it has penetrated the interceptor defense. A Prim-Read deployment d* is defined by the condition

$$p(d^*,1) = p(d^*,2) = \dots = p(d^*,\rho(d^*))$$

and is shown to be given by

$$d^*(i) = -\frac{\log(1+(\rho(d^*)-i)\tilde{p})}{\log q} .$$

Here $\rho(d^*) = \min\{i:d^*(i)=0\}$ is one more than the number of attacking weapons against which interceptors are assigned to be deployed. If $\rho(d^*)$ attacking weapons are committed, the target is destroyed with probability

$$P(d^*,\rho(d^*)) = \frac{\rho(d^*)\tilde{p}}{1 + (\rho(d^*)-1)\tilde{p}},$$

which is less than one if $\tilde{p} < 1$.

4) Theorem (7.16), which gives the form of a Prim-Read deployment (if it exists) for the case of multiple targets and terminally unreliable attacking weapons. The equalization condition defining a Prim-Read deployment d^* is that

$$\begin{aligned} v(j)p(d^*,j,1) &= v(j)p(d^*,j,2) \\ &\vdots \\ &= v(j)p(d^*,j,\rho(d^*,j)) \\ &= v(j')p(d^*,j',\rho(d^*,j')) \\ &\vdots \\ &= v(j')p(d^*,j',1) \end{aligned}$$

for all targets j and j'. If v_0 is the minimum target value and if $k \geq 1$ is an integer for which

$$\frac{v(j)}{\tilde{p}v_0}[1 + (k-1)\tilde{p}] - \frac{1}{\tilde{p}}$$

is an integer for each j, then there exists a Prim-Read deployment d^* such that $\rho(d^*,j) = k$ if $v(j) = v_0$. The deployment d^* is given by

$$\rho(d^*,j) = \frac{v(j)}{\tilde{p}v_0}[1 + (k-1)\tilde{p}] + 1 - \frac{1}{\tilde{p}}$$

$$= \frac{v(j)}{v_0} k + (\frac{v(j)}{v_0} - 1)(\frac{1}{\tilde{p}} - 1)$$

and by

$$d^*(j,i) = - \frac{\log[1 + (\rho(d^*,j)-i)\tilde{p}]}{\log q}$$

for $j = 1, \ldots, T$ and $i = 1, \ldots, \rho(d^*,j)$. Target prices are no longer directly proportional to target values.

5) Theorem (7.19), which gives the form of Prim-Read deployments for the case of a single target and a general attrition function (which must satisfy some mild and plausible restrictions). The more general attrition structure applies only to an interaction involving one attacking weapon and some interceptors; interactions involving different attacking weapons (or occurring at different targets in the multiple target case) continue to be assumed to be independent. Let $r(x)$ be the probability that an attacking weapon is not destroyed (i.e., successfully penetrates) if x interceptors are deployed against it. Under the assumptions that

a) $r(0) = 1$;

b) r is continuous and strictly decreasing (i.e., an inverse function r^{-1} exists);

c) $\lim_{x \to \infty} r(x) = 0$,

and if a Prim-Read deployment d^* is defined by the condition

$$p(d^*,i) = \frac{1}{\rho(d^*)}, \qquad i=1,\ldots,\rho(d^*),$$

then

$$d^*(i) = r^{-1}\left(\frac{1}{\rho(d^*) - i + 1}\right)$$

for $i = 1, \ldots, \rho(d^*)$.

The optimality results of Chapters III, IV, and V extend to include all four phenomena described in this Chapter. Under the assumption that the attrition function r be strictly convex, we believe--but have not verified in detail--that the nonoptimality results of Examples (3.15) and (4.16), and also the results of Chapter VI, remain valid.

Chapter VIII

SUMMARY AND CONCLUSIONS

This Chapter summarizes and synthesizes the main mathematical results in this paper and their principal physical implications and interpretations. We intend it to be somewhat more than a mere re-listing of our most important Theorems, but at the same time we urge the reader not to rely on this Chapter alone in attempting to understand the content of the paper. A Summary, no matter how cogent, cannot convey the full import of the results and examples and comments appearing in the body of the paper. At the very minimum, the general discussions in Chapter I and Chapter V, and the Summaries of Chapters II, III, IV, V, VI, and VII are essential reading in order to obtain a usable knowledge of the paper. After that, we would recommend reading Chapters II, V, IV, VII, VI, and III in that order. Chapter III is anomalous in that it is important to the conceptual and mathematical development of the paper, but treats the physically uninteresting case in which there is only one target.

We have organized this Chapter in the following manner. Section A is a concise re-statement of the important physical and decision-making aspects of the target defense problem treated in this paper. Section B discusses the defining properties and form of the Prim-Read deployments, while Section C treats the important target defense principle that is central to our main results. In Section D we re-state, and once more interpret, the main mathematical results of the paper, all of which appear in Chapter V. A number of secondary results are listed in Section E. In Section F we relate our work to previous research and existing literature on defense allocation problems in general and missile/interceptor allocation problems in particular. Finally, in Section G we mention briefly a few aspects of the problem that are treated not at all or inadequately in this paper. The reader who has read the whole paper may well protest that there couldn't be any, but there assuredly are.

A. THE UNDERLYING PROBLEM AND THE MODEL

The physical problem studied in this paper is the optimal use of limited interceptor resources in nationwide defense of population and production resources against attack by enemy ballistic missiles. While the model and the results of this paper may have applicability to other defense situations involving attacking weapons and interceptors, it is toward analysis and understanding of the nationwide defense question that our research has been directed.

We have represented the optimization aspects of the problem as a sequential, min-max optimization problem in which the attacking side is permitted to optimize its allocation of weapons among targets given full knowledge of the interceptor deployment plans of the defending side. Under the constraint of the attacking side's subsequent opportunity to maximize target damage, the defending side seeks to minimize target damage, where the latter is measured by an appropriate criterion.

Table 2 indicates schematically the decision-making structure of the problem. To summarize that structure one more time, the defending side, knowing that attacking weapons will arrive sequentially at each target and that it must deploy interceptors without knowledge of how many more weapons may follow, seeks a deployment schedule that minimizes the maximum target damage that can be achieved against the chosen deployment, i.e., seeks to minimize the target damage inflicted by

Table 2. DECISION-MAKING STRUCTURE OF THE PROBLEM

DECISION MAKER: Defending Side

CHOICES TO BE MADE:

1. Which targets to defend
2. Allocation of interceptors to targets
3. Allocation of interceptors at each target to sequentially arriving attacking weapons

INFORMATION AVAILABLE:

1. That attacking weapons directed at each target will arrive sequentially
2. Possibly, the size of attacking side's stockpile of weapons

GOAL:

Time ↓

To minimize the target damage that results from an allocation of attacking weapons that is optimized on the basis of full knowledge of the chosen interceptor deployment

* * * * * * * * * *

DECISION MAKER: Attacking side

CHOICES TO BE MADE:

Allocation of attacking weapons among targets

INFORMATION AVAILABLE:

Complete knowledge of allocation of interceptors to sequentially arriving weapons at each target

GOAL:

To maximize either target value destroyed or target value destroyed per attacking weapon committed an allocation of attacking weapons that is optimized on the basis of full knowledge of the chosen deployment.

Two principal target damage criteria are studied in this paper: expected target value destroyed and expected target value destroyed per attacking weapon committed. The former is an absolute criterion, whereas the latter attempts also to account for the generally diminishing yield from commitment of additional attacking weapons. Symbolically, we are studying min-max problems with objective functions of the forms

$$\underset{d}{\text{minimize}} \; \underset{a}{\max} \; \tilde{V}(d,a)$$

and

$$\underset{d}{\text{minimize}} \; \underset{a}{\max} \; \frac{\tilde{V}(d,a)}{\hat{a}} ,$$

where d is the interceptor allocation, a is the allocation of attacking weapons among the targets, $\tilde{V}(d,a)$ is the resultant target value destroyed, and $\hat{a}$ is the number of attacking weapons committed. In both problems we have imposed restrictions on $\hat{a}$, which some of the time are treated parametrically. Similar restrictions on d are crucial to most of our results.

In physical terms, the most important underlying assumptions are the following:

1) Defensive interceptors must be assigned in advance to various targets and cannot be reassigned, adaptively or otherwise, during the course of an attack.

2) Attacking weapons directed at each target arrive there sequentially in time. When an attacking weapon arrives, the defending side does not know how many more attacking weapons will follow it, but must nonetheless allocate interceptors to seek to destroy it.

3) The attacking side must allocate weapons among targets in advance of an attack and cannot adaptively re-allocate weapons as the attack progresses (even though weapons arrive at each target sequentially).

4) Except in unrealistic cases, the defending side's limited interceptor resources will prevent its defending all of the targets. Moreover, those targets that must be left undefended will generally be of differing values.

The additional assumptions we have imposed in order to incorporate these fundamental phenomena into a tractable mathematical model are given in (1.1) and (1.2). Of these, the assumption that a target is destroyed with certainty by an unintercepted attacking weapon has the clearest physical import and is probably the most restrictive.

B. PRIM-READ DEPLOYMENTS

Assume that the defending side has a number T of targets, with respective values $v(1) \geq \ldots \geq v(T)$, all of which are assumed to be positive integers. Let $d(j,i)$ be the number of interceptors allocated to the i^{th} attacking weapon to arrive at target j (of course, because of the allocation chosen by the attacking side, the i^{th} attacking weapon may never be committed). For each j let $\rho(d,j) = \min\{i: d(j,i)=0\}$, which we have defined to be the price of target j imposed by the deployment d. By committing $\rho(d,j)$ weapons to target j the attacking side can be certain of its destruction. Finally, for each j and i, and for an interceptor deployment d, let

$$(8.1) \qquad p(d,j,i) = \left[\prod_{\ell=1}^{i-1} \left(1-q^{d(j,\ell)}\right)\right] q^{d(j,i)} ,$$

where q is the one-on-one penetration probability. Then, $p(d,j,i)$ is the probability that target j is destroyed by the i^{th} attacking weapon committed to it, when the defending side employs the interceptor deployment d. By virtue of our various assumptions, $p(d,j,i) = 0$ if $i > \rho(d,j)$ and

$$\sum_{i=1}^{\rho(d,j)} p(d,j,i) = 1 , \tag{8.2}$$

no matter what the deployment d.

