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Error Probability Characteristics for Multiple Alternative Communication With Diversity, but Without Fading

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Preface

This research was conducted under NUSC IR/IED Project No. A75205, Subproject No. ZR0000101, Applications of Statistical Communication Theory to Acoustic Signal Processing, Principal Investigator Dr. Albert H. Nuttall (Code 3302), Program Manager CAPT David F. Parrish, Naval Material Command (MAT 08L); and under NUSC Project No. C61605, *IACS Systems Engineering*, Principal Investigator Ronald Malone (Code 3222), Sponsoring Activity Naval Sea Systems Command, Program Manager Gary Stringer (SEA 63Y33).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The M-ary character error probability, P_e, for a communications system employing D-fold diversity combination is derived and evaluated for the case of no fading, over the range of parameters M = 2, 3, 4, 8, 16, 32, 64, 128, 256, 512, 1024 and D = 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64 for P_e in the range 10^{-1} to 10^{-5}. The curves presented allow for ready determination of the additional signal-to-noise ratio required for employing diversity when it is not necessary, or when uncertainty forces it upon the receiver processing. Extensions to a thresholding receiver and to a very general class of signal fading are also made, but no numerical results are given for these latter cases.

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List of Symbols

M	Number of multiple alternatives
D	Order of diversity
P_e	M-ary character error probability
P_c	Probability of correct character decision
E_T	Total received signal energy
N_o	Received noise power density level (single-sided)
d_T	Deflection statistic, $(2\,E_T/N_o)^{1/2}$
$B_n(x)$	Auxiliary function (equation 3)
$e_n(x)$	Partial exponential series (equation 4)
P_{e2}	Binary error probability for $M = 2$
P_{CD}	Probability of correct decision
P_{FA}	Probability of false alarm
R_T	Energy-density ratio, $E_T/N_o = d_T^2/2$
$\overline{E}_T$	Average total received signal energy for fading channel
$\overline{R}_T$	Average energy-density ratio, $(\overline{E}_T/N_o)$
ν	Measure of degree of signal fading (equation 9)
μ	Signal fading parameter (equation 10)
$\overline{P}_e$	Average M-ary character error probability
$f(\xi)$	Characteristic function

Error Probability Characteristics for Multiple Alternative Communication With Diversity, but Without Fading

Introduction

The M-ary character error probability, P_e, for orthogonal multiple alternative communication with D-fold diversity, was evaluated in reference 1 for a wide range of parameter values. It was presumed there that the slow signal fading on the D diversity channels was independent and that the received signal energy was distributed exponentially (Rayleigh fading of received signal voltage).

It is of interest here to reconsider the system performance for the case where, although the receiver was designed for D-fold diversity, there is in fact no signal fading. The fractionalization of the received signal into the diversity channels then results in poorer performance than had the signal been confined to one channel and processed coherently. This situation can arise naturally in practice, as for example, as a result of multipath arrivals, or it can come about intentionally in the system design. It can also be of interest in testing a multiple alternative hardware design under controlled laboratory conditions, where the time or cost of simulating actual fading conditions is excessive.

The basic framework and background for the communications technique considered here have already been presented in reference 1 and will not be repeated, for the sake of brevity. The reader is presumed to be familiar with the time-bandwidth duration and separation constraints listed in the above reference, particularly pages 2-4 and appendix B.

Definitions and Technical Results

The source selects one of M equi-probable symbols and transmits a signal on D diversity channels (time and/or frequency) to a receiver. The D channels devoted to each of the M alternatives are disjoint (in time and/or frequency) with each other and with the channels for the other alternatives (see reference 1). The receiver employs matched filtering on each of the totality of MD cells of interest. For each signal alternative, the corresponding D envelope-squared matched filter outputs are sampled appropriately in time and summed. The largest of these M decision variables is then declared to be the transmitted signal alternative.

