AD-A101 610 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 12/1
CHOICE OF TECHNIQUE IN A CONTINUOUS TIME INFINITE HORIZON OPTIM--ETC(U)
JUN 81 C WU N00014-76-C-0134
UNCLASSIFIED ORC-81-14 NL

1 OF 1
AD A
101610

END
DATE
FILMED
8-81
DTIC

ORC 81-14
JUNE 1981

LEVEL II

(12)

CHOICE OF TECHNIQUE IN A CONTINUOUS TIME INFINITE HORIZON OPTIMAL GROWTH MODEL

by

CHSZ-HYEN WU

OPERATIONS
RESEARCH
CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

W

CHOICE OF TECHNIQUE IN A CONTINUOUS TIME INFINITE HORIZON OPTIMAL GROWTH MODEL.

by

Chsz-Hyen Wu
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

JUNE 1981 56 ORC-81-14

This research has been partially supported by the National Science Foundation under Grant SES-7805196 and the Office of Naval Research under Contract N00014-76-C-0134 with the University of California.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: ORC 81-14	2. GOVT ACCESSION NO.: AD-A101610
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): CHOICE OF TECHNIQUE IN A CONTINUOUS TIME INFINITE HORIZON OPTIMAL GROWTH MODEL	5. TYPE OF REPORT & PERIOD COVERED: Research Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Chsz-Hyen Wu	8. CONTRACT OR GRANT NUMBER(s): SES-7805196
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Operations Research Center, University of California, Berkeley, California 94720	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: National Science Foundation, 1800 G Street, Washington, D.C. 20550	12. REPORT DATE: June 1981
	13. NUMBER OF PAGES: 55
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Also supported by the Office of Naval Research under Contract N00014-76-C-0134.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Continuous Mathematical Programming
Strong Duality
Optimal Growth
Dual Price

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude for the advice and encouragement afforded me by my thesis advisor, Professor David Gale, who suggested this topic and had been a constant source of stimulation. Also, I would like to thank Gloria Partee whose efforts transformed an illegible manuscript into a neatly typed report.

ABSTRACT

We consider the choice of technique in a continuous time infinite horizon optimal growth model. There are $n+2$ goods, output, labor and machines $M_1, M_2, \ldots, M_n$. We can convert one unit of labor to q units of output or r_i units of M_i, for each i. Also, we can convert one unit of labor and one unit of M_i to q_i units of output. We prove under some sufficient and necessary conditions that we never build any machines for the general concave utility function. If the condition is not met, we build one machine from beginning to end when the utility function is linear; when the utility function is non-linear life gets complicated. In the one machine case, we give a general algorithm to solve it. In the many machines case, we prove an asymptotic result (as $t \to \infty$, the behavior is similar to that of the linear case) and give examples showing that a simple characterization of the optimal solution is difficult.

TABLE OF CONTENTS

CHAPTER I

MODEL AND DISCUSSION

§1. Introduction

In this paper we investigate the choice of investment in a continuous time optimal growth model. The general model is described as follows: there are m goods, some of which may be provided exogenously (e.g. labor). The technology is described by a set of n activities, each of which consumes various amounts of goods and produces various amounts of goods. To be more specific, the technology is given by a pair of nonnegative $m \times n$ matrices A B and a nonnegative n-vector b, where $A_{ij}(B_{ij})$ denotes the amount of good i used (produced) to operate activity j at unit level, and $b_i(t)$ denotes the amount of good i exogenously provided at time t. There is a utility function which is an increasing concave function of the activity level.

Problem:

Given $b(t)$, find an activity vector $x(t)$ to maximize the discounted integral of future utility.

We can write this as a continuous programming problem:

$$\text{(P)} \qquad \text{Maximize} \int_0^\infty e^{-\alpha t} \cdot U(x(t))dt$$

$$\text{subject to: } A \cdot x(t) \leqq \int_0^t B \cdot x(s)ds + b(t)$$

$$x(t) \geqq 0 .$$

If $\hat{x}(t)$ is ⟨P⟩ feasible $\left(\hat{x}(t) \geqq 0 \ , \ A\hat{x}(t) \leqq \int_0^t B\hat{x}(s)ds + b(t)\right)$, then we have the following dual problem:

$$\langle D(\hat{x}(t)) \rangle: \qquad \text{Minimize} \int_0^\infty w(t)\cdot b(t)dt$$

$$\text{subject to:} \quad w(t)A \geqq \int_t^\infty w(s)Bds + e^{-\alpha t}\cdot \nabla U(\hat{x}(t))$$

$$w(t) \geqq 0 \ .$$

§2. Optimality Theorem for the General Model

In this section we prove that a feasible solution $(\hat{x}(t))$ is optimal if we can find the corresponding dual price $(\hat{w}(t))$ and satisfy some complementary slackness conditions.

Theorem 1-1:

If the following four conditions are satisfied

(1) $\hat{x}(t)$ is ⟨P⟩ feasible $\left(\text{i.e.,} \ A\hat{x}(t) \leqq \int_0^t B\cdot\hat{x}(s)ds + b(t) \ , \ \hat{x}(t) \geqq 0\right)$.

(2) $\hat{w}(t)$ is ⟨D($\hat{x}(t)$)⟩ feasible $\left(\text{i.e.,} \ \hat{w}(t)A \geqq \int_t^\infty \hat{w}(s)Bds + e^{-\alpha t}\cdot\nabla U(\hat{x}(t)) \ , \ \hat{w}(t) \geqq 0\right)$

(3) $\hat{w}(t)\left[A\hat{x}(t) - \int_0^t B\cdot\hat{x}(s)ds - b(t)\right] = 0$

(4) $\left[\hat{w}(t)A - \int_t^\infty \hat{w}(s)Bds - e^{-\alpha t}\cdot\nabla U(\hat{x}(t))\right]\cdot\hat{x}(t) = 0$

then $\hat{x}(t)$ is ⟨P⟩ optimal, $\hat{w}(t)$ is ⟨D($\hat{x}(t)$)⟩ optimal.

Proof:

Let $x(t)$ be any $\langle P \rangle$ feasible solution

$w(t)$ be any $\langle D(\hat{x}(t)) \rangle$ feasible solution.

$$w(t)\cdot b(t) \geqq w(t)\cdot\left[A\cdot x(t) - \int_0^t B\cdot x(s)ds\right].$$

Integrating both sides gives

$$\int_0^\infty w(t)\cdot b(t)dt \geqq \int_0^\infty w(t)\cdot\left[Ax(t) - \int_0^t B\cdot x(s)ds\right]dt$$

$$= \int_0^\infty \left[w(t)A - \int_t^\infty w(s)Bds\right]\cdot x(t)dt \quad \text{(changing the order of integral)}$$

$$\geqq \int_0^\infty e^{-\alpha t}\cdot\nabla U(\hat{x}(t))\cdot x(t)dt \quad (w(t) \text{ is } \langle D \rangle \text{ feasible})$$

$$\geqq \int_0^\infty e^{-\alpha t}[\nabla U(\hat{x}(t))\cdot\hat{x}(t) + U(x(t)) - U(\hat{x}(t))]dt$$

$$= \int_0^\infty e^{-\alpha t}\cdot U(x(t))dt + \int_0^\infty e^{-\alpha t}[\nabla U(\hat{x}(t))\cdot\hat{x}(t) - U(\hat{x}(t))]dt$$

$$= \int_0^\infty e^{-\alpha t}\cdot U(x(t))dt + M .$$

So for any $x(t)$, $w(t)$

$$\int_0^\infty w(t)\cdot b(t)dt \geqq \int_0^\infty e^{-\alpha t}\cdot U(x(t))dt + M \qquad (5)$$

but if we replace $x(t)$, $w(t)$ by $\hat{x}(t)$, $\hat{w}(t)$ *then the above* "$\geqq$" are replaced by "=" (from (3) and (4)) thus

$$\int_0^\infty \hat{w}(t)\cdot b(t)dt = \int_0^\infty e^{-\alpha t}\cdot U(\hat{x}(t))dt + M \,. \tag{6}$$

From (5) and (6)

$\int_0^\infty \hat{w}(t)\cdot b(t)dt$ reaches the lower bound and

$\int_0^\infty e^{-\alpha t}\cdot U(\hat{x}(t))dt$ reaches the upper bound.

So $\hat{w}(t)$ is $\langle P \rangle$ optimal

$\hat{w}(t)$ is $\langle D(x(t)) \rangle$ optimal ■

§3. Description of the Investment Model

We now consider the special case as follows:

(1) There are $n+2$ goods, output (consumption), labor, and n machines $M_1, M_2, \ldots, M_n$.

(2) There are $n+1$ production activities $P_0, P_1, \ldots, P_n$.
P_0 converts one unit of labor to q units of output.
P_i converts one unit of labor and one unit of M_i to q_i units of output $1 \leqq i \leqq n$.

(3) There are n investment activities $I_1, I_2, \ldots, I_n$, where I_j converts one unit of labor to r_j units of M_j $1 \leqq j \leqq n$.

(4) *Labor is given exogenously at constant rate of one.*

Let x be the amount of labor allocated to P_0 ;

z_i be the amount of labor allocated to P_i $1 \leqq i \leqq n$;

y_i be the amount of labor allocated to I_i $1 \leqq i \leqq n$.

Let α be the discount rate and U be a differentiable increasing utility function of the output.

We define $c(t) = q \cdot x(t) + \sum_{i=1}^{n} q_i \cdot z_i(t)$ and $\psi(t) = U'(c(t))$.

Problem:

Given initial stocks k_i of M_i , choose $X(t) = (x(t), z_i(t), y_i(t)) \in \mathbb{R}_+^{2n+1}$ to

$$\text{maximize} \int_0^\infty e^{-\alpha t} \cdot U(c(t)) dt$$

$$\text{subject to:} \quad z_i(t) \leqq r_i \int_0^t y_i(s) ds + k_i \qquad 1 \leqq i \leqq n$$

$$x(t) + \sum_{i=1}^{n} (y_i(t) + z_i(t)) \leqq 1 .$$

By defining $\hat{U}(X(t)) = U(c(t))$, $\nabla\hat{U}(X(t)) = (q, q_1, \ldots, q_n, \underbrace{0, \ldots, 0}_{n \text{ terms}}) \cdot \psi(t) \in \mathbb{R}_+^{2n+1}$ and

$$A = \left[\begin{array}{c|c|c} 0 & & \\ \vdots & I & 0 \\ 0 & & \\ \hline 1 & 1, \ldots, 1 & 1, \ldots, 1 \end{array}\right], \quad B = \left[\begin{array}{c|c|c} 0 & & r_1 \quad\quad \\ \vdots & 0 & \quad r_2 \quad 0 \\ 0 & & 0 \quad \ddots \quad r_n \\ \hline 0 & 0, \ldots, 0 & 0, \ldots, 0 \end{array}\right],$$

A , $B \in \mathbb{R}_+^{(n+1)\times(2n+1)}$, $b(t) = (k_1, \ldots, k_n, 1)$, $b(t) \in \mathbb{R}_+^{n+1}$.