The basis for a Prim-Read deployment can be stated in two ways. In the more physical sense, a deployment d* is a Prim-Read deployment if the payoff function $V(d^*,\cdot)$ defined by

$$V(d^*,i) = \max_{\hat{a}=i} \sum_{j=1}^{T} v(j)P(d^*,j,a(j)) \tag{8.3}$$

is such that its graph is of the form shown in Figure 9: linear on an interval $[0,i^*]$ for some i^* and strictly concave from i^* up to the point i^{**} at which all targets are destroyed. (In (8.3), the maximum is over all allocations $a = (a(1),\dots, a(T))$ of attacking weapons among the targets, for which $\hat{a} = a(1) +\dots+ a(T)$ is equal to i; $P(d^*,j,a(j))$ is the cumulative form of (8.1).) In the more mathematical sense, a deployment d* is said (i.e., defined) to be a Prim-Read deployment if for any two targets j and j' that are *initially defended*,

$$\begin{aligned} v(j)p(d^*,j,1) &= v(j)p(d^*,j,2) \\ &\vdots \\ &= v(j)p(d^*,j,\rho(d^*,j)) \\ &= v(j')p(d^*,j',\rho(d^*,j')) \\ &\vdots \\ &= v(j')p(d^*,j',1). \end{aligned} \tag{8.4}$$

The important question of properly choosing initially defended targets and undefended targets will be discussed in the next Section.

The interpretation of (8.4) is that the expected target value destroyed by an attacking weapon committed to an initially defended target does not vary, either weapon-by-weapon or target-by-target, provided only that no target is certain to have been destroyed. One can infer from (8.2) and (8.4) that the ratio $\rho(d^*,j)/v(j)$ is independent of j (if target j is initially defended). For each choice of scaling parameter α for the common value of these ratios such that $\alpha v(j)$ is an integer for each initially defended target j, there exists a corresponding Prim-Read deployment d^*_α, which is given by

$$\rho(d^*_\alpha,j) = \alpha v(j)$$

and

$$d^*_\alpha(j,i) = -\frac{\log(\alpha v(j)-i+1)}{\log q}, \tag{8.5}$$

where (8.5) is valid for all initially defended targets j and for $i = 1, \ldots, \alpha v(j)$. In Chapters II, IV and V we took α to be an integer k, which, though a simplification, causes no loss of generality.

It can easily be seen from (8.5) that a target of value 1 will not be defended if $\alpha \leq 1$, but may be defended--depending on how the defending side chooses which targets to defend--if $\alpha > 1$. As previously intimated, it is this choice of which targets to defend that is of particular physical importance and interest since in the situations to which this model is envisioned as applicable, the defending side will almost never have sufficient interceptor resources to defend all of the targets. The next Section discusses the mechanism we propose for choosing which targets to defend.

C. THE TARGET DEFENSE PRINCIPLE

Our proposed rule for choosing which targets to defend is the following *target defense principle*, stated first in verbal, then in mathematical form.

Verbal Form. Targets must be defended in order of decreasing value in such a manner that the expected target value destroyed by an attacking weapon committed to an initially defended target, provided that prior destruction of the target not be certain, exceed the value of every initially undefended target.

Mathematical Form. A deployment d satisfies the target defense principle if, whenever j and j' are targets for which $\rho(d,j) \geq 2$ and $\rho(d,j') = 1$ (i.e., target j is initially defended and target j' is not defended at all), then

$$v(j)p(d,j,i) \geq v(j') \tag{8.6}$$

for each $i = 1, \ldots, \rho(d,j)$.

Since it is impossible to satisfy (8.6) if $v(j') > v(j)$, this condition does incorporate the requirement that targets be defended in order of decreasing value.

The reasoning underlying this particular target defense principle is discussed at some length in Chapters I,A and V,A, to which we refer the reader. Briefly, the rationale is that the purpose of defending some targets, but not all, is to force expenditure of as many of the attacking side's resources as possible in order to destroy the defended targets. That is, it is a patent waste of interceptor resources to defend any target so heavily that undefended targets will be attacked instead. We also observe that the stated form of the target defense principle is consistent with the decision-making structure of the problem; under a different decision-making structure the same reasoning might lead to a qualitatively different defense principle.

Recalling that $v(1) \geq \ldots \geq v(T)$, we see that a Prim-Read deployment d* satisfies the target defense principle if:

1) There is some j_0 such that $\rho(d^*,1) \geq 2, \ldots, \rho(d^*,j_0) \geq 2$ but $\rho(d^*,j_0+1) = \ldots = \rho(d^*,T) = 1$; that is, the j_0 most valuable targets are initially defended.

2) For $j \leq j_0$,

$$v(j_0+1) \leq v(j)p(d^*,j,i) \tag{8.7}$$

for $i = 1, \ldots, \rho(d^*,j)$.

Since $v(j_0+1) \geq v(j')$ for all targets j' that are not defended at all, (8.7) does indeed imply (8.6). Since (8.5) implies that

$$p(d^*,j,i) = \frac{1}{\rho(d^*,j)} ,$$

and since by the definition of a Prim-Read deployment, we have $\rho(d^*,j) = \alpha v(j)$ for some α, (8.7) also implies that

$$\alpha \leq \frac{1}{v(j_0+1)} . \tag{8.8}$$

Our principal results, discussed in the next section, are proved under an assumption that equality holds in (8.8).

To conclude this section, we note that a Prim-Read deployment d^* satisfying the target defense principle possesses a payoff function $V(d^*,\cdot)$, as defined by (8.3), that is of the form shown in Figure 9. Somewhat more specifically, $V(d^*,\cdot)$ is linear on an interval $[0,i^*]$ corresponding to destruction of the initially defended targets; here i^* is the number of attacking weapons needed to destroy all of the initially defended targets. Thereafter, $V(d^*,\cdot)$ is strictly concave up to the point i^{**}, which represents the number of attacking weapons necessary (if optimally allocated) to destroy all of the targets. Furthermore, only Prim-Read deployments satisfying the target defense principle have payoff functions of this form.

D. MAIN MATHEMATICAL RESULTS

The most important mathematical results in this paper appear in Chapter V and concern optimality (and nonoptimality) properties of Prim-Read deployments that satisfy the target defense principle, relative to a set of comparable deployments that also satisfy the target defense principle. In Chapter V we impose the hypothesis that

$$v(1) > \ldots > v(T) ,$$

which is not very restrictive, and the hypothesis that $v(j)/v(j+1)$ be an integer for each j. This latter hypothesis is restrictive, but many of our results don't require its full force and we also discuss in Chapter V several ways of minimizing its restrictive effects.

The main results established in Chapter V are the following.

1) Under the hypotheses noted above, for each j_0 there exists a maximally strong Prim-Read deployment d* that satisfies the target defense principle, initially defends targets $1,\ldots,j_0$, and leaves targets $j_0+1,\ldots,T$ undefended. That deployment d* is given by

$$d^*(j,i) = -\frac{\log(n(j)-i+1)}{\log q} , \qquad i=1,\ldots,n(j), \tag{8.9}$$

where $n(j) = v(j)/v(j_0+1)$. In order that this deployment be defined we are forced to assume that $v(j)/v(j_0+1)$ is an integer for each $j \leq j_0$. The result which gives the existence and explicit form of the Prim-Read deployment d* is Theorem (5.7).

2) If d* is the Prim-Read deployment corresponding to initial defense of targets $1, \ldots, j_0$, as given by (8.9), then d* is the unique solution to the optimization problem

(8.10) $$\text{minimize} \max_{1 \le r \le \hat{\rho}(d^*)} \frac{V(d,r)}{r}$$

$$\text{s.t.} \quad I(d) \le I(d^*) \ .$$

In (8.10), the quantity

$$\hat{\rho}(d^*) = \sum_{j=1}^{T} \rho(d^*,j)$$

is the number of attacking weapons needed--against the Prim-Read deployment d^*--in order to destroy all of the targets. Observe that in (8.10) the deployment d is not constrained to satisfy the target defense principle. Therefore, the content of this result, which is Theorem (5.9), is that among all deployments requiring no more interceptors than it does, the Prim-Read deployment uniquely minimizes the maximum target value destroyed per attacking weapon committed.

3) If d^* is the Prim-Read deployment corresponding to defense of targets 1, ..., j_0, as given by (8.9), and if

$$A \le \sum_{j=1}^{j_0} \rho(d^*,j) \ ,$$

then d^* is a solution to the optimization problem

(8.11) minimize $V(d,A)$

s.t. $I(d) \le I(d^*)$

d satisfies the target defense principle and defends targets $1,\ldots,j_0$.

Moreover, if $A \ge \rho(d^*,1) - 1$ (the latter is the maximum number of attacking weapons against which any target is defended), then the Prim-Read deployment d^* is the unique solution to the problem (8.11). This result, which is Theorem (5.17), states that among deployments satisfying the target defense principle and requiring no more interceptors than the Prim-Read deployment d*, the latter minimizes the target value destroyed by an optimized attack consisting of A weapons, provided that A be less than or equal to the total of the Prim-Read prices of the initially defended targets. If, in addition, A is greater than or equal to the maximum number of attacking weapons against which any target is defended, then the Prim-Read deployment uniquely minimizes the target value destroyed.

4) If d* is the Prim-Read deployment corresponding to defense of targets 1, ..., j_0, as given by (8.9), then there exists a deployment d, which can be calculated explicitly, such that d satisfies the target defense principle, such that I(d) = I(d*), and such that

$$V(d,A) = V(d^*,A) - 1 < V(d^*,A)$$

for all A satisfying

$$\sum_{j=1}^{j_0} \rho(d^*,j) < A \leq \sum_{j=1}^{T} \rho(d^*,j) \ .$$

This result, which is Theorem (5.32) and to the proof of which the reader is referred for the explicit construction of the deployment d, states not only that the Prim-Read deployment d* fails to minimize target value destroyed by an attack of size A that is large enough to destroy all of the initially defended targets together with at least one of the initially undefended targets yet not large enough to destroy all the targets, but also that in fact there exists a deployment d satisfying the target defense principle and requiring no more interceptors than does d*, that is uniformly superior to d*--in terms of target value destroyed--for all such attack sizes.

In broad terms, the Prim-Read deployment minimizes target value destroyed by an optimized allocation of attacking weapons if the attack size is at most sufficient to destroy the initially defended targets and does so uniquely if the attack size is not too small. However, the Prim-Read deployment does not minimize target value destroyed by an optimized allocation of attacking weapons if the attack is sufficiently large to be able to destroy at least one of the initially undefended targets. These optimality statements are relative to the set of deployments satisfying the target defense principle and not requiring more interceptors than does the Prim-Read deployment. On the other hand, for the criterion of maximum target value destroyed per attacking weapon committed, the Prim-Read deployment is uniquely optimal relative to the (larger) set of deployments not requiring more interceptors (but also not necessarily satisfying the target defense principle).

We emphasize that in most of the above optimality results the alternative deployments are not constrained to defend precisely the targets 1, ..., j_0.

Table 3 provides a final summary of these important optimality properties.

