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ANNUAL REVIEW OF RESEARCH

under the

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October 1980

Institute for Electronics Science

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Lubbock, Texas 79409

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This report represents the fourth year of research performed under the auspices of the Joint Services Electronics Program at Texas Tech University. The program is concentrated in the "information electronics" area and includes researchers from both the departments of Electrical Engineering and Mathematics. Specific work units deal with Feedback System Design, Nonlinear Control, Nonlinear Fault Analysis, Detection and Estimation in Imagery, Multidimensional System Theory, and Pointing and Tracking.

Each work unit is presented in the report by a summary of the work performed dur-

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) ing the past year, a list of publications and activities in the area, reprints of all papers which have been published during the past year, and abstracts of pending papers. In addition, the report includes lists of all grants and contracts administered by JSEP personnel, the department of Electrical Engineering and the Department of Mathematics; and a list of all publications prepared by JSEP personnel.

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Abstract

This report represents the fourth year of research performed under the auspices of the Joint Services Electronics Program at Texas Tech University. The program is concentrated in the "information electronics" area and includes researchers from both the departments of Electrical Engineering and Mathematics. Specific work units deal with Feedback System Design, Nonlinear Control, Nonlinear Fault Analysis, Detection and Estimation in Imagery, Multidimensional System Theory, and Pointing and Tracking.

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Contents

Significant Accomplishments Report

A. Nonlinear Fault Analysis

During the past year we have made a major change in the direction of our research in the nonlinear fault analysis area which, we believe, will open up the way for a whole new approach to the subject. Hitherto, our research had been directed towards the development of multiple test vector simulation-after-test algorithms. Although such algorithms work well in the linear case they require too much on-line computer time to be effective in the nonlinear case. As such, we have turned our attention toward simulation-before-test algorithms and single test vector simulation-after-test algorithms during the past year with considerable success in both areas.

The basic problem with simulation-before-test algorithms is the large amount of off-line computer time required to generate the fault dictionary which underlies the technique. Although this is also a problem in digital testing it is greatly exaggerated in the analog case by the continuous nature of the failure phenomena, tolerance problems, modeling problems, and the high cost of analog simulation. As such, we have developed a new differential-interpolative approach to the simulation-before-test concept which allows one to locate a failure which lies between the simulated faults and/or failures which have been perturbed by tolerance effects. This, in turn, allows the number of entries in the fault dictionary to be reduced with a commensurate reduction in computer costs.

Our second approach is a single test vector simulation-after-test algorithm which uses a "restricted number of failures" assumption to reduce the number of test points employed. Historically, single test vector simulation-after-test algorithms have always been highly attractive in that they are simple and easy to use and have minimal on-line and off-line computational requirements. Unfortunately, the applicability of these algorithms has been limited by the large number of test points which they require. By taking advantage of the fact that at most three or four components will ever fail simultaneously, however, we have been able to decrease the test point requirements for the algorithm while retaining its other positive attributes.

B. Detection and Estimation in Imagery

During the past year we have completed a study of the detection and estimation problem in imagery. Although the mathematics for such a problem is similar to that encountered in the more classical communications problem the problem is greatly complicated by the nonlinear character of the noise phenomena and the high data rates encountered. In this endeavor we have developed an optimal estimation theory and compared it with various approximate and sub-optimal approaches. In particular, it was shown that one could not approximate the nonlinear noise phenomena by a linear term but one could develop sub-optimal nonlinear algorithms whose performance approximated that of the optimal algorithm while achieving a cost reduction. Indeed, from a practical point of view the sub-optimal algorithms were actually superior to the optimal algorithms because of their greater robustness to modeling errors. Moreover, unlike the optimal algorithms they may be implemented in "real time" at video data rates.

C. Pointing and Tracking

Much of our research in this area during the past year has been devoted to the development of efficient computer algorithms for the implementation of our Lie theoretic pointing and tracking theory developed previously. Specific emphasis has been placed on the development of algorithms which are insensitive to the noise phenomena encountered in video imagery and in algorithms which have the potential for real time implementation. Several of these algorithms have now been experimentally implemented while we are waiting for the delivery of our image processing system to begin experimental "real time" implementation of the theory using actual video tracking data.

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6. Summary:

The goal of the work unit is the development of a theory for the design of general linear feedback systems using ring theoretic techniques. Thus far we have formulated a complete parameterization for the set of compensators which stabilize a given plant and/or cause it to track or reject a prescribed family of inputs.[2,3,7] This theory has, in turn, been applied to the problem of designing robust and adaptive control systems. In particular, we have developed a theory for the simultaneous stabilization of two distinct plants[1] by a single compensator and we have laid the foundations for a new theory of adaptive control.[5]

7. Publications and Activities:

A. Refereed Journal Articles

1. Desoer, C.A., Liu, R.-w., Murray, J., and R. Saeks, "Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis", IEEE Trans. on Auto. Cont., Vol. AC-25, pp. 399-412, (1980).

2. Saeks, R., and J. Murray, "Feedback System Design: The Tracking and Disturbance Rejection Problems", IEEE Trans. on Auto. Cont. (to appear).

B. Conference Papers and Abstracts

1. Desoer, C.A., Liu, R.-w., Murray, J., and R. Saeks, "Feedback System Design: The Fractional Representation to Analysis and

Synthesis", Proc. of the 1979 IEEE Conference on Decision and Control, Ft. Lauderdale, Dec. 1979, pp. 33-37.

2. Karmokolias, C., and R. Saeks, "A Fractional Representation Approach to Adaptive Control", 1980 IEEE Conference on Decision and Control, Albuquerque, Dec. 1980, pp. 272-273.

C. Theses

1. Chua, O., "New Approaches to the Robust Design of Control Systems", M.S. Thesis, Texas Tech Univ., 1980.

2. Iyer, A., Ph.D. Dissertation, Texas Tech Univ., (in preparation).

D. Conferences and Symposia

1. Saeks, R., and Murray, J.J., 1979 IEEE Conference on Decision and Control, Ft. Lauderdale, Dec. 1979.

2. Saeks, R., U.S./Australia Seminar on System Theory, Univ. of Newcastle, March 1980.

3. Saeks, R., and J.J. Murray, 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, April 1980.

4. Saeks, R., and J.J. Murray, Workshop on Mathematics of Networks and Systems, Virginia Beach, May 1980.

E. Lectures

1. Saeks, R., "Feedback System Design", Elec. Engrg. Seminar, Colorado State Univ., April 1980.

2. Saeks, R., "Stability and Homotopy", Decision and Control Seminar, Harvard Univ., May 1980

FEEDBACK SYSTEM DESIGN: THE FRACTIONAL REPRESENTATION APPROACH TO ANALYSIS AND SYNTHESIS

C.A. Desoer, R.-W. Liu, J.Murray
and R. Saeks

IEEE Transactions on Automatic Control

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Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis

C. A. DESOER, FELLOW, IEEE, RUEY-WEN LIU, JOHN MURRAY, AND RICHARD SAEKS, FELLOW, IEEE

Abstract—The problem of designing a feedback system with prescribed properties is attacked via a *fractional representation approach to feedback* system analysis and synthesis. To this end we let H denote a ring of operators with the prescribed properties and model a given plant as the ratio of two operators in H. This, in turn, leads to a simplified test to determine whether or not a feedback system in which that plant is embedded has the prescribed properties and a complete characterization of those compensators which will "place" the feedback system in H. The theory is formulated axiomatically to permit its application in a wide variety of system design problems and is extremely elementary in nature requiring no more than addition, multiplication, subtraction, and inversion for its derivation even in the most general settings.

I. Introduction

INTUITIVELY, the linear feedback system design process may be broken down into three steps: modeling, analysis, and synthesis; each of which may be carried out via a multiplicity of time and frequency domain techniques. In engineering practice, however, the three steps are loosely matched to one another. The purpose of the present paper is to use fractional representation models to

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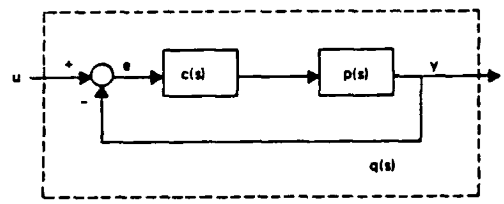

Fig. 1. Single-variate control system.

the analysis and synthesis of feedback systems. Here, if one desires to design a system with prescribed properties the given plant is initially modeled as a quotient of two operators, each of which has the desired properties. Once such a model has been specified a similar model may be formulated for the feedback system constructed from that plant which, in turn, may be used to determine whether or not the feedback system has the desired properties. Moreover, the set of compensators which will cause the feedback system to have the prescribed properties may be completely characterized in terms of such a model. As such, by choosing a model for the plant which is matched to the design criteria the analysis and synthesis processes for a feedback system may be greatly simplified.

These ideas are illustrated by the following derivation of the set of stabilizing compensators for the single variate control system of Fig. 1.

We say that a transfer function $p(s)$ is *exponentially stable* (exp. stable) if $p(s)$ is a *proper rational* function with poles having *negative* real parts. Although the plant may naturally be modeled as a quotient of coprime polynomials [16],[19] $p(s)=a(s)/b(s)$ since our ultimate goal is a stable system we prefer to model $p(s)$ as a quotient of exp. stable rational functions

$$p(s)=n(s)/d(s)=[a(s)/m(s)][b(s)/m(s)]^{-1} \quad (1.1)$$

where $m(s)$ is strictly Hurwitz polynomial of degree equal to the degree of $b(s)$. Moreover, since $a(s)$ and $b(s)$ are coprime, the rational functions $n(s)$ and $d(s)$ are coprime in the sense that there exist exp. stable rational functions $u(s)$ and $v(s)$ such that

$$u(s)n(s)+v(s)d(s)=1. \quad (1.2)$$

Similarly, we assume that our compensator is modeled as a quotient of exp. stable rational functions, $c(s)=x(s)/y(s)$, which are coprime in the above sense. Now, a little algebra will reveal that the closed-loop system transfer function from input u to output y is given by a ratio of exp. stable rational functions in the form

$$h_{yu}(s)=n(s)[y(s)d(s)+x(s)n(s)]^{-1}x(s). \quad (1.3)$$

Moreover, it can be shown[1] that $h_{yu}(s)$ will be stable if and only if

$$[y(s)d(s)+x(s)n(s)]=k(s) \quad (1.4)$$

has an exp. stable inverse. Since $k(s)$ is, itself, exp. stable this implies that the feedback system will be exp. stable if and only if $k(s)$ is nonzero for all $\mathrm{Re}\, s\geq 0$, including ∞. An exp. stable function with these properties is called miniphase. As such, the problem of synthesizing an exp. stable feedback system reduces to the solution of (1.4) for exp. stable rational functions $x(s)$ and $y(s)$ given exp. stable functions $n(s)$ and $d(s)$ and a miniphase function $k(s)$.

By direct substitution one may verify that

$$y^h(s)=r(s)n(s) \quad \text{and} \quad x^h(s)=-r(s)d(s) \quad (1.5)$$

satisfy the homogeneous equation

$$y^h(s)d(s)+x^h(s)n(s)=0 \quad (1.6)$$

for all exp. stable rational functions $r(s)$. Moreover, since $n(s)$ and $d(s)$ are coprime it follows that all exp. stable rational solutions of (1.6) are of this form [15],[18]. On the other hand, a particular solution of (1.4) may be obtained by multiplying (1.2) by $k(s)$, which yields

$$y^p(s)=k(s)v(s) \quad \text{and} \quad x^p(s)=k(s)u(s). \quad (1.7)$$

As such, if we let $r(s)$ vary over the set of exp. stable rational functions and $k(s)$ vary over the set of miniphase functions we obtain a complete parameterization of the stabilizing compensators for our feedback system in the form

$$c(s)=\frac{x(s)}{y(s)}=\frac{[k(s)u(s)-r(s)d(s)]}{[k(s)v(s)+r(s)n(s)]}=\frac{[u(s)-w(s)d(s)]}{[v(s)+w(s)n(s)]} \quad (1.8)$$

where $w(s)=r(s)/k(s)$ ranges over the exp. stable rational functions.

A comparison of (1.8) with the class of stabilizing compensators derived by Youla, Bongiorno, and Jabr [24],[25],[29] will reveal that the two results differ only in that our $u(s)$, $v(s)$, $n(s)$, and $d(s)$ are exp. stable rational functions while theirs are polynomials.[2] Unlike their analytic derivation, however, the above result was obtained via elementary algebraic operations. Indeed, the only properties of the exp. stable rational functions employed are their closure under addition and multiplication together with the fact that the identity is an exp. stable rational function, i.e., the exp. stable rational functions form a ring with identity. As such, if the exp. stable rational functions of the above derivation were to be replaced by any prescribed ring of single-input single-output systems, (1.8) would yield a complete characterization of the compensators which would "place" the feedback system in that ring. If one works with a ring of rational functions with poles in a prescribed region a solution of the pole placement problem is obtained [18], whereas, if one chooses to work with stable transcendental functions a solution to the stabilization problem for distributed systems is obtained [7],[8] etc. Indeed, with minor modifications the derivation can be extended to noncommutative rings thereby including multivariate and time-varying systems. In each case, a simple solution to a fundamental problem of feedback system design is obtained by virtue of choosing a model for the given plant which is matched to the ultimate goal of the design problem. In particular, if we desire to design a feedback system which lies in a prescribed ring of operators we model the plant as a quotient of operators from that ring.

Consistent with the above philosophy the following section of the paper is devoted to the formulation of an axiomatic theory of fractional system representation. Here, a given system is modeled as a quotient of two operators lying in a prescribed ring H. The corresponding feedback system analysis and synthesis problems are then studied in the succeeding sections. In particular, Section III is devoted to the problem of determining whether or not a feedback system lies in H given that its plant is represented as a quotient of systems from H while Section IV is devoted to the problem of characterizing those compensators which will "place" the feedback system in H. The resultant axiomatic theory of feedback system design is applicable to multivariate, time-varying, distrib-

[1] See the axiomatic derivation of Section III for the details.

[2] From a computational point of view, it is more convenient to represent rational functions as ratios of polynomials, as per Youla *et al.*

TABLE I
EXAMPLES OF THE AXIOMATIC SYSTEM (G, H, I, J)

G	$R(s)$	$R_p(s)$	$R(s)^{n\times n}$	$R_p(s)^{n\times n}$	$\hat{B}(\sigma_0)$	$\hat{B}(\sigma)^{n\times n}$	$L_\infty(R)$	$B(H)$
H	$R[s]$	$R(\sigma_0)$	$R[s]^{n\times n}$	$R(\sigma_0)^{n\times n}$	$A_-(\sigma_0)$	$\hat{A}_-(\sigma_0)^{n\times n}$	$H_\infty(R)$	$C(H)$
I	$R[s] \neq 0$	$R^\infty(\sigma_0)$	$M \in R[s]^{n\times n}$ s.t. $\lvert M(s)\rvert \neq 0$	$M \in R(\sigma_0)^{n\times n}$ s.t. $\lvert M(s)\rvert \in R^\infty(\sigma_0)$	$\hat{A}_-^\infty(\sigma_0)$	$M \in \hat{A}_-(\sigma_0)^{n\times n}$ s.t. $\lvert M(s)\rvert \in \hat{A}_-^\infty(\sigma_0)$	$m \in H_\infty(R)$ s.t. $\inf_\omega \lVert m(j\omega)\rVert > 0$	$C_0(H)$
J	$m \in R[s]$ s.t. $m(s) = c \neq 0$	$m \in R^\infty(\sigma_0)$ s.t. $m(s) \neq 0$ for $s \in C_{\sigma_0^+}$	$M \in R[s]^{n\times n}$ s.t. $\lvert M(s)\rvert \neq 0$ for $s \in C_{\sigma_0^+}$	$M \in R(\sigma_0)^{n\times n}$ s.t. $\lvert M(s)\rvert \in R^\infty(\sigma_0)$ & $\lvert M(s)\rvert \neq 0$ for $s \in C_{\sigma_0^+}$	$m \in \hat{A}^\infty(\sigma_0)$ s.t. $m(s) \neq 0$ for $s \in C_{\sigma_0^+}$	$M \in \hat{A}_-(\sigma_0)^{n\times n}$ s.t. $\lvert M(s)\rvert \in \hat{A}_-^\infty(\sigma_0)$ $\lvert M(s)\rvert \neq 0$ for $s \in C_{\sigma_0^+}$	$m \in H_\infty(R)$ s.t. $\inf_\omega \lVert m(j\omega)\rVert > 0$ & m is outer	$CC(H)$
Ref.	19	31	19	15,18	4, 31	6,7,8	12	11,15

$R(s)$ = rational functions with real coeficients

$R_p(s)$ = proper rational functions with real coeficients

$X^{n\times n}$ = n by n matrices of elements in X.

A = distributions of the form $g(t) + \sum g_i \delta(t - t_i)$ where $g(t)$ is an integrable function s.t. $g(t) = 0$ for $t < 0$; g_i is a summable sequence and $0 = t_0 < t_1 < t_2 < \ldots$.

$\hat{A}_-(\sigma_0)$ = Laplace transforms of distributions g such that $g(t)e^{-\sigma_0 t}$ is in A for some $\sigma_1 < \sigma_0$

$\hat{A}_-^\infty(\sigma_0)$ = multiplicative subset of $\hat{A}_-(\sigma_0)$ consisting of elements bounded away from zero at ∞.

$\hat{B}(\sigma_0)$ = quotients of elements of the form m/n where $m \in A_-(\sigma_0)$ and $n \in A_-^\infty(\sigma_0)$.

$R[s]$ = polynomials with real coeficients

$C_{\sigma_0^+}$ = complex numbers with real part greater than or equal to σ_0

$R(\sigma_0)$ = proper rational functions with real coficients which are analytic in $C_{\sigma_0^+}$

$R^\infty(\sigma_0)$ = proper rational functions with real coefficients which are analytic in $C_{\sigma_0^+}$ and nonzero at ∞

$B(H)$ = bounded linear operators on a Hilbert Space H.

$C(H)$ = causal bounded linear operators on a Hilbert space H.

$C_0(H)$ = causal bounded linear operators with a bounded inverse on a Hilbert space H.

$CC(H)$ = causal bounded linear operators with a causal bounded inverse on a Hilbert space H.

$L_\infty(R)$ = essentially bounded Lebesque measurable functions defined on R.

$H_\infty(R)$ = the Hardy space of essentially bounded Lebesque measurable functions defined on R which have an analytic extension into C_{0^+}

uted, and some multidimensional systems and includes the stabilization, pole placement, and feedforward design problems. Several of these applications are illustrated by the examples of Section V. In the final section of the paper a partial generalization of the theory to nonlinear systems is described. This follows the algebraic pattern established in the linear case but is formulated in terms of a left-distributive ring to model the properties of a nonlinear system [23].

II. AXIOMATIC THEORY

Table I displays several examples of the axiomatic system developed below. Reference to it will help in visualizing the breadth and significance of the theory. Additional examples also appear in Section V.

Let G be a (not necessarily commutative) ring with identity and let H be a subring of G which includes the identity. The feedback system and its subsystems will be represented by operators which are elements of G. The compensator will be chosen so that the overall system will be represented by an operator in the subring H.

We define two multiplicative subsets [2],[27] of H,

$$I = \{h \in H \mid h^{-1} \in G\}, \tag{2.1}$$

i.e., I is the set of elements of H which have an inverse in G;

$$J = \{h \in H \mid h^{-1} \in H\}, \tag{2.2}$$

i.e., J is the *subgroup* of H consisting of all invertible elements of H. Note that

$$J \subset I \subset H \subset G. \tag{2.3}$$

Given the above structure we say that a system $g \in G$ has a *right fractional representation* in $\{G,H,I,J\}$ if there exist $n_r \in H$ and $d_r \in I$ such that $g = n_r d_r^{-1}$. Furthermore, we say that the pair $(n_r, d_r) \in H \times H$ is *right coprime* if there exist u_r and v_r in H such that

$$u_r n_r + v_r d_r = 1. \tag{2.4}$$

The right fractional representation $n_r d_r^{-1}$ in $\{G,H,I,J\}$ is said to be *right coprime* if the pair (n_r, d_r) is right coprime.

The relationship between our concept of coprimeness and the usual common factor criterion for coprimeness [28] is given by the following properties.

Property 1: Let the pair $(n_r, d_r) \in H \times H$ be right coprime. Let n_r and d_r have a common right factor $r \in H$, i.e., $n_r = x_r r$, $d_r = y_r r$ for some $x_r \in H$ and $y_r \in H$. Then r has a *left inverse* in H.

Proof: Substitute the assumed factorizations of n_r and d_r into (2.4) and obtain

$$u_r n_r + v_r d_r = (u_r x_r + v_r y_r) r = 1. \tag{2.5}$$

Since H is a ring, $u_r x_r + v_r y_r \in H$. From (2.5) it follows that $r^{-L} = u_r x_r + v_r y_r$ is a left-inverse of r. ■

Property 2: Let $g = n_r d_r^{-1}$ be a right coprime fractional representation of g in $\{G,H,I,J\}$. Let $g = x_r y_r^{-1}$ be a second (not necessarily coprime) right fractional representation of g in $\{G,H,I,J\}$. Then there exists an r in H such that

$$x_r = n_r r \quad \text{and} \quad y_r = d_r r. \tag{2.6}$$

Proof: Given the two factorizations of g, let $r = d_r^{-1}y_r$; hence $r \in G$. Then

$$y_r = d_r r \tag{2.7}$$

and, performing calculations in the ring G, we obtain

$$x_r = gy_r = (n_r d_r^{-1})y_r = n_r(d_r^{-1}y_r) = n_r r. \tag{2.8}$$

From (2.7) and (2.8), r is a common right factor of x_r and y_r. To show that $r \in H$, consider

$$r = d_r^{-1}y_r = (u_r n_r + v_r d_r)d_r^{-1}y_r = u_r n_r d_r^{-1}y_r + v_r y_r$$

$$= u_r g y_r + v_r y_r = u_r x_r + v_r y_r \in H \tag{2.9}$$

where we used the equality $g = x_r y_r^{-1} = n_r d_r^{-1}$ to derive (2.9). ■

Although G is, in general, a noncommutative ring, the entire theory developed above for right fractional representations can be replicated for left fractional representations. In particular, we say that $g \in G$ has a *left fractional representation* in $\{G,H,I,J\}$ if there exist $n_l \in H$ and $d_l \in I$ such that $g = d_l^{-1}n_l$. Furthermore we say that the pair $(n_l,d_l) \in H \times H$ is *left coprime* if there exist u_l and v_l in H such that

$$n_l u_l + d_l v_l = 1. \tag{2.10}$$

The left fractional representation $d_l^{-1}n_l$ is said to be *left coprime* if the pair (n_l,d_l) is left coprime. With these definitions the existence of a *common left factor* for a left fractional representations of g is characterized by the following properties.

Property 1': Let the pair (n_l,d_l) be left coprime. Let n_l and d_l have a common left factor l in H, i.e., $n_l = lx_l$, $d_l = ly_l$ for some $x_l \in H$ and $y_l \in H$. Then l has a *right inverse* $\in H$.

Property 2': Let $g = d_l^{-1}n_l$ be a left coprime fractional representation of g in $\{G,H,I,J\}$. Let $g = y_l^{-1}x_l$ be a second (not necessarily coprime) left fractional representation of g in $\{G,H,I,J\}$. Then there exists an l in H such that

$$x_l = ln_l \quad \text{and} \quad y_l = ld_l. \tag{2.11}$$

The above properties of a coprime fractional representation have all been derived under the assumption that such a representation exists. Of course, if G denotes the rational matrices and H denotes the polynomial matrices the existence of a coprime representation is implied by classical analysis [16],[19]. Indeed, the classical analysis readily extends to the case where H is taken to be the exp. stable rational matrices or the ring of proper rational matrices with poles in a prescribed region [18]. On the other hand for multidimensional [26], distributed [4],[8], and time-varying systems [11],[15] there is no assurance that an arbitrary $g \in G$ will admit a fractional representation nor even that the set of $g \in G$ which admit such a representation will be a linear space. Moreover, all g's which admit a fractional representation may not admit a coprime fractional representation [26]. In general, the set of $g \in G$ which admit a fractional representation in $\{G,H,I,J\}$ will form a subring of G if and only if the Ore condition[3] is satisfied while criteria for coprimeness have been formulated in various special cases though no general theory exists [1],[4],[26]. The standard condition for the existence of fractional representations which are coprime in the sense of (2.4) is that H be a right principal ideal domain.

Reference to Table I shows that in applications it is important to have conditions under which g will be in H and these conditions should be expressed in terms of its fractional representation.

Property 3: Let $g = n_r d_r^{-1}$ with $n_r \in H$ and $d_r \in I$.

a) If $d_r \in J$, then $g \in H$.

b) If $g = n_r d_r^{-1}$ is a *right coprime fractional representation* of g in $\{G,H,I,J\}$, then $g \in H$ implies that $d_r \in J$.

Proof:

a) We have $d_r \in J$; hence by (2.2), $d_r^{-1} \in H$ and thus $n_r d_r^{-1} = g \in H$.

b) We have $g \in H$. Furthermore, $n_r = gd_r$, $d_r = 1d_r$ implies that d_r is a right common factor of n_r and d_r; hence by Property 1, d_r has a left inverse in H. But $d_r \in I$ by assumption, so d_r^{-1} exists and is an element of G; thus $d_r^{-1} = d_r^{-L} \in H$; hence, by (2.2), $d_r \in J$. ■

Property 3': Let $g = d_l^{-1}n_l$ with $n_l \in H$ and $d_l \in I$.

a) If $d_l \in J$, then $g \in H$.

b) If $g = d_l^{-1}n_l$ is a *left coprime fractional representation* of g in $\{G,H,I,J\}$, then $g \in H$ implies that $d_l \in J$.

Property 4: Let $g = n_r d^{-1}n_l$ where $n_r, n_l \in H$, and $d \in I$.

a) If $d \in J$, then $g \in H$.

b) Let, in addition, $n_r d^{-1}$ be a right coprime fractional representation in $\{G,H,I,J\}$ and $d^{-1}n_l$ be a left coprime fractional representation in $\{G,H,I,J\}$; then $g \in H$ implies that $d \in J$.

Proof:

a) By assumption, $d \in J$; hence $d^{-1} \in H$. So $g = n_r d^{-1}n_l \in H$.

b) Since $d^{-1}n_l$ is a left coprime fractional representation there exist $u_l, v_l \in H$ such that

$$n_l u_l + dv_l = 1, \tag{2.12}$$

thus,

$$n_r d^{-1} = n_r d^{-1}(n_l u_l + dv_l) = n_r d^{-1}n_l u_l + n_r v_l = gu_l + n_r v_l. \tag{2.13}$$

Now $g \in H$ hence (2.13) gives $n_r d^{-1} \in H$. By Property 3, $n_r d^{-1} \in H$ together with the fact that the pair (n_r,d) is right coprime implies $d \in J$. ■

III. Analysis

To start with consider the feedback system Σ_p of Fig. 2. Suppose that the plant is described by a right coprime fractional representation $p = n_r d_r^{-1}$ in $\{G,H,I,J\}$. The

[3] $\{G,H,I,J\}$ satisfies the Ore condition for right fractional representations if, whenever $g \in G$ admits a left fractional representation it also admits a right fractional representation and vice versa.[2]

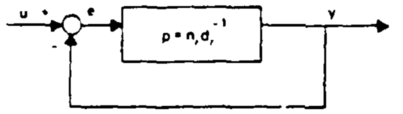

Fig. 2. Unity gain negative feedback system.

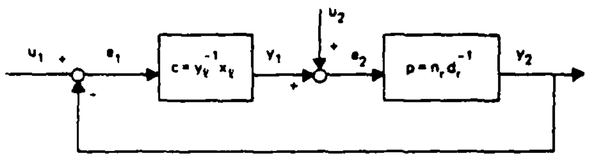

Fig. 3. Feedback system with plant and compensator.

closed-loop dynamics of Σ_p are described by the maps

$$h_{eu}: u \mapsto e\ ; \qquad h_{eu} = (1+p)^{-1} = d_r(d_r+n_r)^{-1} \tag{3.1}$$

$$h_{yu}: u \mapsto y\ ; \qquad h_{yu} = p(1+p)^{-1} = n_r(d_r+n_r)^{-1}. \tag{3.2}$$

Note that

$$h_{eu} + h_{yu} = 1. \tag{3.3}$$

We say that Σ_p is *well defined in* G, (H, respectively), if $h_{eu} \in G$, (H, respectively).

Note that the pairs $(n_r, d_r + n_r)$ and $(d_r, d_r + n_r)$ are right coprime; indeed, the right coprimeness of (n_r, d_r) implies (2.4), hence

$$(u_r - v_r)n_r + v_r(d_r + n_r) = 1 \tag{3.4}$$

while

$$(v_r - u_r)d_r + u_r(d_r + n_r) = 1. \tag{3.5}$$

Theorem 1: Consider the feedback system Σ_p of Fig. 2.

a) Let $p = n_r d_r^{-1}$ be a fractional representation in $\{G,H,I,J\}$ of the element $p \in G$; then Σ_p *is well defined in* G if and only if $d_r + n_r \in I$.

b) Let $p = n_r d_r^{-1}$ be a *right coprime* fractional representation in $\{G,H,I,J\}$ of the element $p \in G$; then Σ_p *is well defined in* H if and only if $d_r + n_r \in J$.

Proof: a) $\Rightarrow$. $h_{eu} \in G$ and $d_r \in I$ imply

$$d_r^{-1}h_{eu} = d_r^{-1}(1+p)^{-1} = d_r^{-1}d_r(d_r+n_r)^{-1} = (d_r+n_r)^{-1} \in G. \tag{3.6}$$

Now $d_r \in I \subset H$ and $n_r \in H$, so $d_r + n_r \in H$. This together with (3.6) implies $d_r + n_r \in I$.

a) $\Leftarrow$. $d_r + n_r \in I$ implies $(d_r+n_r)^{-1} \in G$; hence $h_{eu} = d_r(d_r+n_r)^{-1} \in G$.

b) Follows from Property 3, together with (3.4) and (3.5). ■

Of course, a similar theorem holds for left factorizations.

We now consider the feedback system Σ of Fig. 3 where the plant p is preceded by a compensator c; p and c belong to G and are specified by their coprime fractional representation in $\{G,H,I,J\}$ $n_r d_r^{-1}$ and $y_l^{-1}x_l$, respectively.

To describe the feedback system Σ we consider the map $h_{eu}: (u_1,u_2) \mapsto (e_1,e_2)$. Simple calculations give

$$h_{eu} = \begin{bmatrix} h_{e_1u_1} & h_{e_1u_2} \\ h_{e_2u_1} & h_{e_2u_2} \end{bmatrix} = \begin{bmatrix} (1+pc)^{-1} & -p(1+cp)^{-1} \\ c(1+pc)^{-1} & (1+cp)^{-1} \end{bmatrix}. \tag{3.7}$$

Now let $h_{yu}: (u_1,u_2) \mapsto (y_1,y_2)$. Using the summing node equations it is easy to see that

$$h_{yu} = K(h_{eu} - 1) \quad \text{and} \quad h_{eu} = 1 - Kh_{yu} \tag{3.8}$$

where K is the symplectic matrix

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.9}$$

It is well known that in the case of multivariable rational matrices, one has to consider the four submatrices of h_{eu} in (3.8) because examples show that any one of the submatrices may be unstable while the remaining ones are stable. (For detailed examples, see [30].) Let us calculate

$$\begin{aligned} h_{e_1u_1} &= (1+pc)^{-1} = 1 - pc(1+pc)^{-1} \\ &= 1 - p(1+cp)^{-1}c \\ &= 1 - p\left[y_l^{-1}(y_l d_r + x_l n_r)d_r^{-1}\right]^{-1}c \\ &= 1 - n_r(y_l d_r + x_l n_r)^{-1}x_l \end{aligned} \tag{3.10}$$

$$\begin{aligned} h_{e_2u_1} &= c(1+pc)^{-1} = (1+cp)^{-1}c \\ &= d_r(y_l d_r + x_l n_r)^{-1}x_l \end{aligned} \tag{3.11}$$

$$\begin{aligned} h_{e_2u_2} &= (1+cp)^{-1} = (1+y_l^{-1}x_l n_r d_r^{-1})^{-1} \\ &= \left[y_l^{-1}(y_l d_r + x_l n_r)d_r^{-1}\right]^{-1} \\ &= d_r(y_l d_r + x_l n_r)^{-1}y_l \end{aligned} \tag{3.12}$$

$$h_{e_1u_2} = -p(1+cp)^{-1} = -n_r(y_l d_r + x_l n_r)^{-1}y_l. \tag{3.13}$$

We say that Σ is *well defined in* G, (H, respectively) if and only if each entry of h_{eu} defined in (3.8) belongs to G, (H, respectively).