The dual problem becomes:

Find $\bar{w}(t) = (w_1(t), \ldots, w_n(t), w(t)) \in \mathbb{R}_+^{n+1}$ to

$$\text{minimize} \int_0^\infty \bar{w}(t) \cdot b(t)dt$$

$$\text{subject to:} \quad w(t) \geqq e^{-\alpha t} \cdot q \cdot \psi(t)$$

$$d_i(t) = w(t) - r_i \int_t^\infty w_i(s)ds \geqq 0 \qquad 1 \leqq i \leqq n$$

$$w(t) + w_i(t) \geqq e^{-\alpha t} \cdot q_i \cdot \psi(t) \qquad 1 \leqq i \leqq n .$$

If the utility function is linear, by giving the exact dual price, we prove that (1) if $q \geqq \max_{1 \leqq j \leqq n} \left\{ \frac{r_j q_j}{\alpha + r_j} \right\} = \frac{r_\ell q_\ell}{\alpha + r_\ell}$, the stationary program $(y_i(t) = 0 \ , \ 1 \leqq i \leqq n)$ is optimal; (2) if $q < \frac{r_\ell q_\ell}{\alpha + r_\ell}$, we build M_ℓ $\left(\text{in fact, any } M_t \text{ s.t. } \frac{r_t q_t}{\alpha + r_t} = \frac{r_\ell q_\ell}{\alpha + r_\ell}\right)$. If the utility function is nonlinear, life gets complicated.

To begin with, we do not know whether a right-differentiable (see Perold [5]) optimal solution exists.[1] Secondly, we do not know whether the dual price exists.[2] To simplify the model, we restrict the utility functions to those "well-behaved" differentiable increasing functions such that there exist right-differentiable optimal solutions

---

[1] Since our model is an economic model, we would like an optimal solution which is right-differentiable rather than just Lebesque-measurable.

[2] In 1968, Hanson [3] proved the strong duality of the similar model in the finite horizon case.

and the corresponding dual prices.[3] Under this assumption, we prove that (1) $q \geqq \frac{r_\ell q_\ell}{\alpha + r_\ell}$ is still the sufficient and necessary conditions for the existence of an optimal stationary program; (2) if the condition of (1) is not met, we can prove an asymptotic result, i.e., as $t \to \infty$, the optimal solution has the same behavior as the linear case.

It is difficult to give a simple characterization of the optimal solution of a nonlinear utility function because we have examples in which we switch building machines (Chapter 4, Example 1), or build some machine M_j in time $[T_1, T_2]$ but later never use M_j.

Even in the one machine case, we might have $x(t) = 0$ in the first interval, $x(t) > 0$ in the second interval, and $x(t) = 0$ in the third interval (Chapter 3, Example 2).

In the following we prove that by manipulating the model we can make some assumptions without losing generality.

<u>Assumption 1</u>: $q_1 > q_2 > \dots > q_n$, $r_1 < r_2 < \dots < r_n$

If $q_i = q_j$ and $r_i > r_j$, apparently we never build M_j (for M_i has the same output, but larger r_i).

(i) If $k_j = 0$, then we discard M_j .

(ii) If $k_j > 0$, we define a new problem with $n-1$ machines (without M_j) and the initial stocks $\hat{k}_s$ given by
$\hat{k}_i = k_i + k_j$, $\hat{k}_s = k_s \quad 1 \leqq s \leqq n$, $s \neq i$, j .
It's easy to see that the new problem is in fact the same as the old one.

[3]It is believed that for all "usual" concave functions, these properties are true. Since our model is a simple economic model, there should exist the dual price and a well-behaved optimal solution.

Hence, without losing generality we can assume $q_i \neq q_j$, if $i \neq j$.

Now we rearrange the machines in order of decreasing output.

Assumption 2: $K = \sum_{i=1}^{n} k_i < 1$

If $\sum_{i=1}^{n} k_i \geqq 1$, let $\sum_{i=1}^{j} k_i < 1$, $\sum_{i=1}^{j+1} k_i \geqq 1$. We will show that the general problem is in fact equivalent to this special case.

(1) We never build M_s, $s \geqq j+1$ $\left(\text{for } \sum_{i=1}^{j+1} k_i \geqq 1 \text{ and } q_1 > q_2 > \ldots > q_n\right)$

$z_s(t) = 0$, $s \geqq j+2$ $\forall t \geqq 0$.

(2) By (1) we discard M_s, $s \geqq j+2$.

(3) Consider a new problem with machines $M_1, M_2, \ldots, M_j$ and q replaced by q_{j+1}.

By (2) and (3), it is clear that the new problem is essentially the same as the old one, but $\sum_{i=1}^{j} k_i < 1$ for the new problem.

Before ending this chapter, we prove two theorems which play the key role in the following chapters.

Theorem 1-2:

(1) $x(t) + \sum_{i=1}^{n} (y_i(t) + z_i(t)) = 1$ for all t (full employment).

(2) There always exists the dual price $w_i(t) = \left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+$

$1 \leqq i \leqq n$

which satisfies the conditions of Theorem 1-1.

Proof:

(1) If $x(t) + \sum_{i=1}^{n} (y_i(t) + z_i(t)) = P(t) < 1$

since $x(t)$ has output q, we can increase $x(t)$

by $1 - p(t)$ to produce more output, contradiction.

(2) $w_i(t) \geqq 0$ and $w(t) + w_i(t) \geqq e^{-\alpha t} \cdot q_i \cdot \psi(t)$

implies $w_i(t) \geqq \left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+$.

Define:

$$k_i(t) = k_i + r_i \int_0^t y_i(s) ds .$$

If $k_i > 0$, Case i: $z_i(t) < k_i(t)$

by complementary slackness $w_i(t) = 0 =$

$\left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+$.

Case ii: $z_i(t) = k_i(t)$

$w(t) + w_i(t) = e^{-\alpha t} \cdot q_i \cdot \psi(t)$ implies

$w_i(t) = \left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+$.

If $k_i = 0$, Case i: there exists $T > 0$, such that $k_i(t) = 0$

for $t < T$ and $k_i(t) > 0$ for $t > T$.

$w_i(t) = \left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+ \quad \forall t > T$.

Define $\hat{w}_i(t) = \left[e^{-\alpha t} \cdot q_i \cdot \psi(t) - w(t)\right]_+$ for $t < T$.

$w(t) \geqq r_i \int_t^\infty w_i(s) ds$ ($w_i(t)$ is the dual price)

for $t < T$

$\geqq r_i \int_t^\infty \hat{w}_i(s) ds$ $(w_i(s) \geqq \hat{w}_i(s))$.

$y_i(t) = 0$ for $t < T$ $(k_i(t) = 0)$ implies the complementary slackness is also satisfied for $\hat{w}_i(t)$.

Case ii: if $k_i(t) = 0$ for all t similar proof as above.

Thus there exists an optimal price $w_i(t) = \left[e^{-\alpha t}\cdot q_i \cdot \psi(t) - w(t)\right]_+$ ■

Remark:

(1) If $k_i = 0$, $w_i(t)$ need not be unique.

(2) From now on, we always assume $w_i(t) = \left[e^{-\alpha t}\cdot q_i \cdot \psi(t) - w(t)\right]_+$.

A solution $X(t) = (x(t), y_i(t), z_i(t)) \in \mathbb{R}_+^{2n+1}$ is called stationary if $x(t) = 1 - k$, $y_i(t) = 0$, $z_i(t) = k_i$ $\left(k = \sum_{i=1}^{n} k_i\right)$ for all t $1 \leqq i \leqq n$. Let $\max_{1 \leqq i \leqq n} \frac{r_i q_i}{\alpha + r_i} = v$.

Theorem 1-3:

$q \geqq v$ is the sufficient and necessary condition for the existence of an optimal stationary program.

Proof:

If $q \geqq v$, the stationary program is feasible and $c(t) = q\cdot(1-k) + \sum_{i=1}^{n} k_i \cdot q_i$ = constant implies $\psi(t)$ = constant = $c > 0$.
Define: $w(t) = e^{-\alpha t}\cdot q \cdot \psi(t) = e^{-\alpha t}\cdot q \cdot c$ and $w_i(t) = e^{-\alpha t}\cdot (q_i - q)\psi(t) = e^{-\alpha t}\cdot (q_i - q)\cdot c$.

It is easy to see that if we can prove $d_i(t) \geqq 0$ $1 \leqq i \leqq n$ then the stationary program is optimal.

$$d_i(t) = e^{-\alpha t}\cdot q\cdot c - r_i \int_t^\infty e^{-\alpha s}\cdot(q_i - q)\cdot c\,ds$$

$$= \frac{e^{-\alpha t}\cdot c}{\alpha}\,[\alpha q - r_i(q_i - q)]$$

$$\gtreqqless 0 \quad \left(\alpha q \gtreqqless r_i(q_i - q) \;\text{ is equivalent to }\; q \gtreqqless \frac{r_i q_i}{\alpha + r_i}\right).$$

The other direction is similar. (Here we use the assumption of the existence of the dual price.) ∎

CHAPTER II

THE LINEAR UTILITY FUNCTION

To develop better understanding of the general concave utility function, we start with the linear utility function. There are only two kinds of the optimal program for the linear utility function: one is never building any machine, the other is building only one machine.

For completeness, we list the stationary program in the following.

§1. Stationary Program

Theorem 2-1:

$q \geqq v$ is the sufficient and necessary condition for the existence of an optimal stationary program.

From now on, assume $q < v$.

§2. Nonstationary Program

Machine j is called best, if $\frac{r_j q_j}{\alpha + r_j} = v$. $I = \{i \mid q_i \geqq v ,$ $1 \leqq i \leqq n\} = \{1,2, \ldots, N\}$, $\hat{K} = \sum_{i=1}^{N} k_i$. Machine i is called good, if $i \in I$.

For any best machine j , the following program is called the best program with respect to j : $x(t) = 0$, $y_i(t) = 0 \quad i \neq j$, $y_j(t) = (1 - \hat{k})e^{-r_j t}$, $z_i(t) = k_i \quad i \in I - \{j\}$, $z_j(t) = k_j +$ $(1 - \hat{k})\left(1 - e^{-r_j t}\right)$, $z_r(t) = 0 \quad r \notin I$. (This means that all the good machines are fully utilized, then the rest of the labor is used to build machine j .)