There is one further interesting optimality property presented in Chapter V, namely Theorem (5.22), which asserts that if d* is the Prim-Read deployment that defends targets 1, ..., j_0, as given by (8.9), then d* is the unique solution to the optimization problem

$$(8.12) \quad \text{maximize} \sum_{j=1}^{j_0} \rho(d,j)$$

s.t. d satisfies the target defense principle.

That is, among all deployments satisfying the target defense principle and initially defending targets 1, ..., j_0, the Prim-Read deployment d* uniquely maximizes the resource commitment thereby imposed on the attacking side in order to destroy the initially defended targets.

Table 3. SUMMARY OF OPTIMALITY PROPERTIES OF PRIM-READ DEPLOYMENTS UNDER THE TARGET DEFENSE PRINCIPLE

Criterion	Optimality Properties of d^* [1]
$V(d,A)$	1) Optimal relative to $\{d: I(d) \leq I(d^*)$ and d satisfies the target defense principle$\}$ if $A < \rho(d^*,1) - 1 = \max_j \rho(d^*,j) - 1$. 2) Uniquely optimal relative to $\{d: I(d) \leq I(d^*)$ and d satisfies the target defense principle$\}$ if $\rho(d^*,1) - 1 \leq A \leq \sum_{j=1}^{j_0} \rho(d^*,j)$. 3) Uniformly inferior to a known deployment d such that $I(d) = I(d^*)$ and d satisfies the target defense principle, if $\sum_{j=1}^{j_0} \rho(d^*,j) < A \leq \sum_{j=1}^{T} \rho(d^*,j)$.
$\max\limits_{1 \leq r \leq \hat{\rho}(d^*)} \frac{V(d,r)}{r}$	Uniquely optimal relative to $\{d: I(d) \leq I(d^*)\}$.

(1) d^* = Prim-Read deployment that defends targets $1, \ldots, j_0$; cf. (8.9).

For further information on the results summarized here, the reader is referred to Chapter V.

E. MATHEMATICAL RESULTS OF SECONDARY IMPORTANCE

As we have stated several times previously, we believe that the results of Chapter V are the most important in the paper because they apply to the physically meaningful case wherein many targets of differing values must be left undefended. The results of Chapters IV and VI pertain to the multiple target case, but only when either all targets are defended or only targets of unit value are left undefended. While this latter situation is not necessarily very interesting physically, it is quite interesting mathematically. The results given in Chapters IV and VI are illuminating complements to the results in Chapter V, for they clarify the significance of the undefended targets and of the target defense principle. Furthermore, these results do develop interesting and further optimality properties of Prim-Read deployments. For these reasons we will now give a brief summary of the results of Chapters IV and VI.

Chapter IV is a general exploration of optimality properties of Prim-Read deployments d_k^* of the form

(8.13) $$d_k^*(j,i) = -\frac{\log(kv(j)-i+1)}{\log q}$$

for $j = 1, \ldots, T$ and $i = 1, \ldots, kv(j) = \rho(d_k^*,j)$, where k is a positive integer. Note that all targets are initially defended unless $k = 1$ and that in this latter case only targets of value 1 are left entirely undefended.

The main optimality results established in Chapter IV are the following.

1) If k is fixed, then the Prim-Read deployment d_k^* given by (8.13) is the unique solution to the optimization problem

$$(8.14) \qquad \begin{aligned} &\text{minimize} \quad I(d) \\ &\text{s.t.} \quad \hat{\rho}(d) = \hat{\rho}(d_k^*) \\ &\max_{1 \le r \le \hat{\rho}(d_k^*)} \frac{V(d,r)}{r} \le \frac{1}{k} . \end{aligned}$$

That is, subject to a fixed total of target prices and a constraint on the maximum target value destroyed per attacking weapon committed, the Prim-Read deployment uniquely requires the fewest interceptors; this assertion is Theorem (4.7).

2) For each k, the Prim-Read deployment d_k^* given by (8.13) is the unique solution to the optimization problem

$$(8.15) \qquad \begin{aligned} &\text{minimize} \max_{1 \le r \le \hat{\rho}(d)} \frac{V(d,r)}{r} \\ &\text{s.t.} \quad \hat{\rho}(d) = \hat{\rho}(d_k^*) \\ &I(d) \le I(d_k^*) . \end{aligned}$$

Of all deployments with the same total of target prices and interceptor requirement not exceeding that of the Prim-Read deployment d_k^*, the latter uniquely minimizes the maximum target value destroyed per attacking weapon committed. This result is Theorem (4.8).

Two interesting characterizations (not involving optimality properties) of Prim-Read deployments are also established in Chapter IV; they are the following.

1) For each k, the Prim-Read deployment d_k^* given by (8.13) is the only deployment d such that $\hat{\rho}(d) = \hat{\rho}(d_k^*)$, $I(d) \le I(d_k^*)$ and $V(d,\cdot)$ is convex on $[0,\hat{\rho}(d)]$. This particular result is demonstrated in Theorem (4.3).

2) For a defensive deployment d and a target-by-target allocation of attacking weapons a, let

$$\tilde{V}(d,a) = \sum_{j=1}^{T} v(j)\left[1 - \prod_{\ell=1}^{a(j)}\left(1-q^{d(j,\ell)}\right)\right] ,$$

which is the expected target value destroyed. Then it is shown in Theorem (4.10) that for each k, among all deployments d for which $\hat{\rho}(d) = \hat{\rho}(d_k^*)$ and $I(d) \leq I(d_k^*)$, only the Prim-Read deployment d_k^* given by (8.13) makes $\tilde{V}(d,a)$ dependent on the allocation a only through its total size $\hat{a} = \sum a(j)$. That is, with the total of target prices and the defending side's interceptor resources fixed, only the Prim-Read deployment d_k^* prevents the attacking side from benefitting by its being able to optimize the allocation of attacking weapons among the targets on the basis of full knowledge of the deployment chosen by the defending side.

Finally, it is shown in Example (4.16) that with k fixed the Prim-Read deployment d_k^* given by (8.13) is <u>not</u> a solution to the optimization problem

(8.16) $$\text{minimize } V(d,k\hat{v})$$

$$\text{s.t. } I(d) \leq I(d^*) ,$$

nor even to the more restricted problem in which interceptor resources are constrained on a target-by-target basis to their Prim-Read levels. However, the Prim-Read deployment d_k^* is robust in the sense of being within 10 percent of the uniform deployment for the problem (8.16).

In Chapter VI we consider the effect of fixing individual target prices at Prim-Read levels, fixing the defending side's interceptor resources at the Prim-Read level, and employing the criterion of target value destroyed, where the number of attacking weapons is treated as a parameter in the analysis. Specifically, we show in Theorem (6.13) that if d^* is the Prim-Read deployment given by (8.13) with $k = 1$, then, except for a small number of known exceptional cases, d^* is a solution to the optimization problem

$$\text{(8.17)} \qquad \begin{aligned} &\text{minimize } V(d,i) \\ &\text{s.t.} \quad \rho(d,j) = \rho(d^*,j) = v(j), \qquad j=1,\ldots,T, \\ &\qquad I(d) \le I(d^*) \end{aligned}$$

if and only if there is a set J_0 of targets that can be exactly destroyed by i attacking weapons in the sense that

$$i = \sum_{j \in J_0} v(j) \ .$$

We will not re-state the results of the remaining Chapters, namely Chapters II, III, and VII. In Chapter II we show how to derive the form of Prim-Read deployments from hypotheses of the form (8.4). Chapter III develops optimality properties of Prim-Read deployments in the single target case. While this case is essentially meaningless in terms of the physical problems that motivated our research, it is important because the results and the techniques of proof appearing in Chapter III provide (as it turns out) the correct point of view from which to approach the multiple target case. Finally, Chapter VII derives the forms and investigates some extremely basic properties of Prim-Read deployments when certain of the hypotheses given in (1.1) and (1.2) are weakened. Specifically, we calculate Prim-Read deployments for the cases of:

1) Target dependent intercept probabilities;

2) Initially unreliable attacking weapons that are launched but may or may not reach the vicinity of the target and require interceptors to be deployed against them;

3) Terminally unreliable attacking weapons that may fail to destroy the target despite their not having been intercepted;

4) Alternative forms of attrition equations for the interaction involving one attacking weapon and several interceptors.

We refer the reader to Chapter VII or to its summary for the explicit results obtained.

Table 4 below lists the major and many of the secondary results of the paper grouped by the general form of the result. Using it, the reader may easily compare and contrast analogous results in different cases.

F. RELATED RESEARCH ON INTERCEPTOR ALLOCATION PROBLEMS

We emphasize from the beginning that this section is not intended to be (and is not) a comprehensive survey of the vast existing literature on problems related to allocation of combat resources, or even of problems related to allocation of interceptors as a defense against attacking missiles. Rather, the following discussion

1) concentrates on work related directly or nearly directly to Prim-Read deployments;

2) emphasizes papers dealing primarily with mathematical optimality questions as opposed to the "real-world" decision problem giving rise to those questions;

3) entirely excludes consideration of the physical nature of the interception process itself; and

4) draws almost exclusively upon the open literature.

So far as we can determine, the defensive deployment we have called the Prim-Read deployment was first proposed by W.T. Read, Jr., in [15]. R.C. Prim seemingly performed much of the fundamental mathematical research leading to [15],

Table 4. ANALOGOUS RESULTS FOR DIFFERENT CASES

Kind of Result	Specific Instances
Form of Prim-Read deployment	Single target: Theorem (2.6) Multiple target: Theorem (2.13) Under target defense principle: Theorem (5.7) With target dependent intercept probabilities: Theorem (7.2) With initially unreliable attacking weapons: Theorem (7.5) With terminally unreliable attacking weapons: Theorems (7.11), (7.16) With general attrition function: Theorem (7.19)
Convexity characterization of Prim-Read deployment	Single target: Theorem (3.1) Multiple target: Theorem (4.3)
Optimality properties for target value destroyed	Single target: Example (3.15) Multiple target: Example (4.16); Theorem (6.13) Under target defense principle: Theorems (5.17) and (5.32)
Optimality properties for target value destroyed per attacking weapon committed	Single target: Theorem (3.11) Multiple target: Theorem (4.8) Under target defense principle: Theorem (5.9)
Optimality properties for interceptor requirement	Single target: Theorem (3.9) Multiple target: Theorem (4.7)
Optimality property for target prices	Under target defense principle: Theorem (5.22)

despite his not being named as a co-author. Attribution of the deployment to both men seems as fair as it is established. In [15] the Prim-Read deployment is proposed on the grounds that:

1) Given an interceptor allocation, the attacking side should commit that number of weapons which maximizes the cumulative penetration probability per weapon committed. That is, the attacking side should choose i* such that

$$\frac{P(i^*)}{i^*} = \max_{i} \frac{P(i)}{i} ,$$

where P(i) is the probability that one of the first i attacking weapons destroys the target. This is one of the criteria we have considered in this paper.