Theorem 2: Consider the feedback system Σ of Fig. 3. Let $n_r d_r^{-1}$ and $y_l^{-1}x_l$ be a right and left fractional representations of p and c in $\{G,H,I,J\}$.

a) If $y_l d_r + x_l n_r \in I$, then Σ is well defined in G.

b) If $y_l d_r + x_l n_r \in J$, then Σ is will defined in H.

c) If $h_{e_2u_1} \in G$, then $y_l d_r + x_l n_r \in I$ hence if Σ is well defined in G, then $y_l d_r + x_l n_r \in I$.

d) Assume, in addition, that $n_r(y_l d_r)^{-1}$ and $(y_l d_r)^{-1}x_l$ are right coprime and left coprime fractional representation, respectively; then $h_{e_1u_1} \in H$ implies that $y_l d_r + x_l n_r \in J$, and hence, if Σ is well defined in H, then $y_l d_r + x_l n_r \in J$.

Proof: a) and b). If $y_l d_r + x_l n_r \in I$, (J, respectively), then by the definition (2.1) of I, [(2.2) of J, respectively], the formulas (3.10)–(3.13), and the closure of the ring G, (H, respectively), the conclusion follows.

c) If $h_{e_2u_2} \in G$, then so is $d_r^{-1}h_{e_2u_2}y_l^{-1}$ since $d_r \in I$ and $y_l \in I$. Now,

$$d_r^{-1}h_{e_2u_2}y_l^{-1} = d_r^{-1}(1+cp)^{-1}y_l^{-1}$$
$$= d_r^{-1}(1+y_l^{-1}x_l n_r d_r^{-1})^{-1}y_l^{-1}$$
$$= d_r^{-1}\left[y_l^{-1}(y_l d_r + x_l n_r)d_r^{-1}\right]y_l^{-1} = (y_l d_r + x_l n_r)^{-1} \tag{3.14}$$

hence the fact that $h_{e_2u_2} \in G$ implies that $(y_l d_r + x_l n_r)^{-1} \in G$ and thus $(y_l d_r + x_l n_r) \in I$.

d) First we prove that the pair $(n_r, y_l d_r + x_l n_r)$ is right coprime. Since $(n_r, y_l d_r)$ is right coprime, there exists $\tilde{u}_r$ and $\tilde{v}_r \in H$ such that

$$\tilde{u}_r n_r + \tilde{v}_r y_l d_r = 1; \tag{3.15}$$

hence

$$(\tilde{u}_r - \tilde{v}_r x_l)n_r + \tilde{v}_r(y_l d_r + x_l n_r) = 1 \tag{3.16}$$

and the claim is established. Similarly, we show that $(y_l d_r + x_l n_r, x_l)$ is left coprime. Now consider

$$h_{e_1u_1} = 1 - n_r(y_l d_r + x_l n_r)^{-1}x_l. \tag{3.17}$$

By assumption, $h_{e_1u_1} \in H$; then the special assumption of d) and Property 4 imply that $y_l d_r + x_l n_r \in J$. This completes the proof. ∎

Note, the special assumptions used in d) to the effect that $n_r(y_l d_r)^{-1}$ is right coprime and $(y_l d_r)^{-1}x_l$ is left coprime, imply, in some sense, that p and c have no common factors. More precisely, since J serves as the group of units in our theory these conditions imply that any common factors of p and c must lie in J.

IV. Design

Consistent with our approach of matching the plant model to the goal of the given feedback system design problem the present section is devoted to the problem of characterizing the set of compensators which will "place" a feedback system in a prescribed ring H given that both the plant and compensator are modeled by fractional representations in $\{G, H, I, J,\}$.

Theorem 3: For the feedback system Σ of Fig. 3, let the plant p have a right coprime and a left coprime fractional representation $p = n_r d_r^{-1} = d_l^{-1}n_l$ in $\{G, H, I, J\}$. Let u_r and v_r both in H be such that (2.4) holds. Then for any $w \in H$ such that $wn_l + v_r \in I$, the compensator

$$c = (wn_l + v_r)^{-1}(-wd_l + u_r) \in G \tag{4.1}$$

results in a feedback system Σ well defined in H. For such a compensator, $h_{eu} \in H^{2\times 2}$ and

$$h_{eu} = \begin{bmatrix} 1 - n_r(-wd_l + u_r) & -n_r(wn_l + v_r) \\ d_r(-wd_l + u_r) & d_r(wn_l + v_r) \end{bmatrix}. \tag{4.2}$$

Conversely, if Σ is well defined in H and if the compensator $c = y_l^{-1}x_l$ is such that $(n_r, y_l d_r)$ and $(y_l d_r, x_l)$ are right coprime and left coprime respectively, then c is given by expression (4.1).

Proof:

Step 1: Choose any $k \in J$, (hence $k^{-1} \in H$), and solve for y_l and $x_l \in H$ the equation

$$y_l d_r + x_l n_r = k. \tag{4.3}$$

Observe that if (y_l, x_l) is *any solution in H* of (4.3), then

$$k^{-1}(y_l d_r) + k^{-1}(x_l n_r) = 1 \tag{4.4}$$

and

$$(y_l d_r)k^{-1} + (x_l n_r)k^{-1} = 1, \tag{4.5}$$

hence, $(n_r, y_l d_r)$ is right coprime and $(y_l d_r, x_l)$ is left coprime. Thus, the assumptions of Theorem 2, part d) holds for *any* solution of (4.3).

Step 2: Obtain all solutions of the homogeneous equation

$$y_l^h d_r + x_l^h n_r = 0. \tag{4.6}$$

Since $p = n_r d_r^{-1} = d_l^{-1}n_l$, direct calculation shows that for any $r \in H$,

$$y_l^h = rn_l \qquad x_l^h = -rd_l \tag{4.7}$$

are solutions of (4.6).

It remains to show that all solutions of (4.6) are of the form (4.7); so we *assume* that y_l^h and $x_l^h \in H$ and satisfy (4.6). Let $r = -x_l^h d_l^{-1}$; hence

$$x_l^h = rd_l. \tag{4.8}$$

Now using (4.6)

$$y_l^h = y_l^h d_r d_r^{-1} = -x_l^h n_r d_r^{-1} = -x_l^h p$$
$$= -x_l^h d_l^{-1} n_l = rn_l. \tag{4.9}$$

Equations (4.8) and (4.9) show that any solution of (4.6) has the form of (4.7); it remains, however, to show that $r \in H$,

$$r = -x_l^h d_l^{-1} = -x_l^h d_l^{-1}(d_l v_l + n_l u_l)$$
$$= -x_l^h v_l - x_l^h d_l^{-1} n_l u_l = -x_l^h v_l + y_l^h u_l \in H. \tag{4.10}$$

Step 3: Obtain a particular solution of (4.3). From the right coprimeness condition for (n_r, d_r),

$$kv_r d_r + ku_r n_r = k \tag{4.11}$$

hence

$$y_l^p = kv_r, \quad x_l^p = ku_r. \tag{4.12}$$

Hence any solution of (4.3) is of the form

$$y_l = rn_l + kv_r$$
$$x_l = -rd_l + ku_r \qquad \text{for some } r \in H \tag{4.13}$$

and for any such solution $(n_r, y_l d_r)$ is right coprime and $(y_l d_r, x_l)$ is left coprime.

Step 4: Consider the condition

$$r \in H \text{ and } k \in J \qquad \text{such that } rn_l + kv_r \in I \tag{4.14}$$

or equivalently, if we set $w = k^{-1} r \in H$,

$$w \in H \qquad \text{such that } wn_l + v_r \in I. \tag{4.15}$$

If (4.15) holds,

$$c = (wn_l + v_r)^{-1}(-wd_l + u_r) \in G \tag{4.16}$$

is a compensator in G which can also be written as [see (4.13)]

$$c = (rn_l + kv_r)^{-1}(-rd_l + ku_r). \tag{4.17}$$

If we let $y_l = rn_l + kv_r$ and $x_l = -rd_l + ku_r$, then, by (4.17), $c = y_l^{-1}x$ and, by calculation, we verify that (4.3) holds. Thus for any such compensator, by Theorem 2, the feedback system Σ is well defined in H.

Step 5: Conversely consider a feedback system well defined in H with a compensator $c = y_l^{-1}x_l$ such that $(n_r, y_l d_r)$ and $(y_l d_r, x_l)$ are right coprime and left coprime, respectively. By Theorem 2, (4.3) holds for some $k \in J$, hence by the analysis above, c is also given by (4.1) for some $w \in H$ such that $wn_l + v_r \in I$. The proof is thus complete. ■

The theorem yields a complete parameterization of all possible controllers which will place a plant in H given the existence of:

1) right and left coprime fractional representations of p and

2) a w in H for which $(wn_l + v_r)$ is in I.

In the multivariable case where p is a square matrix whose elements are proper rational functions it is well known that p has left and right coprime fractional representations [19]. In order to obtain a *proper* controller one has to choose w in (4.1) so that $\det[w(s)n_l(s) + v_r(s)] \neq 0$ at infinity. Methods for obtaining such a proper stabilizing controller have been reported in [32] and [33]. Alternatively, one can verify the existence of such a w in our algebraic setting by invoking the fact that n_l and d_r are right coprime and applying linear algebraic arguments thereto. Of course, these arguments apply to distributed systems as well as lumped systems using the formulation of [7] and [8].

In the most general ring theoretic setting neither right nor left coprime fractional representations of p, nor a w such that $(wn + v_r)$ is in I, are assured to exist. At present, the only known counterexample to the latter is, however, in the ring of integers which is of no system theoretic interest.

Conditions 1) and 2) have been conjectured to be both necessary and sufficient conditions for the existence of a compensator, c, which places the feedback system in H [3]. In fact, if c places the feedback system in H, then from (3.7) we obtain left and right fractional representations

$$p = (-h_{e_1u_2})(h_{e_2u_2})^{-1} = (h_{e_1u_1})^{-1}(-h_{e_1u_2}). \tag{4.18}$$

Note that there is no guarantee that these fractional representations are coprime. These representations are, however, coprime when the compensator is in H. Indeed, in that case they satisfy a stronger condition which completely characterizes those plants which can be placed in H by a compensator in H. For an early analogous result, see [10, pp. 85–87].

Corollary 1: For the feedback system Σ of Fig. 3 there exists a c in H which places the feedback system in H if and only if p admits left and right fractional representations $p = d_l^{-1}n_l = n_r d_r^{-1}$ such that n_r is a right factor of $1 - d_r$ and n_l is a left factor of $1 - d_l$.

Proof: If the feedback system is placed in H by a c in H it admits the fractional representations of (4.18). By calculation [see (3.7)]

$$h_{e_2u_2} - ch_{e_1u_2} = 1 \tag{4.19}$$

and

$$h_{e_1u_1} - h_{e_1u_2}c = 1 \tag{4.20}$$

which verifies their coprimeness since c is in H. Moreover, upon rearranging the terms in (4.19) and (4.20) the conditions of the corollary follow. Conversely, if fractional representations exist which satisfy the conditions of the corollary there exists u_r in H such that

$$u_r n_r = d_r = 1 \tag{4.21}$$

(equivalently $p = n_r d_r^{-1}$ is a right coprime fractional representation with $v_r = 1$). Now, by using this right fractional representation in (4.1) (with any left coprime fractional representation) and $w = 0$ we obtain a compensator $c = u_r$ in H, which places the feedback system in H. ■

V. Examples

Example 1: A Single Variate Servomechanism Problem[4]

Here G is the ring of proper rational functions and H is subring of functions analytic in $\mathrm{Re}\, s > -1$. Consider the problem of designing a compensator for the unstable plant $p(s) = (s+1)/(s^2-4)$ which will simultaneously place the poles of the feedback system in the region, $\mathrm{Re}(s) < -1$, and cause the system to asymptotically track a step input. Since our transfer functions are commutative we may adopt common right and left fractional representation for $p(s)$. In particular,

$$p(s) = \frac{(s+1)}{(s^2-4)} = \left[\frac{(s+1)}{(s+2)^2}\right]\left[\frac{(s-2)}{(s+2)}\right]^{-1} = n(s)d(s)^{-1} \tag{5.1}$$

while

$$\left[\frac{16}{3}\right]\left[\frac{(s+1)}{(s+2)^2}\right] + \left[\frac{(s+2/3)}{(s+2)}\right]\left[\frac{(s-2)}{(s+2)}\right] = u(s)n(s) + v(s)d(s) = 1. \tag{5.2}$$

[4]The purpose of this example is merely to give a simple illustration of the theory. In this situation, a much more highly developed theory is available in [29].

Here, each of the four rational functions, $n(s)$, $d(s)$, $u(s)$, and $v(s)$, lie in the ring of operators with poles in the region $\mathrm{Re}(s) < -1$ and hence the set of all compensators which will place the feedback system in this ring is given by Theorem 3 with $w(s)$ also in the ring. Moreover, for an arbitrary $w(s)$ the input–output mapping for the resultant feedback system will take the form

$$h_{y_2u_1}(s) = -\left[\frac{(s+1)(s-2)}{(s+2)^3}\right]w(s) + \left[\frac{16(s+1)}{3(s+2)^2}\right] = -n(s)d(s)w(s) + n(s)u(s). \quad (5.3)$$

By the final value theorem the feedback system will asymptotically track a step input if and only if $h_{y_2u_1}(0) = 1$ (equivalently $c(s)$ has a pole at zero). As such, to simultaneously place the poles of the feedback system in the region, $\mathrm{Re}(s) < -1$, and cause the feedback system to asymptotically track a step input we must find a $w(s)$ with poles in this region such that $h_{y_2u_1}(0) = 1$. Evaluating (5.3) at $s = 0$ and setting it equal to one yields

$$h_{y_2u_1}(0) = \frac{1}{4}w(0) + \frac{4}{3} = 1, \quad (5.4)$$

implying that $w(0) = -4/3$. As such, the simplest $w(s)$ which will achieve our simultaneous goals is the constant $w(s) = -4/3$ whose poles are trivially in the prescribed region. Adopting this $w(s)$, a little algebra with the expressions of Theorem 3 will reveal that the required compensator takes the form

$$c(s) = \frac{(20s+24)(s+2)}{(3s+4)s} \quad (5.5)$$

while the input–output mapping for the feedback system takes the form

$$h_{y_2u_1} = \frac{(s+1)(20s+24)}{3(s+2)^3}. \quad (5.6)$$

Clearly, $c(s)$ has the required pole at zero (for $h_{y_2u_1}(0) = 1$), although it is by no means obvious that this quasi-stable compensator will transfer the unstable poles of $p(s)$ to the prescribed region. Indeed, this illustrates the underlying power of the proposed design technique in that when one designs the system in terms of $w(s)$ rather than $c(s)$ the pole placement or stabilization process is automatically resolved by working with a $w(s)$ whose poles lie in the prescribed region while the remainder of the design process is simplified by the affine relationship between $w(s)$ and the matrices h_{eu} and h_{yu}. Finally, we note that $c(s)$ has a zero at $s = -2$ which may cancel with the pole of $p(s)$ at $s = -2$. This, however, does not contradict the coprimeness assumptions of Theorem 3 since the common factors involved lie in J which serves as the group of units in our theory. Fortunately, such common factors can never lead to an erroneous design since by assumption the poles and zeros of the rational functions in J lie in the prescribed region. As such, any cancellations which may take place are benign.

Since the previous compensator design was achieved with an especially simple $w(s)$ let us add an additional constraint to the problem by requiring that $h_{y_2u_1}(s)$ have zeros at $\pm j$ (so that the system will be insensitive to a noise source at that frequency). Now, from (5.4) it follows that the above design is the only compensator which will make $h_{y_2u_1}(0) = 1$ with a constant $w(s)$; hence to satisfy this additional design constraint we will work with the first order $w(s)$ in the form

$$w(s) = \frac{as-4}{bs+3}. \quad (5.7)$$

Here, by specifying the zeroth-order coefficients of $w(s)$ we assure that $w(0) = -4/3$ while we are left with the parameters a and b to create the required zeros. Of course, to achieve our stability condition we must have $-3/b < -1$. Substituting the $w(s)$ of (5.7) into (5.3) yields

$$h_{y_2u_1}(s) = \frac{(s+1)\left[(16b-3a)s^2 + (60+6a+32b)s + 72\right]}{3(s+2)^3(bs+3)}. \quad (5.8)$$

To obtain the desired zeros at $s = \pm j$ the equation

$$\left[(16b-3a)s^2 + (60+6a+32b)s + 72\right] = k\left[s^2+1\right] \quad (5.9)$$

must be satisfied. Now, this represents three linear equations in three unknowns and has the unique solution

$$a = -17, \quad b = \frac{21}{16}, \quad \text{and } k = 72. \quad (5.10)$$

Moreover, $-3/b = -16/7 < -1$; hence this choice of $w(s)$ will also assure the prescribed degree of stabilization. As such, we take

$$w(s) = \frac{-(17s+4)}{(21s/16+3)} = \frac{-(272s+64)}{(21s+48)} \quad (5.11)$$

which yields

$$c(s) = \frac{128(s+2)(s^2+1)}{(7s^2-56s-60)s} \quad (5.12)$$

and

$$h_{y_2u_1}(s) = \frac{384(s+1)(s^2+1)}{(s+2)^3(21s+48)} \quad (5.13)$$

satisfying all of our design criteria.

Example 2: A Multivariate Lumped-Distributed Decoupling Problem

Consider the multivariate, lumped-distributed plant

$$p(s) = \begin{bmatrix} \dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)}{(s+1)} \\ 0 & \dfrac{1}{(s-1)} \end{bmatrix} \quad (5.14)$$

which we desire to stabilize and simultaneously decouple by feedback. For most lumped-distributed systems one can take H to be a ring of matrices whose elements lie in the algebra $\hat{\mathcal{Q}}_-(\sigma_0)$ of stable transfer functions generated by lumped elements and delays while G is a ring of matrices whose elements lie in $\hat{\mathcal{B}}(\sigma_0)$, the algebra of quotients of elements in $\hat{\mathcal{Q}}_-(\sigma_0)$, as per Table I. In our case, however, although $e^{-1/s}$ is L_2-stable (since it is analytic on the right half-plane and bounded on the imaginary axis [10]) it has a "nasty" singularity at $s=0$ and hence does not lie in $\hat{\mathcal{Q}}(\sigma_0)$ for any $\sigma_0<0$. As such, we take H to be a ring of 2×2 matrices whose elements are transfer functions lying in the Hardy space $H_\infty(R)$ of functions which are (essentially) bounded on the $j\omega$ axis and admit an analytic extension into the right half-plane (thereby making them L_2-stable) [12]. Similarly, we let G be a ring of 2×2 matrices whose entries are transfer functions lying in the Lebesgue space $L_\infty(R)$ [12]. With this setup I becomes the set of H_∞ functions which are uniformly bounded below on the $j\omega$ axis while J is the set of H_∞ functions whose analytic extension is uniformly bounded below in the right half-plane [12]. Equivalently, J is the set of invertible outer functions in $H_\infty(R)$ [12].

Using these spaces a little algebra will reveal that $p(s)$ has the right and left coprime fractional representations in $\{G,H,I,J\}$ shown below:

$$p(s)=\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{1}{(s+1)}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \dfrac{(s-1)}{(s+1)}\end{bmatrix}^{-1}$$

$$=n_r(s)d_r(s)^{-1} \tag{5.15}$$

$$p(s)=\begin{bmatrix}1 & 0\\ 0 & \dfrac{(s-1)}{(s+1)}\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)}{(s+1)}\\ 0 & \dfrac{1}{(s+1)}\end{bmatrix}$$

$$=d_l(s)^{-1}n_l(s) \tag{5.16}$$

where

$$\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{1}{(s+1)}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \dfrac{(s-1)}{(s+1)}\end{bmatrix}$$

$$=u_r(s)n_r(s)+v_r(s)d_r(s)=1 \tag{5.17}$$

and

$$\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)}{(s+1)}\\ 0 & \dfrac{1}{(s+1)}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & \dfrac{(s-1)}{(s+1)}\end{bmatrix}\begin{bmatrix}1 & \dfrac{-2(s-1)}{(s+1)}\\ 0 & 1\end{bmatrix}=n_l(s)u_l(s)+d_l(s)v_l(s)=1. \tag{5.18}$$

Upon substitution of these matrices into the expression for $h_{y_2u_1}(s)$ from Theorem 3 one obtains

$$h_{y_2u_1}(s)=-\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{1}{(s+1)}\end{bmatrix}\begin{bmatrix}w_{11}(s) & w_{12}(s)\\ w_{21}(s) & w_{22}(s)\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & \dfrac{(s-1)}{(s+1)}\end{bmatrix}+\begin{bmatrix}0 & \dfrac{2(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{2}{(s+1)}\end{bmatrix} \tag{5.19}$$

which will be stable if and only if the $w_{ij}(s)$ are stable.

Now, to decouple the system we require that

$$h^{12}_{y_2u_1}(s)=\frac{(s-1)e^{-1/s}}{(s+1)^2}w_{12}(s)+\frac{(s-1)^3}{(s+1)^3}w_{22}(s)+\frac{2(s-1)^2}{(s+1)^2}=0 \tag{5.20}$$

and

$$h^{21}_{y_2u_1}(s)=\frac{1}{(s+1)}w_{21}(s)=0. \tag{5.21}$$

Clearly, $w_{21}(s)=0$ solves (5.21). On the other hand (5.20) has numerous solutions none of which are, however, stable. As such, the system cannot be decoupled and stabilized simultaneously. Note, since our theory guarantees that *all stable feedback systems with plant* $p(s)$ *take the* form of (5.19) if we cannot find stable w's which decouple (5.19) we are assured that it is *impossible* to simultaneously stabilize and decouple $p(s)$ by feedback (using a compensator as specified in Theorem 3) and we need not consider other formulations.

Since we cannot simultaneously stablize and decouple $p(s)$ by feedback the best we can do is to try to stabilize $p(s)$ while preserving its triangularity (which will allow us to sequentially adjust its various outputs). Formally, this can be achieved by taking $w(s)=0$ which yields the input–output mapping

$$h_{y_2u_1}(s)=\begin{bmatrix}0 & \dfrac{2(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{2}{(s+1)}\end{bmatrix}. \tag{5.22}$$

Unfortunately, the first input has been rendered useless by this compensator and hence the goal of being able to sequentially tune the outputs is not achieved. On the other hand, if we take

$$w(s)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \tag{5.23}$$

then

$$h_{y_2u_1}(s)=\begin{bmatrix}\dfrac{e^{-1/s}}{(s+1)} & \dfrac{2(s-1)^2}{(s+1)^2}\\ 0 & \dfrac{2}{(s+1)}\end{bmatrix} \tag{5.24}$$

which has the desired property is obtained. In particular, one can tune the second input to control the second output and then adjust the first input to simultaneously cancel out the effects of the second input on the first output and control the first output. Of course, since $w(s)$ is stable so is $h_{y_2u_1}(s)$.

Finally, we note that as we have formulated our theory one can deal only with square matrices (since rectangular matrices are not closed under multiplication). The extension to rectangular matrices is, however, straightforward [19] and yields an identical theory the details of which are left to the reader.

Example 3: A Multidimensional Image Restoration Problem

Let

$$p(z_1,z_2)=\frac{z_1+z_2}{z_1^2+z_1z_2+3} \tag{5.25}$$

denote the discrete two-dimensional transfer function for a device in a digital image processing system. Since this represents an IIR (infinite impulse response) transfer function the image processing device will tend to "smear" the image with the data observed at any one pixel distorting all other pixels at the output of the device. In an effort to reduce this "smearing" effect we would like to place the device in a feedback system whose input–output transfer function minimizes the "smearing" effect. In particular, that means that the input–output mapping for the feedback system should have an FIR (finite impulse response) transfer function with its "point-spread function" concentrated about a single point as closely as possible.

Since the FIR transfer functions are just the polynomials we let H be the ring of polynomials in two variables and G be the ring of rational functions in two variables [16]. Once again employing only a single fractional representation since these rings are commutative we obtain the coprime fractional representation

$$\begin{aligned}p(z_1,z_2)&=[z_1+z_2][z_1^2+z_1z_2+3]^{-1}\\ &=n(z_1,z_2)d(z_1,z_2)^{-1}\end{aligned} \tag{5.26}$$

where

$$\begin{aligned}\left[-\frac{1}{3}z_1\right][z_1+z_2]+\left[\frac{1}{3}\right][z_1^2+z_1z_2+3]\\ =u(z_1,z_2)n(z_1,z_2)+v(z_1z_2)d(z_1,z_2)=1.\end{aligned} \tag{5.27}$$

As such, the set of all possible FIR transfer functions which can be obtained from $p(z_1,z_2)$ by feedback takes the form

$$\begin{aligned}h_{y_2u_1}(z_1,z_2)=-\left[z_1^3+2z_1^2z_2+z_1z_2^2+3z_1+3z_2\right]\\ \cdot w(z_1,z_2)-\frac{1}{3}\left[z_1^2+z_1z_2\right]\end{aligned} \tag{5.28}$$

where $w(z_1,z_2)$ is an arbitrary polynomial in two variables. Clearly, $w(z_1,z_2)$ should be low order to keep the "point-spread function" of $h_{y_2u_1}(z_1,z_2)$ as concentrated as possible. Indeed, if we take $w(z_1,z_2)=0$ we obtain

$$h_{y_2u_1}(z_1,z_2)=-\frac{1}{3}\left[z_1^2+z_1z_2\right] \tag{5.29}$$

in which the response from a given pixel effects only two adjacent pixels. Note that the fact that these pixels are not centered around the input point does not cause any difficulty since one can always shift the origin of the raster to compensate. Taking this $w(z_1,z_2)$ we obtain the simple compensator $c(z_1,z_2)=-z_1$ which represents a one directional shift and a 180° phase shift.

An alternative design which also yields a "point-spread function" which affects only two pixels, although it is shifted further from the origin, is obtained with $w(z_1,z_2)=-(1/9)z_1$. This yields

$$h_{y_2u_1}(z_1,z_2)=\frac{1}{9}\left[z_1^4+2z_1^3z_2\right] \tag{5.30}$$

and

$$c(z_1,z_2)=\frac{z_1^2(z_1+z_2)}{z_1^2+z_1z_2+3}. \tag{5.31}$$

Since two-thirds of the output energy in this design is concentrated at a single point whereas the energy is equally divided in the previous design it may be argued that this represents a superior design. On the other hand, the shift from the origin is greater and the compensator more complex in this case. Finally, since all FIR transfer functions are stable (in an appropriate sense) the feedback systems obtained via either choice of $w(z_1,z_2)$ are stable. Moreover, both compensators are, themselves, stable as is $p(z_1,z_2)$ [6].

Example 4: A Time-Varying Differential-Delay Stochastic Optimal Control Problem

Consider the feedback system of Fig. 4 where the plant represents a cascade of a time-varying function f with an ideal predictor e^s. The system is driven by a stochastic process a, which is derived from white noise by passing it through a miniphase filter with transfer function $(s+2)/(s+1)$. We desire to choose a compensator which will stabilize the system and minimize the performance measure

$$J=E\|b\|^2+E\|d\|^2 \tag{5.32}$$

under the constraint of stability. Here, d is the stochastic process observed at the output of the system, b is the

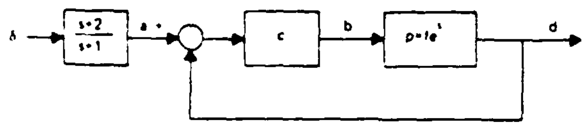

Fig. 4. Stochastic control system.

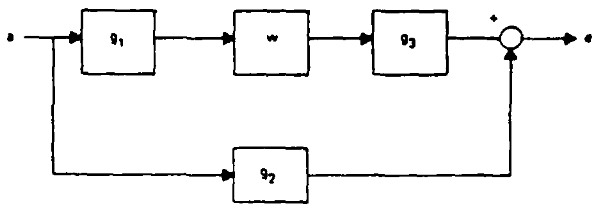

Fig. 5. Open-loop optimization problem.

stochastic process observed at the plant input, and E is the expected value operator.

Since we have a time-varying component, a rational component, and a delay component we formulate our theory in an abstract operator theoretic setting [20] with G taken to be the bounded operators on the Hilbert space $L_2(R)$ and H taken to be the causal bounded operators (which correspond to the stable systems in such a setting) [20],[23]. Note, in this setting we will denote the time-invariant operators by their transfer function and the time-varying multiplication operators by their characteristic function. Of course, one must be careful with such notation since the operational calculus associated with the time-invariant components is only partially valid in such a setting.

Since the inverse of a predictor is the ideal delay which is causal one immediately obtains the right and left coprime fractional representations for p in the form

$$p=[f][e^{-s}]^{-1}=[e^{-s}f^{-1}]^{-1}[1]=n_r d_r^{-1}=d_l^{-1}n_l \quad (5.33)$$

where

$$[f^{-1}][f]+[0][e^{-s}]=u_r n_r+v_r d_r=1 \quad (5.34)$$

and

$$[1][1]+[e^{-s}f^{-1}][0]=n_l u_l+d_l v_l=1. \quad (5.35)$$

Here, we have assumed that f^{-1} exists and is bounded (i.e., f is bounded away from zero) while f and f^{-1} are both causal since multiplication by a function of time is a memoryless operation [20]. From Theorem 3 it now follows that the input–output and input-plant input mappings for our feedback system with compensator defined by a causal operator w will take the form

$$h_{y_2u_1}=-[f]w[e^{-s}][f^{-1}]+1 \quad (5.36)$$

and

$$h_{e_2u_1}=-[e^{-s}]w[e^{-s}][f^{-1}]+[e^{-s}][f^{-1}]. \quad (5.37)$$

As such, our optimization problem reduces to choosing the causal w which minimizes the performance measure of (5.32) where $d=h$ a and $b=h$ a

It is significant to note that even though we are interested in designing an optimal closed-loop system by minimizing over the operator w rather than the compensator we have transformed the problem into the open-loop optimization problem of Fig. 5.

Here we desire to minimize $J=E\|e\|^2$ over all causal operators w, where g_1, g_2, and g_3 are arbitrarily specified bounded operators. In our case we take

$$g_1=[e^{-s}][f^{-1}] \quad (5.38)$$

$$g_2=\begin{bmatrix} -1 \\ \hdashline -[e^{-s}][f^{-1}] \end{bmatrix} \quad (5.39)$$

and

$$g_3=\begin{bmatrix} -[f] \\ \hdashline -[e^{-s}] \end{bmatrix} \quad (5.40)$$

in which case the output of the open-loop system is $e=(d,b)$ in the product space constructed from two copies of the (Hilbert) space on which the given system is defined. Now, if we take the a in our open-loop problem to coincide with the given a in the closed-loop optimization problem then the Pythogorean law (in Hilbert space) implies that

$$J=E\|e\|^2=E\|d\|^2+e\|b\|^2. \quad (5.41)$$

As such, our two optimization problems coincide.