Theorem 2-2:

The best program with respect to j is optimal.

Proof:

$$c(t) = \sum_{i=1}^{n} q_i \cdot z_i(t) \ , \ \psi(t) = \text{constant} = a > 0 \ \text{(because U is linear).}$$

Define:

$$\begin{cases} w(t) = e^{-\alpha t} \cdot [q \cdot a + s] = e^{-dt} \cdot \dfrac{r_j q_j}{\alpha + r_j} a \ , \ S = \left[\dfrac{r_j q_j}{\alpha + r_j} - q \right] \cdot a \geqq 0 \\ w_i(t) = e^{-\alpha t} \cdot \left(q_i - \dfrac{r_j q_j}{\alpha + r_j} \right) \cdot a \qquad \text{if} \quad i \in I \\ w_i(t) = e^{-\alpha t} \cdot \left(q_i - \dfrac{r_j q_j}{\alpha + r_j} \right)_+ \cdot a = 0 \quad \text{if} \quad i \notin I \ . \end{cases}$$

$$d_j(t) = e^{-\alpha t} \cdot a \cdot \frac{r_j q_j}{\alpha + r_j} - r_j \int_t^{\infty} e^{-\alpha s} \cdot a \cdot \left[q_j - \frac{r_j q_j}{\alpha + r_j} \right] ds = 0$$

$$i \in I - \{j\} \ , d_i(t) = e^{-\alpha t} \cdot a \cdot \frac{r_j q_j}{\alpha + r_j} - r_i \int_t^{\infty} e^{-\alpha s} \cdot a \cdot \left(q_i - \frac{r_j q_j}{\alpha + r_j} \right) ds$$

$$= e^{-\alpha t} \cdot a \cdot \frac{\alpha + r_i}{\alpha} \left[\frac{r_j q_j}{\alpha + r_j} - \frac{r_i q_i}{\alpha + r_i} \right] \geqq 0$$

$i \notin I \ , \ d_i(t) = w(t) - 0 \geqq 0 \ .$

If $i \in I \ , \ z_i(t) = k_i(t)$; if $i \notin I \ , \ z_i(t) = 0 \ .$

The complementary slackness is also satisfied ■

Economic Interpretation:

We can think $\frac{r_j q_j}{\alpha + r_j}$ as the present value of machine j. If we only build machine j and fully utilize it, then $k_j(t) = 1 - e^{-r_j t}$ and $\int_0^\infty e^{-\alpha t} \cdot q_j \left(1 - e^{-r_j t}\right) dt = \frac{r_j q_j}{\alpha + r_j}$.

CHAPTER III

GENERAL CONCAVE UTILITY FUNCTION - ONE MACHINE CASE

For the general concave utility, to get deeper insight about the structure of the many machines, we begin with one machine case. In the beginning, we derive some simple properties. Later we prove that if the initial stocks are sufficiently large, we don't use labor alone to produce output. If the utility function is quadratic, we have a simple optimal solution. Otherwise, we have an example with a non-simple solution. We also give an algorithm to solve the general problem. By scaling, we can assume $q = 1$ without losing generality.

Because of the simple structure, we rewrite the primal and dual feasibilities in terms of the following simplified notation:

$$x(t) \leftarrow x(t)\ ,\ y(t) \leftarrow y_1(t)\ ,\ z(t) \leftarrow z_1(t)\ ,\ r \leftarrow r_1\ ,\ \bar{q} \leftarrow q_1\ .$$

$$\text{primal feasibility:} \begin{cases} z(t) \leqq r \displaystyle\int_0^t y(s)ds \\ x(t) + y(t) + z(t) = 1 \\ x(t)\ ,\ y(t)\ ,\ z(t) \geqq 0\ . \end{cases}$$

$$\text{dual feasibility:} \begin{cases} w(t) \geqq e^{-\alpha t}\cdot\psi(t) \\ w(t) \geqq r \displaystyle\int_t^\infty w_1(s)ds \\ w(t) + w_1(t) \geqq e^{-\alpha t}\cdot\bar{q}\cdot\psi(t) \\ w(t)\ ,\ w_1(t) \geqq 0\ . \end{cases}$$

For completeness, we list the stationary program in the following.

§1. Stationary Program

Theorem 3-1:

$1 \geqq \frac{r\bar{q}}{\alpha + r}$ is the sufficient and necessary condition for the existence of an optimal stationary program ∎

From now on, assume $1 < \frac{r\bar{q}}{\alpha + r}$.

§2. Nonstationary Program

We begin with some simple properties.

Property 1:

$y(t) > 0$, for all t[1] .

Proof:

If there exists an interval $(T, T+\varepsilon)$ such that $y(t) = 0$ for $t \in (T, T+\varepsilon)$, then $y(t) = 0$ for all $t > T$, which by stationary property implies $1 \geqq \frac{r\bar{q}}{\alpha + r}$. Contradiction to $1 < \frac{r\bar{q}}{\alpha + r}$ ∎

Corollary:

$w(t) - r\int_t^\infty w_1(s)ds = 0$ for all t ∎

[1] We disregard those isolated time points. When we say $y(t) > 0$ for all t , we mean there is no interval such that $y(t) = 0$ on that whole interval.

Property 2:

$$k(t) = k_0 + r\int_0^t y(s)ds = z(t) \quad \text{(fully utilize the machine).}$$

Proof:

By contradiction. If not true,

Case i: there exists $T > 0$ such that $z(t) < k(t)$ for all $t > T$.

$w_1(t) = 0$ (complementary slackness) implies $w(t) = 0$.

$w(t) \geqq e^{-\alpha t}\cdot\psi(t)$ implies $\psi(t) = 0$, contradiction.

Case ii: there exists T_1, T_2 such that $z(t) < k(t)$ for

$t \in (T_1,T_2)$ and $z(t) = k(t)$ for $t \in (T_2,T_2+\varepsilon)$.

$C(T_1) = x(T_1)+\bar{q}\cdot z(T_1) = 0+\bar{q}\cdot z(T_1) < x(T_2) + \bar{q}\cdot z(T_2) =$

$c(T_2)$ implies $\psi(T_1) \geqq \psi(T_2)$.

$$0 = w(T_1) - r\int_{T_1}^{\infty} w_1(s)ds = w(T_1) - r\int_{T_2}^{\infty} w_1(s)ds \quad (w_1(t) = 0,\ t \in (T_1,T_2))$$

$$= w(T_1) - w(T_2).$$

$$0 \leqq w(T_1) - e^{-\alpha T_1}\cdot\bar{q}\cdot\psi(T_1). \quad (w_1(T_1) = 0)$$

$$w(T_2) \leqq e^{-\alpha T_2}\cdot\bar{q}\cdot\psi(T_2). \quad (z(T_2) > 0)$$

From the above three equations, we have $e^{-\alpha T_2}\cdot\bar{q}\cdot\psi(T_2) \geqq e^{-\alpha T_1}\cdot\bar{q}\cdot\psi(T_1)$ ·, contradiction to $T_1 < T_2$ and $\psi(T_1) \geqq \psi(T_2)$ ■

Corollary:

$k(t) < 1$ for all t. (We never accumulate to 1 unit of machine.)

Proof:

The maximum amount of machine we can have in an optimal solution is given by the following program: $x(t) = 0$, $y(t) = 1 - k(t)$,

$z(t) = k(t)$. $k(t) = r\int_0^t [1-k(s)]ds + k_0$, which implies $k(t) = 1-(1-k_0)e^{-rt}$, thus $k(t) < 1$ for all t ■

Remark:

Property 2 does not hold in the many machines case.

Property 3:

$w(t)$ is continuous.

Proof:

$w(t) = r\int_t^\infty w_1(s)ds$ ■

Property 4:

$\psi(t)$ is continuous.

Proof:

By Property 2 and $\psi(t) = U'(x(t) + \bar{q}\cdot z(t))$, if we can prove for any given $T > 0$, a sequence $t_n \to T$ such that $x(t_n) \to \bar{x}$ we have $\psi(t_n) \to \psi(T)$, then $\psi(t)$ is continuous.

Case i: $x(T) = 0$. If $x(t_n) \to \bar{x} = 0$, apparently $\psi(t_n) \to \psi(T)$.

If $x(t_n) \to \bar{x} > 0$, then $\lim_{n\to\infty} c(t_n) \geqq c(T)$ and $\lim_{n\to\infty} \psi(t_n) \leqq \psi(T)$.

By slackness, $e^{-\alpha t}\cdot\psi(t_n) = w(t_n) \to w(T) \geqq e^{-\alpha t}\cdot\psi(T)$.

Combining the above two equations, we have $\psi(t_n) \to \psi(T)$.

Case ii: $x(T) > 0$. If $x(t_n) > 0$, the same proof as above.

If $x(t_n) = 0$ for all n , $\lim_{n\to\infty} c(t_n) \leqq c(T)$ and

$$\lim_{n\to\infty} \psi(t_n) \geqq \psi(T) .$$

$$e^{-\alpha t}\cdot\psi(t_n) \leqq w(tn) \to w(T) = e^{-\alpha t}\cdot\psi(T) .$$

Combining the above two equation, we have $\psi(t_n) \to \psi(T)$ ∎

Corollary 1:

If $u(x)$ is strictly concave, then (1) $c(t)$ is continuous; (2) $x(t)$, $y(t)$ are continuous.

Proof:

(1) If $c(t)$ is not continuous, then $\psi(t)$ is not continuous, contradiction.

(2) $c(t) = x(t) + \bar{q}\cdot z(t)$. For $c(t)$ and $z(t)$ continuous, $x(t)$ is continuous. $y(t) = 1 - x(t) - z(t)$ implies $y(t)$ is continuous too ∎

Corollary 2:

$w_1(t)$ is continuous.

Proof:

$w_1(t) = \left[e^{-\alpha t}\cdot\bar{q}\cdot\psi(t) - w(t)\right]_+$. Since $\psi(t)$ and $w(t)$ are both continuous, $w_1(t)$ is continuous too ∎

Property 5:

If $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, then $\psi(t) = c\cdot e^{[\alpha-r(\bar{q}-1)]t}$ $c > 0$, $t \in (T_1,T_2)$.