2) Under the assumption that this will be done, the defending side should in some undefined way attempt to make all ways of maximizing P(i)/i the same to the attacking side.

The deployment satisfying (2.5) is then asserted to achieve the defending side's objective. We have seen in the preceding Chapters that, in several senses, it does.

In [15], Read also mentions that his proposed deployment makes the probability of scoring a kill the same for all attacking weapons. It is this property (and optimality properties heuristically inferred to follow from it) that seems to motivate references to and applications of the Prim-Read deployment during the 1960s. As representative examples we mention the monograph of Eckler and Burr [8], the two papers of Berger [2,3] and the paper of Everett [9]. In [8] and [9] the Prim-Read deployment is referred to as the constant value decrement (CVD) deployment, especially in the multiple target case. A recurring conjecture in these works is that uniformity--as expressed in (2.12)--implies optimality according to some objective function. In this paper we have formulated and verified several specific versions of this conjecture.

In the Air Force Syllabus of Equations for Force Effectiveness Analysis [19] there appear several detailed derivations of formulas associated with the Prim-Read deployment, such as the interceptor requirement I* given in (2.8). Consideration is also given to possible use of decoys by the attacking side. In [9], Everett treats the CVD deployment for "dilute" attacks, in which a successfully penetrating attacking weapon is not certain to destroy a target. His results appear to differ from those we present in Chapter VII.

Essentially none of these papers treats the multiple target case under assumption of the target defense principle, and none of them contains the detailed derivations and interpretations of optimality properties that are given in this paper.

The general problem of allocating interceptors and attacking weapons among targets has a lengthy and detailed history; work on it before 1971 is surveyed by Matlin in [11], and much general information is contained in the Air Force Syllabus [19] and in the books of Danskin [6] and Dresher [7]. Our particular problem, with sequential arrivals of attacking weapons and local defenses (i.e., interceptors that cannot be reassigned), is discussed in the papers of Berger [2,3], Gorfinkel [10], Perkins [13], and Shumate and Howard [17]. While some of these authors obtain results similar to ours, none of them is particularly careful about either formulation or establishment of optimality results. Nonetheless, we suggest that these papers be consulted in order that the reader develop a better understanding of the problem.

It is interesting to note, however, that Perkins in [13] asserts that (an ill-defined) optimality obtains when the ratio of attacking strength to defending strength is the same at all targets, which does not occur for Prim-Read deployments except when all targets have the same value.

Several of the papers referred to above put forth concepts of the role of defenses and the goal of committing resources to defensive purposes that are similar to those followed in this paper. This is particularly so in regard to the role of the target defense principle of Chapter V in the situation in which many targets must be left undefended. For example, Read in [15] states that the purpose of a defense is to raise the price of destroying some targets to the point that not many targets can be attacked. The defending side, then, does not benefit in terms of the targets that are defended and attacked but rather in terms of the undefended (and, under the target defense principle, less valuable) targets, which the attacking side--it is hoped--will not have sufficient resources to attack. Brodheim, Herzer, and Russ assert in [5] that "...an active defense system cannot prevent a determined and powerful offense from destroying a given number of targets. Facing such an opponent, the defense objective is generally to maximize the offensive cost of such an attack." It is also well recognized that this entails defending targets in order of decreasing value [19].

The idea that the defending side should choose its deployment under the assumption that the attacking side will thereafter optimize against it is a central idea of min-max theory [6,7]. Specifically in terms of defense deployment problems, one finds in [14] the statement by Pugh that the defending side should "...choose that allocation of defense resources that will give best protection when an attack is optimized against whatever defenses are chosen." The concept that the proper decision criterion is expected target value destroyed is so common and accepted that it defies attribution to any individual, while that the criterion should be (expected) target value destroyed per attacking weapon committed is widely proposed; cf. [2,3,13, 19], for example.

It is virtually universally agreed that appropriate optimality criteria involve expectations. This use of expectations is subtly related to the target defense principle: when the criterion involves expected target value destroyed (regardless of whether normalized by the number of attacking weapons committed) it is sensible to defend the more valuable targets in order to force expenditure of attacking weapons to destroy those targets. It would not make sense to defend any target (even the most valuable one) so heavily that it will not be attacked at all, for the attacking side would then simply attack the undefended targets, which renders the interceptors essentially useless.

Employment of the concept of the price of a target, in the sense of price being a measure of the resources that the attacking side must commit in order to bring about a certain level of target destruction, goes back at least to Read [15]. In [15], the price of the target is defined as

$$\sigma = \left(\max_{i} \frac{P(i)}{i} \right)^{-1} = \left(\frac{P(i^*)}{i^*} \right)^{-1} .$$

For Prim-Read deployments this coincides with price as defined by (2.1). For other deployments it seems best interpreted through the equation

$$P(i^*) = \frac{i^*}{\sigma} ,$$

i.e., σ attacking weapons, each with the same probability $\frac{1}{\sigma}$ of destroying the target, would destroy the target with certainty under a linear payoff function.

Similar, although not always identical, definitions of price may be found in the Air Force <u>Syllabus of Equations for Force Effectiveness Analysis</u> [19] and in the papers of Berger [2,3], Everett [9] and Shumate and Howard [17].

The idea that target prices, in the multiple target case, should be proportional to target values has been harder for us to trace. It appears with no justification in the Air Force

Syllabus [19] and is assumed by Shumate and Howard [17], again with absolutely no justification, to be optimal in the sense of minimizing maximum target damage per attacking weapon. In view of the allegation by some authors that optimal defensive deployments require that interceptor totals (or possibly both interceptor totals and attacking weapon commitments) be proportional to target values, it would be interesting to have a more detailed account of the development of these differing opinions.

Min-max aspects of the missile allocation problem are treated in the books of Danskin [6] and Dresher [7] and in the papers by Pearsall [12] and Soland [18]. None of these authors, however, gives explicit consideration of interceptor deployments against sequentially arriving penetrators.

G. POSSIBLE QUESTIONS FOR FURTHER RESEARCH

The following are a few potentially significant facets of the basic problem treated in this paper that are discussed either not at all or too shallowly. Further research on the problem might profitably be directed to these questions.

1) Except implicitly when dealing with minimizing the number of interceptors required or when imposing on the defending side a constraint on the number of interceptors available, we have not included any discussion of cost structures. Especially, we have not considered how certain expenditures by one side can force the other side into potentially equally or more costly responses. In general this is a neglected part of military modelling: to what extent is a resource expenditure valuable because it imposes a compensating expenditure on the other side, which then may be unable to meet other, important goals? In the context of the problem treated in this paper, the defending side might wish to increase the prices of the targets in order to attempt to force

the attacking side to allocate more attacking weapons to destruction of those targets.

The whole question of attempts to force on the opposing side resource expenditures or allocations it might not otherwise undertake seems very worthy of further study in essentially all combat contexts.

Finally, we do not consider possibly large fixed costs (e.g., for radar) that may render infeasible defense of certain low value targets that would otherwise be defended. Fixed costs could also create difficulties in completing defense of a more valuable target.

2) We have considered only local (i.e., point) defenses in this paper. The case of mixed area/point defenses is physically meaningful and mathematically interesting, and therefore merits some attention.

3) For a Prim-Read deployment, the payoff function is linear down to the origin. However in reality it may be known (or the defense may be willing to believe) that the attack size is certain to exceed some lower bound A_{min}. In this case the Prim-Read deployment is a misuse of resources because it protects against attack sizes less than A_{min}, which cannot occur. Perhaps a different deployment using the same number of interceptors yields a payoff function that is linear over the range of possible attack sizes and lower over the same range than the payoff function of the Prim-Read deployment. Such "modified Prim-Read deployments" are certainly worthy of additional research effort.

4) The assumption that attacking weapons arrive sequentially at each target is so fundamental to our model that a major alteration of it would lead to an entirely new model. Nonetheless, minor but physically meaningful variations of it ought to be investigated.

5) The model does not allow adaptive resource reallocations by either side. Attacking weapons directed at a target that is known to have been destroyed cannot be re-targetted; and interceptors deployed at a destroyed target cannot be reassigned to defense of a target that is still surviving. For many potential applications of the model this seems to be a serious weakness. It would be extremely interesting to investigate the effect of allowing various forms of adaptive reassignment of attacking weapons and interceptors. Representation of shoot-look-shoot capability for the attacking side would be particularly important.

6) Many discreteness (i.e., non-integrality) difficulties have been glossed over, especially in Chapter V. While in some cases these difficulties are minor, in others they may be severe. Further work is necessary in order to elucidate the solutions. An alternative approach worth pursuing is suggested by L. B. Anderson in "Nationwide Defense Against Nuclear Weapons: Definition of Randy-Watch Deployments" (Working Paper WP-11, Project 2371, Institute for Defense Analyses, 1981).

7) Some physical effects that may be of interest in specific applications either are not represented explicitly in the model or are specifically assumed not to occur. Among such effects are decoys, group attacks, partial target damage, collateral damage, clock time, geography, undetected attacking weapons and possible re-use of interceptors that fail to engage (as opposed to engage, but fail to destroy) an attacking weapon.

8) Applicability of the basic model specified by (1.1) and (1.2), or of the variations thereon which we described in Chapter VII, to specific combat situations has been mentioned only briefly. In carrying out our research we have thought of the model as appropriate to defense of (essentially point) targets against attacking missiles. Possibly it is not applicable even to this situation, but possibly it is applicable not only here but also to other combat processes. We have endeavored--by being explicit

about our underlying assumptions and their interpretations--to give the reader at least one basis for judging possible applicability to specific forms of combat. In any event, further study would be required to establish applicability or credibility of the model in specific physical situations.

9) The target defense principle should be studied more carefully. Despite its appeal, we have imposed it as an assumption; if it could be shown to be satisfied (or to be nearly satisfied) by solutions to problems of the form (4.24), the grounds for it would be much more compelling.

REFERENCES

[1] Anderson, L.B., Blankenship, J.W., and Karr, A.F. (1978). "An Attrition Model for Penetration Processes." Submitted to Operations Research.

[2] Berger, H.M. (1965). "Penetration of Terminal Defenses by Interceptor Exhaustion and Leakage" (U). Pen-X Paper 68, Institute for Defense Analyses. CONFIDENTIAL.

[3] Berger, H.M. (1965). "Effect of Preferential Defense of an Urban/Industrial Target on Penetration by Exhaustion and Leakage" (U). Pen-X Paper 72, Institute for Defense Analyses. SECRET.

[4] Blackwell, D., and Girshick, M.A. (1954). Theory of Games and Statistical Decisions. Wiley, New York.

[5] Brodheim, E., Herzer, I., and Russ, L.M. (1967). "A General Dynamic Model for Air Defense." Operations Research 15, 799-796.