Interestingly, an explicit solution has recently been given for the above open-loop optimization problem [9]. Indeed, the optimal causal w is given by

$$w_0=\lambda^{-1}[\lambda^{*-1}g_3^* g_2 Q_a g_1^* \theta^{*-1}]_C\theta^{-1} \quad (5.42)$$

where λ and θ are causal, causally invertible operators such that

$$\lambda^*\lambda=g_3^*g_3 \quad \theta\theta^*=g_1Q_ag_1^*. \quad (5.43)$$

Q_a is the covariance for the stochastic processes a, $[\;]_C$ denotes the causal part of an operator, and "$*$" denotes the adjoint operator. To apply this general theory to our example we represent the adjoint operation when applied to a transfer function by $g(s)^*=g(-s)$ which coincides with the classical adjoint on the $j\omega$ axis. Of course, the memoryless multiplication operators, $[f]$ and $[f^{-1}]$, are self adjoint. Finally, since a is the stochastic process generated from white noise by passing it through the filter $(s+2)/(s+1)$

$$Q_a=\left[\frac{(s+2)}{(s+1)}\right]\left[\frac{(s+2)}{(s+1)}\right]^*=\frac{(s+2)(s-2)}{(s+1)(s-1)}. \quad (5.44)$$

First, we calculate λ and θ via

$$\lambda^*\lambda=[-f \,\vdots\, -e^{s}]\begin{bmatrix} -f \\ \hdashline -e^{-s} \end{bmatrix}=f^2+1 \quad (5.45)$$

and

$$\theta\theta^* = [e^{-s}][f^{-1}]\left[\frac{(s+2)(s-2)}{(s+1)(s-1)}\right][f^{-1}][e^{s}]$$

$$= [f_{-1}^{-1}][e^{-s}]\left[\frac{(s+2)(s-2)}{(s+1)(s-1)}\right][e^{s}][f_{-1}^{-1}]$$

$$= [f_{-1}^{-1}]\frac{(s+2)(s-2)}{(s+1)(s-1)}[f_{-1}^{-1}]. \tag{5.46}$$

Here $f_{-1}(t) = f(t-1)$ and we have used the properties of the delay and predictor to obtain the equalities $[e^{-s}][f^{-1}] = [f_{-1}^{-1}][e^{-s}]$ and $[f^{-1}][e^{s}] = [e^{s}][f_{-1}^{-1}]$. Of course, the exponential transfer functions commute with the rational transfer functions allowing the cancellation of the exponential terms in (5.46). From (5.45) and (5.46) one may now readily obtain the required causal, causally invertible λ and θ operators in the form

$$\lambda = \lambda^* = \sqrt{f^2+1} \text{ and } \lambda^{-1} = \lambda^{*-1} = \frac{1}{\sqrt{f^2+1}} \tag{5.47}$$

while

$$\theta = [f_{-1}^{-1}]\left[\frac{(s+2)}{(s+1)}\right], \quad \theta^* = \left[\frac{(s-2)}{(s-1)}\right][f_{-1}^{-1}],$$

$$\theta^{-1} = \left[\frac{(s+1)}{(s+2)}\right][f_{-1}], \text{ and } \theta^{*-1} = [f_{-1}]\left[\frac{(s-1)}{(s-2)}\right]. \tag{5.48}$$

The next step in evaluating (5.42) is to compute the term in the bracket, i.e.,

$$\lambda^{*-1}g_3^*g_2Q_ag_1^*\theta^{*-1}$$

$$= \frac{1}{\sqrt{f^2+1}}[-f \,\vdots\, -e^{s}]\left[\frac{-1}{[-e^{-s}][f^{-1}]}\right]$$

$$\cdot\left[\frac{(s+2)(s-2)}{(s+1)(s-1)}\right][f^{-1}][e^{s}]\cdot[f_{-1}]\left[\frac{(s-1)}{(s-2)}\right]$$

$$= \frac{1}{\sqrt{f^2+1}}[f+f^{-1}]\left[\frac{(s+2)(s-2)}{(s+1)(s-1)}\right]$$

$$\cdot[f^{-1}][f]\frac{(s-1)}{(s-2)}[e^{s}]$$

$$= [f^{-1}]\sqrt{f^2+1}\left[\frac{(s+2)}{(s+1)}\right][e^{s}] \tag{5.49}$$

whose causal part must now be computed. Recalling that the memoryless term factors through the causal part bracket [9] it suffices to compute the causal part of the time-invariant system with transfer function

$$g(s) = \left[\frac{(s+2)}{(s+1)}e^{s}\right]. \tag{5.50}$$

Taking the inverse Laplace transform we obtain the impulse response of this system in the form

$$g(t) = \delta(t+1) + \bar{e}^{(t+1)}U(t+1) \tag{5.51}$$

where δ is the Dirac delta function and U is the unit step function. Now, the causal part of $g(t)$ is obtained by setting $g(t)$ to zero for t less than zero; hence

$$[g(t)]_C = g(t)U(t) = \bar{e}^{(t+1)}U(t) = \frac{1}{e}e^{-t}U(t) \tag{5.52}$$

or equivalently

$$[g(s)]_C = \frac{1}{e(s+1)}. \tag{5.53}$$

Multiplying through by the memoryless factor from (5.49) we then obtain

$$[\lambda^{*-1}g_3^*g_2Q_ag_1^*\theta^{*-1}]_C = [f^{-1}]\sqrt{f^2+1}\,\frac{1}{c(s+1)} \tag{5.54}$$

and finally

$$w_0 = \lambda^{-1}[\lambda^{*-1}g_3^*g_2Q_ag_1^*\theta^{*-1}]_C\theta^{-1}$$

$$= \frac{1}{\sqrt{f^2+1}}[f^{-1}]\sqrt{f^2+1}$$

$$\cdot\frac{1}{e}\left[\frac{1}{(s+1)}\right]\left[\frac{(s+1)}{(s+2)}\right][f_{-1}]$$

$$= \frac{1}{e}[f^{-1}]\frac{1}{(s+2)}[f_{-1}] \tag{5.55}$$

which is surprisingly simple given the complexity of the derivation.

Substituting the expression of (5.55) into the formula of Theorem 3 now yields an expression for our optimal compensator and the input–output mapping for the resultant feedback system in the form

$$c = e[f_{-1}^{-1}](s+2) - 1 \tag{5.56}$$

and

$$h_{y_2u_1} = \frac{-e^{-s}}{e(s+2)} + 1. \tag{5.57}$$

Note that $h_{y_2u_1}$ is stable, as required, even though both p and c are unstable.

VI. Nonlinear Feedback Systems

From an algebraic point of view the fundamental difference between linear and nonlinear systems is the fact that nonlinear systems fail to satisfy the right-distributive property, $x(y+z) = xy + xz$. They do, however, satisfy all of the other axioms for a ring with identity including the left-distributive property $(y+z)x = yx + zx$. As such, one can attempt to extend the preceding development to nonlinear systems by carrying it out in left-distributive rings, G and H [23]. Indeed, if we define a right coprime fractional representation for a system g in a left-distributive ring G relative to $\{G,H,I,J\}$ precisely as we did in Section II the fundamental properties 1, 2, and 3 go through without modification.

Property 1N: Let $g = n_r d_r^{-1}$ be a right coprime fractional representation of g in $\{G,H,I,J\}$ where G and H are left-distributive rings with identity. Let n_r and d_r have a common right factor $r \in H$, i.e., $n_r = x_r r$, $d_r = y_r r$ for some $x_r \in H$ and $y_r \in H$. Then r has a left inverse in H.

Property 2N: Let $g = n_r d_r^{-1}$ be a right coprime fractional representation of g in $\{G,H,I,J\}$ where G and H are left-distributive rings with identity. Let $g = x_r y_r^{-1}$ be a second (not necessarily coprime) right fractional representation of g in $\{G,H,I,J\}$; then there exists r in H such that

$$x_r = n_r r \quad \text{and} \quad y_r = d_r r. \tag{6.1}$$

Property 3N: Let $g = n_r d_r^{-1}$ with $n_r \in H$ and $d_r \in I$ where G and H are left-distributive rings with identity.

a) If $d_r \in J$, then $g \in H$.

b) If $g = n_r d_r^{-1}$ is a *right coprime fractional representation* of g in $\{G,H,I,J\}$, then $g \in H$ implies $d_r \in J$.

With the aid of property 3N one can do a complete analysis of a nonlinear feedback system $h_{yu} = p(1+p)^{-1} = n_r d_r^{-1}$ where $n_r d_r^{-1}$ is a right coprime fractional representation of h_{yu}. Indeed, h_{yu} is well defined in G if and only if $d_r \in I$ and it is well defined in H if and only if $d_r \in J$. Note, however, that we cannot construct our fractional representation for h_{yu} from a fractional representation for p since the verification that such a representation is coprime appears to require right-distributivity [see (3.4) and (3.5)].

The right coprime fractional representation plays a special role in the *nonlinear* case because $h_{yu} = p(1+p)^{-1}$ holds, whereas $h_{yu} = (1+p)^{-1}p$ does not (even though the latter formula is true for the *linear* case). As such, those results on the analysis of feedback systems which assume a left coprime fractional representation theory fail as does the design theorem since it simultaneously employs both left and right coprime fractional representations. We believe, however, that these results should hold, at least in part, for nonlinear systems with an appropriate modification of the theory. In particular, since the rings G and H are asymmetric we believe that asymmetric concepts of left and right coprimeness will be required to achieve this end.

VII. Conclusions

Although several of our examples are characterized by a deep analytic structure the key to our fractional representation approach to feedback system design is the algebraic nature of the main results. Indeed, the entirety of our modeling, analysis, and synthesis theory was derived with no more sophisticated mathematics than addition, multiplication, subtraction, and inversion. As such, it applies to essentially any class of linear systems and by proper choice of the rings G and H the results are applicable to a variety of systems problems.

Although we believe that the present work represents the first attempt at the formulation of an axiomatic fractional representation theory for systems which may be matched to the feedback system analysis and synthesis problems of interest the work owes much to a number of recent results on the input–output theory of linear systems. The use of a fractional representation theory for multivariate systems, though implicit in a number of classical results, was popularized by Rosenbrock's polynomial matrix fractions [19]. Interestingly, however, Rosenbrock's goal was apparently to permit the powerful analytic and arithmetic theory available for polynomial matrices to be applied to rational matrices whereas the present fractional representation theory is motivated by the desire to formulate a representation theory for systems which is closed under inversion. Over the years numerous generalizations of the polynomial matrix fraction concept have been formulated for distributed systems [4], [5], [13], [21], and multidimensional systems [9],[24] while partial extensions to the time-varying and nonlinear cases have appeared in a number of unpublished reports [11],[22].

For any type of fractional representation theory to be meaningful it must be identified with an appropriate coprimeness concept. Indeed, the key to the present formulation is the use of the algebraic coprimeness concept of (2.4) in lieu of the more classical common factor criterion. Such a criterion has previously been applied by one of the authors in a study of fractional representations for distributed system [4] and was also shown to be the strongest of several possible coprimeness criteria for multidimensional systems by Youla and Gnavi [26]. Of course, it is well known as one of the several equivalent criteria for coprimeness in the polynomial matrix fraction theory [16],[19].

The feedback system analysis theorems of Section III are motivated by the now classical theorems for determining the stability of a multivariate feedback system in terms of its polynomial matrix fraction representation [10]. Moreover, the system synthesis theorem is an outgrowth of the feedback system stabilization theorem of Youla *et al.* [24],[25]. Indeed, the present work began with an attempt to give a simple proof of this most powerful analytic theorem and developed through several stages of generalization and simplification into its present form. Finally, the optimization theory used in Example 4 represents the generalization [9] to an operator theoretic setting of a result originally developed by Youla *et al.* in the frequency domain for use in conjunction with their stabilization theorem [24],[25].

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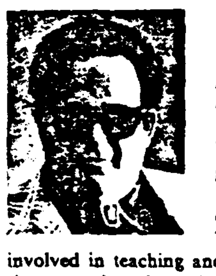

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A FRACTIONAL REPRESENTATION APPROACH TO ADAPTIVE CONTROL

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A FRACTIONAL REPRESENTATION APPROACH TO ADAPTIVE CONTROL†

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The main problem of adaptive control theory is to design a system S which is capable of automatically adjusting the generated control input to the plant P. Such adjustments may be necessary for a variety of reasons, such as insufficient knowledge about the plant, plant perturbations, etc. A multitude of adaptive control techniques have been proposed through the years. A characteristic shared by all of them is the presence of some means of identifying the unknown or perturbed plant. Of course, the design of such a mechanism, termed here the identifier, is an important question in its own right. The design, however, of an adaptive controller is heavily influenced by the particular technique used to generate the control and it therefore inherits the technique's features.

A recent advance in control theory is an approach to feedback control based upon the representation of the plant as the ratio of two operators, both of which belong to an operator ring *H*. (Ref). A brief overview of the approach is as follows. Consider the following ring structure *R*

$$R = \{G, H, I, J\} \quad (1.1)$$

where *G* is a not necessarily commutative ring with identity representing the general class of systems of interest. The subring *H* also contains the identity and represents the class of systems which in some sense are stable. *I* is the set of elements in *H* which admit an inverse in *G* and *J* the set of elements in *H* which admit an inverse in *H*. As shown in (Ref),

$$G \supset H \supset I \supset J \quad (1.2)$$

A plant P is said to have a doubly coprime fractional representation if for $\{N_r, N_l, U_r, U_l, V_r, V_l\} \in H$ and $\{D_r, D_l\} \in I$

$$P = N_r D_r^{-1} = D_l^{-1} N_l \quad (1.3)$$

$$U_r N_r + V_r D_r = 1 \quad (1.4)$$

$$N_l U_l + D_l V_l = 1 \quad (1.5)$$

The aim now is to design a system *S* so that the system's input-output map h is placed in *H*. Consider the system shown in Fig. 1.1 and assume that P has a doubly coprime fractional representation.

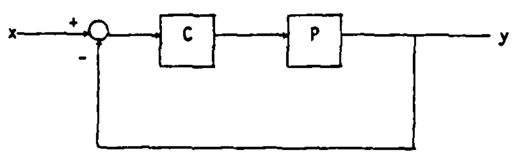

Fig. 1.1. A feedback control system.

For any arbitrary w, let the compensator C be defined as

$$C = (wN_l + V_r)^{-1}(-wD_l + U_r). \quad (1.6)$$

It was shown that if $w \in H$, then the input-output map h also belongs to *H* and

$$h = N_r(-wD_l + U_r). \quad (1.7)$$

An important element of the approach is that it provides a complete characterization of the set of compensators which place h in the ring *H*. It is therefore desirable to investigate the conditions under which fractionally represented feedback systems can be adaptively controlled.

Suppose then that either in the limit as $t \to \infty$, or for all times $t \geq t_0$, an input-output map H in *H* is desired; in other words, suppose that, with the appropriate time interpretation, it is required that

$$h = H. \quad (1.8)$$

Clearly, there exists a choice of three independent variables, namely w, U_r and V_r, to satisfy two linear equations. The decision was made to consider w as a parameter in *H*. Thus the problem can in general be stated as seeking the particular coprimeness operator pair U_r, V_r which for a given w in *H* simultaneously satisfies Eq.s 1.4 and 1.8.

The two main problems to be addressed here are the acquisition and the plant-follower. In the former, the linear, time-invariant plant P is assumed to be insufficiently specified at the initial time t_0. The intention is to provide a feedback system S which consists of an identifier ID and an adaptor AD as shown in Fig. 1.2. The identifier provides the adaptor with estimates p(t) of the plant P such that $\lim_{t \to \infty} p(t) = P$. Then, using these estimates, the adaptor

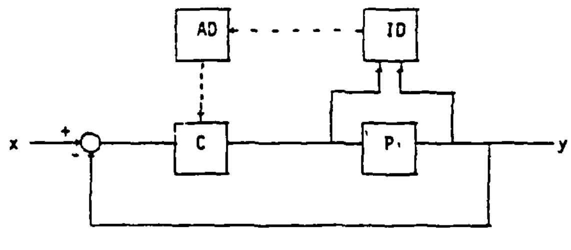

Fig. 1.2. An adaptive control system.

provides the compensator with an operator pair $\{u_r(t), v_r(t)\}$ such that the required coprimeness pair

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(U_r, V_r) is obtained in the limit. The first task is to delineate the class of plants for which such a system is possible. This can be done by deriving the necessary and sufficient conditions for a solution to exist under the assumption of instantaneous identification, (i.e., a perfect identifier). Then it would remain to show that in the non-ideal case the solution can be attained adaptively. In other words it would be required that Eq.s 1.4 and 1.8 are satisfied in the limit.

In the plant-follower problem the linear plant P is perfectly known at the initial time t_0, but it undergoes perturbations thereafter. The intention is to provide the compensator with an operator pair $(U_r(t), V_r(t))$ such that the systems input-output map remains invariant under the plant's perturbations. In other words Eq.s 1.4 and 1.8 are to be satisfied at every point in time. Again the class of plants for which a solution exists is delineated under the perfect identifier assumption. In the non-ideal case it is desirable to examine the extent to which the input-output map is perturbed due to the plant perturbations.

As always, stability is a question of paramount importance. A consequence of the fractional representation approach is the fact that a system is stable in the sense of H whenever the system's input-output map is time-invariant and the coprimeness operators belong to H. This is exploited in the ideal case of both problems. But, whereas, in the acquisition problem the derived stability conditions are time-independent and hence easy to check a priori, in the plant-follower they are time-dependent and thus the task of verifying whether they hold or not is considerabley harder. However, the problem is by-passed by showing that in this case the question of the coprimeness operators belonging to H is equivalent to the classical question of stability in the sense of H of a system with time-invariant feedforward path and memoryless, time-varying feedback path. In the adaptive case of the plant-follower problem stability is resolved by a similar criterion applied to the entire adaptive acquisition problem, the fact that the input-output map converges to a time-invariant element of H suggests that the system is stable as long as the map remains bounded. It is shown that for uniform asymptotic stability this is in fact the case as long as a sufficiently "good" identifier is used. (The quality of the identifier is also shown to be related to the robustness of the adaptive plant-follower system).

The requirement to control the entire input-output map restricts the application to a class of plants which, for all practical purposes, is only slightly larger than the miniphase case. But if a less restrictive requirement is imposed the class becomes considerably larger. The point is demonstrated by the pole positioning problem for plants represented as rational functions (not necessarily proper). It is shown that the problem is equivalent to solving a linear, algebraic equation. Furthermore, a solution to the equation is shown to exist provided that the number of poles to be positioned is sufficiently large. In terms of adaptive control, the equation must be solved repeatedly in time by any of the available methods, (e.g. a continuation algorithm).

FEEDBACK SYSTEM DESIGN: THE FRACTIONAL REPRESENTATION APPROACH TO ANALYSIS AND SYNTHESIS

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FEEDBACK SYSTEM DESIGN:

THE FRACTIONAL REPRESENTATION APPROACH TO ANALYSIS AND SYNTHESIS

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Summary

The problem of designing a feedback system with prescribed properties is attacked via a fractional representation approach to feedback system analysis, and synthesis. To this end we let H denote a ring of operators with the prescribed properties and model a given plant as the ratio of two operators in H. This, in turn, leads to a simplified test to determine whether or not a feedback system in which that plant is embedded has the prescribed properties and a complete characterization of those compensators which will "place" the feedback system in H. The theory is formulated axiomatically to permit its application in a wide variety of system design problems and is extremely elementary in nature requiring no more than addition, multiplication, subtraction, and inversion for its derivation even in the most general settings.

I. Introduction

Intuitively, the linear feedback system design process may be broken down into three steps; modeling, analysis, and synthesis; each of which may be carried out via a multiplicity of time and frequency domain techniques. In engineering practice, however, the three steps are loosely matched to one another. The purpose of the present paper is to use fractional representation models to the analysis and synthesis of feedback systems. Here, if one desires to design a system with prescribed properties the given plant is initially modeled as a quotient of two operators, each of which has the desired properties. Once such a model has been specified a similar model may be formulated for the feedback system constructed from that plant which, in turn, may be used to determine whether or not the feedback system has the desired properties. Moreover, the set of compensators which will cause the feedback system to have the prescribed properties may be completely characterized in terms of such a model. As such, by choosing a model for the plant which is matched to the design criteria the analysis and synthesis processes for a feedback system may be greatly simplified.

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II. Axiomatic Theory

Let G be a (not-necessarily-commutative) ring with identity and let H be a subring of G which includes the identity. The feedback system and its subsystems will be represented by operators which are elements of G. The compensator will be chosen so that the overall system will be represented by an operator in the subring H.

We define two multiplicative subsets[2,27] of H:

$$I = \{h\ H \mid h^{-1} \in G\}$$

i.e., I is the set of elements of H which have an inverse in G;

$$J = \{h \in H \mid h^{-1} \in H\}$$

i.e., J is the *subgroup* of H consisting of all invertible elements of H.

Note that

$$J \subset I \subset H \subset G$$

Given the above structure we say that a system $g \in G$ has a *right fractional representation* in $\{G,H,I,J\}$ if there exist $n_r \in H$ and $d_r \in I$ such that $g = n_r d_r^{-1}$. Furthermore, we say that the pair (n_r, d_r) $\in H \times H$ is *right coprime* if there exist u_r and v_r in H such that

$$u_r n_r + v_r d_r = 1$$

The fractional representation $n_r d_r^{-1}$ in $\{G,H,I,J\}$ is said to be *right coprime* if the pair (n_r, d_r) is right coprime.

The relationship between our concept of coprimeness and the usual common factor criterion for coprimeness[28] is given by the following properties.

Property 1: Let the pair $(n_r, d_r) \in H \times H$ be right coprime. Let n_r and d_r have a common right factor $r \in H$, i.e., $n_r = x_r r$, $d_r = y_r r$ for some $x_r \in H$ and $y_r \in H$. Then r has a *left inverse* in H.

Property 2: Let $g = n_r d_r^{-1}$ be a right-coprime fractional representation of g in $\{G,H,I,J\}$. Let $g = x_r y_r^{-1}$ be a second (not necessarily coprime) right fractional representation of g in (G,H,I,J). Then there exists an r in H such that

$$x_r = n_r r \quad \text{and} \quad y_r = d_r r.$$

Although G is, in general a noncommutative ring, the entire theory developed above for right fractional representations can be replicated for left fractional representations. In particular, we say that $g \in G$ has a *left fractional representation* in (G,H,I,J) if there exist $n_\ell \in H$ and $d_\ell \in I$ such that $g = d_\ell^{-1} n_\ell$. Furthermore we say that the pair $(n_\ell, d_\ell) \in H \times H$ is *left coprime* if there exist u_ℓ and v_ℓ in H such that

$$n_\ell u_\ell + d_\ell v_\ell = 1.$$

The left fractional representation $d_\ell^{-1} n_\ell$ is said to be *left coprime* if the pair (n_ℓ, d_ℓ) is left coprime. With these definitions the existence of a *common left factor* for a left fractional representations of g is characterized by the following properties.

<u>Property 1'</u>: Let the pair (n_ℓ, d_ℓ) be left coprime. Let n_ℓ and d_ℓ have a common left factor ℓ in H, i.e., $n_\ell = x_\ell$, $d_\ell = \ell y_\ell$ for some $x_\ell \in H$ and $y_\ell \in H$. Then ℓ has a *right inverse* $\in H$.

<u>Property 2'</u>: Let $g = d_\ell^{-1} n_\ell$ be a left coprime fractional representation of g in (G,H,I,J). Let $g = y_\ell^{-1} x_\ell$ be a second (not necessarily coprime) left fractional representation of g in (G,H,I,J). Then there exists an ℓ in H such that

$$x_\ell = \ell n_\ell \text{ and } y_\ell = \ell d_\ell$$

The above properties of a coprime fractional representation have all been derived under the assumption that such a representation exists. Of course, if G denotes the rational matrices and H the polynomial matrices the existence of a coprime representation is implied by classical analysis.[16,19] Indeed, the classical analysis readily extends to the case where H is taken to be the *exponentially stable* (exp. stable) rational matrices or the ring of proper rational matrices with poles in a prescribed region.[18] On the other hand for multidimensional,[26] distributed,[4,8] and time-varying systems[11,15] there is no assurance that an arbitrary $g \in G$ will admit a fractional representation nor even that the set of $g \in G$ which admit such a representation will be a linear space. Moreover, all g's which admit a fractional representation may not admit a coprime fractional representation.[26] In general, the set of $g \in G$ which admit a fractional representation in (G,H,I,J) will form a subring of G if and only if the Ore condition* is satisfied while criteria for coprimeness have been formulated in various special cases though no general theory exists.[1,4,26]

<u>Property 3</u>: Let $g = n_r d_r^{-1}$ with $n_r \in H$ and $d_r \in I$:

a) If $d_r \in J$, the $g \in H$;

b) If $g = n_r d_r^{-1}$ is a *right coprime fractional representation* of g in (G,H,I,J), then $g \in H$ implies that $d_r \in J$.

<u>Property 3'</u>: Let $g = d_\ell^{-1} n_\ell$ with $n_\ell \in H$ and $d_\ell \in I$:

a) If $d_\ell \in J$, then $g \in H$;

b) If $g = d_\ell^{-1} n_\ell$ is a *left coprime fractional representation* of g in (G,H,I,J), then $g \in H$ implies that $d_\ell \in J$.

<u>Property 4</u>: Let $g = n_r d^{-1} n_\ell$ where n_r, $n_\ell \in H$, and $d \in I$.

a) If $d \in J$, then $g \in H$.

b) Let, in addition, $n_r d^{-1}$ be a right coprime fractional representation in (G,H,I,J) and $d^{-1} n_\ell$ be a left coprime fractional representation in (G,H,I,J); then $g \in H$ implies that $d \in J$.

III. Analysis

To start consider the feedback system $\sum_p$ of figure 1.

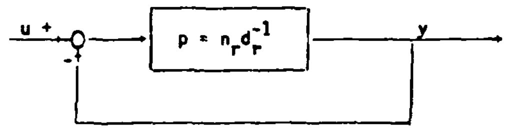

Figure 1: Unity gain negative feedback system.

Suppose that the plant is described by a right coprime fractional representation $p = n_r d_r^{-1}$ in (G,H,I,J). The closed-loop dynamics of $\sum_p$ are described by the maps

$$h_{eu} : u \mapsto e \; ; \; h_{eu} = (1+p)^{-1} = d_r(d_r+n_r)^{-1}$$

$$h_{yu} : u \mapsto y \; ; \; h_{yu} = p(1+p)^{-1} = n_r(d_r + n_r)^{-1}$$

Note that:

$$h_{eu} + h_{yu} = 1$$

We say that $\sum_p$ is *well defined in* G, (H, resp.), if $h_{eu} \in G$, (H, resp.).

Note that the pairs $(n_r, d_r + n_r)$ and $(d_r, d_r + n_r)$ are right coprime; indeed, the right coprimeness of (n_r, d_r) implies

$$(u_r - v_r)n_r + v_r(d_r + n_r) = 1$$

while

$$(v_r - u_r)d_r + u_r(d_r + n_r) = 1$$

<u>Theorem 1</u>: Consider the feedback system $\sum_p$ of figure 2.

a) Let $p = n_r d_r^{-1}$ be a fractional representation in (G,H,I,J) of the element $p \in G$; then $\sum_p$

*(G,H,I,J) satisfies the Ore condition for right fractional representations if, whenever $g \in G$ admits a left fractional representation, it also admits a right fractional representation and vice-versa[2].

is well defined in G if and only if $d_r + n_r \in I$.

b) Let $p = n_r d_r^{-1}$ be a *right coprime* fractional representation in (G,H,I,J) of the element $p \in G$; then Σ_p *is well defined in* H if and only if $d_r + n_r \in J$.

We now consider the feedback system Σ of Figure 2 preceded by a compensator c; p and c belong to G and are specified by their coprime fractional representation in (G,H,I,J) $n_r d_r^{-1}$ and $y_\ell^{-1} x_\ell$, resp.

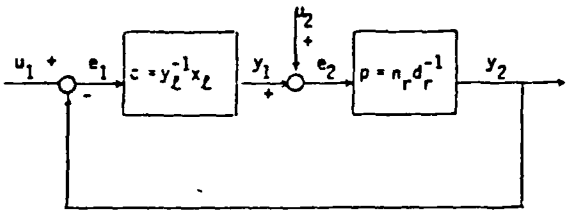

Figure 2. Feedback system with plant and compensator.

To describe the feedback system we consider the map $h_{eu}: (u_1,u_2) \longmapsto (e_1,e_2)$. Simple calculations give

$$h_{eu} = \begin{bmatrix} h_{e_1u_1} & h_{e_1u_2} \\ h_{e_2u_1} & h_{e_2u_2} \end{bmatrix} = \begin{bmatrix} (1+pc)^{-1} & -p(1+cp)^{-1} \\ c(1+pc)^{-1} & (1+cp)^{-1} \end{bmatrix}$$

Now let, $h_{yu}: (u_1,u_1) \longmapsto (y_1,y_2)$. Using the summing node equations it is easy to see that

$$h_{yu} = K(h_{eu} - 1) \text{ and } h_{eu} = 1 - Kh_{yu}$$

where K is the symplectic matrix

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

It is well known that in the case of multivariable rational matrices, one has to consider the four submatrices of h_{eu} because examples show that any one of the submatrices may be unstable while the remaining ones are stable. Let us calculate:

$$h_{e_1u_1} = (1+pc)^{-1} = 1 - pc(1+pc)^{-1}$$
$$= 1 - p\,(1+cp)^{-}\,c$$
$$= 1 - p[y_\ell^{-1}\,(y_\ell d_r + x_\ell n_r)d_r^{-1}]^{-1}\,c$$
$$= 1 - n_r(y_\ell d_r + x_\ell n_r)^{-1} x_\ell$$

$$h_{e_2u_1} = c(1+pc)^{-1} = (1+cp)^{-1}c$$
$$= d_r(y_\ell d_r + x_\ell n_r)^{-1} x_\ell$$

$$h_{e_2u_2} = (1+cp)^{-1} = (1 + y_\ell^{-1} x_\ell n_r d_r^{-1})^{-1}$$
$$= [y_\ell^{-1}(y_\ell d_r + x_\ell n)d^{-1}]^{-1}$$
$$= d_r(y_\ell d_r + x_\ell n_r)^{-1} y_\ell$$

$$h_{e_1u_2} = -p(1+cp)^{-1} = -n_r(y_\ell d_r + x_\ell n_r)^{-1} y_\ell$$

We say that Σ is *well defined in* G, (H, resp.) if and only if each entry of h_{eu} belongs to G, (H, resp.).

Theorem 2: Consider the feedback system Σ of Figure 3. Let $n_r d_r^{-1}$ and $y_\ell^{-1} x_\ell$ be a right and left fractional representations of p and c in (G,H,I,J).

a) If $y_\ell d_r + x_\ell n_r \in I$, then Σ is well defined in G.

b) If $y_\ell d_r + x_\ell n_r \in J$, then Σ is well defined in H.

c) If $h_{e_2u_2} \in G$ then $y_\ell d_r + x_\ell n_r \in I$ *hence if* Σ is well defined in G then $y_\ell d_r + x_\ell n_r \in I$.

d) Assume, in addition, that $n_r(y_\ell d_r)^{-1}$ and $(y_\ell d_r)^{-1} x_\ell$ are right coprime and left coprime factorizational representation, resp.; then $h_{e_1u_1} \in H$ implies that $y_\ell d_r + x_\ell n_r \in J$, and hence *if* Σ is well defined in H then $y_\ell d_r + x_\ell n_r \in J$.

Note, the special assumptions used in d) to the effect that $n_r(y_\ell d_r)^{-1}$ is right coprime and $(y_\ell d_r^{-1}) x_\ell$ is left coprime, imply, in some sense, that p and c *have no* common factors. More precisely, since J serves as the group of units in our theory these conditions imply that any common factors of p and c must lie in J.

IV. Design

Consistent with our approach of matching the plant model to the goal of the given feedback system design problem the present section is devoted to the problem of characterizing the set of compensators which will "place" a feedback system in a prescribed ring H given that both the plant and compensator are modeled by fractional representations in (G,H,I,J).