The additive noise at the receiver is assumed to be white Gaussian over the total band of the possible received signals, with a (single-sided) power density level of N_o watts/Hz. It is shown in appendix A that the M-ary character error probability of this communications technique is given by

$$P_e = 1 - \frac{\exp\left(-d_T^2/2\right)}{d_T^{D-1}} \int_0^\infty dy\, y^D\, e^{-y^2/2}\, I_{D-1}(d_T y) \left[B_{D-1}\left(y^2/2\right)\right]^{M-1} . \tag{1}$$

Here "deflection" statistic d_T is given by

$$\frac{d_T^2}{2} = \frac{E_T}{N_o} = \frac{\text{total received signal energy}}{\text{noise power density level}} , \tag{2}$$

where E_T is the total received signal energy on the D channels; $I_n(x)$ is the modified Bessel function of order n and argument x; auxiliary function $B_n(x)$ is defined as

$$B_n(x) \equiv \int_0^x dt\, \frac{t^n \exp(-t)}{n!} = 1 - e^{-x}\, e_n(x) ; \tag{3}$$

and (reference 2, equation 6.5.11)

$$e_n(x) \equiv \sum_{k=0}^{n} \frac{1}{k!} x^k \tag{4}$$

is the exponential power series through the term x^n.

It is interesting and worthwhile to observe from (1) and (2) that the exact fractionalization of the total received signal energy E_T amongst the D diversity branches is immaterial in so far as system performance is concerned. The available signal energy E_T could be divided equally in the D branches, or it could be concentrated in just one branch; in either case, error probability P_e is identical (assuming M and D are unchanged).

For M = 2, expression (1) can be evaluated in closed form. Several alternative representations for this binary error probability are given in appendix B, along with an efficient program for this special case. The program for evaluation of (1) in general is given in appendix C.

Graphical Results

For a particular value of M, the character error probability P_e in (1) is a function of the number of diversity channels, D, and the total energy-density ratio (signal-to-noise ratio) E_T/N_0. We have plotted P_e on a normal probability ordinate versus E_T/N_0 on a dB abscissa (i.e., $10 \log E_T/N_0$), with D as a parameter, in the range 10^{-5} to 10^{-1} for P_e. Figures 1-11 correspond, respectively, to M = 2, 3, 4, 8, 16, 32, 64, 128, 256, 512, 1024. The curve for D = 1 in each figure corresponds to the NO FADING results in reference 1, figures 1-7.

We have discovered no simple rule-of-thumb for the additional signal energy required to maintain a fixed P_e as D is increased, which covers the whole range of D and M. Nor did we find a simple expression for P_e when it is small ($\sim 10^{-5}$), despite considerable effort. The best we can do is to observe that

$$P_e \approx (M - 1) P_{e2} \quad \text{for small } P_e \quad , \tag{5}$$

and then use result (B-10) for the binary error probability, P_{e2}. D is arbitrary in (5). Further development of this approach appears in appendix B.