[6] Danskin, J.M. (1967). The Theory of Max-Min. Springer-Verlag, Berlin-Heidelberg-New York.

[7] Dresher, M. (1961). Games of Strategy: Theory and Applications. Prentice-Hall, Englewood Cliffs, NJ.

[8] Eckler, R.A., and Burr, S. (1972). Mathematical Models of Target Coverage and Missile Allocation. Military Operations Research Society, Washington, D.C.

[9] Everett, H. (1968). "Defense Models XI: Exhaustion of Interceptor Supplies." Paper 17, Lambda Corporation.

[10] Gorfinkel, M. (1963). "A Decision Theory Approach to Missile Defense. Operations Research 11, 199-209.

[11] Matlin, S. (1970). "A Review of the Literature on the Missile Allocation Problem." Operations Research 18, 334-373.

[12] Pearsall, E.S. (1976). "A Lagrange Multiplier Method for Certain Constrained Min-Max Problems." Operations Research 24, 70-91.

[13] Perkins, F.M. (1961) "Optimum Weapon Deployment for Nuclear Attack." Operations Res. 9, 77-91.

[14] Pugh, G.E. (1964). "Lagrange Multipliers and the Optimal Allocation of Defensive Resources." Operation Res. 12, 543-567.

[15] Read, W.T., Jr. (1958). "Tactics and Deployment for Anti-Missile Defense." (U). Bell Telephone Laboratories, Whippany, NJ. CONFIDENTIAL.

[16] Rudin, W. (1976). Principles of Mathematical Analysis, 3rd ed. McGraw-Hill, New York.

[17] Shumate, K.C., and Howard, G.T. (1974). "A Proportional Defense Model." Naval Res. Logist. Quart. 21, 69-78.

[18] Soland, R.M. (1973). "Optimal Defensive Missile Allocation: A Discrete Min-Max Problem." Operations Res. 21, 590-596.

[19] United States Air Force, Assistant Chief of Staff, Studies and Analysis (1970). A Syllabus of Equations for Force Effectiveness Analysis, 1st ed. UNCLASSIFIED.

[20] (No author listed) 1979 Information Please Almanac, New York, Simon and Schuster.

INDEX OF NOTATION

The following is a list of the notation that appears throughout the paper; the page number associated with each entry is that of its first occurrence. Additional, specialized notation is introduced in (and, in each case, used only within) Chapters III, IV, V, VII and Appendix B.

Notation for the Single Target Case

Notation	Page	Meaning
$d(i)$	23	Number of interceptors assigned to i^{th} attacking weapon by deployment d
d^*	25	Prim-Read deployment
$\tilde{d}$	47	Uniform deployment
$I(d)$	36	Interceptor requirement for deployment d
$p(d,i)$	24	Probability that i^{th} attacking weapon destroys target, given deployment d
$P(d,i)$	35	Probability that one of attacking weapons 1, ..., i destroys target, given deployment d
q	12	One-on-one penetration probability
$\rho(d)$	24	Price of target for deployment d

Notation for the Multiple Target Case

Notation	Page	Meaning
$a(j)$	55	Number of attacking weapons assigned to target j, given attack a
$\hat{a}$	55	Total number of attacking weapons at all targets, given attack a
$d(j,i)$	29	Number of interceptors assigned by deployment d to i^{th} attacking weapon at target j

Notation	Page	Meaning
d^*	30	Prim-Read deployment
$\tilde{d}$	62	Proportional deployment
$I(d)$	31	Interceptor requirement for deployment d
i^*	4	Number of attacking weapons needed to destroy defended targets
i^{**}	4	Number of attacking weapons needed to destroy all targets
j_0	80	Number of defended targets
k	30	Scaling factor for Prim-Read deployments
$n(j)$	82	Ratio of value of target j to value of most valuable undefended target
$p(d,j,i)$	30	Probability that i^{th} attacking weapon at target j destroys it, given deployment d
$P(d,j,i)$	89	Probability that target j is destroyed by one of attacking weapons 1,...,i at that target, given deployment d
p	124	Initial reliability of attacking weapons
$\tilde{p}$	128	Terminal reliability of attacking weapons
q	12	One-on-one penetration probability
$V(d,i)$	55	Maximum target value that can be destroyed by i attacking weapons, given deployment d
$\tilde{V}(d,a)$	59	Target value destroyed, given attack a and interceptor deployment d
$v(j)$	29	Value of target j
$\hat{v}$	54	Total value of all targets
α	147	Scaling factor for Prim-Read deployments
$\rho(d,j)$	29	Price of target j, given deployment d
$\hat{\rho}(d)$	54	Total of all target prices, given deployment d

APPENDIX A. EXAMPLES

EXAMPLES

In this Appendix we construct some Prim-Read deployments for defense of the principal centers of population in the United States and compare them with corresponding uniform deployments.

Table A-1 below lists (according to [20]) the twenty-nine metropolitan regions in the U.S. with populations greater than one million. The data were constructed from a table of populations of SMSA's (Standard Metropolitan Statistical Areas) but SMSA's that are nearby to one another have been combined. The largest metropolitan area with population less than one million has also been included.

Our first example deals with the case where the value of a region as a target is a function of its population in the manner indicated in Figure A-1. That is, a region with population p has value V(p) given by

$$V(p) = 1 \qquad \text{if } p < 10^6$$
$$= 8p/10^6 \qquad \text{if } p \geq 10^6.$$

One might employ such a function on the rationale that regions of size less than one million support insufficient heavy industry to be really important targets; each has the same value, which is one-eighth that of a region of size one million. The value one-eighth is, of course, arbitrary, but is convenient for illustrative purposes.

Table A-1. REGIONS, POPULATIONS, VALUES

	Region	Population in Thousands	Value
1)	New York	6691	134
2)	Los Angeles	9595	77
3)	Chicago-Milwaukee	9014	72
4)	Philadelphia	6531	52
5)	Detroit-Toledo	5700	46
6)	Baltimore-Washington	4978	40
7)	San Francisco	4973	40
8)	Cleveland	3279	26
9)	Cincinnati	3101	25
10)	Boston	2899	23
11)	St. Louis	2410	19
12)	Pittsburgh	2401	19
13)	Dallas-Ft. Worth	2377	19
14)	Buffalo-Rochester	2310	18
15)	Houston	1999	16
16)	Minneapolis-St. Paul	1965	16
17)	Miami	1887	15
18)	Atlanta	1597	13
19)	Tampa-Orlando	1541	12
20)	Seattle	1421	11
21)	San Diego	1357	11
22)	Richmond-Norfolk	1274	10
23)	Kansas City	1271	10
24)	Hartford-Springfield	1261	10
25)	Denver	1237	10
26)	Indianapolis	1109	9
27)	New Orleans	1045	8
28)	Columbus	1017	8
29)	Portland	1009	8
30)	Phoenix	967	7

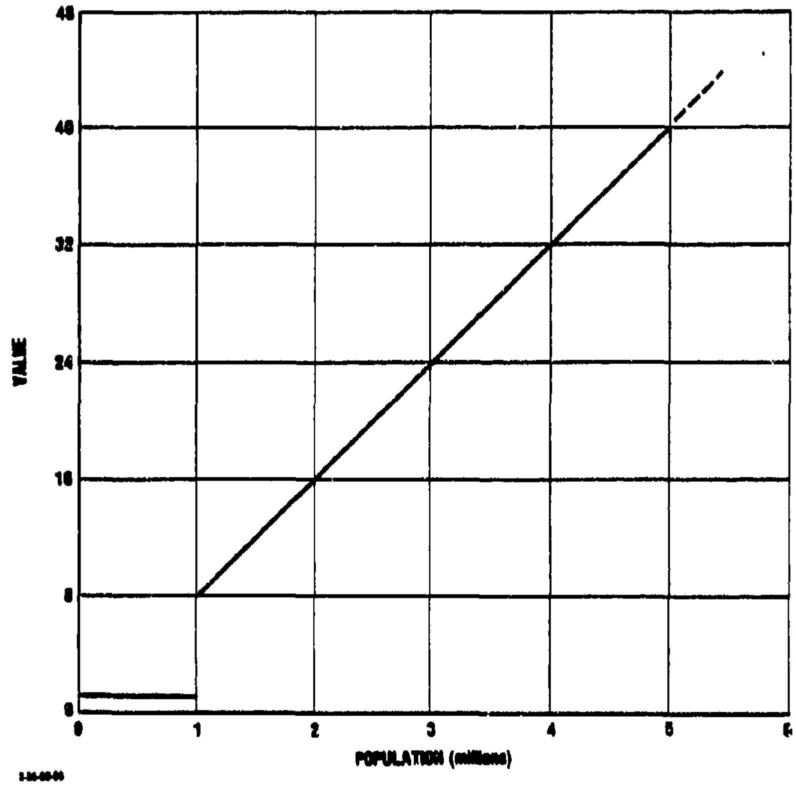


Figure A-1. TARGET VALUE AS A FUNCTION OF POPULATION FOR EXAMPLE 1

Application of this function leads to the target values in Table A-1, each rounded to the nearest integer.

It should be carefully noted that this particular value function has the rather arbitrary cutoff at the population level of 10^6. This has been done in order to ensure that the Prim-Read deployments we construct below will satisfy the target defense principle. In reality, of course, the U.S. might be forced because of limited interceptor resources to implement a deployment that does not satisfy the target defense principle. Some of our theoretical results do not apply to Prim-Read deployments not satisfying the target defense principle, but a Prim-Read deployment might be chosen nevertheless, on the basis that it is likely to be nearly optimal, especially against an attack of unknown size. Finally, we note that these difficulties are a result of the discreteness of targets and attacking weapons, and do not arise in the idealized situation depicted in Figure 5.

Suppose that the U.S., as defending side, chooses to construct a Prim-Read deployment that defends targets 1, ..., 29 and satisfies the target defense principle. To do so, according to the discussion in Sections B and C of Chapter VIII, it will first choose a scaling parameter α such that

1) $\alpha \leq 1/v(30) = 1$;
2) $\alpha v(j)$ is an integer for $j = 1, \ldots, 29$.

Then, it will construct the Prim-Read deployment d_α^* given by

$$d_\alpha^*(j,k) = -\frac{\log(\alpha v(j)-k+1)}{\log q} \tag{A.1}$$

for $1 \leq j \leq 29$ and $1 \leq k \leq \rho(d_\alpha^*,j) = \alpha v(j)$, where the values $v(j)$ are taken from Table A-1.