Theorem 3: For the feedback system Σ of Figure 2, let the plant p have a right coprime and a left coprime fractional representation $p = n_r d_r^{-1} = d_\ell^{-1} n_\ell$ in (G,H,I,J). Then for any $w \in H$ such that $wn_\ell + v_r \in I$, the compensator

$$c = (wn_\ell + v_r)^{-1}(-wd_\ell + u_r) \in G$$

results in a feedback system Σ well-defined in H. For such a compensator, $h_{eu} \in H^{2\times2}$ and

$$h_{eu} = \begin{bmatrix} 1 - n_r(-wd_\ell + u_r) & -n_r(wn_\ell + v_r) \\ d_r(-wd_\ell + u_r) & d_r(wn_\ell + v_r) \end{bmatrix}$$

Conversely, if Σ is well defined in H and if the compensator $c = y_\ell^{-1}x_\ell$ is such that $(n_r, y_\ell d_r)$ and $(y_\ell d_r, x_\ell)$ are right coprime and left coprime resp., then c is given by the above equation.

Finally, we note that although theorem 3 yields a complete design theory for a feedback system given that its plant has both right and left coprime fractional representations in (G,H,I,J) one has no a-priori assurance that such fractional representations exists (except in some known cases). It has been conjectured, however, that this requirement is a necessary condition for the existence of a compensator which will "place" the feedback system in H.[3] Of course, we have already shown that the requirement of coprimeness is a sufficient condition for the existence of such a compensator.

V. Conclusions

The key to our fractional representation approach to feedback system design is the algebraic nature of the main results. Indeed, the entirety of our modeling, analysis, and synthesis theory was derived with no more sophisticated mathematics than addition, multiplication, substraction and inversion. As such, it applies to essentially any class of linear systems and by proper choice of the rings G and H the results are applicable to a variety of systems problems.

Although we believe that the present work represents the first attempt at the formulation of an axiomatic fractional representation theory for systems which may be matched to the feedback system analysis and synthesis problems of interest, the work owes much to a number of recent results on the input-output theory of linear systems. The use of a fractional representation theory for multivariate systems, though implicit in a number of classical results, was popularized by Rosenbrock's polynomial matrix fractions.[19] Interestingly, however, Rosenbrock's goal was apparently to permit the powerful analytic and arithmetic theory available for polynomial matrices to be applied to rational matrices, whereas, the present fractional representation theory is motivated by the desire to formulate a representation theory for systems which is closed under inversion. Over the years numerous generalizations of the polynomial matrix fraction concept have been formulated for distributed systems [4,5,13,21] and multidimensional systems[9,24] while partial extensions to the time-varying and nonlinear cases have appeared in a number of unpublished reports.[11,22]

For any type of fractional representation theory to be meaningful it must be identified with an appropriate coprimeness concept. Indeed, the key to the present formulation is the use of the algebraic coprimeness concept in lieu of the more classical common factor criterion. Such a criterion has previously been applied by one of the authors in a study of fractional representations for distributed systems[4] and was also shown to be the strongest of several possible coprimeness criteria for multidimensional systems by Youla and Gnavi.[26] Of course, it is well known as one of the several equivalent criteria for coprimeness in the polynomial matrix fraction theory.[16,19]

The feedback system analysis theorems of section III are motivated by the now classical theorems for determining the stability of a multivariate feedback system in terms of its polynomial matrix fraction representation.[10] Moreover, the system synthesis theorem is an outgrowth of the feedback system stabilization theorem of Youla, Bongiorno, and Jabr.[24,25] Indeed, the present work began with an attempt to give a simple proof of this most powerful analytic theorem and developed through several stages of generalization and simplification into its present form.

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ABSTRACT OF

FEEDBACK SYSTEM DESIGN:

THE TRACKING AND DISTURBANCE REJECTION PROBLEMS

R. Saeks and J. Murray

Abstract

The problem of designing a compensator for a specified plant which simultaneously stablizes the resultant feedback system and causes it to track a prescribed family of inputs and/or reject prescribed disturbances is considered. A set of linear design equations, in the space of stable systems, is formulated in a general linear systems setting and an explicit parameterization of the resultant solution space is obtained for a class of "generalized multivariate" systems. The theory is illustrated with several single and multivariate examples.

Texas Tech University — Institute for Electronic Science

Joint Services Electronics Program — Research Unit: 2

1. Title of Investigation: Nonlinear Control

2. Senior Investigator: L. Roberts Hunt Telephone: (806) 742-1424

3. JSEP Funds: Current $24,650

4. Other Funds: Current $39,218*

5. Total Number of Professionals: PI's 2 (3 mo.) RA's

6. Summary:

The goal of the proposed work unit is the formulation of a qualitative theory for the analysis and design of nonlinear control systems using differential geometric techniques. Thus far we have formulated a new differential geometric approach to the controllability problem and have laid the foundations for a stabilization theory while we are beginning new investigations of the observability and canonical representation problems.

Although the theory is based on abstract mathematical tools our differential geometric techniques have their origins in the theory of partial differential equations and, as such, are amenable to classical numerical techniques. We therefore believe that the theory can be implemented numerically and have begun an investigation thereof. To this end we have already developed an experimental computer code for implementing the controllability theory in the second order case.

7. Publications and Activities:

   A. Refereed Journal Articles

1. Hunt, L.R., "Controllability of Nonlinear Hypersurface Systems", in Applications of Algebra and Algebraic Geometry to Linear System Theory, AMS Providence, (to appear).

*NASA Grant in support of Professor Hunt's leave of absence at NASA/AMES during the 1980/81 academic year.

2. Hunt, L.R., "Control Theory for Nonlinear Systems in Two Dimensions", Mathematical System Theory, Vol. 13, pp. 361-376(1980)

B. Conference Papers and Abstracts

1. Hunt, L.R., "Reachable Sets for Nonlinear Systems in the Phase Plane", Proc. of the 13th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove Ca., Nov. 1979, pp. 460-462.

2. Hunt, L.R., "Controllability and Stabilizability", Proc. of the 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980, pp. 654-657.

C. Preprints

1. Hunt, L.R., "n-Dimensional Controllability with n-1 Controls", submitted for publication.

2. Hunt, L.R., "Sufficient Conditions for Controllability", submitted for publication.

3. Hunt, L.R., "Controllability and Transversality", submitted for publication.

D. Theses

1. Strangland, R., "Some Aspects of Systems and Control", M.S. Report, Texas Tech Univ., 1980.

E. Conferences and Symposia

1. Hunt, L.R., and J. Murray, 13th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove, CA., Nov. 1980.

2. Hunt, L.R., IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980.

3. Hunt, L.R., Workshop on Mathematical System Theory, Virginia Beach, VA., May 1980.

F. Lectures

1. Hunt, L.R., "Nonlinear Control and Differential Geometry", NASA/AMES Research Center, (a series of 5 lectures given during the Fall of 1980).

CONTROL THEORY FOR NONLINEAR SYSTEMS IN TWO DIMENSIONS

L. R. Hunt

Math. Systems Theory 13, 361–376 (1980)

Mathematical Systems Theory

Global Controllability of Nonlinear Systems in Two Dimensions

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Abstract. Let M be a connected real-analytic 2-dimensional manifold. Consider the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)), x(0) = x_0 \in M,$$

where f and g are real-analytic vector fields on M which are linearly independent at some point of M, and u is a real-valued control. Sufficient conditions on the vector fields f and g are given so that the system is controllable from x_0. Suppose that every nontrivial integral curve of g has a point p where f and g are linearly dependent, $g(p)$ is nonzero, and that the Lie bracket $[f,g]$ and g are linearly independent at p. Then the system is controllable (with the possible exception of a closed, nowhere dense set which is not reachable) from any point x_0 such that the vector space dimension of the Lie algebra L_A generated by f,g and successive Lie brackets is 2 at x_0.

I. Introduction

Suppose we have the system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t)g_i(x(t)), \; x(0) = x_0 \in M, \tag{1.1}$$

where M is a connected real-analytic n-dimensional manifold, $f, g_1, \ldots, g_m$ are real-analytic complete vector fields on M, and $u_1, \ldots, u_m$ are real-valued controls. A theory has recently been developed in [7], [8], and [9] which characterizes the largest subset of M which is reachable from x_0 under assumptions on $f, g_1, \ldots, g_m$ and certain Lie algebras generated by these vector fields.

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We are interested in the implementation of these results for the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)), x(0) = x_0 \in M, \tag{1.2}$$

where M is a connected real-analytic 2-dimensional manifold, f and g are real-analytic vector fields on M, and u is a control. The theory as given in [9] suggests that in examining the controllability of the system from x_0, the important item to check is the direction of the vector field f along the integral curves of g. We make the assumptions that f and g are linearly independent at some point of M (a very natural assumption) and that the Lie algebra L_A generated by f, g and successive Lie brackets has vector space dimension 2 at x_0 (in order that an open set of M be reachable from x_0).

To find sufficient conditions that the system (1.2) be controllable from x_0, we show that the points of interest are those where f and g are linearly dependent and g is nonzero. For each such point there is a control which makes this point an equilibrium point of the system. If every nontrivial integral curve of g has such a point, and the Lie bracket $[f,g]$ and g are linearly independent for at least one such point on each integral curve, then the system (1.2) is controllable from any point x_0 with the vector space dimension of L_A at x_0 being 2. There may be a closed nowhere dense subset of M which is not reachable from x_0, e.g. a common equilibrium (zero) point of f and g is certainly not reachable. Thus controllable means controllable modulo such points. Also equilibrium points of g must be treated separately using the results from [9]. Many examples are given which explain the important geometry near those points where f and g are linearly dependent. These examples also illustrate the ease with which the theory can be implemented.

It is interesting to see the implications of our theory in the linear case. Suppose we consider the linear system

$$\dot{x}(t) = Ax(t) + u(t)B, x(0) = x_0 \in \mathbf{R}^2, \tag{1.3}$$

where A and B are 2×2 constant matrices. The Lie bracket of the vector fields Ax and B is the constant vector field AB. Thus if AB and B are linearly independent at some point of $\mathbf{R}^2$ (i.e., the controllability matrix has rank 2), then the linear system (1.3) is controllable from any $x_0 \in \mathbf{R}^2$.

An interesting expository article giving results on controllability for nonlinear systems is due to Brockett [1]. Related theories can be found in [10], [11], [12], and [13]. We must stress the difference between our results and the nice theory for local controllability along a reference trajectory given by Hermes in [4], [5], and [6]. If $A(t,x_0)$ denotes the set of all points attainable at time t by solutions of (1.2) corresponding to admissible controls and initiating from x_0 at time 0, Hermes [4] examines if the point $\phi(t)$ (the solution to (1.2) at time t starting at x_0 with control $u \equiv 0$) is an interior point of $A(t,x_0)$ or not.

Section 2 of this article contains definitions, examples, and the statement and proof of our main result. Necessary conditions and other problems concerning global controllability are examined in section 3.

II. Definitions, Examples, and a Global Result

For the first part of this section we state definitions and a result for an n-dimensional hypersurface system, but later we restrict our attention to the 2-dimensional system.

If M is a connected-real-analytic n-dimensional manifold, consider the hypersurface system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-1} u_i(t)g_i(x(t)),\ x(0) = x_0 \in M, \tag{2.1}$$

where $f, g_1, \ldots, g_{n-1}$ are real-analytic vector fields, and $u_1, \ldots, u_{n-1}$ are controls.

By $T(M)$ we denote the tangent bundle of M with fiber (tangent space) $T_x(M)$ for $x \in M$. If X is a vector field on M, then α is an *integral curve* of X if α maps the closed interval $I \subset \mathbf{R}$ into M so that $\frac{d\alpha(t)}{dt} = X(\alpha(t))$ for all $t \in I$. If D is a subset of $T(M)$, then an *integral curve of* D is a mapping α from a real interval $[t, t']$ into M such that there exist $t = t_0 < t_1 \ldots t_k = t'$ and vector fields $X_1, \ldots, X_k$ in D with the restriction of α to $[t_{i-1}, t_i]$ being an integral curve of X_i, for each $i = 1, 2, \ldots, k$. A point $x \in M$ is *D-reachable from* x_0 if there is an integral curve α of D and some $T > 0$ in the interval for α such that $\alpha(0) = x_0$ and $\alpha(T) = x$. A subset A of M is *D-reachable from* x_0 if every point $x \in A$ is reachable from x_0.

Since the D under consideration is the subset of $T(M)$ determined by the vector fields in (2.1), we drop the D from D-reachable. If an open subset of M is reachable from x_0, then the largest open subset U of M which is reachable from x_0 is called the *region of reachability from* x_0. If $U = M$, we say that the system is *controllable from* x_0, and controllability from every $x_0 \in M$ gives us a *controllable system*.

Let O be an open set in M and let $x \in \partial O$. The vector field f *points in the direction of* $\bar{O}$ *(or towards* $\bar{O}$*) at* x if there is an open neighborhood W of x in M such that the integral curve of f starting at x and intersected with W is contained in $\bar{O}$. In addition if ∂O is $\mathcal{C}^1$ near x and $f(x)$ is not tangent to ∂O at x, then f *points in the direction of* O *(or towards* O*) at* x. If f points towards $\bar{O}$ (or O) for all $x \in \partial O$, then f *points in the direction of* $\bar{O}$ *(or* O*) on* ∂O. Given two $\mathcal{C}^\infty$ vector fields h_1 and h_2 on M, the *Lie bracket* of h_1 and h_2 is defined by

$$[h_1, h_2] = \frac{\partial h_2}{\partial x} h_1 - \frac{\partial h_1}{\partial x} h_2,$$

where $\frac{\partial h_1}{\partial x}$ and $\frac{\partial h_2}{\partial x}$ denote Jacobian matrices. Of course other Lie brackets like $[h_1, [h_1, h_2]], \ldots$ can be taken.

By L_A we denote the Lie algebra generated by $f, g_1, \ldots, g_{n-1}$ and successive Lie brackets, and by L'_A we denote the Lie algebra generated by $g_1, \ldots, g_{n-1}$ and successive Lie brackets.

Our first theorem, which characterizes the region of reachability from x_0 for the system (2.1) is taken from [9]. It is this theorem that we are interested in implementing for the 2-dimensional case.

Theorem 2.1. *Assume the vector space dimension of L_A at x_0 is n and that $f, g_1, \ldots, g_{n-1}$ are linearly independent at some point of M. Let U be the smallest open subset of M with $x_0 \in \bar{U}$ satisfying ∂U contains the integral manifolds of L'_A which intersect it (and which are given by Chow's Theorem* [2]) *and f points in the direction of $\bar{U}$ on ∂U. Then U is the region of reachability from x_0 for the system* (2.1).

In the statement of this theorem, we should add the assumption that if $U \neq M$, every open neighborhood of any point $p \in \partial U$ contains points from U and the complement of $\bar{U}$. G. Stefani and A. Baccioti have pointed out that the correct conclusion to the theorem as stated above is $U \subset$ region of reachability $\subset$ interior of $\bar{U}$. The author wishes to thank Professors Stefani and Baccioti for their comments.

The set P of points in M where the vector space dimension of L_A is less than n is a closed nowhere dense subset of M if the dimension at one point x_0 is n (see [9]). If there is no proper open set $U \subset M$ with ∂U as in the theorem and with ∂U disconnecting M, then the system (2.1) is controllable from x_0 (as stated in the introduction, there may be a subset of P, e.g. common equilibrium points of f and g, which is not reachable). If such a set U exists, then the system is certainly not controllable.

Unless otherwise noted for the remainder of this paper we restrict our attention to the 2-dimensional system

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \end{bmatrix} + u(t)\begin{bmatrix} g_1(x(t)) \\ g_2(x(t)) \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)), x(0) = x_0 \in M,$$

where M is our connected 2-dimensional manifold. We assume that the equilibrium points of the system $\dot{x}(t) = f(x(t))$ are isolated. The set of points where f and g are linearly dependent are given by the equation $f_1(x)g_2(x) - f_2(x)g_1(x) = 0$.

To obtain a perspective on the global controllability of our system (2.2) we consider a linear and a nonlinear example.

Example 2.1. On $\mathbf{R}^2$ let

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)) = Ax(t) + u(t)B.$$

Since the matrix $[B, AB]$ has rank 2, this system is controllable from any point x_0 in $\mathbf{R}^2$. However, the important fact geometrically is how the vector field f behaves along the integral curves of g near the points where f and g are linearly dependent. The set of points (x_1, x_2) in $\mathbf{R}^2$ where f and g are dependent are on the straight line $x_1 = 0$. We take an arbitrary integral curve of g, say $x_2 = c =$ constant, which divides $\mathbf{R}^2$ into two connected components $\{x \in \mathbf{R}^2 : x_2 > c\}$ and $\{x \in \mathbf{R}^2 : x_2 < c\}$. Let ϵ be a small positive number. At a point $(-\epsilon, c)$ on $x_2 = c$, f points towards $\{x \in \mathbf{R}^2 : x_2 < c\}$, and at a point (ϵ, c) on $x_2 = c$, f points towards

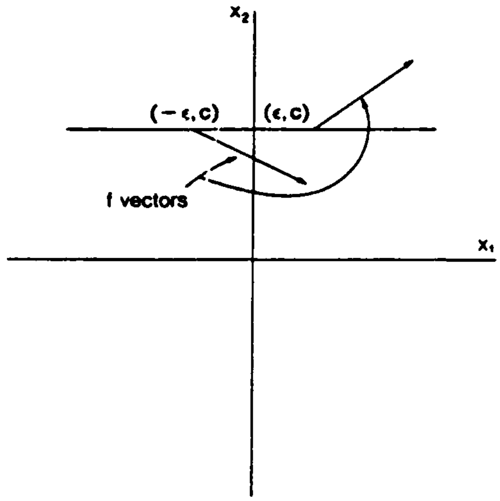

Fig. 1.

$\{x \in \mathbf{R}^2 : x_2 > c\}$. Thus the vector field f "turns through" the integral curve of g at the point on $x_2 = c$ where f and g are linearly dependent. Since this occurs for every integral curve of g, we have by Theorem 2.1 that no integral curve of g can be the boundary of the region of reachability from any $x_0 \in \mathbf{R}^2$. Hence this system is controllable from every $x_0 \in \mathbf{R}^2$.

Since the standard linear methods for proving controllability of a linear system will not generalize to the nonlinear case, it is the vector field f "turning through" the integral curves of g at points where f and g are linearly dependent that becomes the essential item in the nonlinear theory. We show later that this "turning" occurs at points where f and g are linearly dependent and where $[f,g]$ and g are linearly independent.

Example 2.2. Consider

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 \end{bmatrix} + u(t)\begin{bmatrix} -x_2 \\ 1 \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbf{R}^2.$$

This example satisfies the hypotheses of Theorem 2.1 since f and g are linearly dependent if and only if $x_2 = 0$ and the dimension of L_A at every point $x_0 \in \mathbf{R}^2$ is 2. Every integral curve of g intersects the line $x_2 = 0$ (note that $\begin{bmatrix} -x_2 \\ 1 \end{bmatrix}$ is not horizontal in the plane). Computing

$$[f,g] = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -x_2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

and $[f,g]$ and g are linearly independent at points where $x_2 = 0$.

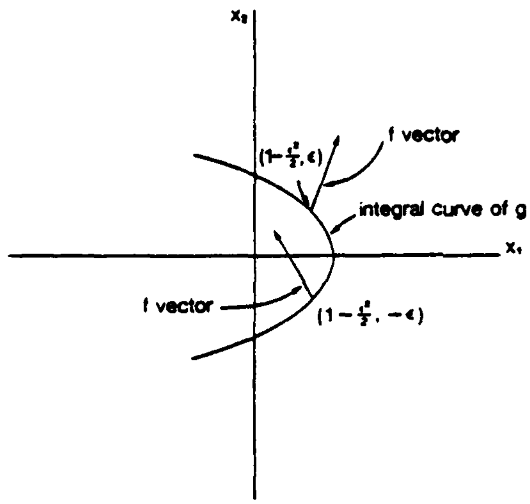

Fig. 2.

An integral curve of g divides $\mathbf{R}^2$ into two connected components. Let ϵ be a small positive number. From the picture we see that the vector field f "turns through" the integral curve of g at the point where $x_2 = 0$. Since this happens for all such integral curves, the system is controllable on $\mathbf{R}^2$. Again, this "turning" is implied by the computation on $[f,g]$ as we show in our main result, Theorem 2.2.

Now we return to our general 2-dimensional system (2.2). Let $p \in M$ be a point where $f(p)$ and $g(p)$ are nonzero and $f(p)$ and $g(p)$ are linearly dependent. For a sufficiently small open neighborhood V of p in M, the integral curve of g through p divides V into two connected open components. We say that *f lies on one side of g near p* if the integral curve of f through p (with the point p deleted) in some small open neighborhood, say V, of p is contained in one of the two connected components of V determined by the integral curve of g through p.

This property of f lying on one side of g near p is of course invariant under rotations on $\mathbf{R}^2$ if we are working in $\mathbf{R}^2$. Since M is a real-analytic 2-dimensional

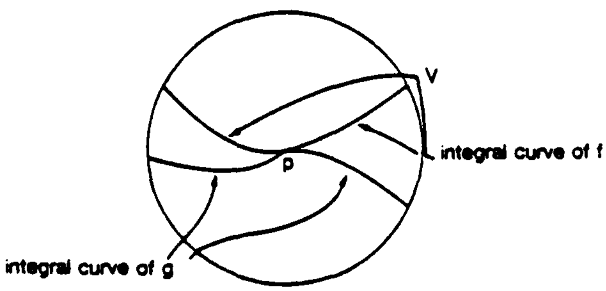

Fig. 3.

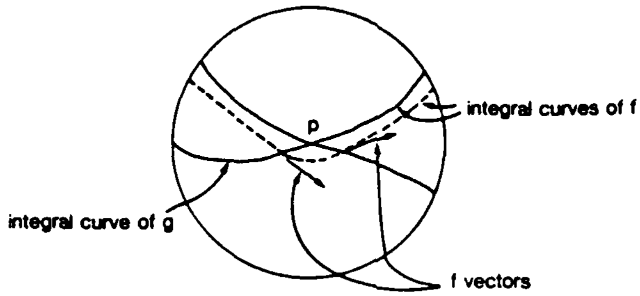

Fig. 4.

manifold there is an open neighborhood of p in M which is real-analytically homeomorphic to an open neighborhood of the origin in $\mathbf{R}^2$. Hence there is no generality loss in assuming that $g(p)=\begin{bmatrix}1\\0\end{bmatrix}$ and that we are working locally in some neighborhood of the point $p=(0,0)$ in $\mathbf{R}^2$. Then to show that f lies on one side of g near p we need only show that the integral curve of f through p minus the integral curve of g through p has a local maximum or local minimum at p. If this occurs then the flow generated by f through certain points near p must intersect the integral curve of g through p in such a way that the vector field f "turns through" the integral curve of g through p.

We now state and prove the main result. In the definition of integral curve of g we could include an integral curve (where g is nonzero) together with an equilibrium point of g and another integral curve of g (with g nonzero). For example, we could call the positive x_2 axis together with the origin and the positive x_1 axis in the system

$$\dot{x}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+u(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \text{ on } \mathbf{R}^2$$

an integral curve of g. However, we assume that g is nonzero in the statement of our main result, and in certain examples given after the theorem we show how to handle equilibrium points of g and common equilibrium points of f and g using Theorems 2.1 and 2.2 together.

Theorem 2.2. *Assume that f and g are linearly independent at some point of M and that g never vanishes on M. Suppose every integral curve of g which disconnects M has a point p where f and g are linearly dependent and $[f,g]$ and g are linearly independent. Then the system (2.2) is controllable from any point x_0 such that the vector space dimension of L_A at x_0 is 2.*

Proof. Suppose that $f(p)$ is nonzero, the case $f(p)=\begin{bmatrix}0\\0\end{bmatrix}$ being considered later in the proof, and assume that $g(p)=\begin{bmatrix}1\\0\end{bmatrix}$. As before, we are working in a neighborhood of the point $p=(0,0)$ in $\mathbf{R}^2$. We must show that $[f,g]$ and g being linearly independent at p (which is invariant under coordinate changes) implies that the integral curve of f through p minus the integral curve of g through p

(both considered as functions of the x_1 variable) has a local maximum or local minimum at p. By the discussion preceding the statement of the theorem, this integral curve of g cannot be the boundary of the region of reachability U from x_0 in Theorem 2.1.

Since

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \end{bmatrix} + u(t)\begin{bmatrix} g_1(x(t)) \\ g_2(x(t)) \end{bmatrix}, g(p) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f(p) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $f(p)$ and $g(p)$ linearly dependent, we must have $f_1(p) \neq 0$, $g_1(p) \neq 0$ and $f_2(p) = g_2(p) = 0$.

The first derivative of the integral curve of f minus the integral curve of g with x_2 considered as a function of x_1 is given by

$$\frac{f_2(x(t))}{f_1(x(t))} - \frac{g_2(x(t))}{g_1(x(t))}.$$

The second derivative is given as

$$\frac{1}{f_1(x(t))g_1(x(t))}\left[g_1(x(t))\frac{\partial f_2(x(t))}{\partial x_2} - f_1(x(t))\frac{\partial g_2(x(t))}{\partial x_1} \right].$$

By the second derivative test we have the desired local maximum or local minimum provided

$$g_1(x)\frac{\partial f_2(x)}{\partial x_1} - f_1(x)\frac{\partial g_2(x)}{\partial x_1} \neq 0 \text{ at } p.$$

Computing

$$[f,g] = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix} \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

Then $[f,g]$ and g are linearly dependent at p if and only if

$$\frac{\partial f_2(x)}{\partial x_1}g_1(x) + \frac{\partial f_2(x)}{\partial x_2}g_2(x) - \frac{\partial g_2(x)}{\partial x_1}f_1(x) - \frac{\partial g_2(x)}{\partial x_2}f_2(x) = 0$$

at p since $g_2(p) = 0$. Because $f_2(p) = g_2(p) = 0$, $[f,g]$ and g are linearly independent at p if and only if

$$\frac{\partial f_2(x)}{\partial x_1}g_1(x) - \frac{\partial g_2(x)}{\partial x_1}f_1(x) \neq 0$$

at p. Thus we must have the desired local maximum or local minimum from our assumption on $[f,g]$ and g.

It remains to consider the case where $f(p)=\begin{bmatrix}0\\0\end{bmatrix}$. Recall that f and g are linearly dependent at the set of points $f_1(x)g_2(x)-f_2(x)g_1(x)=0$. Again we assume that $g(p)=\begin{bmatrix}1\\0\end{bmatrix}$, implying $f_2(p)=g_2(p)=0$. We apply the implicit function theorem to the set of points where $f_1(x)g_2(x)-f_2(x)g_1(x)=0$, near p. Since

$$\frac{\partial f_1(x)}{\partial x_1}g_2(x)+\frac{\partial g_2(x)}{\partial x_1}f_1(x)-\frac{\partial f_2(x)}{\partial x_1}g_1(x)-\frac{\partial g_1(x)}{\partial x_1}f_2(x)$$

$$=\frac{\partial g_2(x)}{\partial x_1}f_1(x)-\frac{\partial f_2(x)}{\partial x_1}g_1(x) \text{ at } p,$$

our assumption on $[f,g]$ and g implies $\frac{\partial g_2(p)}{\partial x_1}f_1(p)-\frac{\partial f_2(p)}{\partial x_1}g_1(p)\neq 0$ at p. Thus the zero set of $f_1(x)g_2(x)-f_2(x)g_1(x)$ defines a real-analytic 1-dimensional submanifold S of M near p.

Recall that we assumed the equilibrium points of

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}f_1(x(t))\\ f_2(x(t))\end{bmatrix}$$

are isolated. Hence we take all points of S except p to be points at which $f(x)$ is not equal to $\begin{bmatrix}0\\0\end{bmatrix}$. We choose an open neighborhood W of p in M such that

(i) p is only point in W with $f(p)=\begin{bmatrix}0\\0\end{bmatrix}$,

(ii) the only points in W where f and g are linearly dependent are those in $S\cap W$,

(ii) given any point $q\in S\cap W$ with $q\neq p$, the integral curve of g through q divides W into two connected open components and the vector field f points in the direction of one component on one side of q in the integral curve and in the direction of the other component on the other side of q in the integral curve, and

(iv) $[f,g]$ and g are linearly independent on W.

Part (iii) follows from the first part of this proof since $f(q)\neq\begin{bmatrix}0\\0\end{bmatrix}$ and (ii). The set W can also be chosen so that the integral curve of g through p divides W two connected open components. Suppose f points in the direction of the closure of one of these components along the integral curve of g in W. Since the integral curves of g vary smoothly (as we move from curve to curve), the only way the preceding statement can hold in view of (iii) is to have the integral curves of f and g coincide on one side of p along the integral curve of g through p. But $[f,g]$ must vanish along such a set, a contradiction to (iv).

Hence we have that the vector field f must "turn through" every integral curve of g which disconnects M at some point in S on each such curve. Since M

is connected, the only set U satisfying Theorem 2.1 for our x_0 must be equal to M itself. Thus the system (2.2) is controllable from any such x_0. □

As an easy application of this theory we could prove the known controllability theorem for a linear system in $\mathbf{R}^2$, but remarks in this direction have already been made.

We now give several examples of nonlinear systems to illustrate how to apply Theorem 2.2.

Example 2.3. Consider

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 4 \\ x_2 \end{bmatrix} + u(t)\begin{bmatrix} x_2 \\ 1 \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbf{R}^2.$$

The set of points in $\mathbf{R}^2$ where f and g are linearly dependent is defined by $4 - x_2^2 = 0$, which gives two lines $x_2 = 2$ and $x_2 = -2$. Computing

$$[f,g] = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ 1 \end{bmatrix}$$

Thus the vector space dimension of L_A at any point x_0 is 2 and $[f,g]$ and g are linearly independent on the straight lines $x_2 = 2$ and $x_2 = -2$. Since every integral curve of g intersects these lines, we have a controllable system by Theorem 2.2.

Example 2.4. Consider

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 \end{bmatrix} + u(t)\begin{bmatrix} x_2^2 \\ 1 \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbf{R}^2.$$

The set of points in $\mathbf{R}^2$ where f and g are linearly dependent is defined by the straight lines $x_2 = 0$ and $x_2 = 1$. Computing

$$[f,g] = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2^2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2x_2 \\ 0 \end{bmatrix}.$$

Then $[f,g]$ and g are linearly independent on $x_2 = 0$ and $x_2 = 1$ and the vector space dimension of L_A is 2 everywhere. Since g has no horizontal tangent vectors in the plane, all integral curves of g must intersect these lines. Hence this system is controllable.

Example 2.5. Let

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t)\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + u(t)\begin{bmatrix} x_1 \\ -4x_2 \end{bmatrix}$$
$$= f(x(t)) + u(t)g(x(t)),$$

where $x(0) = x_0 \in \mathbf{R}^2 - (0,0)$.

The origin in $\mathbf{R}^2$ is a common equilibrium point of f and g. The vector fields f and g are linearly dependent on the set of points defined by $4x_2^2 - x_1^2 = 0$ or $x_1 = \pm 2x_2$. We have

$$[f,g] = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ -4x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = 5\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

Thus $[f,g]$ and g are linearly independent on $x_1 = 2x_2$ and $x_1 = -2x_2$ except at the point $(0,0)$. Also the vector space dimension of L_A at every point in $\mathbf{R}^2 - (0,0)$ is 2.

Let's apply Theorem 2.2 to the open first quadrant. It is easy to show that any integral curve of g which starts in the first quadrant intersects the line $x_1 = 2x_2$ (we can move forward and backward in time along integral curves of g since it is the vector field we control). By Theorem 2.2 we know the first quadrant is reachable from any point x_0 in the first quadrant. Similar arguments imply the same result for the remaining three quadrants.