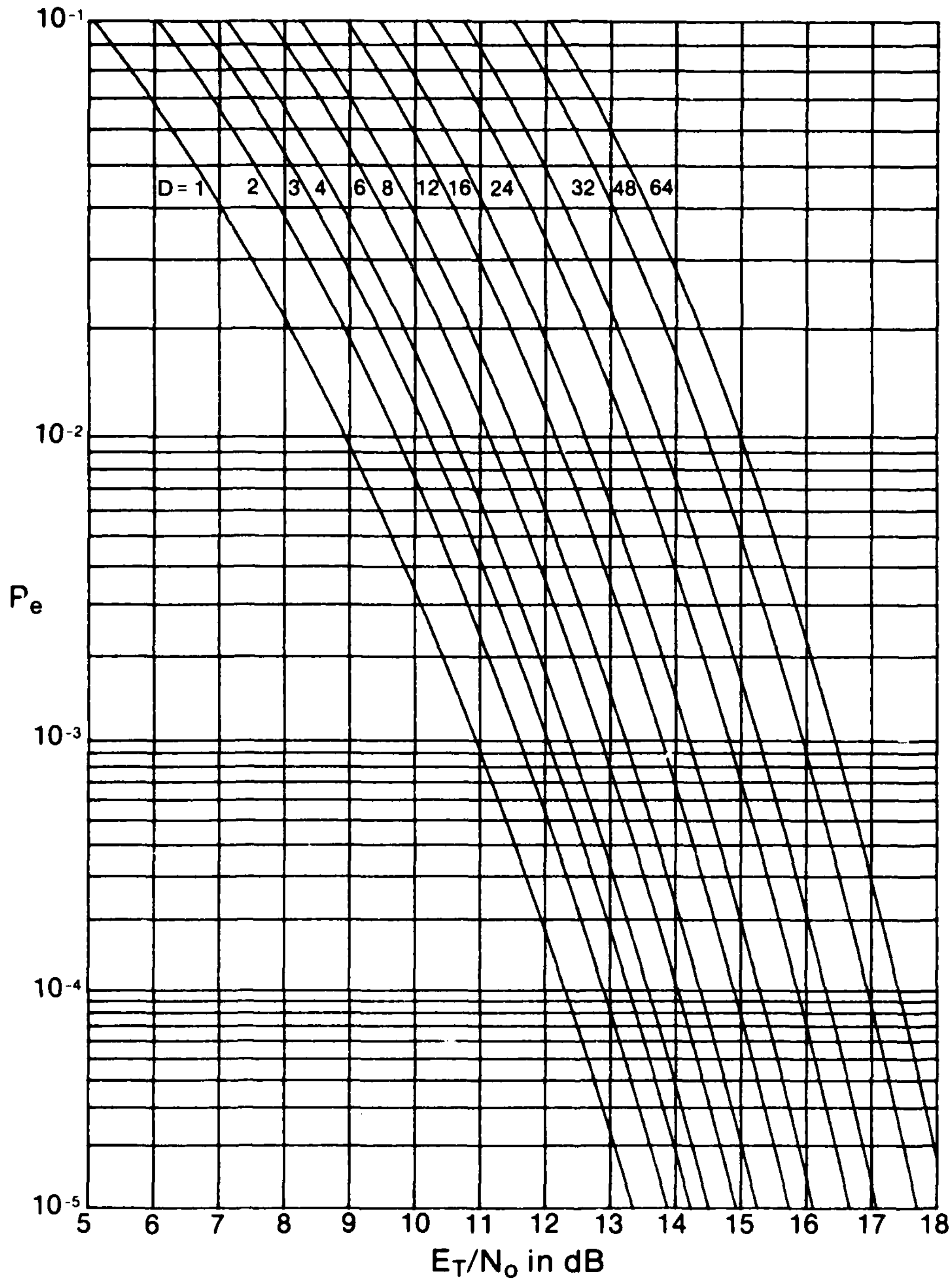


Figure 1. M-ary Character Error Probability for M = 2