Below we present results for the cases $\alpha = 1$, $\alpha = 1/2$ and $\alpha = 1/4$; these three cases appear as Tables A-2, A-3, and A-4, respectively. In all cases, $q = 0.8$. In each Table we give target prices, Prim-Read interceptor totals for each target, and target-by-target interceptor allocations if the total number of interceptors (at all targets) for the Prim-Read deployment were allocated among the initially defended targets in direct proportion to their respective values. In symbols,

$$I(d_\alpha^*,j) = \sum_{k=1}^{\rho(d_\alpha^*,j)} d_\alpha^*(j,k)$$

Table A-2. RESULTS FOR EXAMPLE 1 WHEN $\alpha = 1$

Target	Price	$I(d_\alpha^*, j)$	$I(\tilde{d}, j)$
1	134	2356	1648
2	77	1168	947
3	72	1071	886
4	52	701	640
5	46	596	566
6	40	494	492
7	40	494	492
8	26	275	320
9	25	260	308
10	23	231	283
11	19	176	234
12	19	176	234
13	19	176	234
14	18	163	221
15	16	137	197
16	16	137	197
17	15	125	185
18	13	101	160
19	12	90	148
20	11	79	135
21	11	79	135
22	10	68	123
23	10	68	123
24	10	68	123
25	10	68	123
26	9	57	111
27	8	48	98
28	8	48	98
29	8	48	98
30	1	0	0
Totals	778	9558	9559

Table A-3. RESULTS FOR EXAMPLE 1 WHEN $\alpha = 0.5$

Target	Price	$I(d_\alpha^*, j)$	$I(\tilde{d}, j)$
1	67	977	635
2	38	461	360
3	36	429	341
4	26	275	246
5	23	231	218
6	20	190	190
7	20	190	190
8	13	101	123
9	12	90	114
10	12	90	114
11	10	68	95
12	10	68	95
13	10	68	95
14	9	57	85
15	8	48	76
16	8	48	76
17	8	48	76
18	6	29	57
19	6	29	57
20	6	29	57
21	5	21	47
22	5	21	47
23	5	21	47
24	5	21	47
25	5	21	47
26	4	14	38
27	4	14	38
28	4	14	38
29	4	14	38
30	1	0	0
Totals	390	3687	3687

Table A-4. RESULTS FOR EXAMPLE 1 WHEN α = 0.25

Target	Price	$I(d^*_\alpha, j)$	$I(\tilde{d}, j)$
1	34	397	233
2	19	176	130
3	18	163	123
4	13	101	89
5	12	90	82
6	10	68	68
7	10	68	68
8	7	38	48
9	6	29	41
10	6	29	41
11	5	21	34
12	5	21	34
13	5	21	34
14	5	21	34
15	4	14	27
16	4	14	27
17	4	14	27
18	3	8	21
19	3	8	21
20	3	8	21
21	3	8	21
22	3	8	21
23	3	8	21
24	3	8	21
25	3	8	21
26	2	3	14
27	2	3	14
28	2	3	14
29	2	3	14
30	1	0	0
Totals	200	1361	1364

is the number of interceptors deployed at target j by the Prim-Read deployment d* and

$$\text{(A.2)} \qquad I(\tilde{d},j) = \frac{v(j)}{\sum_{\ell=1}^{j_0} v(\ell)} \sum_{m=1}^{j_0} I(d_\alpha^*,m)$$

is the number of interceptors allotted to target j by the proportional deployment $\tilde{d}$.

A few explanatory comments are in order before the Tables are discussed.

1) There are both inter- and intra-Table discrepancies arising from our rounding target prices and interceptor allocations to be integers. For example, the sum of target prices for $\alpha = 1$ should be twice that for $\alpha = 1/2$, which in turn should be twice that for $\alpha = 1/4$. Also, within Tables A-2 and A-4 we have

$$\sum_j I(d_\alpha^*,j) \neq \sum_j I(\tilde{d},j) \ ,$$

even though (A.2) implies that equality must hold in principle.

2) Except when $\alpha = 1$, the constraint that $\alpha v(j)$ be an integer for $j = 1, \ldots, 29$ is not satisfied. What we have done, as would be done (without any significant harm) in any practical situation, is to round the values $\alpha v(j)$ to the nearest integer. (An alternative would be to always round to the next higher integer, which would prevent the target defense principle from being violated simply because of rounding.)

3) The value $q = 0.8$ (recall that q is the one-on-one penetration probability) is possibly too high. Later we will consider the case $q = 0.2$.

Perhaps the most striking features of Tables A-2, A-3 and A-4 are the significant but systematic differences between the Prim-Read deployments d^*_α and the corresponding proportional allocations $\tilde{d}$. Although a theoretical discussion of the exact nature of these differences is deferred to Appendix B, the following aspects are especially noteworthy.

1) The Prim-Read deployment devotes relatively more interceptor resources to defense of valuable targets and relatively fewer to defense of less valuable targets. For example, for $\alpha = 1$ the ratio $I(d^*_\alpha,j)/I(\tilde{d},j)$ ranges from 1.43 for $j = 1$ to 0.49 when $j = 29$.

2) As α decreases the discrepancy between d^*_α and $\tilde{d}$ increases so that, for example, when $\alpha = 0.25$ the ratios $I(d^*_\alpha,j)/I(\tilde{d},j)$ range over the interval [0.21,1.71], as compared to [0.49,1.43] when $\alpha = 1$.

3) For a fixed value of α, the ratio $I(d^*_\alpha,j)/I(\tilde{d},j)$ is monotonically decreasing as j increases.

4) The value of j at which the ratios $I(d^*_\alpha,j)/I(\tilde{d},j)$ pass from a greater than one to less than one is independent of α.

To illustrate these points in the numerical sense, we present in Table A-5 a listing of ratios $I(d^*_\alpha,j)/I(\tilde{d},j)$ for the cases $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0.25$ treated above. The Table fully confirms the four points just listed.

As previously mentioned, we give in Appendix B theoretical statements of these properties and mathematical verifications of those statements. Here, however, we wish to discuss some heuristic arguments for their validity.

Table A-5. COMPARISON OF PRIM-READ AND PROPORTIONAL DEPLOYMENTS

Target	$I(d_\alpha^*,j)/I(\tilde{d},j)$ for $\alpha = 1$	$I(d_\alpha^*,j)/I(\tilde{d},j)$ for $\alpha = 0.5$	$I(d_\alpha^*,j)/I(\tilde{d},j)$ for $\alpha = 0.25$
1	1.4296	1.5385	1.7039
2	1.2334	1.2806	1.3538
3	1.2088	1.2581	1.3252
4	1.0953	1.1179	1.1348
5	1.0530	1.0596	1.0976
6	1.0041	1.0000	1.0000
7	1.0041	1.0000	1.0000
8	.8594	.8211	.7911
9	.8442	.7895	.7073
10	.8163	.7895	.7073
11	.7521	.7158	.6176
12	.7521	.7158	.6176
13	.7521	.7158	.6176
14	.7376	.6706	.6176
15	.6954	.6184	.5185
16	.6954	.6184	.5185
17	.6758	.6184	.5185
18	.6312	.5088	.3810
19	.6081	.5088	.3810
20	.5852	.5088	.3810
21	.5852	.4468	.3810
22	.5528	.4468	.3810
23	.5528	.4468	.3810
24	.5528	.4468	.3810
25	.5528	.4468	.3810
26	.5135	.3684	.2143
27	.4898	.3684	.2143
28	.4898	.3684	.2143
29	.4898	.3684	.2143

Concerning the first (and, we believe, most important) point, we believe that it holds as a consequence of the target defense principle and of Theorem (5.17). Pareto optimality of the Prim-Read deployment for attack sizes less than or equal to $i^*(d^*)$, which is the content of Theorem (5.17), means that a Prim-Read deployment is effective at protecting against an attack of *unknown* size provided that the size not exceed $i^*(d^*)$. The proportional deployment $\tilde{d}$, on the other hand, can be shown (cf. Chapter IV for some related results) to be optimal against an attack of *known* size $i^*(d^*)$ when attacking weapons are allocated among the targets as shown in the "Price" columns of Tables A-2, A-3 and A-4, but may not be optimal against an attack of smaller size. The Prim-Read deployment, however, protects against precisely that possibility (of an attack of size less than $i^*(d^*)$), and does so by devoting more resources to defense of the more valuable targets. That is, the Prim-Read response to uncertainty about the size of a potential attack is to defend the higher value targets more heavily.

To illustrate numerically, suppose that under the proportional deployment $\tilde{d}$, those interceptors assigned to each target j are deployed uniformly against the first $\alpha v(j)(=\rho(d^*_\alpha,j))$ attacking weapons arriving there. That is,

$$\tilde{d}(j,k) = \frac{I(\tilde{d},j)}{\alpha v(j)}, \qquad k=1,\ldots,\alpha v(j),$$

which implies in particular that

$$\rho(\tilde{d},j) = \rho(d^*_\alpha,j) + 1,$$

for $j = 1, \ldots, 29$. Suppose that $\alpha = 1$ (Table A-2) and consider an attack of size $a = 211$ $(=\rho(d^*_\alpha,1) + \rho(d^*_\alpha,2))$. Then, of course,

$$V(d^*_\alpha,a) = 211,$$

while by trial and error one can show that

(A.3) $$V(\tilde{d},a) \geq 408 .$$

Similar, but more pronounced, results hold for smaller values of α. When $\alpha = 0.5$ and $a = 105$ $(=\rho(d^*_\alpha,1) + \rho(d^*_\alpha,2))$ then

$$V(d^*_\alpha,a) = 105 ,$$

but

(A.4) $$V(\tilde{d},a) \geq 211 .$$

We also emphasize that the bounds in (A.3) and (A.4) are far from sharp; they were obtained simply by experimental calculations.

The preceding results demonstrate rather graphically the extent to which the Prim-Read deployment mitigates the undesirable effects that could otherwise ensue when the defending side lacks knowledge of the size of an attack.

To make the point once again, the differences between Prim-Read and proportional deployments are consequences of the fact that the former protect effectively against a range of attack sizes without being optimal against any single attack size, whereas the latter are optimal for certain attack sizes but extremely inefficient against other, smaller attack sizes.

The remaining three points will be considered only briefly. We do not have a really good heuristic explanation of why the discrepancy between d^*_α and $\tilde{d}$ increases as α decreases; however, the effect is not spurious, since we give a mathematical verification in Appendix B. Monotonicity of $I(d^*_\alpha,j)/I(\tilde{d},j)$ as a function of j is not unexpected in view of the discussion of the first point. Finally, independence of the cross-over point from the value of α is pleasant but seems to have no clear physical basis.