We need to show that $\mathbf{R}^2 - (0,0)$ is reachable from any point x_0 in $\mathbf{R}^2 - (0,0)$. The positive x_1 axis, the positive x_2 axis, the negative x_1 axis, and the negative x_2 axis are all integral curves of g. The vector field f points towards the first quadrant along the positive x_1 axis, towards the second quadrant along the positive x_2 axis, towards the third quadrant along the negative x_1 axis, and towards the fourth quadrant along the negative x_2 axis.

Thus we are able to move from one quadrant to the next by using the f vectors. Since each quadrant is controllable, the system is controllable, ignoring the point $(0,0)$.

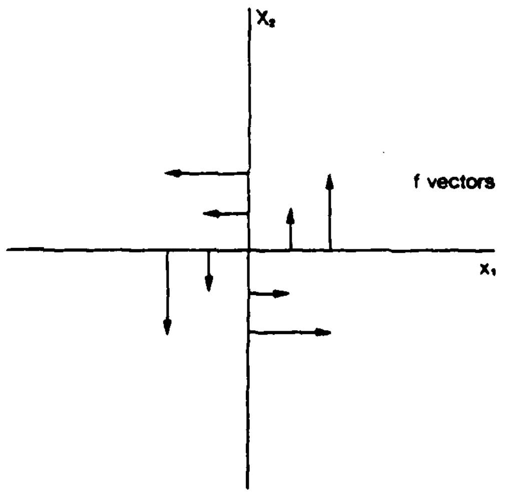

Fig. 5.

Example 2.6. Let

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t)\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + u(t)\begin{bmatrix} x_1 \\ 4x_2 \end{bmatrix}$$

$$= f(x(t)) + u(t)g(x(t)),$$

where $x(0) = x_0 \in \mathbf{R}^2 - (0,0)$.

The point (0,0) is a common equilibrium point of f and g. Points where f and g are linearly dependent are given by $x_1 = 2x_2$ and $x_1 = -2x_2$. As in the preceding example, each open quadrant is controllable since

$$[f,g] = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 3\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}.$$

However along the positive x_1 axis (an integral curve of g), f points towards the first quadrant, and along the positive x_2 axis (an integral curve of g), f points towards the first quadrant also.

Thus once we reach the first quadrant, we cannot leave it by Theorem 2.1. This system is not globally controllable.

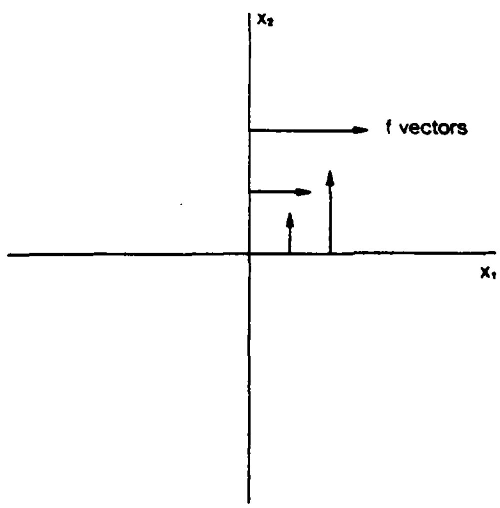

Fig. 6.

Example 2.7[1]. Let

$$\dot{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + u(t) \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$= f(x(t)) + u(t)g(x(t)),$$

where $x(0) = x_0 \in \mathbf{R}^2 - (0,0)$.

The point (0,0) is a common equilibrium point of f and g. The vector fields f and g are linearly dependent when $x_1^2 + x_2^2 = 0$, i.e. at the point (0,0) only.

The positive x_1 axis and the positive x_2 axis are both integral curves of g. Along the positive x_1 axis f points toward the first quadrant, and the same is true for the positive x_2 axis. By Theorem 2.1, this system is not controllable. If we restrict our attention to any one of the open quadrants, we find it also is not controllable by Theorem 2.1. This occurs because an integral curve of g, say in the first quadrant, disconnects the first quadrant. Moreover, along such a curve f points in the direction of one of the components bounded by the integral curve.

The above examples serve to show the practical applications of Theorem 2.2 together with Theorem 2.1.

III. Other Problems

We can of course ask if the sufficient condition concerning $[f,g]$ in Theorem 2.2 for the system (2.2) to be controllable is also necessary. It is known in the linear case that it is necessary. We show later in this section an example of a nonlinear system which is controllable but for which the condition on $[f,g]$ and g does not hold.

First we state some necessary conditions for controllability, which do not involve computations of $[f,g]$. For an example of an application of the following theorem, take the system of Example 2.7 restricted to the open first quadrant in $\mathbf{R}^2$.

Theorem 3.1. *Assume that our connected real-analytic 2-dimensional manifold M is also simply connected. If there exists an integral curve of g which disconnects M and which does not intersect the set of points where f and g are linearly dependent, then the system (2.2) is not controllable.*

Proof. Suppose this integral curve of g gives us two connected open components O and O' in M. Since M is connected and simply connected, if f points in the direction of O at some points along the curve and in the direction of O' at other points along the curve, then there must be a point p in this integral curve where f and g are linearly dependent, a contradiction. By Theorem 2.1, if we assume f points towards O on the integral curve of g, then M is not reachable from any point x_0 in O. □

Returning to our question concerning the necessity of the sufficient condition in Theorem 2.2, we offer the following three examples. In the first two, $[f,g]$ is a multiple of g at all points where f and g are linearly dependent and the systems are not controllable. The third example involves a controllable system in which $[f,g]$ is a multiple of g at all points where f and g are linearly dependent, showing our sufficient conditions are not necessary.

Example 3.1. Consider

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + u(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= f(x(t)) + u(t) g(x(t)), \\ x(0) &= x_0 \in \mathbf{R}^2 - (0,0). \end{aligned}$$

The set of points in $\mathbf{R}^2$ where f and g are linearly dependent are given by $x_2 = x_1$ and $x_2 = -x_1$. Also $(0,0)$ is a common equilibrium point of f and g. The Lie bracket

$$[f,g] = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and there is no hope of using Theorem 2.2.

Disregarding the point $(0,0)$ the set of points where $x_1 = x_2$ is a common integral curve of f and g (it is this type of behavior that occurs in a noncontrollable linear system in $\mathbf{R}^2$). The set of points in $\mathbf{R}^2$ where $x_1 = x_2$ disconnects $\mathbf{R}^2$ and f points in the direction of this set when $x_1 = x_2$. By Theorem 2.1, there is no way of moving from one side of the line $x_1 = x_2$ to the other, and we cannot have controllability.

Example 3.2. Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2x_2^2 \\ 1 \end{bmatrix} + u(t) \begin{bmatrix} x_2^2 \\ 1 \end{bmatrix} = f(x(t)) + u(t) g(x(t)), \\ x(0) &= x_0 \in \mathbf{R}^2. \end{aligned}$$

We show that this system is not controllable, but the reasons given cannot occur in the linear case as in Example 3.1.

The line $x_2 = 0$ is the set of points where f and g are linearly dependent. Computing

$$[f,g] = -\begin{bmatrix} 0 & -4x_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2^2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2x_2^2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6x_2 \\ 0 \end{bmatrix},$$

which vanishes on $x_2 = 0$, and Theorem 2.2 is not applicable.

The integral curve of the system $\dot{x}(t)=g(x(t))$ through $(0,0)$ is given by $x_1=\frac{x_2^3}{3}$, and the integral curve of the system $\dot{x}(t)=f(x(t))$, through $(0,0)$ is given by $x_1=\frac{-2x_2^3}{3}$. These integral curves intersect only at the point $(0,0)$ where they have a common tangent. The integral curve of g through $(0,0)$ separates $\mathbf{R}^2$ into two connected open components, and f points towards only one of these components along the integral curve near $(0,0)$ and hence along the entire curve for g (except at $(0,0)$). There is one side of this integral curve from which we cannot move to the other, showing the system is indeed not controllable.

Example 3.3. Let

$$\dot{x}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}x_2^3\\ 1\end{bmatrix}+u(t)\begin{bmatrix}-x_2^3\\ 1\end{bmatrix}=f(x(t))+u(t)g(x(t)),$$

$$x(0)=x_0\in\mathbf{R}^2.$$

The straight line $x_2=0$ defines the set of points where f and g are linearly dependent. Computing

$$[f,g]=-\begin{bmatrix}0 & 3x_2^2\\ 0 & 0\end{bmatrix}\begin{bmatrix}-x_2^3\\ 1\end{bmatrix}+\begin{bmatrix}0 & -3x_2^2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_2^3\\ 1\end{bmatrix}=\begin{bmatrix}6x_2^2\\ 0\end{bmatrix},$$

which is not linearly independent from g when $x_2=0$. If we take the integral curve of f and the integral curve of g through any point p on the line $x_2=0$, we find the differences of these curves (with x_1 as a function of x_2) has a maximum at the point p. This is exactly the desired behavior we need for the vector field f to "turn through" the integral curve of g at such points. Hence along any integral curve of g the vector field f points in the direction of one component (given by the integral curve of g) at some points and in the direction of the other component at different points. Then this system is controllable despite the relationship of $[f,g]$ and g when $x_2=0$.

This last example suggests the existence of some "higher order" sufficient conditions for controllability which may also be necessary. Hermes [4] has "higher order" conditions for the local controllability problem. Two other immediate problems are suggested. Find a Theorem 2.2 which implements Theorem 2.1 for hypersurface systems in the case where the dimension of the manifold M is greater than two. Find a Theorem 2.2 to implement the controllability theory in [7] and [8] for general nonlinear systems of the form (see (1.1))

$$\dot{x}(t)=f(x(t))+\sum_{i=1}^{m}u_i(t)g_i(x(t)),\ x(0)=x_0\in M.$$

Professor D. L. Elliott has pointed out to the author a paper of Y. Gerbier [3]. Gerbier shows that controllable systems of two vector fields in $\mathbf{R}^2$ without equilibria are topologically equivalent (in a sense defined in his paper) to $\left\{\frac{\partial}{\partial x},-\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right\}$.

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REACHABLE SETS FOR NONLINEAR SYSTEMS
IN THE PHASE PLANE

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REACHABLE SETS FOR NONLINEAR SYSTEMS IN THE PHASE PLANE

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Abstract

Consider the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbb{R}^2,$$

where f and g are real-analytic vector fields on $\mathbb{R}^2$ which are linearly independent at some point of $\mathbb{R}^2$, and u is a real-valued control. Sufficient conditions on f and g are known so that this system is controllable from x_0. The purpose of this article is to implement these conditions in the bilinear case

$$\dot{x}(t) = Ax(t) + u(t)Bx(t),\ x(0) = x_0 \in \mathbb{R}^2,$$

where A and B are constant 2×2 matrices. The process involves finding the set of points where Ax and Bx are linearly dependent and computing the Lie bracket [Ax, Bx] at all such points. This is a generalization of the well known controllability results for a linear system on $\mathbb{R}^2$.

1. Introduction

We are interested in the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in M, \quad (1)$$

where M is a connected real-analytic 2-dimensional manifold, f and g are real-analytic vector fields on M, and u is a control. In examining the controllability of this system from x_0, the important item to check is the direction of the vector field f along the integral curves of g.[2] We assume that f and g are linearly independent at some point of M and that the Lie algebra L_A generated by f,g and successive Lie brackets has vector space dimension 2 at x_0. The points of interest are those where f and g are linearly dependent and g is nonzero. If every integral curve of g which is nontrivial has such a point, and the Lie bracket [f,g] and g are linearly independent for at least one such point on each integral curve, then the system (1) is controllable from any point x_0 with the vector space dimension of L_A at x_0 being 2.[3] There may be a closed nowhere dense subset of M which is not reachable from x_0, and controllable means controllable modulo such points.

We restrict our attention to the bilinear system

$$\dot{x}(t) = Ax(t) + u(t)Bx(t),\ x(0) = x_0 \in \mathbb{R}^2, \quad (2)$$

where A and B are constant 2×2 matrices. First we compute the set of points S where Ax and Bx are linearly dependent. If there exists an integral curve of Bx which disconnects $\mathbb{R}^2$ and which does not intersect S, then the system (2) is not controllable.[3] Next we compute the Lie bracket [Ax,Bx] at all points where Ax and Bx are linearly dependent except the origin. This computation yields a constant (along each manifold part of an algebraic variety) which if nonzero can indicate we have controllability. It remains only to check the direction of the vector field Ax on the integral curves of Bx that approach the origin.

We illustrate our method by applying it to several examples. Results on controllability are useful in the study of the problem of stabilization.[1]

2. Definitions and Results

If X is a vector field on $\mathbb{R}^2$, then α is an <u>integral curve</u> of X if α maps the closed interval $I \subset \mathbb{R}$ into $\mathbb{R}^2$ so that $\frac{d\alpha(t)}{dt} = X(\alpha(t))$ for all $t \in I$. If D is a set of vector fields on $\mathbb{R}^2$, then an <u>integral curve of D</u> is a mapping α from a real interval [t,t'] into $\mathbb{R}^2$ such that there exist $t = t_0 < t_1 < \ldots < t_k = t'$ and vector fields $X_1,\ldots,X_k$ in D with the restriction of α to $[t_{i-1},t_i]$ being an integral curve of X_i, for each $i = 1,2,\ldots,k$. The set D we consider is the one determined by the vector fields in the bilinear system (2).

A point $x \in \mathbb{R}^2$ is <u>reachable from x_0</u> if there is an integral curve α of D and some $T \geq 0$ in the interval for α such that $\alpha(0) = x_0$ and $\alpha(T) = x$. A subset of $\mathbb{R}^2$ is <u>reachable from x_0</u> if every point in this set is reachable from x_0. We shall make assumptions so that an open subset of $\mathbb{R}^2$ is reachable from x_0 for (2), and the largest open subset U of $\mathbb{R}^2$ which is reachable from x_0 is called the <u>region of reachability from x_0</u>. If $U = \mathbb{R}^2$, we say that the system is controllable from x_0, and controllability from every point $x_0 \in \mathbb{R}^2$ gives us a <u>controllable system</u>.

Let O be an open subset of $\mathbb{R}^2$ and let $x \in \partial O$. The vector field f <u>points in the direction of $\bar{O}$</u>, or <u>towards $\bar{O}$, at x</u> if there is an open neighborhood W of x in $\mathbb{R}^2$ such that the integral curve of f starting at x and intersected with W is contained in $\bar{O}$. In addition if ∂O is C^1 near x and f(x) is not tangent to ∂O at x, then f <u>points in the direction of O</u>, or <u>towards O</u>, at x. If f <u>points in the direction of $\bar{O}$ (or O)</u> for all $x \in \partial O$, then f <u>points in the direction of $\bar{O}$ (or O) on ∂O</u>.

If h_1 and h_2 are C^∞ vector fields on $\mathbb{R}^2$, the <u>Lie bracket</u> of h_1 and h_2 is defined by

$$[h_1,h_2] = \frac{\partial h_2}{\partial x}h_1 - \frac{\partial h_1}{\partial x}h_2,$$

where $\frac{\partial h_1}{\partial x}$ and $\frac{\partial h_2}{\partial x}$ denote Jacobian matrices. Of course other Lie brackets like $[h_1,[h_1,h_2]],\ldots$ can be taken.

The Lie algebra generated by Ax, Bx, and successive Lie brackets is denoted by L_A.

Our first result concerns the region of reachability of our system (2). It is proved for a hypersurface system of dimension n.[2]

Theorem 1. Assume the vector space dimension of L_A at x_0 is 2 and that Ax and Bx are linearly independent at some point of R^2. Let U be the smallest open subset of R^2 with $x_0 \in \bar{U}$ satisfying ∂U contains the integral curves of Bx which intersect it and Ax points in the direction of $\bar{U}$ on U. Then U is the region of reachability from x_0 for our system (2).

It is our goal to implement the following theorem, which has been proved for a general nonlinear system in two dimensions.[3] Here V is a domain in R^2, and the statement is for our bilinear system.

Theorem 2. Assume that Ax and Bx are linearly independent at some point in V. Suppose every integral curve of Bx which disconnects $V(V - \{(0,0)\}$ if (0,0) is in V) contains a point where Ax and Bx are linearly dependent and [Ax,Bx] and Bx are linearly independent. Then the system (2) is controllable on V from any point $x_0 \in V$ such that the vector space dimension of L_A at x_0 is 2.

3. Computation of Lie Brackets

We compute [Ax,Bx] at all points (except (0,0)) where Ax and Bx are linearly dependent. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

Then Ax and Bx are linearly dependent on the algebraic variety given by

$$\det\begin{pmatrix} a_{11}x_1 + a_{12}x_2 & b_{11}x_1 + b_{12}x_2 \\ a_{21}x_2 + a_{21}x_2 & b_{21}x_1 + b_{22}x_2 \end{pmatrix} = 0,$$

i.e. $(a_{11}b_{21} - a_{21}b_{11})x_1^2 + (a_{11}b_{22} + a_{12}b_{21} - a_{21}b_{12} - a_{22}b_{11})x_1x_2 + (a_{12}b_{22} - a_{22}b_{12})x_2^2 = 0$. Let

$$a_{11} - a_{21}b_{11} = c_1$$
$$a_{11}b_{22} + a_{12}b_{21} - a_{21}b_{12} - a_{22}b_{11} = c_2$$
$$a_{12}b_{22} - a_{22}b_{12} = c_3.$$

Then we have no nonzero real solutions (x_1,x_2) if and only if

$$\det\begin{pmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{pmatrix} > 0.$$

Thus we assume that this determinant is ≤ 0.

If $c_1 = c_2 = c_3 = 0$, then Ax and Bx are linearly dependent everywhere, contrary to our assumptions of Theorems 1 and 2. If $c_1 = 0$ and at least one of c_2 and $c_3 \neq 0$, then Ax and Bx are linearly dependent on the lines $x_2 = 0$ and $c_2x_1 + c_3x_2 = 0$. Similarly, if $c_3 = 0$ and at least one of c_1 and $c_2 \neq 0$, then Ax and Bx are linearly dependent on the lines $x_1 = 0$ and $c_1x_1 + c_2x_2 = 0$. If both c_1 and $c_3 = 0$ and $c_2 \neq 0$, then Ax and Bx are linearly dependent when $x_1 = 0$ and $x_2 = 0$. Finally, if c_1 and $c_3 \neq 0$, then Ax and Bx are linearly dependent when

$$x_1 = \frac{-c_2x_2 \pm x_2\sqrt{c_2^2 - 4c_1c_3}}{2c_1}.$$

Hence there are four cases we must consider.

Computing we find

$$[Ax,Bx] = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + b_{22}x_2 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

$$= -\begin{pmatrix} a_{11}b_{12}x_2 + a_{12}b_{21}x_1 + a_{12}b_{22}x_2 - a_{12}b_{11}x_2 - a_{21}b_{12}x_1 - a_{22}b_{12}x_2 \\ a_{21}b_{11}x_1 + a_{21}b_{12}x_2 + a_{22}b_{21}x_1 - a_{11}b_{21}x_1 - a_{12}b_{21}x_2 - a_{21}b_{22}x_1 \end{pmatrix}.$$

We are given

$$Bx = \begin{pmatrix} b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + b_{22}x_2 \end{pmatrix}.$$

Let

$$d = \det\begin{pmatrix} a_{11}b_{12}x_2 + a_{12}b_{21}x_1 + a_{12}b_{22}x_2 - a_{12}b_{11}x_2 - a_{21}b_{12}x_1 - a_{22}b_{12}x_2 & b_{11}x_1 + b_{12}x_2 \\ a_{21}b_{11}x_1 + a_{21}b_{12}x_2 + a_{22}b_{21}x_1 - a_{11}b_{21}x_1 - a_{12}b_{21}x_1 - a_{21}b_{22}x_1 & b_{21}x_1 + b_{22}x_2 \end{pmatrix}.$$

Case I. $c_1 = 0$ and at least one of c_2 and $c_3 \neq 0$.

Plug $x_2 = 0$ into d and factor out x_1 to get a constant d^1_+, and plug $x_1 = -\frac{c_3}{c_2}x_2$ if $c_2 \neq 0$ (or $x_2 = -\frac{c_2}{c_3}x_1$ if $c_3 \neq 0$) into d and factor out x_2 (or x_1) to get a constant d^1_-.

Case 2. $c_3 = 0$ and at least one of c_1 and $c_2 \neq 0$.

Plug $x_1 = 0$ into d and factor out x_2 to get a constant d^2_+, and plug $x_1 = -\frac{c_2}{c_1}x_2$ if $c_1 \neq 0$ (or $x_2 = -\frac{c_1}{c_2}x_1$ if $c_2 \neq 0$) into d and factor out x_2 (or x_1) to get a constant d^2_-.

Case 3. $c_1 = c_3 = 0$ and $c_2 \neq 0$.

Plug $x_1 = 0$ into d and factor out x_2 to get a constant d^3_+, and plug $x_2 = 0$ into d and factor out x_1 to get a constant d^3_-.

Case 4. c_1 and $c_3 \neq 0$.

Plug $x_1 = \frac{x_2}{2c_1}(-c_2 + \sqrt{c_2^2 - 4c_1c_3})$ into d and factor out x_2 to get a constant d^4_+, and plug $x_1 = \frac{x_2}{2c_1}(-c_2 - \sqrt{c_2^2 - 4c_1c_3})$ into d and factor out x_2 to get a constant d^4_-.

Our next result follows directly from Theorem 2. Again V is a domain in R^2.

Theorem 3. Suppose Ax and Bx are linearly independent at some point in V. Assume that every integral curve of Bx which disconnects $V(V - \{(0,0)\}$ if (0,0) is in V) contains a point where Ax and Bx are linearly dependent and that d^i_+ and d^1_- are both nonzero if we are under case i, i = 1,...,4. Then the system (2) is controllable on V from any point $x_0 \in V$ such that the vector space dimension of L_A at x_0 is 2.

4. Examples

We apply Theorems 1, 2 and 3 to the following three examples.

Example 1. Let

$$\dot{x}(t) = \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u(t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= Ax(t) + u(t)Bx(t),\ x(0) = x_0 \in \mathbb{R}^2 - \{(0,0)\}.$$

The vector fields Ax and Bx are linearly dependent only on the set S defined by $x_1^2 + x_2^2 = 0$, i.e. at the origin (0,0). Any integral curve of Bx in the open first quadrant disconnects $\mathbb{R}^2$ and does not intersect S. By a statement in the introduction (which follows from Theorem 1[2,3]), this system is not controllable from every $x_0 \in \mathbb{R}^2 - \{(0,0)\}$, even though the vector space dimension of L_A at each such x_0 is 2.

Example 2. Let

$$\dot{x}(t) = \begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u(t)\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= Ax(t) + Bx(t),\ x(0) = x_0 \in \mathbb{R}^2 - \{(0,0)\}.$$

The set S of points where Ax and Bx are linearly dependent is given by $x_1 = 2x_2$ and $x_1 = -2x_2$ and we are in case 4. We find $d_+^4 = -24$ on $x_1 = 2x_2$ and $d_-^4 = -24$ on $x_1 = -2x_2$. Applying Theorem 3 with V equal to the open first quadrant in $\mathbb{R}^2$ shows that this quadrant is controllable for the system. The same is also true for the second, third, and fourth open quadrants in $\mathbb{R}^2$.

Since the positive x_1-axis and the positive x_2-axis are integral curves of Bx, and since Ax points toward the open first quadrant along these two curves and is zero at the origin, Theorem 1 implies that our system is not controllable for $\mathbb{R}^2 - \{(0,0)\}$. Once we are in the open first quadrant, it is impossible to leave it.

Even in this example which is not controllable on $\mathbb{R}^2 - \{(0,0)\}$, our theory and technique give us much insight into the behavior of the system under controls.

Example 3. Let

$$\dot{x}(t) = \begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u(t)\begin{pmatrix}1 & 0\\ 0 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= Ax(t) + u(t)Bx(t),\ x(0) = x_0 \in \mathbb{R}^2 - \{(0,0)\}.$$

The set S where Ax and Bx are linearly dependent is given by $x_1 = 2x_2$ and $x_1 = -2x_2$. Also the vector space dimension of L_A at each point of $\mathbb{R}^2 - \{(0,0)\}$ is 2.

We apply Theorem 3 to the open first quadrant. Any integral curve of Bx which starts in the first quadrant intersects the line $x_1 = 2x_2$. Since $d_+^4 = +40$ on $x_1 = 2x_2$, the first quadrant is controllable from any x_0 in it. Similar arguments show that each of the other three quadrants is also controllable.

The positive x_1-axis, the positive x_2-axis, the negative x_1-axis, and the negative x_2-axis are all integral curves of Bx. The vector field Ax points towards the first quadrant along the positive x_1-axis, towards the second quadrant along the positive x_2-axis, towards the third quadrant along the negative x_1-axis, and towards the fourth quadrant along the negative x_2-axis. Thus we are able to move from one quadrant to the next, and our system is controllable on $\mathbb{R}^2 - \{(0,0)\}$.

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CONTROLLABILITY AND STABILIZABILITY

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CONTROLLABILITY AND STABILIZABILITY

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ABSTRACT

Consider the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbb{R}^2,$$

where f and g are real-analytic vector fields on $\mathbb{R}^2$. If this is a controllable linear system, then it is well known the system is stabilizable by linear feedback. We want to consider a similar problem for nonlinear systems, with emphasis on bilinear systems. Sufficient conditions for the above system to be controllable have been found, and implementation for bilinear systems has been discussed. If a bilinear system is controllable under these conditions, we show that we can move from any point $x_0 \in \mathbb{R}^2 - \{(0,0)\}$ to the origin.

1. INTRODUCTION

If the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)),\ x(0) = x_0 \in \mathbb{R}^2, \quad (1)$$

where f and g are real-analytic vector fields on $\mathbb{R}^2$, is a linear system, then the relationship between controllability and stabilizability are known [14]. For nonlinear systems there are some results on "controlled stability" [2]. Recently, theorems giving sufficient conditions for the system (1) to be controllable have been proved [9] and implemented in the bilinear case [10].

For the bilinear system

$$\dot{x}(t) = Ax(t) + u(t)Bx(t),\ x(0) = x_0 \in \mathbb{R}^2 - \{(0,0)\}, \quad (2)$$

where A and B are constant 2×2 matrices, we show that these sufficient conditions for controllability imply that we can choose controls to drive from the point x_0 towards the origin along the solution curves corresponding to these controls. Since we are presently interested in only qualitative results, we assume that it is possible to move along the integral curves of the vector field Bx if necessary.

Two examples are used to illustrate our method.

2. RESULTS

We give the following definitions, where f and g are the vector fields from system (1).

Let O be an open subset of $\mathbb{R}^2$ and let $x \in \partial O$. Then f <u>points in the direction of $\bar{O}$, or toward $\bar{O}$, at x</u> if there exists an open neighborhood W of x in $\mathbb{R}^2$ such that the integral curve of f starting at x and intersected with W is contained in $\bar{O}$. Here $\bar{O}$ denotes the closure of O in $\mathbb{R}^2$. In addition if ∂O is C^1 near x and f(x) is not tangent to ∂O at x, then f <u>points in the direction of O, or toward O, at x</u>. If f points toward $\bar{O}$ (or O) for all $x \in \partial O$, then f <u>points in the direction</u> of $\bar{O}$ (or O) on ∂O.

The Lie bracket of the vector fields f and g is defined by

$$[f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g,$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial x}$ denote Jacobian matrices. Other Lie brackets $[f,[f,g]]$, $[g,[f,g]]$,... can also be taken. The Lie algebra generated by f, g, and successive Lie brackets of f and g is denoted by L_A.

In system (2) we set

$$Ax = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and}$$

$$Bx = \begin{pmatrix} B_1(x) \\ B_2(x) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

If V is a domain in $\mathbb{R}^2$, then the following results have been proved [9].

<u>Theorem 1</u>. Assume that Ax and Bx are linearly independent at some point in V. Suppose every integral curve of Bx which disconnects $V(V - \{(0,0)\}$ if (0,0) is in V) contains a point p where Ax and Bx are linearly dependent and [Ax,Bx] and Bx are linearly independent. Then the system (2) is controllable on V from any point $x_0 \in V$ such that the vector space dimension of L_A at x_0 is 2.

Thus under the above assumptions we can reach any point in $V(V - \{(0,0)\}$ if (0,0) is in V) from x_0 by choosing a finite number of controls and following the solution curves of the corresponding differential equations.

Let us understand the geometric meaning of the Lie bracket [Ax,Bx] and the vector field Bx being linearly independent at p [9]. There is an open neighborhood W of p in V such that the integral curve of Bx through p divides W into open connected components W_1 and W_2 and f points toward W_1 at all points in this integral curve on one side of p and toward W_2 at all points in this curve on the other side of p. In other words, the vector field f "turns through" the integral curve of Bx through p at p.

The set of points where Ax and Bx are linearly dependent is the algebraic variety S defined by

$$\det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix} = 0.$$

The interesting case occurs when we have an algebraic variety consisting of two straight lines that intersect at the origin. It is shown that the bracket [Ax,Bx] and the vector field Bx are linearly independent on one of these straight lines if and only if a computable constant (depending on the line) is nonzero [10]. However, we do not use those computations here for the sake of brevity.

We want to apply our results on controllability to prove that we can move from point x_0 in $\mathbb{R}^2$ to the origin in $\mathbb{R}^2$.

Theorem 2. Assume that Ax and Bx are linearly independent at some point in $\mathbb{R}^2$ and that the vector space dimension of L_A is 2 at every point in $\mathbb{R}^2 - \{(0,0)\}$. Suppose every integral curve of Bx (or -Bx) either approaches the origin or disconnects $\mathbb{R}^2$ and contains a point *p in the set* $S - \{(0,0)\}$ where [Ax,Bx] and Bx are linearly independent. For any $x_0 \in \mathbb{R}^2 - \{(0,0)\}$ we can choose controls to drive from x_0 to (0,0) along the solution curves corresponding to these controls.

Proof. As mentioned previously, we assume that we can move along the integral curves of Bx. If x_0 is contained in one of these curves which approach the origin, then we certainly can move along this curve (either in forward or backward time since we can control Bx) to (0,0). Thus if all integral curves of Bx approach the origin we are through. Otherwise, we have that the algebraic variety S defined by

$$\det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix} = 0$$

consists of two straight lines intersecting at the origin.

Suppose x_0 is contained in an integral curve C of Bx which disconnects $\mathbb{R}^2$ and intersects the set of points S. By moving along this integral curve in forward or backward time, we can reach a point $p \in S$. This curve C divides $\mathbb{R}^2$ into two open connected components U_1 and U_2, one of which, say U_1, contains the origin. The point p divides C into two components C_1 and C_2. By our geometric interpretation of the linear independence of [Ax,Bx] and Bx, there is an open neighborhood W of p in $\mathbb{R}^2$ such that Ax points toward U_1 on $W \cap C_1$ and U_2 on $W \cap C_2$.

Take a line L through the origin and a point $p_1 \in W \cap C_1$ of slope m. We can reach p_1 by moving along the integral curve C (if x_0 is in $W \cap C_1$, there is no need in driving from x_0 to a point like p_1). We wish to move from p_1 to the origin along the straight line L. We can choose the slope m so that

i) the line L is not in the set S,

ii) $mB_1(x) - B_2(x) \neq 0$ on L.

Condition i) is obvious but condition ii) requires justification. The set of points defined by $mB_1(x) - B_2(x) = 0$ is a straight line through the origin and therefore either intersects L only at (0,0) or coincides with L. If these two lines coincide then L is an integral curve of Bx which intersects C, another integral curve of Bx, transversally at p_1, a contradiction.

It remains to take the control u in our system (2) to move from p_1 to (0,0) along L. Thus we must have $\dot{x}_2 = m\dot{x}_1$ or $A_2(x) + uB_2(x) = mA_1(x) + muB_1(x)$. Solving for u we have

$$u = \frac{A_2(x) - mA_1(x)}{mB_1(x) - B_2(x)}, \text{ and } mB_1(x) - B_2(x) \neq 0 \text{ on L.}$$

If we substitute this u into (2) we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{-\det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix}}{mB_1(x) - B_2(x)} \\ \dfrac{-m\det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix}}{mB_1(x) - B_2(x)} \end{pmatrix}.$$

Since $L \cap S = (0,0)$ and S is defined by

$$\det\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix} = 0,$$

this control will push us from p_1 to (0,0). Hence we can move from x_0 toward (0,0) by moving along C to p_1 (if necessary) and along L from p_1 to the origin. Q.E.D.