Proof:

The dual equations are: $w(t) = e^{-\alpha t} \cdot \psi(t)$, $w_1(t) = e^{-\alpha t} \cdot (\bar{q} - 1)\psi(t)$, $w(t) = r \int_t^{\infty} w_1(s)ds$. The solution is $\psi(t) = c \cdot e^{[\alpha - r(\bar{q}-1)]t}$, $c > 0$ ■

Corollary 1:

If $u'(\bar{q}) > 0$ and $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, then $\psi(t) \leqq \beta < 0$.

Proof:

$$\dot{\psi}(t) = [\alpha - r(\bar{q} - 1)]\psi(t) \leqq U'(\bar{q})[\alpha - r(\bar{q} - 1)] = \beta < 0$$ ■

Corollary 2:

If $\psi(t)$ = constant for $t \in (T, T + \varepsilon)$, then $x(t) = 0$ for $t \in (T, T + \varepsilon)$ ■

Corollary 3:

$c(t)$ is increasing in (T_1, T_2) .

Proof:

$\dot{\psi}(t) = c \cdot [\alpha - r(\bar{q} - 1)] \cdot e^{[\alpha - r(\bar{q}-1)]t} < 0$, so $\psi(t)$ is decreasing in t . Since $U'(x)$ is decreasing in x , $c(t)$ is increasing in (T_1, T_2) ■

Property 6:

If the utility function is strictly concave, then $c(t)$ is a strictly increasing function.

Proof:

If there exist T_1, T_2 $T_1 < T_2$ such that $c(T_1) \geqq c(T_2)$, by Property 4 there must exist T , $T \in [T_1, T_2)$ such that

$$c(T) = \max_{t\in[T_1,T_2]} c(t) .$$

If $x(T) > 0$, by Property 5 $c(t)$ is strictly increasing in $(T-\varepsilon, T+\varepsilon)$, for ε small enough, contradiction to

$$c(T) = \max_{t\in[T_1,T_2]} c(t) .$$

If $x(T) = 0$, by Property 2 $c(t)$ is strictly increasing in $[T, T+\varepsilon)$, for ε small enough, contradiction ■

In the following, we study the asymptotic behavior.

Theorem 3-2:

If the utility function is nonsaturate $(U'(\bar{q}) > 0)$, then for t sufficiently large, $x(t) = 0$.

Before proving this theorem, we need the following lemmas.

Lemma 1:

If the utility function is nonsaturate, k_0 is the initial stock and $u'(\bar{q}\cdot k_0) \leqq U'(\bar{q}) \cdot \frac{r\bar{q}}{\alpha + r}$. Then $x(t) = 0$, $y(t) = (1-k_0)e^{-rt}$, $z(t) = 1 - (1-k_0)e^{-rt}$ is optimal.

Proof:

Define

$$\begin{cases} w(t) = e^{-\alpha t}\cdot[\psi(t) + s(t)] \\ w_1(t) = e^{-\alpha t}[(\bar{q}-1)\psi(t) - s(t)] \end{cases}$$

where $s(t) = -\psi(t) + r\bar{q}\int_t^\infty e^{-(\alpha+r)x}\cdot\psi(x)dx\cdot e^{(\alpha+r)t}$.

$$w_1(t) = e^{-\alpha t}\cdot\bar{q}\cdot\psi(t) - r\bar{q}\int_t^{\infty} e^{-(\alpha+r)x}\psi(x)dx\cdot e^{(\alpha+r)t}$$

$$\geqq e^{-\alpha t}\cdot\bar{q}\cdot\psi(t) - r\bar{q}\cdot\psi(t)\int_t^{\infty} e^{-(\alpha+r)x}dx\cdot e^{(\alpha+r)t} \qquad (\psi(t) \text{ is decreasing})$$

$$= e^{-\alpha t}\cdot\bar{q}\cdot\psi(t)\cdot\frac{\alpha + r - r}{\alpha + r} > 0 .$$

Apparently, $w(t) \geqq 0$ too.

By calculation, $w(t) = r\int_t^{\infty} w_1(s)ds$ for all t, which implies the complementary slackness is satisfied.

If we can show $w(t) \geqq e^{-\alpha t}\cdot\psi(t)$, then the dual feasibility is also satisfied, thus optimal.

$$U'(\bar{q}\cdot k_0) \geqq \psi(t) \geqq U'(\bar{q})$$

$$s(t) = -\psi(t) + r\bar{q}\int_t^{\infty} e^{-(\alpha+r)x}\psi(x)dx\cdot e^{(r+\alpha)t}$$

$$\geqq -U'(\bar{q}\cdot k_0) + \frac{r\bar{q}}{\alpha + r}\cdot U'(\bar{q}) \geqq 0$$

implies $w(t) \geqq e^{-\alpha t}\cdot\psi(t)$ ∎

<u>Lemma 2</u>:

$k(t) \to 1$ for t sufficiently large.

<u>Proof</u>:

Since $k(t)$ is increasing and bounded, $k(t) \to K$, $y(t) \to 0$ as $t \to \infty$.

If $k < 1$, as $t \to \infty$, $x(t) = 1 - y(t) - z(t) = 1 - k(t) - y(t) \to 1 - k > 0$. By Property 5 $\psi(t) \to 0$, contradiction. Thus $k(t) \to 1$ for t sufficiently large ■

Now we are ready to prove the theorem.

Proof:

$\frac{r\bar{q}}{\alpha + r} > 1$ and $U'(\bar{q}) > 0$ imply the existence of $k < 1$ such that $U'(\bar{q} \cdot k) \leqq U'(\bar{q}) \cdot \frac{r\bar{q}}{\alpha + r}$

By Lemma 2, there exists $T > 0$ such that $k(T) \geqq k$.

Now by Lemma 1, for t sufficiently large, we have $x(t) = 0$ ■

In the next section, we give a general algorithm to solve the one machine problem completely. This algorithm comprises of solving two subproblems (one with three activities, the other with two activities) backward by turns. It turns out that we can partition the machine stock into disjoint intervals such that on each interval either two activities or three activities are optimal, and alternately.

PROBLEM I (k)

(1) Objective: To find the largest interval $[\tilde{k},k]$ such that $x(t) > 0$, $y(t) > 0$, $z(t) > 0$ is optimal if the machine stock is between $[\tilde{k},k]$.

(2) This method is similar to the dual simplex method. By maintaining the dual feasibility and complementary slackness, we try to find the largest interval such that the primal feasibility is also satisfied.

(3) $z(t) = k - r \int_t^0 y(s)ds \quad t < 0$

$$c(t) = 1 - y(t) - z(t) + \bar{q}\cdot z(t) = 1 - k + \bar{q}k - y(t) - r(\bar{q} - 1) \int_t^0 y(s)ds \ .$$

Step 1: Solve equation $U'(k\cdot\bar{q})\cdot e^{[\alpha-r(\bar{q}-1)]t} = U'(c(t))$ with boundary condition $y(0) = 1 - k$.

Step 2: Compute $z(t)$.

Find T $(T < 0)$ such that $z(T) = k_0$. (T always exists, for as $T \to -\infty$, $e^{[\alpha-r(\bar{q}-1)]T} \to +\infty$ implies the equation in Step 1 has no solution.)

Step 3: If for all $t \in [T,0)$, $0 \leqq y(t) + z(t) \leqq 1$, $y(t) \geqq 0$, $z(t) \geqq 0$.

Let $\tilde{k} \leftarrow k_0$, $\Delta \leftarrow -T$. Return.

Otherwise, go to Step 4.

Step 4: Find smallest $\hat{T}$ $(\hat{T} < T)$, such that for $t \in (\hat{T},0)$

$0 \leqq y(t) + z(t) \leqq 1$, $y(t) \geqq 0$, $z(t) \geqq 0$.

Let $\tilde{k} \leftarrow z(\hat{T})$, $\Delta \leftarrow -\hat{T}$. Return.

PROBLEM II (k)

(1) Objective: To find the largest interval $[\tilde{k},k]$ such that $x(t) = 0$, $y(t) > 0$, $z(t) > 0$ is optimal if the machine stock is between $[\tilde{k},k]$.

(2) This method is similar to the simplex method. By maintaining the primal feasibility and complementary slackness, we try to find the largest interval such that the dual feasibility is also satisfied.

(3) $x(t) = 0$, $y(t) = (1-k)e^{-rt}$, $z(t) = 1 - (1-k)e^{-rt}$,

$c(t) = \bar{q}[1 - (1-k)e^{-rt}]$.

$$s(t) = -U'(c(t)) + \left[r\bar{q} \int_t^0 e^{-(r+\alpha)s} \cdot U'(c(s))ds + U'(\bar{q}\cdot k)\right] \cdot e^{(r+\alpha)t} .$$

Remark: $s(t)$ is the solution of the following integral equation with boundary condition $s(0) = 0$.

$$e^{-\alpha t} \cdot [U'(c(t)) + s(t)] = r \int_t^\infty e^{-\alpha x}[(\bar{q}-1)U'(c(x)) - s(x)]dx .$$

Step 1: Find T $(T < 0)$ such that $k_0 = 1 - (1-k)e^{-rt} = z(T)$.

(T always exists, for $z(T) \to -\infty$ as $T \to -\infty$.)

Step 2: If for all $t \in [T,0)$, $s(t) \geqq 0$.

Let $\tilde{k} \leftarrow k_0$, $\Delta \leftarrow -T$, return.

Otherwise, go to Step 3.

Step 3: Find smallest $\hat{T}$ $(\hat{T} < T)$ such that $s(t) \geqq 0$ for $t \in [\hat{T},0)$.

Let $\tilde{k} \leftarrow z(\hat{T})$, $\Delta \leftarrow -\hat{T}$. Return.

ALGORITHM

Initialize: Find $\hat{k}$ decided from Theorem 3-2.

$$k_1 \leftarrow \hat{k} \ , \ i \leftarrow 1 \ , \ k \leftarrow k_1 \ .$$

Step 1: Solve Problem I (k) .

$i \leftarrow i+1$.

$x^i(t) \leftarrow x(t)$, $y^i(t) \leftarrow y(t)$, $z^i(t) \leftarrow z(t)$, $k_i \leftarrow \tilde{k}$, $\Delta_i \leftarrow \Delta$.

If $k_i = k_0$, stop.

Or else, $k \leftarrow k_i$, and go to Step 2.

Step 2: Solve Problem II (k) .

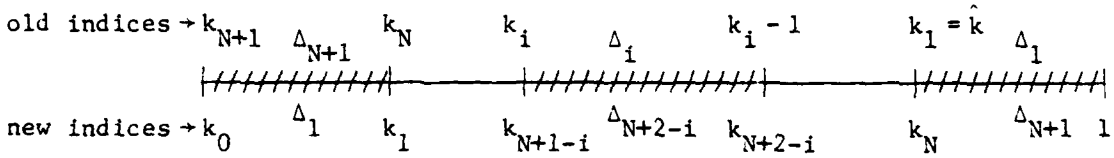
$i \leftarrow i+1$.