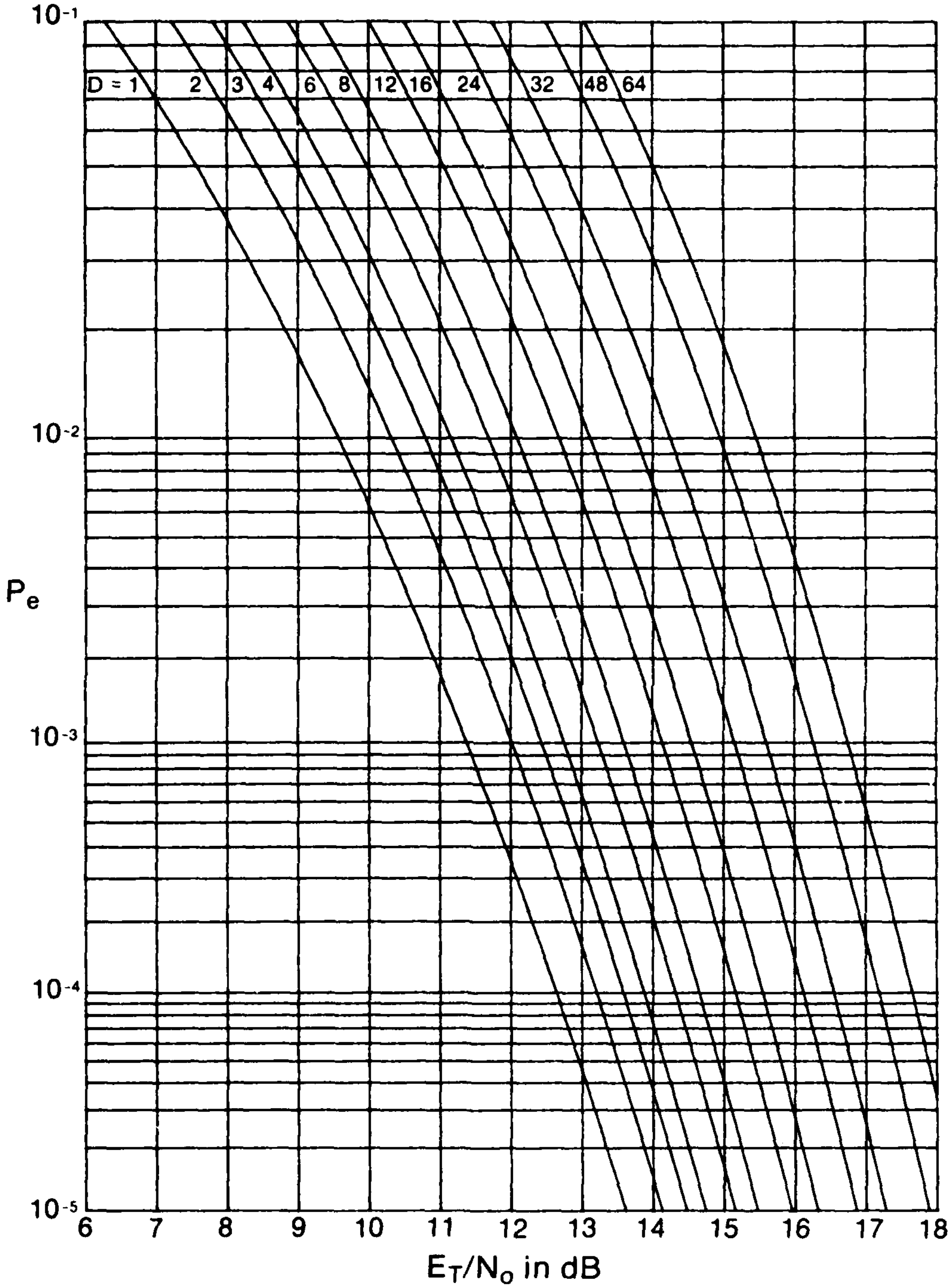


Figure 2. M-ary Character Error Probability for M = 3

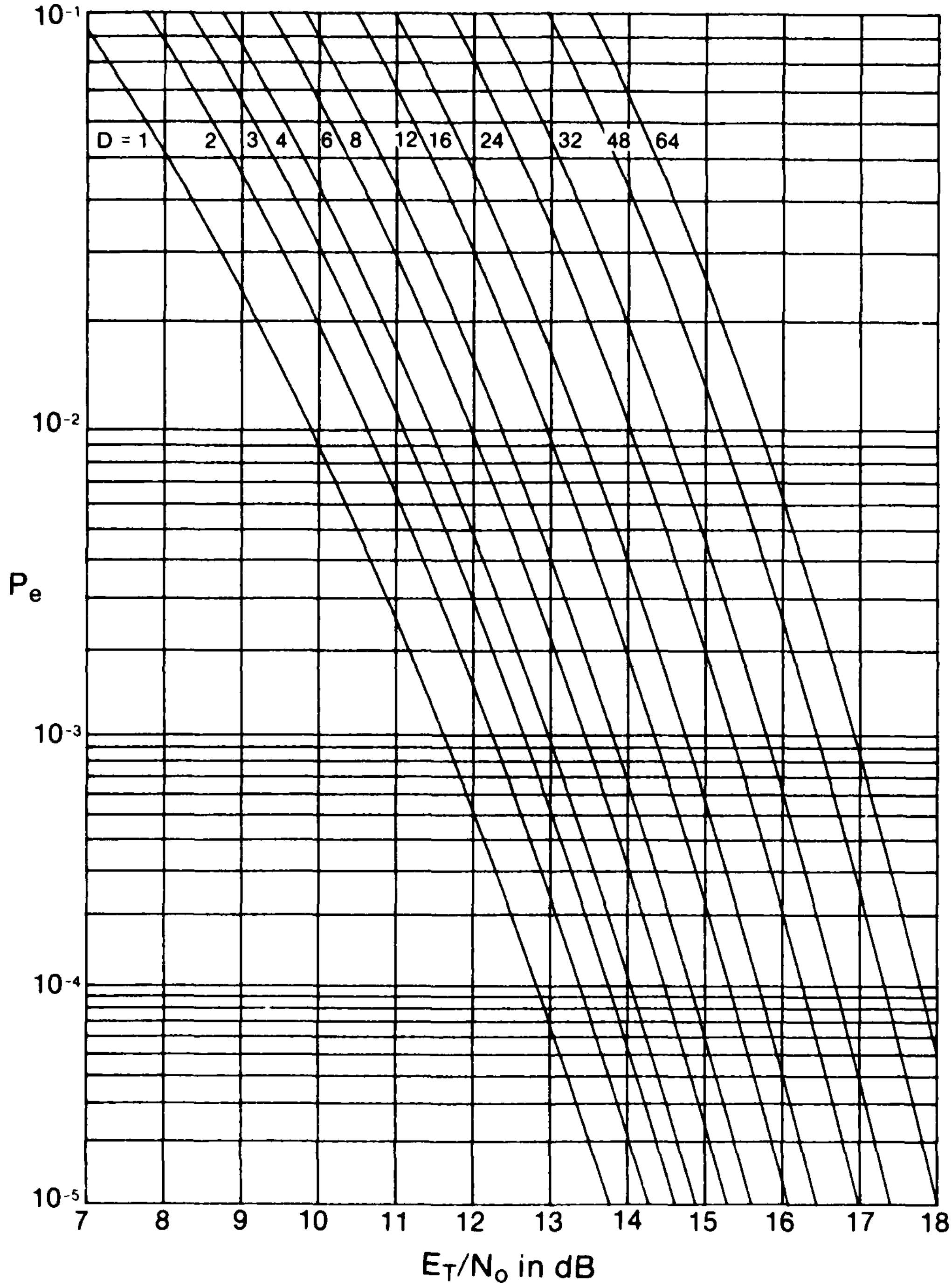