3. EXAMPLES

It is well known that all controllable linear systems on $\mathbb{R}^n$ are stabilizable by linear feedback [14]. Our first example is a controllable linear system $\dot{x} = Ax + uB$ in $\mathbb{R}^2$. We use the ideas of this paper to move from a point $x_0 \in \mathbb{R}^2 - \{(0,0)\}$ to the origin along an integral curve of B (if necessary) and then along a straight line to the origin. It is interesting that the control u we choose to move along the straight line is a linear feedback control.

<u>Example 1</u>.

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u\begin{pmatrix}0\\ 1\end{pmatrix} = Ax + uB,\ x_0 \in \mathbb{R}_2 - \{(0,0)\}.$$

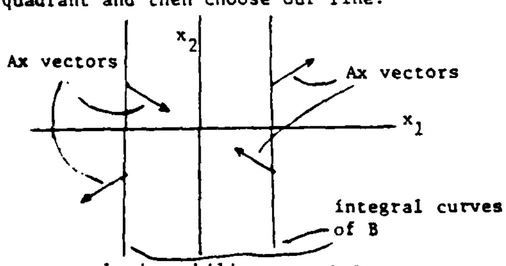
The integral curves of B are vertical lines in the (x_1, x_2) phase plane. The set S where Ax and B are linearly dependent is given by $x_2 = 0$. Since the controllability matrix $\{B, AB\}$ has rank 2 (this is equivalent to the Lie bracket $[Ax, B]$ and the vector field B being linearly independent) our system is controllable. This implies that Ax turns through an integral curve of B at the point where B intersects S.

If x_0 is in the integral curve of B through (0,0), we simply move along this curve to (0,0). If x_0 is in the open first quadrant or the positive x_1 axis we drive along the integral curve of B through x_0 until we reach the open fourth quadrant. Then we choose a line L of slope m and the control $u = \dfrac{A_2(x) - mA_1(x)}{-1}$, where

$$Ax = \begin{pmatrix}A_1(x)\\ A_2(x)\end{pmatrix} \text{ and } B = \begin{pmatrix}0\\ 1\end{pmatrix},$$

to approach the origin. Since $A_2(x) = x_1$ and $A_1(x) = x_2$, $u = -x_1 + mx_2$ is simply linear feedback.

If x_0 is the second or fourth quadrants we simply take such a straight line L and then find our control u. If x_0 is in the third quadrant or in the negative x_1 axis we first move to the second quadrant and then choose our line.



Our next example is a bilinear system.

<u>Example 2</u>. Let

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u\begin{pmatrix}1 & 0\\ 0 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= Ax + uBx,\ x(0) = x_0 \in \mathbb{R}^2 - \{(0,0)\}.$$

The set S of points where Ax and Bx are dependent is defined by the straight lines $x_1 = 2x_2$ and $x_1 = -2x_2$. The Lie bracket $[Ax, Bx]$ and the vector field Bx are linearly independent at all points in $S - \{(0,0)\}$, and this is a controllable system [9]. The vector space dimension of L_A at all points of $\mathbb{R}^2 - \{(0,0)\}$ is 2. Also the integral curves of Bx that approach the origin are the positive and negative x_1 and x_2 axes. All other integral curves of Bx intersect the set S and Theorem 2 applies.

If x_0 is not contained in the x_1 axis or the x_2 axis, then we move along an integral curve of $B\bar{x}$ to the "correct side of S" (if necessary) and choose our line L of slope m through the origin. In this case our control is given by $\dfrac{x_1 + mx_2}{mx_1 + 4x_2}$, and with it we can move toward the origin.

The results of this paper for bilinear systems in two dimensions should generalize to more complicated systems in higher dimensions. Of course instead of moving along straight lines we probably will want to move along certain smooth surfaces. The development of a theory for higher dimensional problems has begun [6],[7],[8]. Several important papers on nonlinear controllability [1],[3],[4],[5],[11],[12],[13] contain results which may prove useful in this theory.

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Abstract of

CONTROLLABILITY OF NONLINEAR HYPERSURFACE SYSTEMS

L. R. Hunt

Abstract

Consider the nonlinear system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-1} u_i(t)g_i(x(t)), x(o) = x_o \; M$$

where M is a connected real-analytic n-dimensional manifold, $f, g_1, \ldots, g_{n-1}$ are real-analytic vector fields on M, and $u_1, \ldots u_{n-1}$ are real-valued controls. We are interested in characterizing the largest open subset U of M, if any, which is reachable from x_o and which we call the region of reachability of our system from x_o. If the Lie algebra L_A generated by $f, g_1, \ldots, g_{n-1}$ and successive Lie brackets has vector space dimension n at x_o, and if $g, g_1, \ldots, g_{n-1}$ are linearly independent at some point in M, we find the region of reachability from x_o. Suppose U is the smallest open subset of M with $x_o \in \overline{U}$ so that ∂U contains the integral manifolds of the Lie algebra L'_A generated by $g_1, \ldots, g_{n-1}$ that intersect it and f assigns vectors on U which point in the direction of $\overline{U}$. Then U is the region of reachability from x_o for our system. Much of the work is involved in proving a similar result in the more general C^∞ case under the stronger assumption that $f, g_1, \ldots, g_{n-1}$ are linearly independent on the connected C^∞ n-dimensional manifold M.

Abstract of

n-DIMENSIONAL CONTROLLABILITY WITH (n-1) CONTROLS

L. R. Hunt

Abstract

Let M be a connected real-analytic n-dimensional manifold, $f, g_1, \ldots g_{n-1}$ be complete real-analytic vector fields on M which are linearly independent at some point of M, and $u_1, \ldots, u_{n-1}$ be real-valued controls. Consider the controllability of the system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-1} u_i(t) g_i(x(t)),\ x(0) = x_0 \varepsilon M.$$

Necessary and sufficient conditions are given so that this system is controllable on any simply connected domain D contained in M on which $g_1, \ldots, g_{n-1}$ are linearly independent. These conditions depend on the computation of Lie brackets at those points where $f, g_1, \ldots, g_{n-1}$ are linearly dependent.

Abstract of

SUFFICIENT CONDITIONS FOR CONTROLLABILITY

L. R. Hunt

Abstract

The problem is to find sufficient conditions for the system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \quad x(0) = x_0 \quad M$$

to be controllable. Here M is a connected C^∞ n dimensional manifold, $f, g_1, \ldots, g_m$ are *complete* vector fields C^∞ vector fields on M, *and* $u_1, \ldots, u_m$ are real-valued controls. If $m = n - 1$, M, f, $g_1, \ldots, g_{n-1}$ are real-analytic, M is simply connected, and $g_1, \ldots, g_{n-1}$ are linearly independent on M, then necessary and sufficient conditions are known. For the case of our C^∞ system with general m, we assume that the Lie algebra L_A generated by f, $g_1, \ldots, g_m$ and successive Lie brackets has constant dimension p on M and the algebra L_A' generated by $g_1, \ldots, g_m$ and successive Lie brackets has constant dimension $p' \leq p$ on M. If $p' = p$, Chow's Theorem implies controllability for a p-dimensional submanifold of M containing x_0. If $p' < p$, sufficient conditions are found involving the computation of certain Lie brackets at points where the vector field f is tangent to the integral manifolds of L_A'. Here we assume that every integral manifold of L_A' contains such a point. In many cases it is impossible for every integral manifold of L_A' to contain a point where f is tangent to it. Therefore, we illustrate a method which can yield controllability results if this occurs.

Abstract of

CONTROLLABILITY AND TRANSVERSALITY

L. R. Hunt

Abstract

Consider the system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t)g_i(x(t)), \quad x(0) = x_0 \in m,$$

where M is a C^∞ real n-dimensional manifold, $f, g_1, \ldots, g_m$ are C^∞ vector fields on M, and $u_1, \ldots, u_m$ are real-valued controls. For linear systems, it is known that the controllable systems are dense in the set of all systems on $\mathbb{R}^n$. If m = 1 and our system is nonlinear, this is not true, but it is shown that the set of systems whose reachable sets contain open subsets of M is dense in the set of systems. If $m \geq 2$, then the systems which are controllable from any point $x_0 \in M$ form a dense set, for the proper topology, in the set of all such systems. The technique used to prove the last two statements involves the use of Thom Tranversality Theory.

These results have the obvious effect in applications. In modeling by a system or in numerically solving a system, it is important to know if slight variations in a system or approximations of a system by other systems can radically change the controllability properties of the given system. In the literature these types of problems are found in the study of structural stability.

Texas Tech University | Institute for Electronic Science

Joint Services Electronics Program | Research Unit: 3

1. Title of Investigation: Nonlinear Fault Analysis
2. Senior Investigator: Richard Saeks Telephone: (806) 742-3528
3. JSEP Funds: Current $24,650
4. Other Funds:
5. Total Number of Professionals: PI's 2 (3 mo.) RA's
6. Summary:

The objective of the research program is the formulation of commutationally efficient algorithms for fault diagnosis in nonlinear electronic circuits. The resultant algorithm will be implemented in the form of two software packages: an automatic test program generator (ATPG) which runs in a mainframe computer and a fault diagnosis system (FDS) which runs in an appropriate minicomputer based automatic test set. From an algorithmic point of view the primary factor underlying the design of the fault analysis package is that the ATPG is used only once for each type of circuit in inventory while the FDS is used each time a circuit of that type fails. As such, one can justify a complex and long running ATPG but the FDS must use both computer time and storage efficiently. The goal of the proposed research program is, therefore, the formulation of fault diagnosis algorithms which can be run efficiently in this dual mode environment rather than simply the solution of the problem.

7. Publications and Activities:

A. Refereed Journal Articles

1. Saeks, R., and R.-w. Liu, "Fault Diagnosis in Electronic Circuits", Japan Jour. of Electrical Engineering, (to appear).

B. Conference Papers and Abstracts

1. Wu, C.-c., Sangiovanni-Vencentelli, A., and R. Saeks, "A Differential-Interpolative Approach to Analog Fault Diagnosis", Proc. of the 1981 IEEE Inter. Symp. on Circuits and Systems, (to appear).

2. Wu, C.-c., Nakajima, K., and R. Saeks, "Post-Test Fault Simulation with Failure Limitations", Proc. of the 24th Midwest Symp. on Circuits and Systems, (to appear).

C. Preprints

1. Wu, C.-c., and R. Saeks, "A Data Base for Symbolic Network Analysis, (submitted for publication).

2. Sangiovanni-Vencentelli, A., and R. Saeks, "Multitest Diagnosibility of Nonlinear Circuits and Systems", (submitted for publication).

D. Theses

1. Hsieh, M., "A Fault Diagnosis Algorithm for Nonlinear Circuits and Systems", Ph.D. Dissertation, Texas Tech Univ., 1980.

2. Ngo, Q.-d., "A Dual Mode Fault Diagnosis Technique for Nonlinear Analog Electronic Systems", M.S. Thesis, Texas Tech Univ., 1980.

3. Wu, C.-c., Ph.D. Disseration, Texas Tech Univ., (in preparation).

E. Conferences and Symposia

1. Saeks, R., NSF Workshop on Nonlinear Circuits and Systems, Rice Univ., Jan. 1980.

2. Saeks, R., 17th Allerton Conf. on Communications, Control and Conputing, Univ. of Illinois, Oct. 1979.

3. Chao, K.-s., 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980.

F. Lectures

Saeks, R., "Fault Analysis - The Missing Circuit Theory", Univ. of Notre Dame, Oct. 1979.

Saeks, R., "Fault Diagnosis - The Missing Circuit Theory", Duke Univ., May 1980.

Abstract of

A Differential-Interpolative Approach to Analog Fault Simulation

C.-c. Wu, A. Sangiovani-Vencentelli, and R. Saeks

A Differential-Interpolative Approach to Analog Fault Simulation

C.-c. Wu, A. Sangiovani-Vencentelli, and R. Saeks

Abstract

After a half century of neglect by the circuits and systems community the past decade has witnessed the emergence of a research effort in the analog circuit maintenance area. The various algorithms which have been thus far proposed for the analog fault diagnosis problem may naturally be subdivided into two classes termed "simulation-before-test" and "simulation-after-test". The former are commonly used in digital system test algorithms and require an automatic test program generator (ATPG) which simulates the responses of "all possible" failures. This is typically done at a maintenance depot with the simulated responses being recorded and shipped to the field where the response of the unit under test (UUT) is compared with the simulated responses to determine the failure. The major advantage of simulation-before-test is that it is ideally matched to the depot/field maintenance environment with the largest part of the computation done only once. As such, the technique is ideally suited for digital testing where the binary nature of the problem keeps the number of failures to be simulated within bounds and eliminates tolerence problems. Unfortunately, in the analog problem we must cope with a continuum of possible failures and simultaneously deal with good components which are in tolerance but not nominal. As such, a tremendous number of simulations are required by a simulation-before-test algorithm, while some type of decision algorithm is required to cope with the tolerance effects.
The purpose of the present paper is to describe a research effort directed at alleviating some of the difficulties in developing a simulation-before-test algorithm for analog fault diagnosis.

Abstract of

Fault Diagnosis in Electronic Circuits

R. Saeks and R.-w. Liu

Fault Diagnosis in Electronic Circuits

R. Saeks and R.-w.Liu

Abstract

The state-of-the-art in analog fault diagnosis is surveyed. The specific economic criteria which must be met by a viable fault diagnosis algorithm are discussed and the various fault diagnosis algorithms which have been proposed are reviewed in the context of these economic constraints.

Abstract of

Post-Test Fault Simulation with Failure Limitations

C.-c. Wu, K. Nakajima, and R. Saeks

Post-Test Fault Simulation with Failure Limitations

C.-c. Wu, K. Nakajima, and R. Saeks

Abstract

Although numerous algorithms have been proposed for fault diagnosis in analog circuits and systems they may naturally be subdivided into three classes:

i) Simulation-before-test

ii) Simulation-after-test using a single test vector

iii) Simulation-after-test using multiple test vectors

At the present time none of the three approaches has been shown to yield satisfactory performance. Simulation-before-test requires an extremely costly ATPG and some type of decision algorithm to compensate for the discretization of component parameters and tolerance effects. Simulation-after-test using a single test vector circumvents these problems but requires too many points while one must solve an extremely complex set of nonlinear equations to implement a simulation-after-test algorithm using multiple test vectors.

Unlike the simulation-before-test algorithms, simulation-after-test algorithms do not exploit any type of failure limitation assumption restricting the number of simultaneous failures. For instance, if a system contains 100 components, but it is assumed that no more than 3 fail simultaneously, such an assumption can, at least conceptually, reduce a 100 dimensional problem to a 3 dimensional problem. The open question is to find trackable methods by which to exploit such an assumption.

The purpose of the present paper is to describe a new single test vector simulation-after-test algorithm which exploits a failure limitation assumption

to bring the test point requirements into line without significantly increasing its computational complexity. The procedure:

i) is applicable to both linear and nonlinear systems

ii) tests a system up to any specified shop replacable assembly

iii) can be applied to a sub-system in-situe

iv) and is computationally efficient both with respect to ATPG and on-line requirements.

Abstract of

A Data Base for Symbolic Network Analysis

C.-c. Wu and R. Saeks

A Data Base for Symbolic Network Analysis

C.-C. Wu and R. Saeks

Abstract

Historically, symbolic network analysis has been motivated by the problems of circuit design and, as such, the emphasis has been placed on quickly and efficiently obtaining a symbolic transfer function from a given set of circuit specifications. In an operational or maintenance environment, however, one is typically given a prescribed nominal circuit and desires determine the effect of various (possibly large) perturbations thereon. This is the case in a power system where one is given a fixed network and desires to determine the effect of proposed modifications thereto. Alternatively, in the problem of analog circuit fault diagnosis one desires to simulate the effect of a number of alternative failures to compare the simulated data with the observed failure data.

In such an operational or maintenance environment numerous perturbations of the nominal circuit are studied and, as such, significant computational efficiencies can be obtained if one first generates a data base in terms of the nominal circuit parameters and then extracts the appropriate symbolic transfer function from the data base each time a different symbolic transfer is required. Of course the benefit to be achieved via such an approach is dependent on the size of the data base and the ease with which a symbolic transfer function may be retrieved therefrom.

The obvious manner in which to generate such a data base is to simply pre-compute the coefficients of all required symbolic transfer functions and store them in the data base. Retreival from such a data base is, of course, immediate but the data base may become overly large. Indeed, the number of transfer functions which must be stored is $0(\underline{k}^{\underline{p}})$ where $\underline{k}$ is the total number of potentially variable circuit parameters and $\underline{p}$ is the maximum number of circuit parameters which may vary simultaneously. An alternative approach is to store the nominal transfer function information and then use Householder's formula to compute the required symbolic transfer functions. In such a data base we need only store $0(\underline{n}^2)$ transfer functions where $\underline{n}$ is the total number of component output terminals but retreival requires $0(\underline{n}^3+p^3)$ multiplications where p is the actual number of circuit parameters which vary simultaneously. Since, in practice, $\underline{n} >> p$ the retreival process requires approximately $0(\underline{n}^3)$ multiplications and is dominated by the large dimensional matrix multiplication required by Householder's formula rather than the low dimensional inverse.

In the present paper we will formulate an alternative data base for the symbolic transfer functions which also requires $0(\underline{n}^2)$ entries, but for which retreival requires only $0(p^3)$ multiplications. Since p is typically small this is tantamount to immediate retreival.

Abstract of

Multitest Diagnosibility of Nonlinear Circuits and Systems

A. Sangiovanni-Vencentelli and R. Saeks

Multitest Diagnosibility of Nonlinear Circuits and Systems

A. Sangiovanni-Vencentelli and R. Saeks

Abstract

During the past decade a considerable research effort has been devoted to the analog fault diagnosis problem wherein one desires to locate faulty circuit components given the overall circuit response to one or more test vectors. Conceptually the process may be described by a nonlinear equation

$$y - f(\alpha,u)$$

where y represents the measured response to the test vector u given the faulty parameter vector, . Since us is know and y is a measureable quantity the fault diagnosis problem may be resolved by simply solving the above equation for α given u and y. Unfortunately, in practice, the dimension of y is limited by the number of accessible test points in the circuit and is typically smaller than the dimension of the parameter vector thereby precluding direct solution of the above equation. To alleviate this difficulty a set of test vectors; $\{u_1,u_2, \ldots ,u_n\}$; is employed yielding the set of simultaneous equations

$$y_i = f(\alpha,u_i) \; ; \; i=1,2, \ldots ,m$$

Since the parameter vector, α, is independent of the choice of test vector this process effectively increases the number of available equations without increasing the number of unknowns. More concisely, if we let $\underline{y} = \mathrm{col}(y_i)$ and $F(\alpha) = \mathrm{col}\, f(,u_i)$ the "multi-test vector" fault diagnosis problem reduces to the solution of

$$\underline{y} = F(\alpha)$$

Needless to say once this equation has been formulated its solution is amenable to standard algorithms. The problem, however, is to determine whether or not there exists a set of test vectors $\{u_1, u_2, \ldots, u_m\}$ such that equation is solvable in an appropriate sense. To this end we will formulate a diagnosibility criterion directly in terms of the function f which determines the degree to which the equation $\underline{y} = F(\alpha)$ will be solvable given an "optimal" choice of the test vectors. Since this criterion is a property of the circuit rather than the test algorithm it can therefore be used as a design aid with which to choose test points and/or to aid in designing "testable circuits".

Texas Tech University — Institute for Electronic Science

Joint Services Electronics Program — Research Unit: 4

1. Title of Investigation: Multidimensional System Theory

2. Senior Investigator: John J. Murray Telephone: (806) 742-3506

3. JSEP Funds: Current $24,650

4. Other Funds:

5. Total Number of Professionals: PI's 1 (1 mo.) RA's 1 (1/2 time)

6. Summary:

The objective of the work unit is the formulation and exploitation of a one dimensional scanning model for the digital image processing problem. For a system with an n-point raster width the resultant model is periodically time-varying and is characterized by an n-by-n matrix of rational functions in one variable rather than the classical two variable image processing model. The scanning model includes edge effects and distortion phenomena inherent in the physical scanning process. Moreover, it is amenable to the standard analytic design techniques which have been developed for multivariate systems. The major difficulty to be overcome in the approach is that one must work with large matrices (n is typically a power of two between 64 and 1024). Fortunately, these matrices are also quite degenerate and, as such, our main effort has been directed at the development of techniques for working with these large but degenerate matrices. If the matrix is degenerate is its inverse also degenerate? its spectral factors? etc.?

7. Publications and Activities

A. Refereed Journal Articles

1. Murray, J., "Some Comments on Lumped-Distributed Networks and Differential-Delay Systems", in Applications of Algebra and Algebraic Geometry to Linear System Theory, Providence, AMS, (to appear).

B. <u>Conference Papers and Abstracts</u>

1. Murray, J., "A Design Method for 2-D Recursive Digital Filters", Proc. of the 13th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove, CA., Nov. 1979, pp. 104-107.

2. Murray, J., "A New Approach to 2-D Digital Filtering", Proc. of the 24th Midwest Symp. on Circuits and Systems, Univ. of New Mexico, Albuquerque, (to appear).

C. Preprints

1. Murray, J., "A Design Method for Two-Dimensional Recursive Digital Filters", submitted for publication.

D. Theses

1. Chen, S-H, M.S. Thesis (in preparation).

E. Conferences and Symposia

1. Murray, J., 13th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove, CA., Nov. 1979.

2. Murray, J., Workshop on Multidimensional System Theory, Berkeley, CA., Nov. 1979.

A DESIGN METHOD FOR 2-D RECURSIVE DIGITAL FILTERS

PROCEEDINGS OF THE 13TH ASILOMAR CONFERENCE
ON CIRCUITS, SYSTEMS, AND COMPUTERS

PACIFIC GROVE, CA., PP. 104-107, NOV. 1979

A DESIGN METHOD FOR 2-D RECURSIVE DIGITAL FILTERS

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Abstract

A method is described for the design of two-dimensional half-plane recursive digital filters, in the form of a cascade connection of filters which are of second order in the (principal) direction of recursion, and of arbitrarily high order in the other direction. The filters thus derived are shown to be automatically stable, but yield poor responses in the vicinity of very wide or very narrow bandwidths. Some techniques for tackling these difficulties are discussed, and the results of applying these design procedures are shown.

2. INTRODUCTION

Although several excellent design procedures for two-dimensional recursive digital filters are known, the experience of classical one-dimensional filtering (both digital and analog) strongly suggests that no one technique is best for all problems likely to be encountered. In particular, it appears that a design procedure which sacrifices accuracy or implementation efficiency to simplicity of design would be of value. This is especially the case in some image processing applications, where the classical design objectives of low ripple, narrow transition bands, etc., can be sacrificed to some extent without significant loss of performance. In a continuation of some previous work [1], such a procedure is presented here. We will assume that the filter specification is given in the form of a frequency response to be approximated on the square $[-\pi,\pi]\times[-\pi,\pi]$, and that this response has quadrantal symmetry. The design will be in the form of a cascade of recursive, symmetric half-plane filters.

2. SYMMETRIC HALF-PLANE FILTERS

Although the idea of a nonrecursive symmetric half-plane filter has been known for a considerable length of time [2], and has recently been used as the basis for a very successful design algorithm [3], we will confine ourselves here to recursively implementable symmetric half-plane filters. The most general such filter has a denominator of the form

$$A(Z_1,Z_2) = 1 + \sum_{m=1}^{M} \sum_{n=-N}^{N} a_{mn} Z_1^m Z_2^n$$

The price paid for restricting the denominator in this way is that one can not approximate an arbitrary magnitude specification using such denominators alone; one must also use a one-dimensional "compensating" filter in the Z_2-direction.

We further restrict our denominators by requiring that they be products of second-order factors in Z_1; thus our "elementary" filters are of the form

$$H(Z_1,Z_2) = \frac{1}{1+p(\theta_2)Z_1 + q(\theta_2)Z_1^2} \text{.......} \quad (1)$$

where $Z_2 = e^{j\theta_2}$, and $p(\theta)$ and $q(\theta)$ are trigonometric polynomials of order N and have real coefficients (because of the assumption of quadrantal symmetry).

For recursive symmetric half-plane filters, the stability conditions are given by

$$A(Z_1,Z_2) \neq 0 \text{ for } |Z_1|<1, |Z_2| = 1.$$

In the second-order case this is equivalent to (in the notation in (1))

$$|p(\theta_2)| < 1 + q(\theta_2) < 2, \ \forall \theta_2 \text{......} \quad (2)$$

This is the stability condition with which we will work.

3. THE DESIGN PROCEDURE

We assume that a frequency specification $h(\theta_1,\theta_2)$ is given; we want to design a stable filter whose denominator is a product of factors of the form (1). (We will actually take the numerators to be of this form also). We proceed as follows:

a) For each value of θ_2, we get a one-dimensional frequency specification in θ_1:

$$h_{\theta_2}(\theta_1) = h(\theta_1, \theta_2)$$

b) For each value of θ_2, we use any of the design procedures available in one dimension to design a stable, one-dimensional recursive filter, in the form of a cascade of second-order sections, to approximate the specification $h_{\theta_2}(\theta_1)$. A single section would look like:

$$H(Z_1,\theta_2) = k(\theta_2)\frac{1 + r(\theta_2)Z_1 + s(\theta_2)Z_1^2}{1 + p(\theta_2)Z_1 + q(\theta_2)Z_1^2}$$

c) If we now ignore the factor $k(\theta_2)$ (which goes to form the one-dimensional compensating filter), what we have is a two-dimensional symmetric half-plane filter; unfortunately, however, it is transcendental as a function of Z_2. Thus the final step in the design procedure is the following:

d) Approximate $r(\theta_2)$ and $s(\theta_2)$ by trigonometric polynomials, and approximate $p(\theta_2)$ and $q(\theta_2)$ by trigonometric polynomials in such a way that the inequalities (2) continue to hold. (Since the one-dimensional filters designed in step b) are stable, the transcendental functions $p(\theta_2)$ and $q(\theta_2)$ obtained in step b) satisfy the inequalities (2) automatically).

Approximating $r(\theta_2)$ and $s(\theta_2)$ is easy, since in these cases the approximation is unconstrained. In order to approximate $p(\theta_2)$ and $q(\theta_2)$ while preserving stability, we proceed as follows:

Pick any trigonometric polynomial $P(\theta)$ of order N with the following properties:

i) $P(\theta) \geq 0 \ , \forall \theta$

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)d\theta = 1$

iii) $P(\theta)$ is a good approximation to $\delta(\theta)$ (Dirac Delta).

Then the functions

$$\hat{p}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta-\phi)p(\phi)\, d\phi$$

and

$$\hat{q}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta-\phi)q(\phi)\, d\phi$$

can be seen to satisfy the inequalities (2) (by using the properties i) and ii) above, and the fact that $p(\theta)$ and $q(\theta)$ satisfy (2)). Further, $\hat{p}(\theta)$ and $\hat{q}(\theta)$ are trigonometric polynomials of order N, since $P(\theta)$ is, and by property iii), $\hat{p}(\theta)$ and $\hat{q}(\theta)$ should be good approximations to $p(\theta)$ and $q(\theta)$, respectively. In more familiar terms, this procedure consists of truncating the Fourier series for $p(\theta)$ and $q(\theta)$ and windowing with the Fourier coefficients of $P(\theta)$. The simplest choice of window function whose Fourier transform satisfies i), ii) and iii) is probably the triangular window, whose coefficients are given by

$$w_N(n) = \begin{cases} 1 - \frac{|n|}{N+1} & |n| < N \\ 0 & |n| > N. \end{cases}$$

In this case $P(\theta)$ is the Fejer kernel.

When the weighting and windowing procedure was applied in practice (with a triangular window) the amplitude response of the resulting filters was found to deviate enormously from the desired response at points where the bandwidth was close to 0 or close to π. This deviation took the form of immensely underdamped response. Further analysis showed that this could be cured by a variation of the above procedure. This consisted of applying the truncation and windowing procedure to the functions

$$\sqrt{1 + q + p}$$

and $$\sqrt{1 + q - p}$$

to obtain two trigonometric polynomials α and β.

The functions $\dot{p}(\theta)$ and $\dot{q}(\theta)$ are then calculated from

$$\dot{p}(\theta) = \tfrac{1}{2}(\alpha^2 + \beta^2) - 1$$

and $$\dot{q}(\theta) = \tfrac{1}{2}(\alpha^2 - \beta^2).$$

It is easy to see that the $\dot{p}$ and $\dot{q}$ given by this procedure are again stable, and as shown in the next section, they yield quite satisfactory responses. However, this procedure does have the disadvantage of doubling the order of the filter in Z_2.

4. EXAMPLES

In order to make the above more concrete, an example consisting of a $90°$ fan filter will be presented. For simplicity, we will design our one-dimensional filters in Z_1 as Butterworth filters, and will develop only the case of a second-order Butterworth filter in detail. (Higher-order sections are virtually identical; only a single constant needs to be changed.)

Our ideal response is given by

$$h(\theta_1,\theta_2) = \begin{cases} 1 & |\theta_1| < |\theta_2| \\ 0 & |\theta_1| > |\theta_2| \end{cases}$$

For each fixed θ_2, this gives a one-dimensional lowpass filter in θ_1, with cutoff frequency equal to $|\theta_2|$. The bilinear transform of a second-order lowpass Butterworth filter is

$$\frac{W_c^2(1+Z_1)^2}{W_c^2+\sqrt{2}W_c+1+2(W_c^2-1)Z_1+(W_c^2-\sqrt{2}W_c+1)Z_1^2} \quad \ldots\ldots (3)$$

and in order to make the cutoff frequency of this filter equal to θ_2, the usual frequency warping relationship indicates that we must take

$$W_c = |\tan \theta_2/2| \quad \ldots\ldots\ldots\ldots (4)$$

Now (3) can be written in the form

$$k(\theta_2)\frac{1+r(\theta_2)Z_1+s(\theta_2)Z_1^2}{1+p(\theta_2)Z_1+q(\theta_2)Z_1^2}$$

where

$$k(\theta_2) = \frac{W_c^2}{W_c^2+\sqrt{2}W_c+1}$$

$$r(\theta_2) = 2$$

$$s(\theta_2) = 1$$

$$p(\theta_2) = \frac{2(W_c^2-1)}{W_c^2+\sqrt{2}W_c+1}$$

and $$q(\theta_2) = \frac{W_c^2-\sqrt{2}W_c+1}{W_c^2+\sqrt{2}W_c+1}$$

and W_c is given by (4) in all of the above formulas.

Thus, in order to design a filter which is of order 2 in Z_1 and order N in Z_2, it is necessary only to find the first N Fourier coefficients of each of the functions

$$\sqrt{1+q+p} = \frac{2W_c}{\sqrt{W_c^2+\sqrt{2}W_c+1}}$$

and $$\sqrt{1+q-p} = \frac{2}{\sqrt{W_c^2+\sqrt{2}W_c+1}}$$

and to window these coefficients with a triangular window. The polynomials $\hat{p}(\theta)$ and $\hat{q}(\theta)$ may then be easily calculated by use of the formulas in section 3. The one-dimensional compensating filter can be designed using any standard one-dimensional design procedure. The amplitude response of a second-order filter with N=8 is shown in Fig. 1, and that of an eighth-order filter with N=20 is shown in Fig. 2. In each of these filters, the one-dimensional compensating filter is a FIR filter of order N.