$x^i(t) \leftarrow 0$, $y^i(t) \leftarrow y(t)$, $z^i(t) \leftarrow z(t)$, $k_i \leftarrow \tilde{k}$, $\Delta_i \leftarrow \Delta$.

If $k_i = k_0$, stop.

Or else, $k \leftarrow k_i$, and go to Step 1.

To find the optimal solution, we have to reverse the order of the indices. Let $N+1$ be the last index.



$$\Delta_i \leftarrow \Delta_{N+2-i} \ , \ 2 \leq i \leq N+1$$

$$k_i \leftarrow k_{N+1-i} \ , \ 1 \leq i \leq N+1$$

$$x^i \leftarrow X^{N+2-i}(t) \ , \ y^i(t) \leftarrow y^{N+2-i}(t) \ , \ z^i(t) \leftarrow z^{N+2-i}(t) \ ,$$

$$2 \leq i \leq N+1 \ .$$

Define $T_i = \sum_{j=1}^{i} \Delta_j$, $1 \leqq i \leqq N$, $T_0 = 0$. Then

(1) $x^i(t)$, $y^i(t)$, $z^i(t)$ is optimal for $t \in [T_{i-1}, T_i)$ $1 \leqq i \leqq n$.

(2) $x(t) = 0$, $y(t) = (1-\hat{k})e^{-rt}$, $z(t) = 1-(1-\hat{k})e^{-rt}$ is optimal for $t \geqq T_n$.

§3. Quadratic Utility Function

When the utility function is quadratic, $U(x) = -\frac{(M-x)^2}{2}$, $M \geqq \bar{q}$, we have a simple optimal solution. First we would like to compute $\hat{k}$.

$$\psi(t) = M - \bar{q} + \bar{q}(1-\hat{k})e^{-rt}$$

$$s(t) = -\psi(t) + r\bar{q}\int_t^\infty e^{-(r+\alpha)s}\cdot\psi(s)ds\cdot e^{(r+\alpha)t}$$

$$= \frac{r(\bar{q}-1)-\alpha}{\alpha+r}(M-\bar{q}) - \frac{\alpha - r(\bar{q}-1)+r}{\alpha+2r}\cdot\bar{q}\cdot(1-\hat{k})e^{-rt} .$$

Case i: $M > \bar{q}$

If $\alpha - r(\bar{q}-1) + r \leqq 0$, then $\hat{k} = 0$.

If $\alpha - r(\bar{q}-1) + r > 0$, let $\bar{k}$ satisfy $\frac{r(\bar{q}-1)-\alpha}{\alpha+r}(M-\bar{q}) - \frac{\alpha - r(\bar{q}-1)+r}{\alpha+2r}\bar{q}(1-\bar{k}) = 0$, then $\hat{k} = \max\{0,\bar{k}\}$.

Case ii: $M = q$

If $\alpha - r(\bar{q}-1) + r \leqq 0$, then $\hat{k} = 0$.

If $\alpha - r(\bar{q}-1) + r > 0$, then $\hat{k} = 1$. (By Property 2, this means we never reach this situation.)

Remark:

$\alpha - r(\bar{q}-1) + r \leqq 0$ is not a sufficient condition for $\hat{k} = 0$ in a general concave utility function. If $\hat{k} = 0$, then we have $x(t) = 0$ for all t. If $\hat{k} > 0$, we would like to find the optimal program by using Algorithm.

Solve Problem I $(\hat{k})$:

$$\psi(t) = M - 1 + \hat{k} - \bar{q}\cdot\hat{k} + y(t) + r(\bar{q} - 1)\int_t^0 y(s)ds \qquad t < 0 .$$

Solve $U'(\hat{k}\cdot\bar{q})\cdot e^{[\alpha - r(\bar{q}-1)]t} = \psi(t)$ with $y(0) = 1 - \hat{k}$. Using the equation $\frac{r(\bar{q}-1) - \alpha}{\alpha + r}(M - \bar{q}) - \frac{\alpha - r(\bar{q}-1) + r}{\alpha + 2r}\bar{q}(1 - \hat{k}) = 0$ to simplify $y(t)$, we get

$$y(t) = \frac{(1-\hat{k})[\alpha - r(\bar{q}-1) + r]^2}{[\alpha - 2r(\bar{q}-1)](2r+\alpha)}\cdot e^{r(\bar{q}-1)t} + \frac{(M - q\cdot\hat{k})[\alpha - r(\bar{q}-1)]}{\alpha - 2r(\bar{q}-1)}\cdot e^{[\alpha - r(q-1)]t} .$$

Fact 1:

$y(t)$ is decreasing and $y(t) \geqq 0$ for all $t < 0$.

Proof:

$\alpha - r(\bar{q}-1) < 0$ implies $\dot{y}(t) < 0$, thus $y(t)$ is decreasing. Also $y(0) = 0$, thus $y(t) \geqq 0$ for $t < 0$ ■

Fact 2:

$y(t) + z(t)$ is a concave function.

Proof:

$\frac{d^2}{dt^2}[y(t) + z(t)] = \frac{d}{dt}[\dot{y}(t) + \dot{z}(t)] < 0$ (for $\alpha - r(\bar{q}-1) < 0$ and $\alpha - r(\bar{q}-1) + r > 0$) ■

Fact 3:

$y(t) + z(t) \leqq 1$ for all $t < 0$.

Proof:

$y(t) + z(t)$ is increasing as $t \to 0_-$. (Otherwise, violate $y(t) + z(t) \leqq 1$ for $y(0) + z(0) = 1$.) By Fact 2, $y(t) + z(t)$ is increasing for all $t < 0$, thus $y(t) + z(t) \leqq 1$ for all $t < 0$ ■

From Fact 3 and Fact 1, Algorithm stops.

Summarize the results as follows:

(1) If $M > \bar{q}$:

(i) If $k_0 \geqq \hat{k}$, then $x(t) = 0$, $y(t) = (1-k_0)e^{-rt}$, $z(t) = 1 - (1-k_0)e^{-rt}$ is optimal.

(ii) If $k_0 < \hat{k}$, $x(t) > 0$, $y(t) > 0$, $z(t) > 0$ is optimal if $t < T$, where $k(T) = \hat{k}$.
$x(t) = 0$, $y(t) > 0$, $z(t) > 0$ is optimal if $t \geqq T$.

(2) If $M = \bar{q}$:

(i) If $\alpha - r(\bar{q}-1) + r > 0$ $(\hat{k} = 1)$, then $x(t) > 0$, $y(t) > 0$, $z(t) > 0$ is optimal for all t .

(ii) If $\alpha - r(\bar{q}-1) + r \leqq 0$ $(\hat{k} = 0)$, then $x(t) = 0$, $y(t) = (1-k_0)e^{-rt}$, $z(t) = 1-(1-k_0)e^{-rt}$ is optimal.

Example 1:

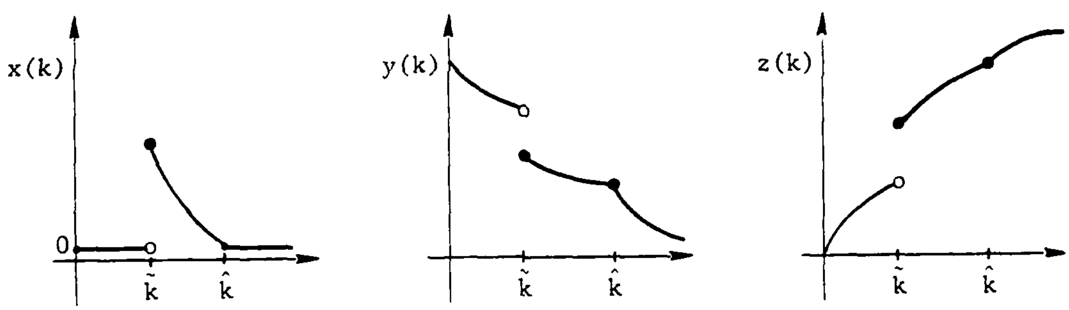
$\bar{q} = 2$, $\alpha = 0.9$, $r = 1$, $k_0 = 0$, and the utility function is $U(x) = -\frac{(5-x)^2}{2}$. $r(\bar{q}-1) - \alpha = 0.1 > 0$, $\alpha - r(\bar{q}-1) + r = 0.9 > 0$.
$\frac{0.1}{1.9} \times 3 - \frac{0.9}{2.9} \times 2 \times (1-\hat{k}) = 0$ implies $\hat{k} \doteqdot 0.73$. We can compute $y(t)$, $z(t)$, $x(t)$, when $k(t) < 0.73$, using the formula derived before ■

Let T be such that $x(T) + 2 \cdot z(T) = 1$, and $z(T) = \tilde{k}$.

Example 2:

Consider a modified quadratic utility function $U(x) = 4x - 12$ if $x \leqq 1$, $U(x) = -\frac{(5-x)^2}{2}$ if $x > 1$. From Example 1, if $\hat{k} \geqq k(t) \geqq \tilde{k}$ ($k(t)$ is the capital stock at time t), $x(t) > 0$; if $k(t) \geqq \hat{k}$, $x(t) = 0$. If $k(t) < \tilde{k}$, the output is located on the linear part of the utility function, thus $x(t) = 0$ (by Property 5, Corollary 2).

The following pictures show the discontinuous behavior of the optimal solution.



Remark:

We can remodify the utility function to be strictly concave by $U(x) = 4x - 12 - \varepsilon \cdot (x-1)^2$, $\varepsilon > 0$ if $x \leqq 1$. As $\varepsilon > 0$, $U''(x) \to 0$ if $x \leqq 1$. By Corollary 1, Property 5, $x(t) > 0$ for all t , where $k(t) < \hat{k}$, is not optimal.

CHAPTER IV

GENERAL CONCAVE UTILITY FUNCTION - MANY MACHINES CASE

In this chapter we generalize the idea of Chapter 2 and Chapter 3 to obtain the asymptotic result. Also, we give an example which shows the off-beat nature of this general problem. Again, we list the stationary program for completeness.

§1. Stationary Program

Theorem 4-1:

$1 \geqq \max_{1 \leqq i \leqq n} \frac{r_i q_i}{\alpha + r_i}$ is the sufficient and necessary condition for the existence of an optimal stationary program ■

From now on, assume $1 < \max_{1 \leqq i \leqq n} \frac{r_i q_i}{\alpha + r_i}$.