Figure 3. M-ary Character Error Probability for M = 4

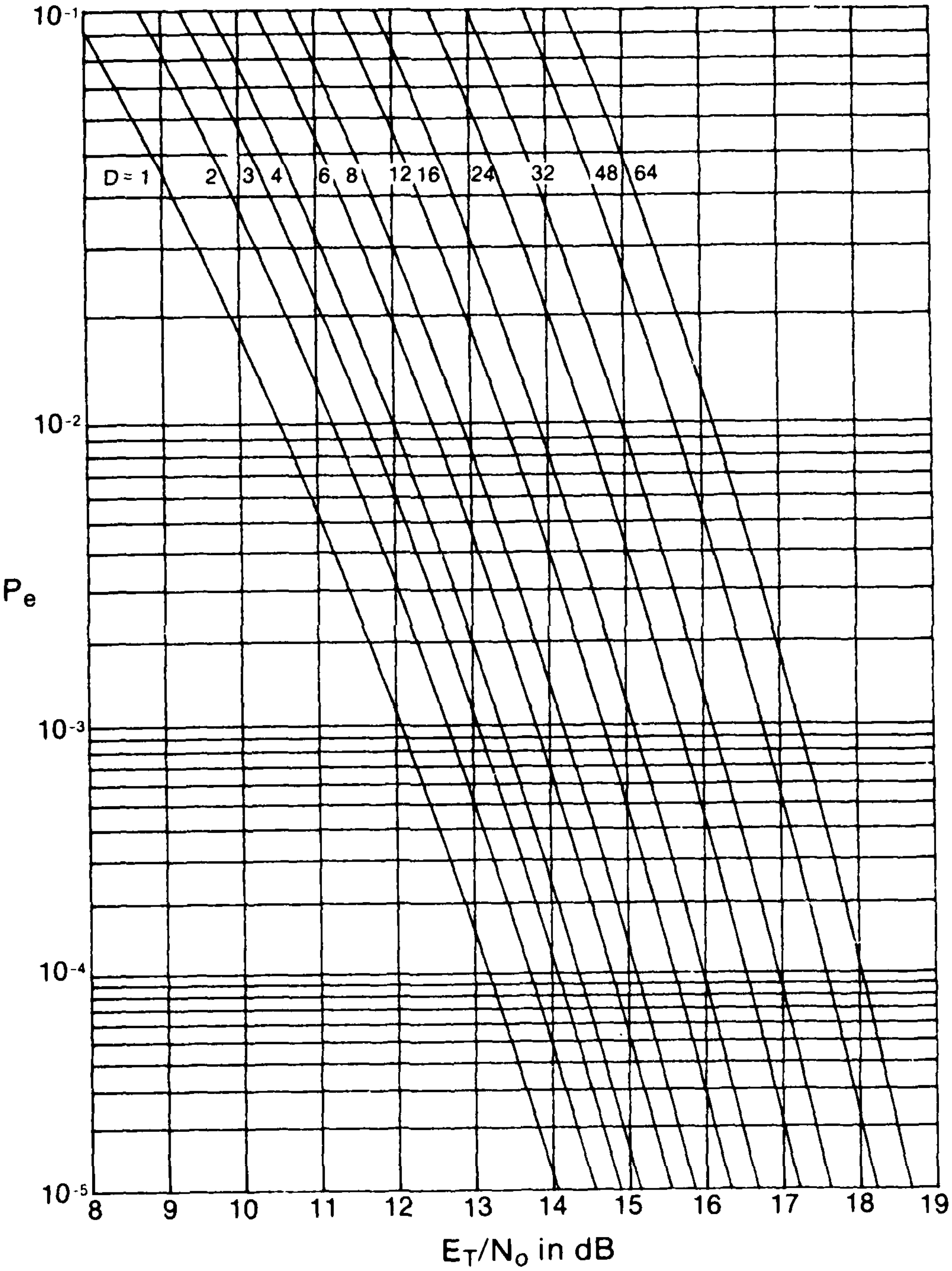