To complete this example we give in Table A-6 the results corresponding to those in Table A-2, except that now $q = 0.2$. No further comments are really necessary. Table A-7 shows

Table A-6. RESULTS FOR EXAMPLE 1 WHEN $q = 0.2$ AND $\alpha = 1$

Target	Price	$I(d^*_\alpha,j)$	$I(\tilde{d},j)$
1	134	327	228
2	77	162	131
3	72	148	123
4	52	97	88
5	46	83	78
6	40	69	68
7	40	69	68
8	26	38	44
9	25	36	43
10	23	32	39
11	19	24	32
12	19	24	32
13	19	24	32
14	18	23	31
15	16	19	27
16	16	19	27
17	15	17	26
18	13	14	22
19	12	12	20
20	11	11	19
21	11	11	19
22	10	9	17
23	10	9	17
24	10	9	17
25	10	9	17
26	9	8	15
27	8	7	14
28	8	7	14
29	8	7	14
30	1	0	0
Totals	778	1324	1322

Table A-7. PRIM-READ INTERCEPTOR REQUIREMENTS FOR DIFFERING PENETRATION PROBABILITIES

Penetration Probability q	$I(d^*_\alpha)$ for $\alpha = 1$
.95	41571
.90	20238
.85	13120
.80	9558
.75	7412
.70	5978
.60	4174
.50	3076
.40	2327
.30	1771
.20	1324
.10	926
.05	712

the dependence of only the total Prim-Read interceptor deployment $I(d^*_\alpha) = \sum_j I(d^*_\alpha, j)$ on the penetration probability q. Incidentally, as will be shown in Appendix B, all the observations above concerning the ratios $I(d^*_\alpha, j)/I(\tilde{d}, j)$ are entirely independent of the value of q. However, $I(d^*_\alpha)$ does depend on q as illustrated in the Table.

Table A-7 shows that once interceptors are sufficiently effective that q < .6, there is relatively less payoff from further improvements in effectiveness. However, if q is as high as .9, then it seems wise to pursue improvements in the interceptors. Of course in reality any such decision must be made in light of the costs of various alternative choices.

In our second example, which we present much more briefly than the preceding example, we assume that target value as a function of population is given by the function depicted in Figure A-2. Symbolically, we have

$$V(p) = 1 \qquad \text{if } p < 10^6$$
$$= \left(\frac{p}{10^6}\right)^2 \qquad \text{if } p \geq 10^6 .$$

The essential difference between the function and that used in the first example is that this one is quadratic, whereas that of Example 1 is linear.

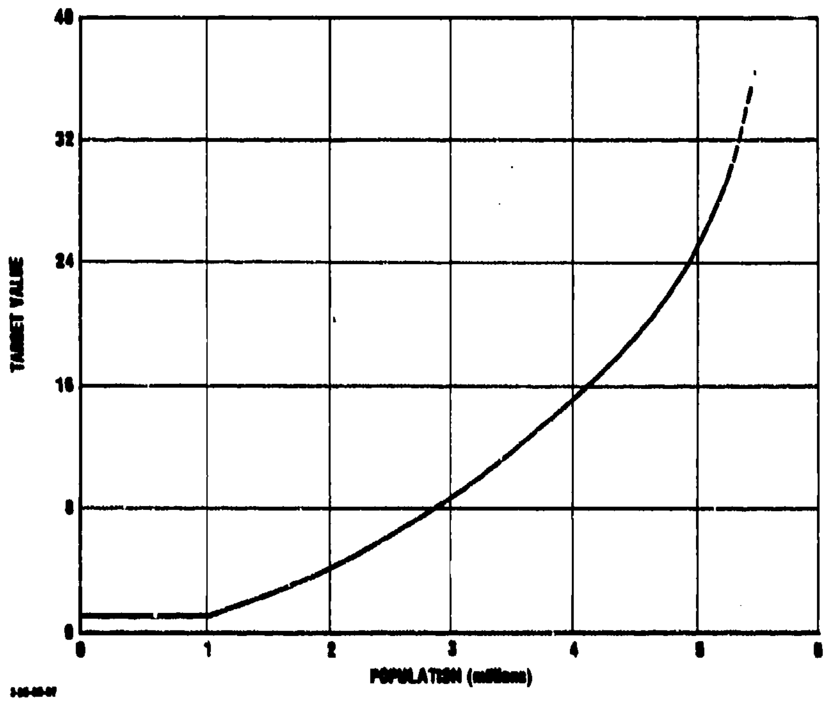


Figure A-2. TARGET VALUE AS A FUNCTION OF POPULATION FOR EXAMPLE 2

From this definition of the value function one obtains the target values listed in Table A-8.

Table A-8. TARGET VALUES FOR EXAMPLE 2

Target	Value	Target	Value
1	279	16	4
2	92	17	4
3	81	18	3
4	43	19	2
5	33	20	2
6	25	21	2
7	25	22	2
8	11	23	2
9	10	24	2
10	8	25	2
11	6	26	1
12	6	27	1
13	6	28	1
14	5	29	1
15	4	30	1

Given the target values listed in Table A-8, the only Prim-Read deployment defending targets 1, ..., 29 and satisfying the target defense principle is that corresponding to $\alpha = 1$ in the expression (A.1). In Table A-9 we present this Prim-Read deployment. For the sake of comparability with Table A-2, we continue to use $q = 0.8$ as the value of the one-on-one penetration probability. In view of our extended discussion of Example 1 we do not feel that further comments are really required at this point. Observe, however, that in order to satisfy the target defense principle, targets 26, 27, 28 and 29 must also be left undefended.

Table A-9. RESULTS FOR EXAMPLE 2 (α = 1, q = 0.8)

Target	Price	$I(d^*_\alpha, j)$	$I(\tilde{d}, j)$
1	279	5805	4376
2	92	1466	1443
3	81	1246	1270
4	43	545	674
5	33	381	518
6	25	260	392
7	25	260	392
8	11	79	173
9	10	68	157
10	8	48	125
11	6	29	94
12	6	29	94
13	6	29	94
14	5	21	78
15	4	14	63
16	4	14	63
17	4	14	63
18	3	8	47
19	2	3	31
20	2	3	31
21	2	3	31
22	2	3	31
23	2	3	31
24	2	3	31
25	2	3	31
26	1	0	0
27	1	0	0
28	1	0	0
29	1	0	0
30	1	0	0
Totals	664	10337	10333

To serve as one final comparison, we present in Table A-10 the same results as in Table A-9, except that the penetration probability is lowered to 0.5.

Table A-10. RESULTS FOR EXAMPLE 2 ($\alpha = 1$, $q = 0.5$)

Target	Price	$I(d_\alpha^*, j)$	$I(\tilde{d}, j)$
1	279	1870	1411
2	92	472	465
3	81	401	410
4	43	175	217
5	33	123	167
6	25	84	126
7	25	84	126
8	11	25	56
9	10	22	51
10	8	15	40
11	6	10	30
12	6	10	30
13	6	10	30
14	5	7	25
15	4	5	20
16	4	5	20
17	4	5	20
18	3	3	15
19	2	1	10
20	2	1	10
21	2	1	10
22	2	1	10
23	2	1	10
24	2	1	10
25	2	1	10
26	1	0	0
27	1	0	0
28	1	0	0
29	1	0	0
30	1	0	0
Totals	664	3333	3329

APPENDIX B. COMPARISON OF PRIM-READ AND PROPORTIONAL DEPLOYMENTS

COMPARISON OF PRIM-READ AND PROPORTIONAL DEPLOYMENTS

This Appendix contains mathematical formulations and verifications of various relationships between Prim-Read deployments and corresponding proportional deployments. Most of these relationships were observed and commented upon in the discussion in Appendix A of the numerical results appearing in Tables A-2, A-3, and A-4.

To set the stage for the results in this Appendix, let d_α^* denote by Prim-Read deployment with scaling factor α, given by

$$(B.1) \qquad d_\alpha^*(j,k) = -\frac{\log(\alpha v(j)-k+1)}{\log q}, \qquad \begin{array}{l} j = 1,\ldots,j_0 , \\ k = 1,\ldots,\alpha v(j) , \end{array}$$

where j_0 is the number of initially defended targets and α is chosen so that the target defense principle is satisfied and so that $\alpha v(j)$ is (or is taken to be) an integer for each $j=1,\ldots,j_0$. The total interceptor requirement at target j is

$$(B.2) \qquad I(d_\alpha^*,j) = -\frac{\log \alpha v(j)!}{\log q} .$$

Denote by $\tilde{d}_\alpha$ any deployment that allocates the total Prim-Read interceptor requirement, namely,

$$I(d_\alpha^*) = -\frac{1}{\log q} \sum_{j=1}^{j_0} \log \alpha v(j)! ,$$

among the targets $1,\ldots,j_0$ in direct proportion to their respective values. That is,

$$(B.3)\qquad I(\tilde{d}_\alpha, j) = \frac{v(j)}{\sum_{k=1}^{j_0} v(k)} \; I(d^*_\alpha)$$

$$= \frac{v(j)}{\sum_{k=1}^{j_0} v(k)} \; \frac{\sum_{k=1}^{j_0} \log \alpha v(k)!}{-\log q}$$

for $j=1,\ldots,j_0$. It is not relevant to the results below precisely how interceptors assigned to each target are allocated among potential incoming weapons there. In order to compare the deployments d^*_α and $\tilde{d}_\alpha$, we introduce the ratios $r(j,\alpha)$ given by

$$(B.4)\qquad r(j,\alpha) = \frac{I(d^*_\alpha, j)}{I(\tilde{d}_\alpha, j)}$$

$$= \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{\log \alpha v(j)!}{\sum_{k=1}^{j_0} \log \alpha v(k)!}$$

whose behavior, as functions of both j and α, we will analyze in this Appendix. We observe, incidentally, that these ratios are independent of the one-to-one penetration probability q, so that our results are valid for all values of q.

By way of motivation we recall the following *empirical* properties of these ratios $r(j,\alpha)$, as observed in the various Tables in Appendix A:

1) With α fixed, $r(j,\alpha)$ appears to be decreasing as j increases (i.e., as target value decreases), with $r(j,\alpha)>1$ for high value targets (low j) and $r(j,\alpha)<1$ for low value targets (high j).

2) As α decreases, $r(j,\alpha)$ increases and deviates farther from 1 provided that j be a high value target and decreases, but still deviates farther from 1, when j corresponds to a low value target.

3) The value of j for which

$$r(j,\alpha) > 1 > r(j+1,\alpha)$$

(i.e., at which the decreasing ratios $r(j,\alpha)$ pass from greater than one to less than one) appears to be nearly independent of α.

The principal results of this Appendix, which we now proceed to develop, are confirmations of the observed behaviors noted above. We first verify monotonicity of the ratios $r(j,\alpha)$ in j with α fixed.