§2. Nonstationary Program

To begin with, we prove some simple properties.

Property 1:

$w(t)$ is continuous.

Proof:

We only need to consider those $T > 0$ such that there exist i , j $(i \neq j)$ and $y_i(t) > 0$ for $t \in (T - \varepsilon, T)$ and $y_j(t) > 0$ for $t \in (T, T + \varepsilon)$.

If $t \in (T-\varepsilon, T)$ $w(t) = r_i \int_t^\infty w_i(s)ds$, $w(t) \geqq r_j \int_t^\infty w_j(s)ds$.

If $t \in (T, T+\varepsilon)$ $w(t) = r_j \int_t^\infty w_j(s)ds$, $w(t) \geqq r_i \int_t^\infty w_i(s)ds$.

If $w(T) > r_i \int_T^\infty w_i(s)ds$, then as $t \to T_-$, $w(t) \to r_i \int_T^\infty w_i(s)ds <$ $w(T) = r_j \int_T^\infty w_j(s)ds$, contradiction. Thus $w(T) = r_i \int_T^\infty w_i(s)ds$ and $w(t)$ is continuous ■

Remark:

This property gives us the boundary condition.

Property 2:

$\psi(t)$ is continuous.

Proof:

If we can prove that for any sequence $t_n \to T$, such that $x(t_n) \to \beta$ or there exists i such that $z_i(t_n) \to \beta$ and $z_s(t_n) = 0$ $s \leqq i-1$, we have $\psi(t_n) \to \psi(T)$, then $\psi(t)$ is continuous.

The proof is similar to that of Property 4, Chapter 3 ■

Corollary:

If $u(x)$ is strictly concave, then (1) $c(t)$ is continuous; (2) $x(t)$ and $z_i(t)$ are all continuous $1 \leqq i \leqq n$ ■

Below we study the asymptotic behavior. We follow the notation of Chapter 2, and let $L = \left\{ (k_1, \ldots, k_N) \mid \sum_{i=1}^{N} k_i \leqq 1 , k_i \geqq 0 \right\}$.

For any best machine j , $\bar{k} = (k_1, \ldots, k_N) \in L$, define the following:

$$c(t,\bar{k}) = \sum_{i=1}^{N} k_i \cdot q_i + q_j\left(1 - \sum_{i=1}^{N} k_i\right)\cdot\left(1 - e^{-r_j t}\right)$$

$$s(t,\bar{k}) = -U'(c(t,\bar{k})) + r_j q_j \int_t^{\infty} e^{-(\alpha+r_j)s} U'(c(s,\bar{k}))ds \cdot e^{(\alpha+r_j)t}$$

$$G = \left\{ \bar{K} \;\middle|\; \begin{array}{l} \bar{K} \in L \text{ , and } r_j q_j \displaystyle\int_t^{\infty} e^{-(\alpha+r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot e^{(\alpha+r_j)t} - \\ \qquad\qquad q_{N+1} \cdot U'(c(t,\bar{k})) \geqq 0^{*} \\ \displaystyle\int_t^{\infty} e^{-\alpha s} \cdot U'(c(s,\bar{k}))ds \cdot r_i(q_j - q_i) + \int_t^{\infty} e^{-(\alpha+r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot \\ \qquad\qquad e^{r_j t} \cdot q_j(r_j - r_i) \geqq 0 \text{ for } i \in I - \{j\} \\ \text{for all } t \geqq 0 \text{ .} \end{array} \right\}$$

Given any initial stocks k_i of machine i, $1 \leqq i \leqq n$, let $\bar{k} = (k_1, \ldots, k_N)$, and we have the following theorem.

<u>Theorem 4-2</u>:

If $\bar{K} \in G$, then the best program with respect to j is optimal.

---

*If $N = n$, then replace q_{N+1} by 1 ; * implies $s(t,\bar{k}) \geqq 0$ for all t .

Proof:

The output rate function is $c(t,\bar{k})$. Define:

$$w(t) = e^{-\alpha t}\cdot[U'(c(t,\bar{k})) + s(t,\bar{k})]$$

$$w_i(t) = e^{-\alpha t}\cdot[(q_i - 1)U'(c(t,\bar{k})) - s(t,\bar{k})]_+ \qquad 1 \leqq i \leqq n \; .$$

If $i \in I$,

$$(q_i - 1)U'(c(t,\bar{k})) - s(t,\bar{k})$$

$$= q_i\cdot U'(c(t,\bar{k})) - r_j q_j \int_t^\infty e^{-(\alpha+r_j)s}\cdot U'(c(s,\bar{k}))ds\cdot e^{(\alpha+r_j)t}$$

$$\geqq q_i\cdot U'(c(t,\bar{k})) - \frac{r_i q_j}{\alpha + r_j} U'(c(t,\bar{k})) \qquad (U'(c(t,\bar{k})) \text{ is decreasing})$$

$$\geqq 0 \qquad (\text{definition of } I)$$

implies $w_i(t) = e^{-\alpha t}\cdot[(q_i - 1)U'(c(t,\bar{k})) - s(t,\bar{k})]$.

If $i \notin I$,

$$(q_i - 1)U'(c(t,\bar{k})) - s(t,\bar{k})$$

$$\leqq q_{N+1}\cdot U'(c(t,\bar{k})) - r_j q_j \int_t^\infty e^{-(\alpha+r_j)s}\cdot U'(c(s,\bar{k}))ds\cdot e^{(\alpha+r_j)t}$$

$$(q_i \leqq q_{N+1})$$

$$\leqq 0 \qquad (\text{definition of } G)$$

implies $w_i(t) = 0$.

$w(t) = r_j \int_t^\infty w_j(s)ds$ (by the result in the one machine case).

If $i \notin I$,

$$w(t) - r_i \int_t^\infty w_i(s)ds = w(t) \geqq 0 .$$

If $i \in I$,

$$w(t) - r_i \int_t^\infty w_i(s)ds$$

$$= \int_t^\infty e^{-\alpha s} \cdot U'(c(s,\bar{k}))ds \cdot r_i(q_j - q_i) + \int_t^\infty e^{-(\alpha + r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot q_j(r_j - r_i) \cdot e^{r_j t}$$

$\geqq 0$ (definition of G) .

Complementary slackness is also satisfied. Thus the best program with respect to j is optimal■

Let $a = \sum_{i=1}^{N} k_i \cdot q_i + b$, $b = q_j\left(1 - \sum_{i=1}^{N} k_i\right)$, then $c(t,\bar{k}) = a - b \cdot e^{-r_j t}$, $U'(a) \leqq U'(c(t,\bar{k})) \leqq U'(a-b)$.

<u>Corollary 1</u>:

If

$$U'(a-b) \leqq \min \begin{cases} U'(a) \cdot \dfrac{r_j q_j}{(\alpha + r_j)q_{N+1}} \ , \ U'(a) \cdot \dfrac{q_j(r_j - r_i) \cdot \alpha}{(\alpha + r_j)r_i \cdot (q_i - q_j)} \\ \qquad \text{if } i \in I - \{j\} \text{ and } r_j > r_i \ , \\ U'(a) \cdot \dfrac{(\alpha + r_j)r_i \cdot (q_j - q_i)}{q_j(r_i - r_j) \cdot \alpha} \quad \text{if } i \in I - \{j\} \text{ and } r_i > r_j \end{cases}$$

then the best program with respect to j is optimal.

Proof:

$$r_j q_j \int_t^\infty e^{-(\alpha+r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot e^{(\alpha+r_j)t} - q_{N+1} \cdot U'(c(t,\bar{k}))$$

$$\geqq \frac{r_j q_j}{\alpha + r_j} U'(a) - q_{N+1} \cdot U'(a-b) \geqq 0 \quad \text{(by assumption).}$$

If $r_j > r_i$ and $i \in I - \{j\}$

$$\int_t^\infty e^{-\alpha s} \cdot U'(c(s,\bar{k}))ds \cdot r_i \cdot (q_j - q_i) + \int_t^\infty e^{-(\alpha+r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot e^{r_j t} \cdot q_j \cdot (r_j - r_i)$$

$$\geqq \left[\frac{U'(a-b) \cdot r_i \cdot (q_j - q_i)}{\alpha} + \frac{U'(a) \cdot q_j \cdot (r_j - r_i)}{\alpha + r_j}\right] e^{-\alpha t} \quad (q_i > q_j)$$

$$\geqq 0 \quad \text{(by assumption).}$$

If $r_i > r_j$ and $i \in I - \{j\}$

$$\int_t^\infty e^{-\alpha s} \cdot U'(c(s,\bar{k}))ds \cdot r_i \cdot (q_j - q_i) + \int_t^\infty e^{-(\alpha+r_j)s} \cdot U'(c(s,\bar{k}))ds \cdot e^{r_j t} \cdot q_j \cdot (r_j - r_i)$$

$$\geqq \left[\frac{U'(a) \cdot r_i \cdot (q_j - q_i)}{\alpha} + \frac{U'(a-b) \cdot q_j \cdot (r_j - r_i)}{\alpha + r_j}\right] e^{-\alpha t} \quad (q_j > q_i)$$

$$\geqq 0 .$$

Thus $\bar{K} \in G$ and the best program is optimal ■

Corollary 2:

If $U'(a) > 0$ and $\dfrac{r_j q_j}{\alpha + r_j} > \dfrac{r_i q_i}{\alpha + r_i}$ for $i \neq j$, then when $\sum_{i=1}^{N} k_i \to 1$, $\bar{K} \in G$.

Proof:

$$\frac{r_j q_j}{\alpha + r_j} > q_{N+1} \quad (N+1 \notin I) \text{ , implies } \frac{r_j q_j}{(\alpha + r_j)q_{N+1}} > 1 \text{ .}$$

If $i \in I - \{j\}$ and $r_j > r_i$,

$$q_j(r_j - r_i)\alpha - (\alpha + r_j)r_i(q_i - q_j) = r_j q_j(\alpha + r_i) - r_i q_i(\alpha + r_j)$$

$$> 0 \left(\frac{r_j q_j}{\alpha + r_j} > \frac{r_i q_i}{\alpha + r_i}\right) ,$$

$$\text{implies } \frac{q_j(r_j - r_i)\alpha}{(\alpha + r_j)r_i(q_i - q_j)} > 1 \text{ .}$$

If $i \in I - \{j\}$ and $r_i > r_j$,

$$(\alpha + r_j)r_i(q_j - q_i) - q_j(r_i - r_j)\alpha = r_j q_j(\alpha + r_i) - r_i q_i(\alpha + r_j) > 0 \text{ ,}$$

$$\text{implies } \frac{(\alpha + r_j)r_i \cdot (q_j - q_i)}{q_j(r_i - r_j)\alpha} > 1 \text{ .}$$

As $\sum_{i=1}^{N} k_i \to 1$, $U'(a) \to U'(a-b)$; thus as $\sum_{i=1}^{N} k_i \to 1$, Corollary 1 is true, which implies $\bar{K} \in G$ ■

Remark:

If the best machine is not unique, we can use Taylor Expansion to get the similar result.