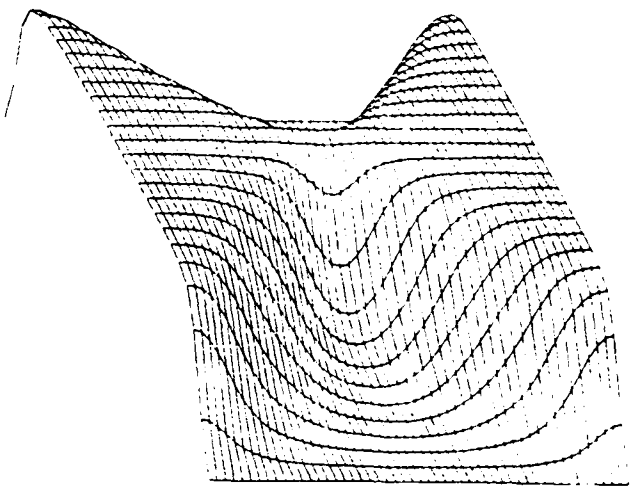

Fig. 1. M = 2; N = 8

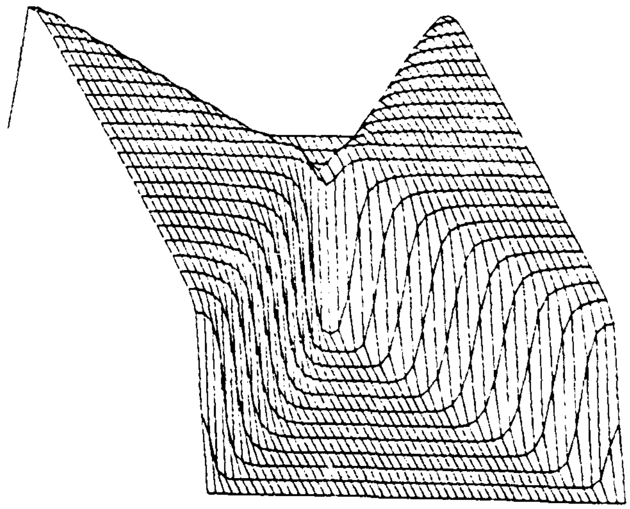

Fig. 2. M = 8; N = 20

5. CONCLUSIONS

A quick, simple method for designing a class of two-dimensional recursive digital filters has been presented. Although the designs achieved using this method are not optimal, they are guaranteed to be stable (apart from possible numerical error), and can yield respectable results for sufficiently high orders. The computation time required is somewhat greater than that required for the calculation of (M+1)(N+1) Fourier coefficients, where M is the order of the filter in z_1, and N is the order in z_2.

References:

[1] J. Murray, "Symmetric Half-Plane Filters", Proc. 20th Midwest Symp. Circ. and Syst., Lubbock, 1977.

[2] J.H. Justice and J.L. Shanks, "Stability Criterion for N-Dimensional Digital Filters", IEEE Trans. Autom. Contr., Vol. AC-18, pp. 284-286, June 1973.

[3] D.B. Harris, Dissertation, Elec. Engrg. Dept., M.I.T., 1979.

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ANNUAL REVIEW OF RESEARCH UNDER THE JOINT SERVICES ELECTRONICS --ETC(U)
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Abstract of

A NEW APPROACH TO 2-D DIGITAL FILTERING

John Murray

Abstract

A new approach to two-dimensional digital filtering is presented. It is based on a periodically time-varying model which accurately reflects the scanning process inherent in most recursive multidimensional signal processing. Such models are essentially equivalent to multi-input, multi-output, one-dimensional time-invariant systems, and therefore permit the application of classical techniques to design and analysis problems. Two further advantages of this approach are its flexibility and the fact that it by-passes the problem of boundary conditions.

Abstract of

A DESIGN METHOD FOR 2-D RECURSIVE DIGITAL FILTERS

John Murray

Abstract

A method is described for the design of two-dimensional half-plane recursive digital filters, in the form of a cascade connection of filters which are of second order in the (principal) direction of recursion, and of arbitrarily high order in the other direction. The filters thus derived are shown to be automatically stable, but yield poor responses in the vicinity of very wide or very narrow bandwidths. Some techniques for tackling these difficulties are discussed, and the results of applying these design procedures are shown.

Abstract of

A DESIGN METHOD FOR TWO-DIMENSIONAL RECURSIVE DIGITAL FILTERS

J. Murray

Abstract

A method for designing two-dimensional, symmetric half-plane recursive digital filters is presented: a filter is first designed as a parameterized family of one-dimensional filters; a simple approximation is then used to find a rational, stable two-dimensional filter. Some advantages and disadvantages of the method are discussed, and several examples are given.

Abstract of

SOME COMMENTS ON LUMPED-DISTRIBUTED NETWORKS AND DIFFERENTIAL-DELAY SYSTEMS

J. Murray

Abstract

An analytic approach to the similarities and differences between lumped-distributed networks and differential-delay systems is presented. This approach is based on the calculation of the spectrum of a commutative Banach Algebra of appropriate convolution operators; it is shown that this calculation naturally involves the two complex variables approach of lumped-distributed circuit theory, and thus gives a link between this and the convolution approach. Further, when this spectrum is drawn, it gives some intuition for the systems in question; for example, it becomes clear that the passive synthesis problem is two-dimensional, while the stability problem is one-dimensional, unless delays of arbitrary length are considered. It also shows that the analog of the Nyquist criterion in this situation involves two "winding numbers".

Texas Tech University | Institute for Electronic Science

Joint Services Electronics Program | Research Unit: 5

1. Title of Investigation: Detection and Estimation in Imagery
2. Senior Investigator: Telephone: (806) 742-3500
3. JSEP Funds: Current $24,650
4. Other Funds: ________
5. Total Number of Professionals: PI's 2 (1 mo.) RA's 1 (1/2 time)
6. Scientific Objective:

Although the estimation problem in image processing is conceptually similar to the estimation problem which arises in a communications context, in reality the two problems have little in common. In particular, the optical noise phenomena encountered in image processing are highly nonlinear while the immense quantity of data associated with an image (typically ranging from 1/4 Megabyte to 16 Megabytes per frame) precludes the use of many classical detection and estimation algorithms. The purpose of the present work unit is to develop an alternative class of estimation algorithms designed to cope with the reality of the image processing problem.

7. Publications and Activities

A. Conference Papers and Abstracts

1. Froehlich, G., Walkup, J., and T.F. Krile, "Some Effects of Signal-Dependent Noise on Estimator Structures", 1980 OSA Meeting, Chicago, Oct. 1980.

B. Preprints

1. Froehlich, G., Walkup, J., and T. Krile, "Estimation in Signal Dependent Film-Grain Noise", submitted for publication.

2. Froehlich, G., Walkup, J., and T. Krile, "Multiple Parameter Estimation in Signal-Dependent Noise" submitted for publication.

C. Theses

1. Froehlich, G., "Estimation in Signal Dependent Noise", Ph.D. Dissertation, Texas Tech Univ., 1980.

2. Kasturi, R., Ph.D. Thesis, (in preparation).

D. Conferences and Symposia

1. Walkup, J.F., Krile, T., Froehlich, G., and R. Kasturi, 1980 OSA Conf., Chicago, Oct. 1980.

2. Walkup, J.F., and T.F. Krile, "Gordon REsearch Conf. on Holography and Optical Information Processing", Ventura, CA., June 1980.

ABSTRACT OF

SOME EFFECTS OF SIGNAL-DEPENDENT NOISE ON ESTIMATOR STRUCTURES

Gary K. Froehlich, John F. Walkup and Thomas F. Krile

Abstract

Optimal estimators are derived for a very general measurement model which can be made to include (or exclude) a signal-dependent noise term. The estimators include minimum mean-square error (MMSE), maximum *a posteriori* (MAP), and maximum likelihood (ML) estimators. Then, for the specific case of photographic film-grain noise, the sensitivity of the estimators' structures to the strength of the signal-dependent noise term is described. In addition, the performance of each estimator is found by simulation, and compared with the performance under various mismatched conditions wherein certain *a priori* assumptions about the signal statistics are violated.

ABSTRACT OF

ESTIMATION IN SIGNAL-DEPENDENT FILM-GRAIN NOISE

G. Froehlich, J. Walkup and T. Krile

Abstract

Optimal estimators are derived for a signal-dependent film grain noise model, and the effect of signal-dependence on the estimators' structures is investigated. Due to the mathematical complexity of these optimal estimators, various suboptimal estimators are proposed. Computer simulations are then presented which compare the optimal and suboptimal estimators with regard to mean-square estimation error, sensitivity to signal-dependence, and robustness (with respect to the *a priori* probability density of signal).

ABSTRACT OF

MULTIPLE PARAMETER ESTIMATION IN SIGNAL-DEPENDENT NOISE

G. Froehlich, J. Walkup, and T. Krile

Abstract

A general model incorporating signal-dependence noise is introauced. Joint maximum *a posteriori* (MAP) and joint maximum likelihood (ML) estimators are derived, followed by a discussion of the effects of statistical coupling between adjacent measurements and nonstationarity on the part of the signal. An alternate approach, using state-space methods, is also discussed.

Texas Tech University — Institute for Electronic Science

Joint Services Electronics Program — Research Unit: 6

1. Title of Investigation: Pointing and Tracking
2. Senior Investigator: Thomas G. Newman Telephone: (806) 742-2571
3. JSEP Funds: Current $24,650
4. Other Funds: Current $19,983*
5. Total Number of Professionals: PI's 1 (1 mo.) RA's 1 (1/2 time)
6. Summary:

The goal of the program is the formulation of a group theoretic approach to the pointing and tracking problem. Typically, one is given a scene containing several objects moving in different directions and at different velocities; say an airplane, a missile, and a cloud, all in front of a fixed background. The solution of the pointing and tracking problem requires that we distinguish between the various objects and simultaneously track the motion of a prescribed object.

Although the motion of an object as seen in the plane of a camera (radar, sonor, etc.) can clearly be characterized by a pair of Cartesian coordinates, this results in an extremely complex equation of motion for the image of a rigid body which is, in fact, moving with six degrees of freedom in three space. Rather, we choose to model the motion of the image by a Lie group (of tr.nslations, rotations, magnifications) which results in a greatly simplified equation of motion.

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*ARO Contract for a study of the numerical problems associated with the extraction of multiple moving paterms from imagery.

7. Publications and Activities

A. Conference Papers and Abstracts

1. Newman, T.G., and D.A. Davis, "Lie Theoretic Methods in Video Tracking", Proc. of the MICOM Workshop on Imaging Trackers and Autonomous Acquisition Applications for Missle Guidance, Redstone Arsenal, Nov. 1979, pp. 166-174 (GACIAC-PR-80-01).

B. Preprints

1. Newman, T.G., "Lie Groups and Lie Algebras in Video Tracking", submitted for publication.

2. Fredricks, G.A., and T.G. Newman, "Method in Differential Geometry with Application to Video Tracking", submitted for publication.

3. Fredricks, G.A., "Canonical Forms for Nondegenerate Second Order Linear Partial Differential Operators and Equations" submitted for publication.

C. Theses

1. Zlobec, L. "Pattern Matching by Means of Adaptive Control", M.S. Thesis, Texas Tech Univ., May 1980.

2. Demus, D.A., M.S. Thesis, Texas Tech Univ., (in preparation).

D. Conferences and Symposia

1. Newman, T.G., Inter. Symp. on Ill-Posed Problems: Theory and Practice, Univ. of Delaware, Oct. 1979.

2. Newman, T.G., Workshop on Imaging Trackers and Autonomous Acquisition Applications for Missle Guidance, Redstone Arsenal, Nov. 1979.

E. Lectures

1. Newman, T.G., "An Inverse Problem Related to Video Tracking", Univ. of Delaware, Oct. 1979.

2. Newman, T.G., "Application of Lie Theory to Viedo Tracking", Invited Address at the Advanced Technology Center, Voight Corp., Dec. 1979.

LIE THEORETIC METHODS IN VIDEO TRACKING

Thomas G. Newman and David A. Demus

LIE THEORETIC METHODS IN VIDEO TRACKING

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Department of Mathematics
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ABSTRACT

Consider a 2-dimensional image in which objects are in motion through trajectories describable by translation (both horizontal and vertical), rotation, and magnification. The trajectory of such an object can be completely described by a 4-vector of parameters $\lambda(t)=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ which determine the velocities with respect to the four possible motions. If the data at time t and position x in the view plane is written as F(t,x), then we can show that

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{4} \lambda_i(t) X_i F,$$

where X_1, X_2, X_3 and X_4 are certain (known) differential operators associated with the group of motions.

The derivatives appearing above may be evaluated numerically at various points in a given time slice to produce a system of linear equations which may be solved for the motion parameters. Evaluation at points within a moving rigid body leads to a vector of motion parameters unique to that particular body. In principle, at least, this technique permits application to tracking as well as segmentation of images based on relative motion of various objects.

The paper concludes by presenting the results of having implemented the above method on digitized video images.

INTRODUCTION

A complex three dimensional scene may contain an arbitrary number of objects, each of which is in motion relative to a stationary background. The trajectories of the various objects may or may not be the same. When such a scene is projected on a viewing plane (for example, through the use of a television camera), the various objects appear as moving regions which vary in time in a complex fashion as a result of their actual trajectories in space. Variations due to certain trajectories, such as rotation about a line parallel to the image plane, are not readily predictable. Previously unseen patches of the surface of an object may be brought into view for the first time, while others may disappear. In addition, a near object may pass between the camera and a distant object, occluding all or part of the latter.

The situation is further complicated in case mobility is provided at the camera. Motion of the camera results in an opposing change in the apparent motion of all of the objects in the scene, including background. In many applications camera mobility is desirable or even necessary. For instance, in tracking applications the motion of the camera is required to stabilize a particular portion of the scene within the viewing field. Although this may in general be impossible, as with the rotating objects mentioned above, a fair degree of stabilization with respect to position, size, and orientation can be achieved.

In the following sections we present a model for describing motion in images which is valid in a large number of practical applications and which is a reasonable approximation in many others. A novel feature is that camera motion and relative motion of objects within a scene are both described within the model.

THEORETICAL MODEL

Let G be a Lie group of transformations on an analytic manifold M. Suppose G has dimension n while M has dimension m. Let x and y denote the coordinates of elements f and g in G, respectively, in a patch containing the identity element e of G. Also, let p denote coordinates of an element u of M in some patch in M. We may then express the coordinates z of the product h = fg and the coordinates q of the element v = gu, relative to suitable patches, by means of analytic functions

$$z = J(x,y) \tag{1}$$

$$q = K(y,p) \tag{2}$$

K and J are vector-valued, having values in n-dimensional space R^n or C^n and m-dimensional space R^m or C^m. Hereafter we shall assume that these underlying spaces are real. We denote the ith component of J by J_i and the jth component of K by K_j.

In order to define the Lie algebra of G we first introduce real-valued maps on G by

$$P_{ij}(x) = \frac{\partial J_i}{\partial y_j}(x,y)\Big|_{y=e}, \tag{3}$$

where i and j each range from 1 to n. The cross-section $P*_j$, which consists of the P_{ij} as i ranges from 1 to n, and j is fixed, may be thought of as a vector field in R^n. Such a vector field attaches to a point x the vector $P*_j(x)$. As such, $P*_1, P*_2, \ldots, P*_n$ form a basis for the tangent space at the point x [1,2]. In view of the correspondence between elements f in G and the coordinates in R^n, the tangent vectors are implicitly attached to the elements of G.

In terms of the above vector fields we may express the infinitesimal transformations of G by defining, for each $j = 1,2,\ldots,n$,

$$X_j = \sum_{i=1}^{n} P_{ij}(x)\frac{\partial}{\partial x_i} . \tag{4}$$

The differential operators so defined are to be considered as linear operators on the space of analytic functions on G, or, more generally, on the space of differentiable functions on G. The Lie algebra of G is simply the n-dimensional vector space consisting of all linear combinations of these operators, and will be denoted by L(G) [2].

Now it is a surprising and useful fact that the Lie algebra of G may be defined in terms of its actions on the manifold M. Analogous to (3) we define

$$Q_{\alpha j}(p) = \frac{\partial K_\alpha}{\partial y_j}(y,p)\Big|_{y=e} \tag{5}$$

for $\alpha = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Finally, as in (4) above we set

$$X'_j = \sum_{\alpha=1}^{m} Q_{\alpha j} \frac{\partial}{\partial p_\alpha} . \tag{6}$$

The operators $X'_1,\ldots,X'_n$ span a Lie algebra L'(G) which is also of dimension n. Note that these operators act on functions defined on the manifold M.

Many interesting relationships may be shown to hold between the two representations of the Lie algebra of G as given above. However, the following property is of immediate interest to our application:

<u>Theorem 1</u>: Let $f: M \to R$ be differentiable and define $F: G \times M \to R$, in terms of coordinates by

$$F(x,p) = f(K(x,p)). \tag{7}$$

Then for each $j = 1,2,\ldots,n$ we have

$$X_j F = X'_j F. \tag{8}$$

Proof: First we shall show that for each $j = 1,2,\ldots,n$ we have

$$X_jK = X_j'K. \tag{9}$$

We note that from the action of G on M we obtain

$$K(J(x,y),p) = K(x,K(y,p)) \tag{10}$$

for all x,y and p in suitable coordinate patches. Application of the operator

$$\frac{\partial}{\partial y_i}\Big|_{y=e}$$

to both sides of (10) gives

$$\frac{\partial K_\alpha(J(x,y),p)}{\partial y_i}\Big|_{y=e} = \sum_{k=1}^{n} \frac{\partial J_k(x,y)}{\partial y_i}\Big|_{y=e} \cdot \frac{\partial K_\alpha(x,p)}{\partial x_k} =$$

$$\sum_{k=1}^{n} P_{kj}(x)\frac{\partial K_\alpha(x,p)}{\partial x_k} = X_jK_\alpha(x,p)$$

for the left hand side and

$$\frac{\partial K_\alpha(x,K(y,p))}{\partial y_j}\Big|_{y=e} = \sum_{\beta=1}^{m} \frac{\partial K_\beta(y,p)}{\partial y_i}\Big|_{y=e} \frac{\partial K_\alpha(x,p)}{\partial p_\beta} =$$

$$\sum_{\beta=1}^{m} Q_{\beta j}(p)\frac{\partial K_\alpha(x,p)}{\partial p_\beta} = X_j'K_\alpha(x,p)$$

on the right hand side. From this it follows that $X_jK = X_j'K$ as desired. Now setting $q = K(x,p)$ and performing a computation similar to that above, we find that

$$X_jF(x,p) = \sum_{\alpha=1}^{m} X_jK_\alpha(x,p) \cdot \frac{\partial f(q)}{\partial q_\alpha}$$

and that

$$X_j'F(x,p) = \sum_{\alpha=1}^{m} X_j'K_\alpha(x,p) \cdot \frac{\partial f(q)}{\partial q_\alpha} .$$

The result of the theorem follows immediately from this and our preliminary result.

Now let us consider a curve $t \to g(t)$ in G satisfying $g(0) = e$. In terms of a coordinate patch at e, g(t) may be described by a curve x(t) in R^n satisfying $x(0) = 0$. We shall consider the case in which x(t) is given as the solution of an evolution equation of the form

$$\dot{x}(t) = \sum_{i=1}^{n} \lambda_i(t) P_{*i}(x(t)), \; x(0) = 0, \tag{11}$$

where $P_{*1}, \ldots, P_{*n}$ are cross-sections of the array of functions given by (3), and the control functions $\lambda_1(t), \ldots, \lambda_n(t)$ are suitable continuous functions. The latter are the parameters of motion, and have the characteristics associated with velocity, thereby providing a basis for the continuity assumption.

Now let p denote the coordinates of a point u in some coordinate patch. For a differentiable map $f: M \to R$ we may define $H: R \times M \to R$ by setting

$$H(t,p) = f(g(t)u). \tag{12}$$

We recognize that $H(t,p) = F(x(t),p)$ where F is the extension of f to $G \times M$ as in Theorem 1 above. From the point of view of application, if we regard $f: M \to R$ as an image, then H(t,p) represents the moving image obtained by translation due to the curve g(t). We may now present our main result.

Theorem 2: In the context described above we have

$$\frac{\partial H}{\partial t} = \sum_{i=1}^{n} \lambda_i(t) X_i' H. \tag{13}$$

Proof: We have

$$\frac{\partial H}{\partial t}(t,p) = \frac{\partial F(x(t),p)}{\partial t} = \sum_{j=1}^{n} \dot{x}_j(t) \frac{\partial F}{\partial x_j}(x(t),p) =$$

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_i(t) P_{ji}(x(t)) \right) \frac{\partial F}{\partial x_j}(x(t),p) =$$

$$\sum_{i=1}^{n} \lambda_i(t) \left(\sum_{j=1}^{n} P_{ji}(x(t)) \frac{\partial F}{\partial x_j}(x(t),p) \right) =$$

$$\sum_{i=1}^{n} \lambda_i(t) X_i F(x(t),p).$$

By Theorem 1 we have $X_i F = X_i' F$. But clearly $X_i' F(x(t),p) = X_i' H(t,p)$, so that

$$\frac{\partial H}{\partial t}(t,p) = \sum_{i=1}^{n} \lambda_i(t) X_i' H(t,p),$$

as desired.

We should observe that the results above are presented as local properties which hold in suitable neighborhoods and appear to be highly coordinate dependent. As a matter of fact, though we shall not attempt to prove it here, the underlying vector fields continue globally throughout both G and M to give corresponding global analogues of these theorems.

The primary importance of Equation (13) lies in the fact that it gives a linear equation in the control parameters $\lambda_1, \ldots, \lambda_n$ with coefficients that are in principle observable, since the values $H(t,p)$ constitute the data.

In the next section this result will be applied to the problem of tracking spatial objects through the use of two-dimensional projections.

APPLICATIONS TO VIDEO TRACKING

The control system for the Real-Time Videotheodolite (RTV) permits four basic motions of the camera [3]. These are azimuth, elevation, electronic rotation of the view plane, and lens zoom. When the effects of these motions on the viewing plane are scrutinized, we see that they correspond, respectively, to horizontal translation, vertical translation, rotation, and magnification - at least to a satisfactory degree of approximation. Moreover, inspection of a number of real images reveals that a surprisingly large number (but not all) motions of spatial objects, when projected on the viewing plane, are likewise well approximated by these four motions in the plane.

Thus with only a mild apology we restrict our attention in what follows to the group G generated by horizontal and vertical translations, rotation, and magnification. The corresponding generators for the Lie algebra of G are as follows:

$$X_1 = \frac{\partial}{\partial x} \qquad (14a)$$

$$X_2 = \frac{\partial}{\partial y} \qquad (14b)$$

$$X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad (14c)$$

$$X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad (14d)$$

In these equations we are using x and y as coordinates in the view plane $M = R \times R$ and have represented the infinitesimal transformations as they act on M.

Let us note that in the theorems of the previous section it was assumed that the trajectories of all of the points of M were derived from the same evolution equations. However, for complex scenes we find that various objects may be present which have different trajectories. A little reflection reveals, nevertheless, that the conclusions of Theorem 2 remains valid as long as we avoid the boundaries between objects or regions having different trajectories. In the present context, we may paraphrase the results of Theorem 2 as follows:

Theorem 3: Let H(t,x,y) be a time varying two dimensional image. Within the interior of each object in the image which is moving along a G-trajectory, we have

$$\frac{\partial H}{\partial t} = \sum_{i=1}^{4} \lambda_i(t) X_i H, \tag{15}$$

where $\lambda_1,\ldots,\lambda_4$ are continuous functions and $X_1,\ldots,X_4$ are given in (14).

Upon evaluation of the various derivatives appearing in (15) at each point of a suitable grid, within a given time slice, we obtain a system of linear equations which may be solved for the parameters of motion, $\lambda_1,\ldots,\lambda_4$. In the example to be presented, a 3×3 grid was used.

A sequence of digitized video images showing the launch of a Hawk missile were obtained from the U.S. Army White Sands Missile Range. The images were trimmed to 128×128 pixels from full frame interlaced video in which each raster line was sampled 512 times.

One of the frames is shown in the upper left of the illustration below. Of noteworthy interest, we mention the "cold plume" region (lower left) which can be seen billowing out behind the missile. Although hardly discernible, the foreground contains several buildings and other ground clutter.

By evaluation of Equation (15) at each point of a 3×3 neighborhood of each pixel, nine equations in the four parameters $\lambda_1,\ldots,\lambda_4$ were obtained. In the upper right frame of the illustration, we see the results of scaling the horizontal translation component, λ_1 for display. The effect of image noise and truncation error is apparent from the rapid transition from white to black in this view. This component of the velocity profile was passed through a median filter to obtain the image shown in the lower left of the illustration. Finally, in the lower right we see the results of thresholding, about $\lambda_1 = 0$. In this image the dark region indicates points which are at rest relative to the camera (which was apparently successfully tracking the missile), while the white regions appear to be moving with respect to the camera.

It is interesting to note that the cold region of the plume has been correctly classified with the background, while the hot region of the plume appears to be moving with the missile.

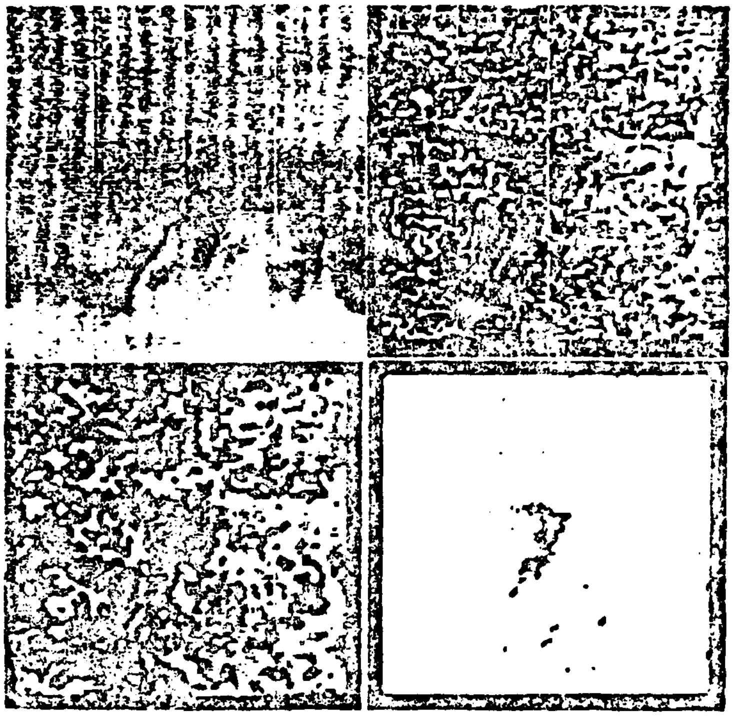

Figure 1. Processing the launch of a Hawk missile.

Similar results were obtained with other parameters and with other images. These results are encouraging, although the numerical methods employed are clearly too susceptible to noise and truncation. Better computational procedures are being explored, including one technique which is based on integration rather than differentiation.

SUMMARY AND CONCLUSIONS

We have developed a fundamental equation satisfied by moving images which uses Lie theory to determine the trajectories of various objects within an image. The theory has been implemented on real data with some success. While the implementation suffers from the effects of random noise and truncation errors, the results obtained have shown sufficient success as to be encouraging. We feel that the computations can be greatly improved by the incorporation of better numerical methods.

ACKNOWLEDGEMENT

This research was conducted under the auspices of the Joint Services Electronics Program at Texas Tech University and in collaboration with the Advanced Technology Office, Instrumentation Directorate, U.S. Army White Sands Missile Range. Support was provided by the Office of Naval Research under contract N00014-76-C-1136.

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ABSTRACT OF

LIE GROUPS AND LIE ALGEBRAS IN VIDEO TRACKING

T.G. Newman

Abstract

Motion of objects in time-varying images can sometimes be described by the action of a group of transformations on the image plane, regarded as a manifold. Moreover, the transformation groups occurring in applications can generally be described analytically in terms of a finite number of parameters; that is to say, they are Lie groups. In this situation we show that that data satisfies a linear partial differential equation in which the parameters of motion appear as linear coefficients. More or less standard numerical methods permit these parameters to be determined.

The parameters of motion determined as indicated above may be regarded as a velocity profile. This profile has the useful property of being spatially constant for each moving object in the image. In principle, at least, this permits detection and tracking of various objects having different trajectories.

Following development of the appropriate theory, the paper concludes by presenting the results of applying the technique to a number of real images in the form of digitized video.

ABSTRACT OF

RESULTS IN DIFFERENTIAL GEOMETRY WITH APPLICATION TO VIDEO TRACKING

G.A. Fredricks and T.G. Newman

Abstract

We present some results concerning the interplay between various vector fields arising from the action of a Lie group on a smooth manifold. Although the proofs are elementary, the results are both surprising and applicable. In the last section we show that the fundamental partial differential equation in the main theorem is at the mathematical foundation of video tracking.

ABSTRACT OF

CANONICAL FORMS FOR NONDEGENERATE SECOND ORDER LINEAR PARTIAL DIFFERENTIAL OPERATORS AND EQUATIONS

G.A. Fredricks

Abstract

The classical canonical forms theorems for second order linear partial differential operators and equations in two variables are generalized to n variables for nondegenerate operators. These generalizations are geometric, involving the Riemann curvature tensor and the conformal curvature tensor of Weyl and Schouten. A Sylvester Theorem for symmetric matrices with smooth entries is also proved.

Texas Tech University | Institute for Electronic Science

Joint Services Electronics Program | Research Unit: 7

1. Title of Investigation: Image Processing System
2. Senior Investigator: John F. Walkup Telephone: (806) 742-3500
3. JSEP Funds: Current $33,025+
4. Other Funds: Current $10,000*
5. Total Number of Professionals: None#
6. Summary:

The purpose of the work unit is to partially fund the purchase of an image processing system to be used in support of the research associated with work units 4, 5, and 6. Each of these work units deals with an aspect of the image processing problem and in each case experimental validation of the various theoretical investigations is required.

i. Budget: Total funding for the purchase of the image processing system will be approximately $247,000 derived over the three year contract using capital equipment funds derived from this work unit, work units 4, 5, and 6 together with College of Engineering and University matching funds. We have also negotiated an agreement with the university for financing the system with the equipment being ordered at the beginning of the contract period but billed to ONR in three separate federal fiscal years as required by the contract.

*State of Texas matching funds for this work unit.

#This work unit represents a request for capital equipment funds. Personnel using the equipment will be supported by work units 4,5, and 6.

+In addition to this supplemental, capital equipment funds from regular work units 4,5, and 6 will be used for the purchase of the image processing system in the amount of $31,000 for the year.

ii. Host Computer: We have recently completed the purchasing process for the host computer for the image processing system and submitted a purchase order to Digital Equipment Corp. for a "Unibus VAX". This is essentially a VAX 11/780 CPU with PDP 11/70 peripherals. As such, we obtain the power and expandability of the VAX CPU at a price close to that of the PDP 11/70. The VAX CPU will have a 1 1/4 MB of random access memory, two 28MB disks, and a 1600 bpi tape drive.

iii. Image Display/Array Processor: A Comtal/3M Vision 120 display system has been ordered. The system includes memory for 3 image displays and four graphics planes, full arithmetic capability and a high level firmware operating system as well as interfaces to the VAX 11/780.

iv. Delivery: Both the computer and display are scheduled for delivery in the late spring or early summer of 1981 and, as such, we expect to have the system up and operating during the summer of 1981.

7. Publications and Activities:

A. Conferences and Symposia

1. Saeks, R., 1980 ACM Computer Graphics Conference (SIGGRAPH/80), Seattle, July, 1980.

Texas Tech University — Institute for Electronic Science

Joint Services Electronics Program — Research Unit: 8

1. Title of Investigation: Director's Discretionary Fund
2. Senior Investigator: R. Saeks Telephone: (806) 742-3528
3. JSEP Funds: Current $19,075
4. Other Funds:
5. Total Number of Professionals: To be Determined
6. Summary:

During the past year the directors discretionary fund has been used to complete work on a large scale systems work unit from the 1978/79 JSEP program (mainly running examples of the theory which was developed previously and preparing publications), to initiate work on a new approach to integrated circuit design, and to begin a preliminary investigation of the potential for parallel processing in system theory.