Figure 4. M-ary Character Error Probability for M = 8

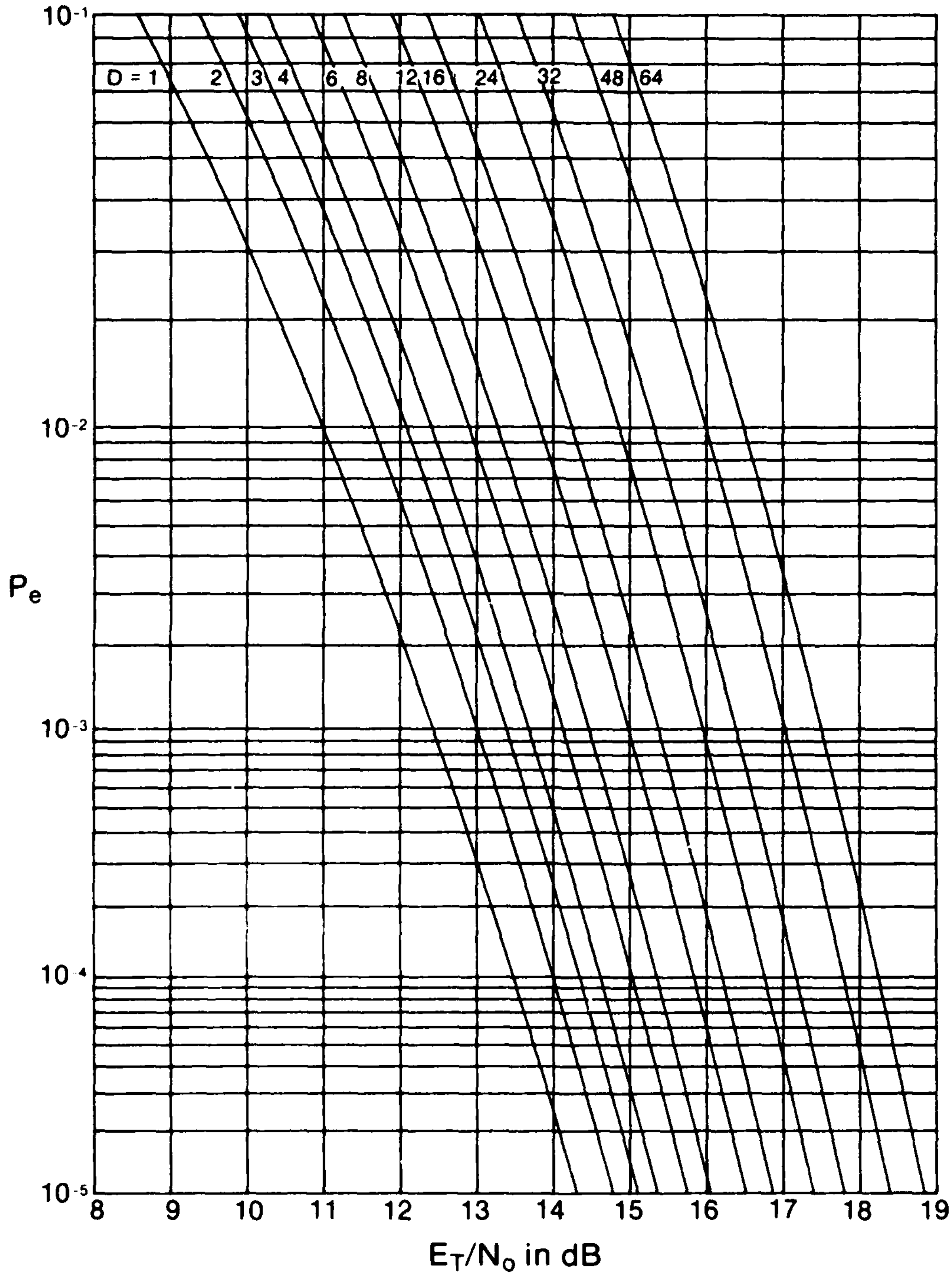


Figure 5. M-ary Character Error