Lemma 1:

As $t \to \infty$, $c(t) \to$ constant, $z_i(t) \to$ constant, $y_i(t) \to 0$, $x(t) \to$ constant.

Proof:

Since $k_i(t)$ is increasing and bounded, $k_i(t) \to c_i$ as $t \to \infty$, which implies $y_i(t) \to 0$, $\sum_{i=1}^{n} z_i(t) + x(t) \to 1$ as $t \to \infty$.

Since $q_1 > q_2 > \ldots > q_n > 1$, so $x(t)$ and $z_i(t) \to$ constant as $t \to \infty$. $c(t) = x(t) + \sum_{i=1}^{n} q_i \cdot z_i(t) \to$ constant as $t \to \infty$ ■

Theorem 4-3:

As $t \to \infty$, $\sum_{i=1}^{N} k_i(t) \to 1$.

Proof:

For any $\varepsilon > 0$, there exist $T, P_1, P_2, \ldots, P_n$, $P > 0$, such that $x(t) \geqq P$, $z_i(t) \geqq P_i$ $1 \leqq i \leqq n$ and $\sum_{i=1}^{n} P_i + P \geqq 1 - \varepsilon$ *for all* $t \geqq T$. After time T we can solve the original problem in the following two steps:

Step 1: Put $P_i(P)$ units of labor to work with machine i (alone).

Step 2: Consider a new problem with $1 - \left(\sum_{i=1}^{n} P_i + P \right)$ units of labor: let $\tilde{c}(t)$ be the output rate function, $\tilde{U}(x)$ be the utility function, where $\tilde{U}(\tilde{c}(t)) =$

$$U\left(\tilde{c}(t) + \sum_{i=1}^{n} p_i q_i + p\right).$$

As $\varepsilon \to 0$, $\tilde{u}(\tilde{c}(t)) \to$ constant for all t, thus the optimal program for Step 2 is the best program with respect to j, which implies that we only build machine j as $t \to \infty$. Combining Step 1 and Step 2, we find that as $t \to \infty$ we only build machine j. From the proof of Theorem 4-2, we must have $z_i(t) = 0 \quad i \notin I$ as $t \to \infty$, which implies $\sum_{i=1}^{N} k_i(t) \to 1$ as $t \to \infty$ ■

Corollary:

If $U'(\bar{q}_i) > 0$, $\frac{r_j q_j}{\alpha + r_j} > \frac{r_i q_i}{\alpha + r_i}$ for $i \neq j$, then for t sufficiently large, the best program with respect to j is optimal ■

Property 3:

$\psi(t)$ is decreasing.

Proof:

By contradiction. If not, let $(T - \varepsilon, T)$ be the last time interval such that $\psi(t)$ is increasing. (T exists, for $\psi(t)$ is continuous and as $t \to \infty$ the best program is optimal.)

There are two possibilities: $x(t) > 0$ or $0 < z_i(t) < k_i(t)$ for some i. If $x(t) > 0$ and $y_\ell(t) > 0$, $t \in (T - \varepsilon, T)$,

$$w(t) = e^{-\alpha t} \cdot \psi(t) = r_\ell \int_t^\infty w_\ell(s) ds$$

$$\leq r_\ell \int_t^T w_\ell(s) ds + r_\ell (q_\ell - 1) \cdot \psi(T) \int_T^\infty e^{-\alpha s} ds$$

($\psi(t)$ is decreasing for $t \geq T$ and $w_\ell(t) \leq e^{-\alpha t} \cdot \psi(t) \cdot (q_\ell - 1)$).

As $t \to T_-$, $c^{-\alpha T} \cdot \psi(T) \leqq \frac{r_\ell(q_\ell - 1)}{\alpha} \psi(T) \cdot e^{-\alpha T}$ implies $\alpha \leqq r_\ell(q_\ell - 1)$.

But $\psi(t) = c \cdot e^{[\alpha - r_\ell(q_\ell - 1)]t}$ (by Property 5, Chapter 3) implies $\psi(t)$ is decreasing, contradiction. In the case of $0 < z_i(t) < k_i(t)$, the proof is similar ■

Corollary 1:

If $y_i(t) > 0$ then $\alpha \leqq r_i(q_i - 1)$.

Proof:

$$e^{-\alpha t} \cdot \psi(t) \leqq w(t) \blacksquare$$

Corollary 2:

If $u(x)$ is strictly concave, then $c(t)$ is strictly increasing ■

§3. Example

Example 1:

The following two-machines problem shows that we build different machines. (Time $[0,T)$ we build machine 2, time $[T,\infty)$ we build machine 1.)

$$U(x) = -\frac{(51 - x)^2}{2} , \ \alpha = 0.9 , \ r_1 = 1 , \ q_1 = 50 , \ r_2 = 4 , \ q_2 = \frac{50}{1.9} , \ q = 1 .$$

$\frac{r_1 q_1}{\alpha + r_1} = q_2 > \frac{r_2 q_2}{\alpha + r_2}$ implies machine 1 is best, machine 2 is good.

And we can find the exact dual price, using the technique in Chapter 3, such that there exist T , $k > 0$ $\left(k = 1 - e^{-r_2 \cdot T} \approx 0.0932\right)$ and

$x(t) = 0$, $y_1(t) = (1-k)e^{-r_1(t-T)}$, $z_1(t) = 1 - k - (1-k)e^{-r_1(t-T)}$, $z_2(t) = k$, $y_2(t) = 0$ is optimal for $t \geq T$; $x(t) = 0$, $y_1(t) = 0$, $z_1(t) = 0$, $y_2(t) = e^{-r_2 t}$, $z_2(t) = 1 - e^{-r_2 t}$ is optimal for $t < T$.

Example 2:

This two machines example shows that we build machine 2 in some time interval (T_1, T_2) , but as $t \to \infty$ we never use machine 2. $M = 51$, $r_1 = 1$, $q_1 = 50$, $k_1 = k_2 = 0$, $\alpha = 0.9$, $q_2 = \frac{40}{1.9}$, r_2 = some large number (decided later), $u(x) = -\frac{(51-x)^2}{2}$, $q = 1$. $\frac{r_1 q_1}{\alpha + r_1} = \frac{50}{1.9} > \frac{40}{1.9} = q_2 \geq \frac{r_2 q_2}{\alpha + r_2}$, implies $z_2(t) = 0$ as $t \to \infty$ (machine 2 is not good). $\alpha - r_1(q_1 - 1) + r_1 = 0.9 - 49 + 1 \leq 0$, so if we can prove that $x(t) = 0$, $y_1(t) = e^{-r_1 t}$, $z_1(t) = 1 - e^{-r_1 t}$, $y_2(t) = 0$, $z_2(t) = 0$ is not optimal, then there must exist some time interval in which we build machine 2. If $x(t) = 0$, $y_1(t) = e^{-r_1 t}$, $z_1(t) = 1 - e^{-r_1 t}$ is optimal, then $w_2(t) = e^{-\alpha t}\left[\left(q_2 - \frac{r_1 q_1}{\alpha + r_1}\right)(M - q_1) + e^{-r_1 t} \cdot q_1 \cdot \left(q_2 - \frac{r_1 q_1}{\alpha + 2r_1}\right)\right]_+$. If $q_2 - \frac{r_1 q_1}{\alpha + r_1} < 0$, $q_2 - \frac{r_1 q_1}{\alpha + 2r_1} > 0$ and $\left(q_2 - \frac{r_1 q_1}{\alpha + r_1}\right)(M - q_1) + q_1\left(q_2 - \frac{r_1 q_1}{\alpha + 2r_1}\right) > 0$, then there exists $T > 0$ such that $\left(q_2 - \frac{r_1 q_1}{\alpha + r_1}\right)(M - q_1) + e^{-r_1 T} \cdot q_1 \cdot \left(q_2 - \frac{r_1 q_1}{\alpha + 2r_1}\right) = 0$, which implies $w_2(t) = 0$ $t \geq T$, $w_2(t) > 0$ $t < T$.

$$r_2 \int_{\frac{T}{2}}^{\infty} w_2(s)ds = r_2 \int_{\frac{T}{2}}^{T} w_2(s)ds = r_2 \cdot k \quad \text{where } k = \int_{\frac{T}{2}}^{T} w_2(s)ds > 0 .$$

$$> w\left(\frac{T}{2}\right) \quad \text{as } r_2 \to \infty .$$

Contraction to $w\left(\frac{T}{2}\right) \gtreqqless r_2 \int_{\frac{T}{2}}^{\infty} w_2(s)ds$. Check:

$$q_2 - \frac{r_1 q_1}{\alpha + r_1} \approx -5 < 0 \ .$$

$$q_2 - \frac{r_1 q_1}{2r + \alpha} \approx 20 - 16.3 \approx 3.7 > 0 \ .$$

$$\left(q_2 - \frac{r_1 q_1}{\alpha + r_1}\right)(M - q_1) + q_1\left(q_2 - \frac{r_1 q_1}{2r_1 + \alpha}\right) = -5 + 50 \times 3.7 > 0 \ .$$

CHAPTER V

VARIANT OF THE MODEL

In this chapter we introduce the depreciation factor δ into the model. Difficulty arises here because the capital stocks are no longer monotonic over time in the many machines case. For simplicity, we concentrate on the one machine case.

The model with the depreciation factor δ is:

$$\langle P \rangle \quad \max \int_0^\infty e^{-\alpha t}\cdot U(x(t))dt \quad \text{s.t.} \quad A\cdot x(t) \leqq \int_0^t e^{-\delta(t-s)}\cdot B\cdot x(s)ds + b(t) \ ,$$

$$x(t) \geqq 0 \ .$$

$$\langle D \rangle \quad \min \int_0^\infty w(t)\cdot b(t)dt \quad \text{s.t.} \quad w(t)A \geqq \int_t^\infty e^{-\delta(s-t)}\cdot w(s)\cdot Bds +$$

$$e^{-\alpha t}\cdot \nabla U(\hat{x}(t))dt \ , \ w(t) \geqq 0 \ .$$

In the one machine case, if we compare the depreciation and non-depreciation cases, the differences are the following two equations:

$$z(t) \leqq r \int_0^t e^{-\delta(t-s)}\cdot y(s)ds + k_0\cdot e^{-\delta t} \ .$$

$$w(t) \geqq r \int_t^\infty e^{-\delta(s-t)}\cdot w_1(s)ds \ .$$

The following program is called contraction: $x(t) = 1 - k_0 e^{-\delta t}$, $y(t) = 0$, $z(t) = k_0\cdot e^{-\delta t}$.