7. Publications and Activities:

A. Refereed Journal Articles

1. Karmokolias, C., Portnoy, W., and R. Saeks, "Optimal Selection of IC Fabrication Parameters", Inter. Jour. of Circuit Theory and its Applications (to appear).

B. Conference Papers and Abstracts

1. Green, B., Saeks, R., and K.S. Chao, "Continuation Algorithms for the Eigenvalue Problem", Proc. of the 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980, p. 775, (abstract only).

2. Iyer, A., and R. Saeks, "Numerical Implementation of a Continuation Algorithm for the Eigenvalue Problem", 1980 IEEE Inter. Conf. on Circuits and Computers, Port Chester, Oct. 1980, pp. 437-440.

C. Preprints

1. Green, B., Saeks, R., Chao, K.S., and A. Iyer, "Continuation Algorithms for the Eigenvalue Problem", subjitted for publication.

D. Theses

1. Iyer, A., "Numerical Implementation of a Continuation Algorithm for the Eigenvalue Problem", M.S. Thesis, Texas Tech Univ., 1980.

E. Conferences and Symposia

1. Saeks, R., 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980.

2. Iyer, A., and R. Saeks, 1980 IEEE Inter. Conf. on Circuits and Computers, Port Chester, Oct. 1980.

NUMERICAL IMPLEMENTATION OF A CONTINUATION ALGORITHM FOR THE EIGENVALUE PROBLEM

A. Iyer and R. Saeks

NUMERICAL IMPLEMENTATION OF A CONTINUATION ALGORITHM FOR THE EIGENVALUE PROBLEM

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ABSTRACT: An algorithm for the solution of the eigenvalue problem for a continuous parameterized family of sparse matrices is presented. A continuous LU (or LR) algorithm is implemented recursively. The sparsity of the given matrices is preserved throughout the numerical process.

I. Introduction

In recent years a number of stability tests for linear systems have been proposed which require the evaluation of the eigenvalues of a continuously parameterized family of sparse matrices for their implementation. Most notably of these are the multivariate Nyquist test of MacFarlane, et al.,[7] the application of the multivariate Nyquist test in an interconnected systems context,[3] and a "root locus like" formulation for interconnected systems.[3,4] Typically, one employs a classical eigenvalue package at a sequence of parameter values, possibly with special software to exploit the common sparsity pattern of the various matrices. Alternatively, one can compute the eigenvalues at an initial parameter value and "continue the result" by integrating an appropriate differential equation whose trajectors define the eigenvalue loci of the given family of matrices. The most common such differential equation[5] for a continuously parameterized family of matrices, $M(r)(=M)$, takes the form

$$\frac{d\lambda_i}{dr} = \frac{\langle \frac{dM}{dr}e_i, f_i \rangle}{\langle e_i, f_i \rangle} \tag{1.1}$$

$$\frac{de_i}{dr} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\langle \frac{dM}{dr} e_i, f_j \rangle}{(\lambda_i-\lambda_j)\langle e_j, f_j \rangle} e_j \tag{1.2}$$

$$\frac{df_i}{dr} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\langle f_i, \frac{dM}{dr}e_j \rangle}{(\bar{\lambda}_i-\bar{\lambda}_j)\langle f_j, e_j \rangle} f_j \tag{1.3}$$

where λ_i is the ith eigenvalue of M, e_i is the corresponding eigenvector, $\bar{\lambda}_i$ is the complex conjugate of λ_i, and f_i is the eigenvector of the matrix M^* associated with the eigenvalue $\bar{\lambda}_i$ of M^*. Here, all vectors and matrices nat be complex, $\langle \; , \; \rangle$ denotes the complex inner product, "*" denotes the complex conjugate/transpose, and the set of differential equations 1.1 through 1.3 are well defined whenever M has distinct eigenvalues.[5]

The major difficulty with the above described "continuation algorithm" is that the array of eigenvectors for a sparse matrix is typically non-sparse.[10] As such, the computational benefits of working with the sparse matrix M will be lost if one attempts to integrate equations 1.1 through 1.3. This is mostly readily illustrated by letting $S = \text{col}(f_i^T)$ be the n by n matrix whose rows are defined by the eigenvectors of M^*. Then, assuming that the eigenvectors are properly normalized $S^{-1} = \text{row}(e_i)$ is an n by n matrix whose columns are the eigenvectors of M, allowing us to transform the simultaneous differential equations 1.1 through 1.3 into a matrix differential equation[5] in the form

$$\dot{S} = W[S\dot{M}S^{-1}, SMS^{-1}]S \tag{1.4}$$

$$T = SMS^{-1} \tag{1.5}$$

where $W[\;,\;]$ is in an appropriate matrix valued function of two matrix valued variables (which defines the coefficients of 1.3) and T is a diagonal matrix of eigenvalues. As such, the simultaneous differential equations 1.1 through 1.3 may be viewed as a differential equation in the similarity transformation which diagonalizes M. Unfortunately, this similarity transformation is typically non-sparse, even when M is sparse and therefore fails to yield a computationally viable continuation algorithm.[10]

This difficulty is alleviated in the present paper by formulating a continuation algorithm around similarity transformations which triangularizes, rather than diagonalizes, M. Such similarity transformations preserve the sparseness of M while the eigenvalues of M are given by the diagonal entries of the resultant triangular matrix. In the following section we formulate a continuation algorithm, which may be viewed as a continuous LU algorithm.[9] This algorithm employs a unit upper triangular matrix to transform M into a lower triangular form.

In the continuation algorithm, the required differential equation takes the form of 1.4 where $W[\;,\;]$ is the solution of an appropriate <u>triangular commutant equation</u>

$$^u[K] = {}^u[\,TW - WT\,] \tag{1.6}$$

Here $K = S\dot{M}S^{-1}$, $T = SMS^{-1}$, and $^u[\;]$ is the operator which zeros out all entries on or below the diagonal of a matrix. This solution of the resultant triangular commutant equation is discussed in section III. An analytic expression for the solution is given which is amenable to a simple recursive computational procedure which preserves the sparseness of the given matrices. Several examples of the continuous LU algorithm are discussed in section IV.

II. Continuation Algorithm

We are interested in a decomposition of the form

$$T = SMS^{-1} \qquad (2.1)$$

where M (=M(r)) is our given parameterized family of matrices S (=S(r)) is an appropriate family of similarity transformations, and T (=T(r)) is lower triangular. In this case $^u[\dot{T}] = 0$ which together with the matrix equality

$$(\dot{S^{-1}}) = -S^{-1}\dot{S}S^{-1} \qquad (2.2)$$

yields

$$S\dot{M}S^{-1} - \dot{T} = -\dot{S}MS^{-1} - SM(\dot{S^{-1}})$$

$$= SMS^{-1}\dot{S}S^{-1} - \dot{S}S^{-1}SMS^{-1} = TW - WT$$

where $W = \dot{S}S^{-1}$. Finally, since $^u[\dot{T}] = 0$ this reduces to the desired triangular commutant equation

$$^u[S\dot{M}S^{-1}] = {}^u[TW - WT] \qquad (2.3)$$
$$\dot{S} = WS$$

A Continuous LU Algorithm: In the classical LU (or LR) algorithm for computing the eigenvalues of a single sparse matrix, a unit upper triangular similarity transformation, U, which triangularizes the given, M, via

$$L = UMU^{-1} \qquad (2.4)$$

is computed. As such, the triangular equation reduces to

$$^u[U\dot{M}U^{-1}] = {}^u[LX - XL]$$

$$\dot{U} = XU \qquad (2.5)$$

where $X = \dot{U}U^{-1}$ is strictly upper triangular (since $\dot{U}$ is strictly upper triangular and U^{-1} is upper triangular). Since X is strictly upper triangular the above triangular commutant equation represents n(n-1)/2 equations in n(n-1)/2 unknowns which must be solved to compute $X = X[U\dot{M}U^{-1}, UMU^{-1}]$ and U. Of course, once $\dot{U}$ is known, any standard numerical integration technique can be used to compute U(r) and $L(r) = U(r)M(r)U^{-1}(r)$ given appropriate conditions (which may be obtained via the classical LU algorithm).

III. Solution of the Triangular Commutant Equations

The key to the viability of the continuation algorithm described in the preceeding section is the existence of an easily computed solution to the triangular commutant equation. For this algorithm we must solve

$$^u[D] = {}^u[LX - XL] \qquad (2.5)$$

for a strictly upper triangular X given $D = U\dot{M}U^{-1}$ and a lower triangular $L = UMU^{-1}$. Although no matrix algebraic solution to 2.5 is apparent, a recursive algorithm for the solution of 2.5 may be obtained by expanding the i-j entry on both sides of the equality. Since $^u[\ ]$ zeros out all entries on or below the diagonal, it suffices to consider the case $1 \leq i < j$. Upon invoking the fact that X is strictly upper triangular and L is lower triangular we then obtain

$$D_{ij} = \sum_{k=1}^{i} L_{ik}X_{kj} - \sum_{k=j}^{n} X_{ik}L_{kj}$$
$$= (\lambda_i - \lambda_j)X_{ij} + \sum_{k=1}^{i-1} L_{ik}X_{kj}$$
$$- \sum_{k=j-1}^{n} X_{ik}L_{kj} \ ; \quad 1 \leq i < j \qquad (3.1)$$

Here we have used the fact that diagonal entries of L are the eigenvalues of M, i.e., $L_{ii} = \lambda_i$. Assuming distinct eigenvalues, this equation may be solved for X_{ij} yielding

$$X_{ij} = [D_{ij} + \sum_{k=j+1}^{n} X_{ik}L_{kj} - \sum_{k=1}^{i-1} L_{ik}X_{kj}]$$
$$/ \ (\lambda_i - \lambda_j) \ ; \ 1 \leq i < j \qquad (3.2)$$

The resultant X_{ij} is clearly linear and continuous in D_{ij}. Moreover, the equation can be solved recursively by starting with i = 1 and j = n to compute X_{1n}. Then $X_{1,n-1}$ may be computed in terms of X_{1n} and the given matrices. This information is then used to compute $X_{1,n-2}$, etc. In general, we may compute X_{ij} in terms of X_{rs} where $r < i = i$ and $s > j$. As such, X_{ij}, $1 \leq i < j$, may be computed recursively by increasing i and decreasing j. Of course, since X is strictly upper triangular

$$X_{ij} = 0 \ , \ j \leq i \leq n \qquad (3.3)$$

while the formula of equation 3.2 is readily implemented in a sparse matrix algorithm and preserves the sparsity of the given matrices.

THEOREM 1: Let M have distinct eigenvalues and $L = UMU^{-1}$ be lower triangular. Then the triangular commutant equation

$$^u[D] = {}^u[LX - XL]$$

admits a unique strictly upper triangular solution which may be computed recursively via eqiatopms 3.2 amd 3.3. □

Finally, we note that the above triangular commutant equation is a special case of the general equation

$$C = AX + XB \qquad (3.4)$$

Because of the triangular nature of our arrays, however, the above described recursive formula for the triangular commutant equation is far simpler than the various algorithms which have been proposed for the solution of the general equation. See for instance the paper by Bartels and Stewart[2]

IV. Examples

To illustrate the numerical accuracy of the continuation algorithms presented, the LU algorithm was employed to compute the eigenvalues of families of matrices.

EXAMPLE 1: The given matrix M,

$$M(r) = T(r)\lambda(r)T(r)^{-1}$$

where T(r) and (r) are n dimensional matrices whose elements are given by

$$T_{ii} = 1 \quad i = 1,2, \ldots, n.$$

$$T_{2i} = 1 \quad i = 1,2, \ldots, n/2.$$

$$T_{1n} = r$$

$$T_{ij} = 0 \text{ elsewhere.}$$

and

$$\lambda_{ii} = i(1+r+\sqrt{r}\ ) \quad i = 1,2, \ldots, n$$

and

$$\lambda_{ij} = 0 \text{ elsewhere.}$$

'r' was allowed to vary from 0 to 1. Table 1 illustrates the results for various matrix dimensions. In general, as the dimension of the matrix increased, the step size decreased.

Table 2 compares step sizes. R was varied from 0 to 0.01 in 1 step, 10 steps and 100 steps. The numerical error resulting from these computations seem to decrease linearly with step size.

TABLE 1

EIGENVALUES COMPUTED AT R=1 FOR EXAMPLE 1

ORDER STEP SIZE ITERATIONS	22 0.001 1000	14 0.01 100	10 0.01 100	6 0.01 100	ACTUAL VALUES
λ_1	2.940447909930	2.999241283978	2.952493308439	2.999999999651	3.000000000000
λ_2	6.003348248312	5.998064658837	5.993798281765	5.999999999302	6.000000000000
λ_3	9.004499671515	9.000000003725	9.000000003725	9.000000003725	9.000000000000
λ_4	12.006644267240	11.963118391580	12.111909089730	11.999999998660	12.000000000000
λ_5	15.007499404600	15.000000000290	15.000000000290	15.000000000290	15.000000000000
λ_6	18.008999343150	18.000000007450	18.000000007450	18.000000007450	18.000000000000
λ_7	21.010499287510	21.000000015480	21.000000015480		21.000000000000
λ_8	24.603002869990	24.052942127460	23.087992529200		24.000000000000
λ_9	27.013498917220	26.999999998950	26.999999998950		27.000000000000
λ_{10}	30.014998809200	30.000000000580	30.000000000580		30.000000000000
λ_{11}	33.016498733200	33.000000005700			33.000000000000
λ_{12}	36.017998686220	36.000000014900			36.000000000000
λ_{13}	39.019498638810	39.000000022350			39.000000000000
λ_{14}	42.020998575030	42.000000030970			42.000000000000
λ_{15}	45.022498482140				45.000000000000
λ_{16}	47.397466558030				48.000000000000
λ_{17}	51.025497935480				51.000000000000
λ_{18}	54.026997834440				54.000000000000
λ_{19}	57.028497742720				57.000000000000
λ_{20}	60.029997618400				60.000000000000
λ_{21}	63.031497519700				63.000000000000
λ_{22}	66.032997466390				66.000000000000

TABLE 2

STEPSIZE COMPARISON AT R=0.01 FOR EXAMPLE 1

ORDER STEP SIZE ITERATIONS	14 0.010 1	14 .001 10	14 .0001 100	ACTUAL VALUES
λ_1	1.10999993	1.10999988	1.11000000	1.11000000
λ_2	2.22000210	2.22000098	2.22000008	2.22000000
λ_3	3.33000000	3.33000000	3.33000000	3.33000000
λ_4	4.45059493	4.44088749	4.44007096	4.44000000
λ_5	5.55000000	5.55000000	5.55000000	5.55000000
λ_6	6.66000000	6.66000000	6.66000000	6.66000000
λ_7	7.77000000	7.77000000	7.77000000	7.77000000
λ_8	8.86946893	8.87911251	8.87992903	8.88000000
λ_9	9.99000000	9.99000000	9.99000000	9.99000000
λ_{10}	11.10000000	11.10000000	11.10000000	11.10000000
λ_{11}	12.21000000	12.21000000	12.21000000	12.21000000
λ_{12}	13.32000000	13.32000000	13.32000000	13.32000000
λ_{13}	14.43000000	14.43000000	14.43000000	14.43000000
λ_{14}	15.54000000	15.54000000	15.54000000	15.54000000

EXAMPLE 2: The matrix M(r) is given by

$$M_{ii} = 1 \quad i = 1,2, \ldots, n.$$

$$M_{i+1,i} = 1 \quad i = 1,2, \ldots, n-1.$$

$$M_{1n} = r.$$

$$M_{ij} = 0 \quad \text{elsewhere.}$$

Table 3 compares the results for different dimension matrices with the eigenvalues calculated by solving the characteristic equation of the matrix at r = 1.

TABLE 3
EIGENVALUES COMPUTED AT R=0.1 FOR EXAMPLE 2

ORDER STEP SIZE ITERATIONS	5 0.01 100	ACTUAL VALUES
λ_1	1.04643575	1.04592039
λ_2	1.82951692	1.84701523
λ_3	3.28129404	3.27522419
λ_4	3.78183756	3.79073431
λ_5	5.03693025	5.03849529
ORDER STEP SIZE ITERATIONS	9 0.01 100	ACTUAL VALUES
λ_1	1.0000263932	1.0000248031
λ_2	1.9997909827	1.9998016543
λ_3	3.0007158106	3.0006948691
λ_4	3.9935944345	3.9986121406
λ_5	5.0017435055	5.0017356740
λ_6	5.9986086449	5.9998108814
λ_7	7.0006940215	7.0006935027
λ_8	7.9998318476	7.9998016393
λ_9	9.0000245405	9.0000247858
ORDER STEP SIZE ITERATIONS	15 0.01 100	
λ_1	1.0000000000127362	
λ_2	1.9999999958290635	
λ_3	3.0000000010701078	
λ_4	3.9999999957922932	
λ_5	5.0000000115050053	
λ_6	5.9999999770255821	
λ_7	7.0000000344495097	
λ_8	7.9999999608378462	
λ_9	9.0000000344467222	
λ_{10}	9.9999999770255607	
λ_{11}	11.0000000011482214	
λ_{12}	11.9999999958874657	
λ_{13}	13.0000000001043816	
λ_{14}	13.9999999999939449	
λ_{15}	15.0000000000011435	

V. Conclusion

Although continuation algorithms have historically proven their usefulness in the solution of "small" numerical problems the classical differential equations modeling the various numerical processes are not compatible with sparse matrix techniques. The present work coupled with a previous paper in which a continuation algorithm for the inversion of sparse matrices is formulated,[8] however, indicate that the concept can be made compatible with sparse matrix techniques.[8,6]

VI. References

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CONTINUATION ALGORITHMS FOR THE EIGENVALUE PROBLEM

B. Green, R. Saeks and K.-S. Chao

CONTINUATION ALGORITHMS FOR THE EIGENVALUE PROBLEM*

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Abstract

The eigenvalue problem for a continuously parameterized family of sparse matrices, M(r), often arises in stability analysis. Typically, one employs a classical eigenvalue package at a sequence of parameter values, possibly with special software to expolit the common sparsity pattern of the various matrices. Alternatively, one can compute the eigenvalues at an initial parameter value and "continue the result" by integrating an appropriate differential equation whose trajectories define the eigenvalue loci of the given family of matrices. The most common such differential equation for this purpose, however, employs the eigenvectors as an auxiliary variable which destroys the sparseness of the problem since the array of eigenvectors for a sparse matrix is typically non-sparse. As such, the computational benefits of working with the sparse matrix M will be lost if one attempts to integrate such an equation.

This difficulty is alleviated in the present paper by formulating continuation algorithms around a family of similarity transformations, S(r), which triangularize M(r). Such similarity transformations preserve the sparseness of M while the eigenvalues of M are given by the diagonal entries of the resultant family of triangular matrices, T(r). We formulate three such continuation algorithms. The first, which may be viewed as a continuous LU (or LR) algorithm, employs a unit upper triangular matrix, S to transform M into lower triangular form. The second, which may be viewed as a continuous QR algorithm, uses a unitary matrix to transform M into lower triangular form. Finally, our third algorithm uses an upper triangular matrix to transform M into lower Hessenberg form.

In each of the three continuation algorithms the required differential equation takes the form

$$\frac{ds}{dr} = W[S\frac{dM}{dr}S^{-1}, SMS^{-1}]S$$

$$T = SMS^{-1}$$

where W[,] is the solution of an appropriate *triangular commutant equation*

$$^{u}[S\frac{dM}{dr}S^{-1}] = {}^{u}[SMS^{-1}W - WSMS^{-1}]$$

and $^{u}[\ ]$ is the operator which zeros out all entries on or below the diagonal of a matrix. In each case an analytic expression for the solution of the required triangular commutant equation is given which is amenable to a simple recursive computational procedure which preserves the sparseness of the given matrices.

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Abstract of

OPTIMAL SELECTION OF IC FABRICATION PARAMETERS

C. Karmokolias, W. Portnoy, and R. Saeks

Abstract

A procedure is described in which the output characteristics of an integrated circuit are optimized with respect to a set of variable fabrication parameters. A simple RC coupled audio amplifier is used as an example. The gain-bandwidth product is obtained as a function of oxidation and diffusion times and temperatures, and the optimization is performed by way of a line search using these variables as the parameters of the optimization. The values established for the process parameters are consistent with those employed for conventional fabrication, and desired changes in performance can be obtained, in general, by a straightforward readjustment of the values of the process variables. Although limited by certain assumptions and a relatively primitive circuit, the results demonstrate the validity of the procedure.

Grants and Contracts Administered by JSEP Personnel

A. Funded

Murray, J., AFOSR Grant, "The Application of Crossed Products to the Stability and Design of Time-Varying Systems", 1 yr. $22,091, SORF Matching, $500.

Walkup, J.F., and T. Krile, ARO Grant, "Optical Signal Processing Workshop", 1 yr., $11,867.

Saeks, R., ONR Contract, "Joint Services Electronics Program", 3 yrs, $630,000, SORF Matching, $30,000.

Chao, K.-S., NSF Grant, "Continuation Algorithms in Computer-Aided Design", 2 yrs., $37,421.

Saeks, R., NSF Grant, "Frequency Domain-Like Methods for the Analysis and Design of Time-Varying and Nonlinear Systems", 3 yrs. $79,240.

Walkup, J.F., AFOSR Grant, "Space Variant Optical Systems, 1 yr., $95,070.

Gustafson, D., and T. Krile, E-Systems Corp. Contract, "Digital and Optical Signal Processing and Detection" 3/4 yr. $19,988.

Walkup, J.F., SPIE Grant, "Optical Engineering Education", 1 yr., $2,000.

Krile, NSF Grant, "Fibre Optic Experiments for Undergraduate Education", 2 yrs., $21,533, SORF Matching $11,896.

Newman, T.G., ARO Grant, "Lie Groups and Lie Algebras in Video Tracking", 1 yr., $10,983.

Hunt, L.R., NASA Grant, "Support of Professor Hunt's Leave of Absence at NASA/AMES", 3/4 yr., $39,218.

Newman, T.G., SORF Grant, $Synthesis of Digital Filters for Differentiation of Digitized Images", 1 yr., $700.

Total Annual Funding $483,254 .

B. Proposed

Chao, K.-S., Proposal to NSF, "Continuation Algorithms in Nonlinear Circuits and Systems", 2 yrs., $53,000.

Walkup, J.F., and T. Krile, Proposal to RADC, "Estimation and Detection in Communication Systems", 1 yr., $50,000.

Grants and Contracts in Electrical Engineering

A. Systems

Murray, J., AFOSR Grant, "The Application of Crossed Products to the Stability and Design of Time-Varying Systems", 1 yr., $22,091, SORF Matching, $500.

Walkup, J.F., and T. Krile, ARO Grant, Optical Signal Processing Workshop", 1 yr., $11,867.

Saeks, R., ONR Contract, "Joint Services Electronics Program", 3 yrs., $630,000, SORF Matching $30,000.

Chao, K.-S., NSF Grant, "Continuation Algorithms in Computer-Aided Design", 2 yrs, $37,421.

Saeks, R., NSF Grant, "Frequency Domain-Like Methods for the Analysis and Design of Time-Varying and Nonlinear Systems", 3 yrs. $79,240.

Walkup, J.F., AFOSR Grant, $Space Variant Optical Systems", 1 yr., $95,070.

Gustafson, D., and T. Krile, E-Systems Corp. Contract, "Digital and Optical Signal Processing and Detection", 3/4 yr., $19,988.

Total Annual Funding in Systems, $404,639.

B. Electro-Physics

Hagler, M.O., NSF-Grant, "Investigation of RF Plasma Heating in Toroidal Geometry", 2 yrs., $110,000.

Portnoy, W.M., DOE Grant, "Deep Traps in AlGaAs Layers at AlGaAs-GaAs Interfaces", 1 yr., $37,414, SORF Matching, $9,600.

Portnoy, W.M., NRL Contract, "Reliability Study of Gallium Arsenide Devices", 1 yr., $21,000.

Williams, P.F., Texas Instruments Contract, "Laser Spectroscopy", 1 yr., $10,000.

Trost, T., NASA Grant, "Lightning Sensors and Data Interpretation", 1 yr., $50,000.

Portnoy, W.M., Masterie Corp., "Semiconductor Device Physics and Reliability", 1 yr., $1,401.

Total Annual Funding in Electro-Physics $184,415.

C. Pulsed Power Research

Kristiansen, M., AFOSR Grant, "Pulsed Power Research Colloquium, 1/2 yr., $8,000.

Kunhardt, E., NSWC Contract, "Breakdown at High Voltages", 1/2 yr., $41,043.

Kunhardt, E., Hatfield, L. and M. Kristiansen, AFWL Contract, "An Opening Using a Diverter", 1 yr., $24,968.

Kristiansen, M., AFOSR Contract, "Coordinated Research Program in Pulsed Power Physics", 1 yr., $666,263.

Kristiansen, M., ARO Contract, "Coordinated REsearch Program in Pulsed Power Physics", 1 yr., $100,000.

Kristiansen, M., AFOSR Grant, "Special Equipment Grant", 1 yr., $100,000, SORF Matching, $20,000.

Kristiansen, M., ARO Grant, "Opening Switch Meeting", 1 yr., $10,000.

Total Annual Funding in Pulsed Power $970,274 .

D. Power Systems

Craig, J.P., Texas Power and Light Co., "Power System Studies", 1 yr., $11,438.

Reichert, J.D., DOE Contract, "Crosbyton Solar Power Project", 5/6 yr., $950,000.

Total Annual Funding in Power Systems $961,438 .

E. Educational Activities

Williams, P.F., NSF Grant, "Innovative Undergraduate Laboratory Program in Optical Communications", 2 yrs., $8,700, SORF Matching, $8,700.

Seacat, R., SORF Grant, "Research and Development in Electrical Engineering", 1 yr., $19,689.

Kunhardt, E., NSF Grant, "Undergraduate Research Participation", 1 yr., $19,931.

Walkup, J.F., SPIE Grant, "Optical Engineering Education", 1 yr., $2,000.

Krile, T., NSF Grant, "Fibre Optic Experiments for Undergraduate Engineers", 2 yrs., $21,533, SORF Matching $11,896.

Total Annual Funding of Educational Activities $67,034 .

F. Sources of Funding in Electrical Engineering

Air Force	$916,392
Navy	262,043
Army	121,867
DOE	987,414
NASA	50,000
SORF	70,087
Industry	44,827
NSF	135,170
Total Annual Funding in Electrical Engineering	$2,587,800

Grants and Contracts in Mathematics

Anderson, R., and W. Ford, SORF Grant, "Fixed Point Formulations of Porous Media Problems", 1 yr., $8,000.

Barnard, R., NSF Grant, "Some Extremal Problems in Complex Function Theory", 2 yrs., $16,046.

Ford, W., and R. Anderson, DOE Contract, "Mathematical Methodology for Evaluating Simulations of Flow in Porous Media", 2 yrs., $141,367.

Ford, W., GURC Grant, "Porous Media Survey", 1 yr., $2,500.

Harris, G., NSF Grant, "Local Function Theory and Geometry of Real Submanifolds of C_n", 2 yrs., $11,642.

Strauss, M., NSF Grant, "Uniqueness and Norm Convexity in the Cauchy Problem", 1 yr., $8,832.

Nelson, P., AFOSR Grant, "Applications of Invariant Imbedding", 1 yr., $45,021.

Newman, T.G., ARO Grant, "Lie Groups and Lie Algebras in Video Tracking", 1 yr., $19,983.

Hunt, L.R., NASA Grant, "Support of Professor Hunt's Leave of Absence at NASA/AMES", 3/4 yr., $39,218.

Newman, T.G., SORF Grant, "Synthesis of Digital Filters for Differentiation of Digitized Images", 1 yr., $700.

Sources of Funding in Mathematics

Air Force	$45,021
Navy	-0-
Army	19,983
DOE	70,683
NASA	39,218
SORF	8,700
Industry	2,500
NSF	22,676
Total Annual Funding in Mathematics	$208,781

Publications by JSEP Personnel*

A. Refereed Journal Articles

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Saeks, R., and R.-w. Liu, "Fault Diagnosis in Electronic Circuits", Japan Jour. of Electrical Engineering, (to appear, JSEP).

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* Includes all publications by JSEP personnel with sourse of support.

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Walkup, J.F., Williams, P.F., and M. Gundersen, "Optics at Texas Tech University: Learning by Doing", IEEE Trans on Education, Vol. E-23, pp. 118-124, (1980, NSF).

B. <u>Conference Papers and Abstracts</u>

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Froehlich, G., Walkup, J., and T.F. Krile, "Some Effects of Signal-Dependent Noise on Estimator Structures", 1980 OSA Meeting, Chicago, Oct. 1980, (JSEP).

Green, B., Saeks, R., and K.S. Chao, "Continuation Algorithms for the Eigenvalue Problem", Proc. of the 1980 IEEE Inter. Symp. on Circuits and Systems, Houston, May 1980, p. 775, (abstract only, JSEP).

Iyer, A., and R. Saeks, "Numerical Implementation of a Continuation Algorithm for the Eigenvalue Problem", 1980 IEEE Conference on Circuits and Computers, Port Chester, Oct. 1980, pp. 437-440, (JSEP).

Karmokolias, C., and R. Saeks, "A Fractional Representation Approach on Adaptive Control", 1980 IEEE Conference on Decision and Control, Albuquerque, Dec. 1980. pp. 272-273, (JSEP).

Murray, J., "A Design Method for 2-D Recursive Digital Filters", Proc. of the 13th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove, CA., Nov. 1979, pp. 104-107, (JSEP).

Murray, J., "A New Approach to 2-D Digital Filtering", Proc. of the 24th Midwest Symp. on Circuits and Systems, Univ. of New Mexico, Albuquerque, (to appear, JSEP).

Newman, T.G., and D.A. Davis, "Lie Theoretic Methods in Video Tracking", Proc. of the MICOM Workshop on Imaging Trackers and Autonomous Acquisition Applications for Missile Guidance, Redstone Arsenal, Nov. 1979, pp. 161-174 (GACIAC-PR-80-01, JSEP).

Wu, C.-c., Sangiovanni-Vencentelli, A., and R. Saeks, "A Differential-Interpolative Approach to Analog Fault Diagnosis", Proc. of the 1981 IEEE Inter. Symp. on Circuits and Systems, (to appear, JSEP).

Wu, C.-c., Nakajima, K., and R. Saeks, "Pos.-Test Fault Simulation with Failure Limitations", Proc. of the 24th Midwest Symp. on Circuits and Systems, (to appear, JSEP).

C. Preprints

Fredricks, G.A., "Canonical Forms for Nondegenerate Second Order Linear Partial Differential Operators and Equations" (submitted for publication,JSEP).

Fredricks, G.A., and T.G. Newman, "Method in Differential Geometry with Application to Video Tracking", (submitted for publication, JSEP).

Froehlich,G., Walkup, J., and T. Krile, "Multiple Parameter Estimation in Signal-Dependent Noise" (submitted for publication, JSEP).

Froehlich,G., Walkup, J., and T. Krile, "Estimation in Signal Dependent Film-Grain Noise", (submitted for publication, JSEP).

Green, B., Saeks, R., Chao, K.S., and A. Iyer, "Continuation Algorithms for the Eigenvalue Problem", (submitted for publication, JSEP).

Jones, M.I., Walkup, J.F., and M.O. Hagler, "Multiplex Hologram Representations of Space Variant Optical Systems Using Ground-Glass Encoded Reference Beams", (submitted for publication, AFOSR).

Newman, T.G., "Lie Groups and Lie Algebras in Video Tracking", (submitted for publication, JSEP).

Sangiovanni-Vincentelli, A., and R. Saeks, "Multitest Diagnosibility of Non-linear Circuits and Systems", (submitted for publication, JSEP).

Wu, C.-c., and R. Saeks, "A Data Base for Symbolic Network Analysis, (submitted for publication, JSEP and ONR).

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