(B.5) THEOREM. Assume that $v(1) > \ldots > v(j_0)$ and let $r(j,\alpha)$ be given by (B.4). Then for each fixed value of α,

(B.6) $$r(1,\alpha) > r(2,\alpha) > \ldots > r(j_0,\alpha).$$

PROOF. To show that $r(1,\alpha)>r(2,\alpha)$ it suffices by virtue of (B.4) to show that

$$\frac{\log \alpha v(1)!}{\alpha v(1)} > \frac{\log \alpha v(2)!}{\alpha v(2)},$$

and this expression holds by induction provided that

(B.7) $$\frac{\log (\ell+1)!}{\ell+1} > \frac{\log \ell!}{\ell}$$

for each integer $\ell \geq 2$. To obtain (B.7) we proceed as follows:

$$\frac{\log(\ell+1)!}{\ell+1} - \frac{\log \ell}{\ell} = \frac{\ell \log(\ell+1)! - (\ell+1)\log \ell!}{\ell(\ell+1)}$$

$$= \frac{\ell[\log(\ell+1) + \log \ell!] - \ell \log \ell! - \log \ell!}{\ell(\ell+1)}$$

$$= \frac{\ell \log(\ell+1) - \log \ell!}{\ell(\ell+1)}$$

$$= \frac{1}{\ell(\ell+1)} \sum_{k=1}^{\ell} [\log(\ell+1) - \log k]$$

$$> 0 .$$

This completes the proof of the Theorem. ∎

We note that if we assume only that $v(1) \geq v(2) \geq \ldots \geq v(j_0)$, then the conclusion (B.6) of Theorem (B.5) must be weakened correspondingly, and becomes

$$r(1,\alpha) \geq r(2,\alpha) \geq \ldots \geq r(j_0,\alpha) ,$$

where the inequality is strict whenever $v(j) > v(j+1)$.

The next result is not direct confirmation of any of the properties listed above, but will be used in the process of confirming the second property, and is also of some independent interest.

(B.8) PROPOSITION. For each $j=1,\ldots,J$, we have

$$\lim_{\alpha\to\infty} r(j,\alpha) = 1 \tag{B.9}$$

PROOF. We recall Stirling's approximation [16, p. 194]

$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n}$$

as $n\to\infty$, and substitute into each factorial term in (B.4) to obtain

$$r(j,\alpha) \sim \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{\log(2\pi)^{1/2} + (\alpha v(j)+1/2)\log \alpha v(j) - \alpha v(j)}{\sum_{k=1}^{j_0} [\log(2\pi)^{1/2} + (\alpha v(k)+1/2)\log \alpha v(k) - \alpha v(k)]}$$

$$\sim \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{(\alpha v(j) + 1/2)\log \alpha v(j) - \alpha v(j)}{\sum_{k=1}^{j_0} [(\alpha v(k)+1/2)\log \alpha v(k) - \alpha v(k)]}$$

$$= \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{(1+\frac{1}{2\alpha v(j)})\log \alpha v(j) - 1}{\sum_{k=1}^{j_0} [(\frac{v(k)}{v(j)} + \frac{1}{2\alpha v(j)})\log \alpha v(k) - \frac{v(k)}{v(j)}]}$$

$$\sim \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{\log \alpha v(j)}{\sum_{k=1}^{j_0} \frac{v(k)}{v(j)} \log \alpha v(k)}$$

$$= \frac{\sum_{k=1}^{j_0} v(k)}{v(j)} \; \frac{v(j)}{\sum_{k=1}^{j_0} v(k) \frac{\log \alpha v(k)}{\log \alpha v(j)}}$$

$$\to 1$$

since

$$\lim_{\alpha\to\infty} \sum_{k=1}^{j_0} v(k) \frac{\log \alpha v(k)}{\log \alpha v(j)} = \sum_{k=1}^{j_0} v(k) \lim_{\alpha\to\infty} \frac{\log \alpha + \log v(k)}{\log \alpha + \log v(j)}$$

$$= \sum_{k=1}^{j_0} v(k). \qquad ∎$$

One interpretation of Proposition (B.8) is that for large values of α, the Prim-Read deployment d^*_α and the proportional deployment $\tilde{d}_\alpha$ do not differ greatly, at least in the relative sense as described by the ratios $r(j,\alpha)$.

We can now confirm the second observed property of the ratios $r(j,\alpha)$; in fact, we shall prove more.

(B.10) THEOREM. Assume that $v(1)>v(j_0)$. Then,

a) For all sufficiently *small* j (i.e., for all targets of sufficiently high value), $r(j,\alpha)$ is a strictly *decreasing* function of α, and therefore $r(j,\alpha) > 1$ for all α.

b) For all sufficiently *large* j (i.e., for all initially defended targets of sufficiently low value), $r(j,\alpha)$ is a strictly *increasing* function of α, and therefore $r(j,\alpha) < 1$ all of α.

PROOF. We begin by noting that the second part of each conclusion follows from the first part together with (B.9). For example, in a), if $\alpha \to r(j,\alpha)$ is decreasing as α increases and (B.9) holds, then $r(j,\alpha) > 1$ for all α. The first parts of the two statements are proved using arguments that are essentially identical in pattern, so we can prove both parts simultaneously.

Monotonicity of $\alpha \to r(j,\alpha)$ is equivalent to monotonicity in the *same* direction of $\alpha \to g(j,\alpha)$, where

$$g(j,\alpha) = \frac{\log \Gamma(\alpha v(j)+1)}{\sum_{k=1}^{j_0} \log \Gamma(\alpha v(j)+1)} ,$$

and where Γ is Euler's gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

which has the property that

$$\Gamma(n+1) = n!$$

for each integer $n \geq 0$. Moreover, monotonicity of $\alpha \to g(j,\alpha)$ is equivalent to monotonicity in the opposite direction of $\alpha \to h(j,\alpha)$, where

$$\text{(B.11)} \qquad h(j,\alpha) = \frac{1}{g(j,\alpha)} - 1 = \sum_{\substack{k=1 \\ k \neq j}}^{j_0} \frac{\log \Gamma(\alpha v(k)+1)}{\log \Gamma(\alpha v(j)+1)} .$$

If we define

$$f(j,x) = \log \Gamma(xv(j)+1),$$

then $x \to f(j,x)$ is convex by [16, Theorem 8.18] and it follows from (B.11) that

$$\text{(B.12)} \qquad h(j,\alpha) = \sum_{k \neq j} \frac{f(j,\alpha \frac{v(k)}{v(j)})}{f(j,\alpha)} .$$

From (B.12) and convexity of $f(j,\cdot)$ the following conclusions are immediate:

1) $\alpha \to h(1,\alpha)$ is strictly increasing since each of the ratios $\alpha \to f(1,\alpha v(k)/v(j))/f(1,\alpha)$ is nondecreasing (by convexity of $f(1,\cdot)$ and the fact that $v(k)/v(1) \leq 1$) and at least one of these ratios is strictly increasing.

2) $\alpha \to h(j_0,\alpha)$ is strictly decreasing by an inversion of the argument just used.

3) If $\alpha \to h(j,\alpha)$ is strictly increasing, so is $\alpha \to h(j-1,\alpha)$, whereas, if $\alpha \to h(j,\alpha)$ is strictly decreasing, then the same is true of $\alpha \to h(j+1,\alpha)$.

The Theorem follows at once from these statements. ∎

Theorem (B.10) confirms that as α decreases, the deviation between the Prim-Read deployment d^*_α and the proportional deployment $\tilde{d}_\alpha$ increases, at least for all the highest value targets and all the lowest value targets. For numerical evidence we refer the reader to Table A-5 in Appendix A.

Our final result establishes that the cross-over point for the ratios $r(j,\alpha)$ (from greater than one to less than one) is the same for all α. Table A-5 suggests that this might be so, but fails to illustrate the property precisely because of rounding.

(B.13) THEOREM. Assume that $v(1)>v(2)>\ldots>v(j_0)$. Then there exists an integer j^* such that

(B.14) $$r(j^*,\alpha) > 1 > r(j^*+1,\alpha)$$

for all α.

We emphasize that both inequalities in (B.14) are strict.

PROOF. For each α, let j_α be the unique integer such that

$$r(j_\alpha,\alpha) \geq 1 > r(j_\alpha+1,\alpha) ;$$

existence is clear and uniqueness follows from Theorem (B.5). Since $\alpha \to r(j,\alpha)$ is a continuous function for each j it follows that the $\alpha \to j_\alpha$ is continuous at every point α_0 for which

$$r(j_\alpha,\alpha) > 1 > r(j_\alpha+1,\alpha) .$$

Therefore, if $r(j,\alpha) \neq 1$ for all j and all α, then the function $\alpha \to j_\alpha$ is everywhere continuous and integer-valued and, consequently, must be constant, which suffices to demonstrate (B.14).

Thus, it remains only to prove that

(B.15) $$r(j,\alpha) \neq 1$$

for all j and α. If, on the contrary, (B.15) fails for some j and α, then by (B.4) we have that

(B.16) $$\frac{v(j)}{\sum_{k=1}^{j_0} v(k)} = \frac{\log \alpha v(j)!}{\sum_{k=1}^{j_0} \log \alpha v(k)!} .$$

That is, since $\Gamma(n+1) = n!$ for each integer n, (B.16) implies that

$$\frac{1}{v(j)} \sum_{k=1}^{j_0} v(k) = \frac{1}{\log \Gamma(\alpha v(j)+1)} \sum_{k=1}^{j_0} \log \Gamma(\alpha v(k)+1) .$$

Consider now the functions

$$f_1(\alpha) = \log \Gamma(\alpha v(j)+1) \sum_{k=1}^{j_0} v(k)$$

and

$$f_2(\alpha) = v(j) \sum_{k=1}^{j_0} \log \Gamma(\alpha v(k)+1) .$$

Then,

$$f_1'(\alpha) = v(j)\psi(\alpha v(j)+1) \sum_{k=1}^{j_0} v(k)$$

and

$$f_2'(\alpha) = v(j) \sum_{k=1}^{j_0} v(k)\psi(\alpha v(k)+1) ,$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$. Suppose that $f_1'(\alpha)$ were equal to $f_2'(\alpha)$ for some α; this would imply that

(B.17) $$\psi(\alpha v(j)+1) = \frac{1}{\sum_{k=1}^{j_0} v(k)} \sum_{k=1}^{j_0} v(k)\psi(\alpha v(k)+1) .$$

However, (B.17) violates strict convexity of the function $x \to \psi(\alpha x+1)$. Therefore (once more with the $v(k)$ fixed), we have $f_1'(\alpha) \neq f_2'(\alpha)$ for all α, which implies that either $f_1'(\alpha) > f_2'(\alpha)$ for all α or $f_1'(\alpha) < f_2'(\alpha)$ for all α, since these functions are continuous. Finally, since $f_1(0) = f_2(0)$ it follows that f_1 and f_2 can be equal for no other value of α, which completes the proof of the Theorem. $\square$