Theorem 5-1:

$1 \geqq \frac{r\bar{q}}{\alpha + r + \delta}$ is a necessary condition for the existence of an optimal contraction program.

Proof:

$$c(t) = \bar{q}\cdot k_0 \cdot e^{-\delta t} + 1 - k_0 \cdot e^{-\delta t} \text{ , } \psi(t) \text{ is increasing.}$$

Define: $w(t) = e^{-\alpha t}\cdot\psi(t)$, $w_1(t) = e^{-\alpha t}(\bar{q}-1)\psi(t)$.

$$0 \leqq w(t) - r\int_t^\infty e^{-\delta(s-t)}\cdot w_1(s)ds \quad \text{-----} \ (*)$$

$$\leqq e^{-\alpha t}\cdot\psi(t) - r(\bar{q}-1)\psi(t)\cdot e^{\delta t}\cdot \int_t^\infty e^{-(\alpha+\delta)s}ds \quad (\psi(t) \text{ is increasing})$$

$$= e^{-\alpha t}\cdot\psi(t) \ \frac{\alpha + \delta - r(\bar{q}-1)}{\alpha + \delta}$$

implies $\alpha + \delta - r(\bar{q}-1) \geqq 0$, $1 \geqq \frac{r\bar{q}}{\alpha + r + \delta}$ ■

Remark:

(1) (*) is the sufficient and necessary condition for the existence of an optimal contraction program.

(2) In the linear utility function case, the above condition is also sufficient.

From now on, assume $1 < \frac{r\bar{q}}{\alpha + r + \delta}$ and $U'(\bar{q}) > 0$.

In the following, we prove an asymptotic result similar to the nondepreciation case.

Lemma 1:

If $x(t) = 0$, $z(t) = r\int_0^t e^{-\delta(t-s)}\cdot y(s)ds + k\cdot e^{-\delta t}$, $x(t) + y(t) + z(t) = 1$, then $y(t) = \frac{\delta}{r+\delta} + \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}$, $z(t) = \frac{r}{r+\delta} - \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}$.

Proof:

$$r\int_0^t e^{-\delta(t-s)}y(s)ds + y(t) = 1 - k\cdot e^{-\delta t}\text{ , implies}$$

$$y(t) = \frac{\delta}{r+\delta} + \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}\text{ , } z(t) = \frac{r}{r+\delta} - \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t} \blacksquare$$

Define:

$$\begin{cases} c(t,k) = \bar{q}\left[\frac{r}{r+\delta} - \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}\right] \\ s(t,k) = -U'(c(t,k)) + r\bar{q}\int_t^\infty e^{-(\alpha+r+\delta)x}\cdot U'(c(x,k))dx\cdot e^{(\alpha+r+\delta)t} \\ \theta(t,k) = U'(c(t,k)) - r\int_t^\infty e^{-(\alpha+r+\delta)x}\cdot U'(c(x,k))dx\cdot e^{(\alpha+r+\delta)t} \\ G = \{k \mid 0 \leqq k \leqq 1 \text{ , } s(t,k) \geqq 0 \text{ , } \theta(t,k) \geqq 0 \text{ for all } t \geqq 0\} \text{ .} \end{cases}$$

If $k = \frac{r}{r+\delta}$, $s\left(t, \frac{r}{r+\delta}\right) = -U'\left(\frac{r\bar{q}}{r+\delta}\right) + \frac{r\bar{q}\cdot U'\left(\frac{\bar{q}\cdot r}{r+\delta}\right)}{\alpha+r+\delta} =$

$U'\left(\frac{r\bar{q}}{\delta+r}\right)\cdot\frac{r\bar{q}-r-\alpha-\delta}{\alpha+r+\delta} > 0$, $\theta\left(t, \frac{r}{r+\delta}\right) = U'\left(\frac{r\bar{q}}{r+\delta}\right)\left(1 - \frac{r}{\alpha+r+\delta}\right) > 0$.

Thus $\frac{r}{r+\delta} \in G$, $G \neq \emptyset$.

Lemma 2:

$s(t,k)$ is the solution of the following equations:

$$w(t) = e^{-\alpha t}\cdot[U'(c(t,k)) + s(t,k)] \ , \ w_1(t) = e^{-\alpha t}\cdot[(\bar{q}-1)U'(c(t,k)) - s(t,k)],$$

$$w(t) = r\int_t^\infty e^{-\delta(s-t)}\cdot w_1(s)ds \ .$$

Proof:

It is a straightforward calculation ■

Theorem 5-2:

If $k \in G$, then $x(t) = 0$, $y(t) = \frac{\delta}{r+\delta} + \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}$,
$z(t) = \frac{r}{r+\delta} - \left(\frac{r}{r+\delta} - k\right)e^{-(r+\delta)t}$ is optimal.

Proof:

Define:

$$\begin{cases} w(t) = e^{-\alpha t}\cdot[U'(c(t,k)) + s(t,k)] \\ w_1(t) = e^{-\alpha t}\cdot[(\bar{q}-1)U'(c(t,k)) - s(t,k)]_+ \ . \end{cases}$$

$$w_1(t) = e^{-\alpha t}\cdot\bar{q}\cdot\left[U'(c(t,k)) - r\int_t^\infty e^{-(\alpha+r+\delta)s}\cdot U'(c(s,k))ds\cdot e^{(r+\alpha+\delta)t}\right]_+$$

$$= e^{-\alpha t}\cdot\bar{q}\cdot\theta(t,k) = e^{-\alpha t}\cdot[(\bar{q}-1)U'(c(t,k)) - s(t,k)] \ .$$

By Lemma 2 and the definition of G, the rest of the proof is straightforward ■

Corollary:

If $k_0 \to \frac{r}{r+\delta}$, then $k_0 \in G$.

Proof:

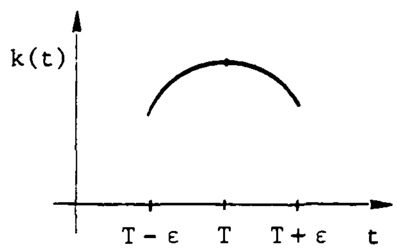
As $k_0 \to \frac{r}{r+\delta}$, $s(t,k_0) \to s\left(t, \frac{r}{r+\delta}\right) > 0$, $\theta(t,k_0) \to \theta\left(t, \frac{r}{r+\delta}\right) > 0$ implies $k_0 \in G$ ■

Lemma 3:

$$k(t) = k_0 \cdot e^{-\delta t} + r \int_0^t e^{-\delta(t-s)} \cdot y(s)ds \quad \text{is monotonic.}$$

Proof:

If $k(t)$ is not monotonic, say increasing in $(T-\varepsilon, T)$, decreasing in $(T, T+\varepsilon)$. Since $k(t)$ is continuous, some moment right before T should have the same capital stock as some moment right after T , thus the same behavior. (See the illustration below.)



Contradict to the fact that $k(t)$ is increasing before T and $k(t)$ is decreasing after T ■

Lemma 4:

As $t \to \infty$, $c(t) \to$ constant, $\psi(t) \to$ constant.

Proof:

As $t \to \infty$ $z(t) = k(t) \to$ constant ($k(t)$ is monotonic and bounded);

$y(t) \to$ constant ($k(t) \to$ constant);

$$x(t) \to \text{constant} \quad (x(t) + y(t) + z(t) = 1) \ ;$$

$$c(t) = x(t) + \bar{q} \cdot z(t) \to \text{constant}, \quad \psi(t) \to \text{constant} \ \blacksquare$$

<u>Lemma 5</u>:

If $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, then $\psi(t) = c \cdot e^{[\alpha+\delta-r(\bar{q}-1)]t}$.

<u>Proof</u>:

Straightforward $\blacksquare$

<u>Lemma 6</u>:

As $t \to \infty$, $y(t) \to \beta > 0$.

<u>Proof</u>:

By contradiction. If $y(t) \to 0$ as $t \to \infty$, then $z(t) \to 0$, $x(t) \to 1$ as $t \to \infty$. If $y(t) \neq 0$ as $t \to \infty$, contradiction to Lemma 4 and Lemma 5. If $y(t) = 0$ as $t \to \infty$, contradiction to Theorem 5-1 $\blacksquare$

From Lemmas 4, 5 and 6, we know for t sufficiently large, $x(t) = 0$. Thus $k(t) \to \frac{r}{r + \delta}$, $k(t) \in G$ as $t \to \infty$. We state the above results in the following theorem.

<u>Theorem 5-3</u>:

(1) If $k_0 < \frac{r}{r + \delta}$ then $k(t)$ is increasing, and for t sufficiently large $k(t) \to \frac{r}{r + \delta}$, $x(t) = 0$.

(2) If $k_0 > \frac{r}{r + \delta}$ then $k(t)$ is decreasing, and for t sufficiently large $k(t) \to \frac{r}{r + \delta}$, $x(t) = 0$ $\blacksquare$

REFERENCES

[1] Bellman, R. E., DYNAMIC PROGRAMMING, Princeton University Press, Princeton, NJ, 1957.

[2] Grinold, R. C., "Continuous Programming, Part One: Linear Objective," Journal of Mathematical Analysis and Applications, Vol. 28, No. 1 (1969).

[3] Hanson, M., "A Class of Continuous Convex Programming Problems," Journal of Mathematical Analysis and Applications, Vol. 22, pp. 427-437 (1968).

[4] Lehman, R. S., "On the Continuous Simplex Method," RAND Research Memorandum RM-1386, Santa Monica, California (1954).

[5] Perold, A. F., "Continuous Simplex Method," Ph.D. Dissertation, Stanford University, Stanford, California (1979).

[6] Tyndall, W. F., "A Duality Theorem for a Class of Continuous Linear Programming Problems," SIAM Journal on Applied Mathematics, Vol. 13, No. 3 (1965).

If $U'(a) > 0$ and $\frac{[illegible]}{\alpha + r_j} > \frac{[illegible]}{\alpha + r_i}$ for $i \neq j$, then when $\sum_{i=1}^{N} k_i \to 1$, $\bar{K} \in G$.