Probability for M = 16

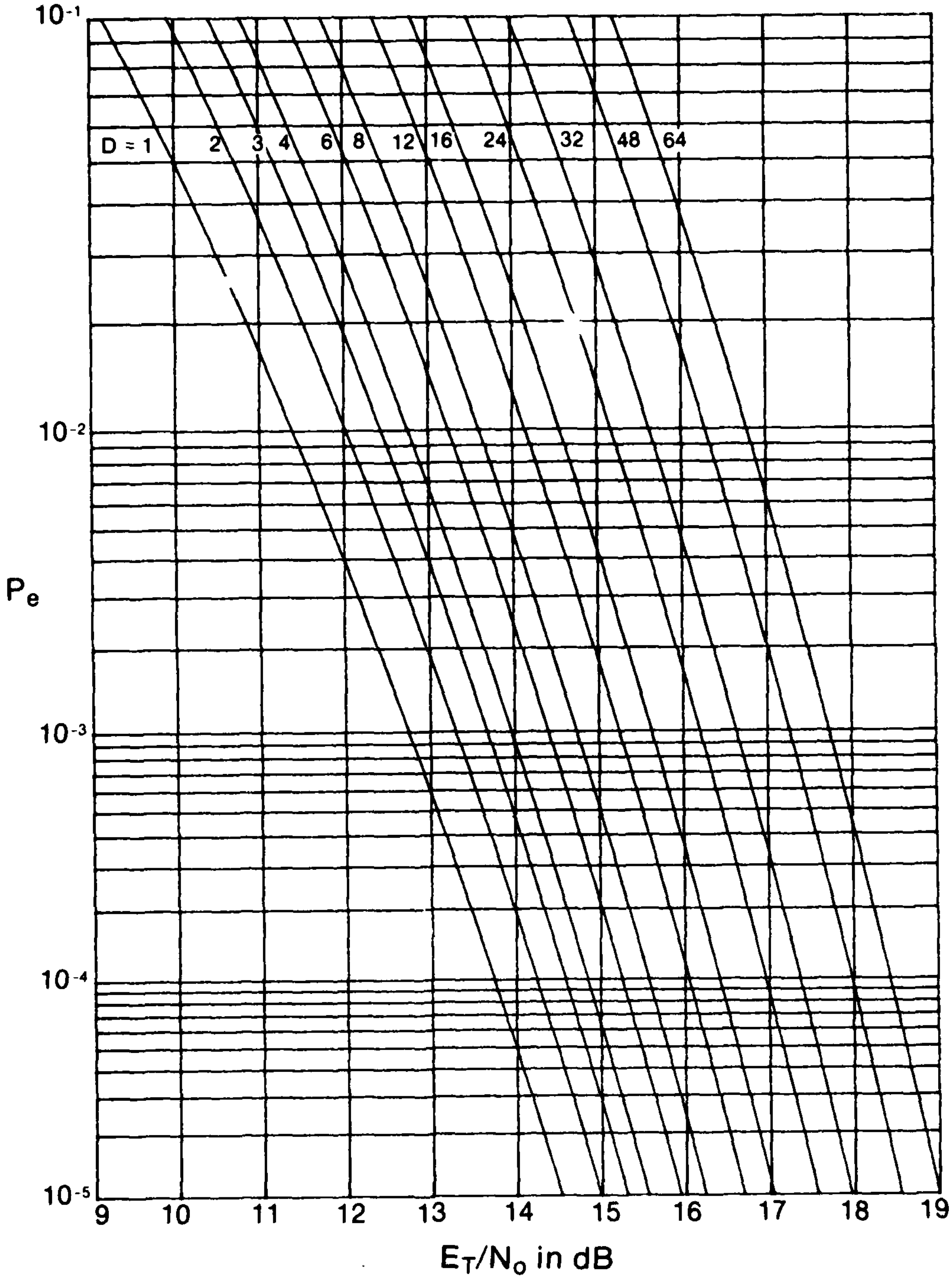


Figure 6. M-ary Character Error Probability for M = 32

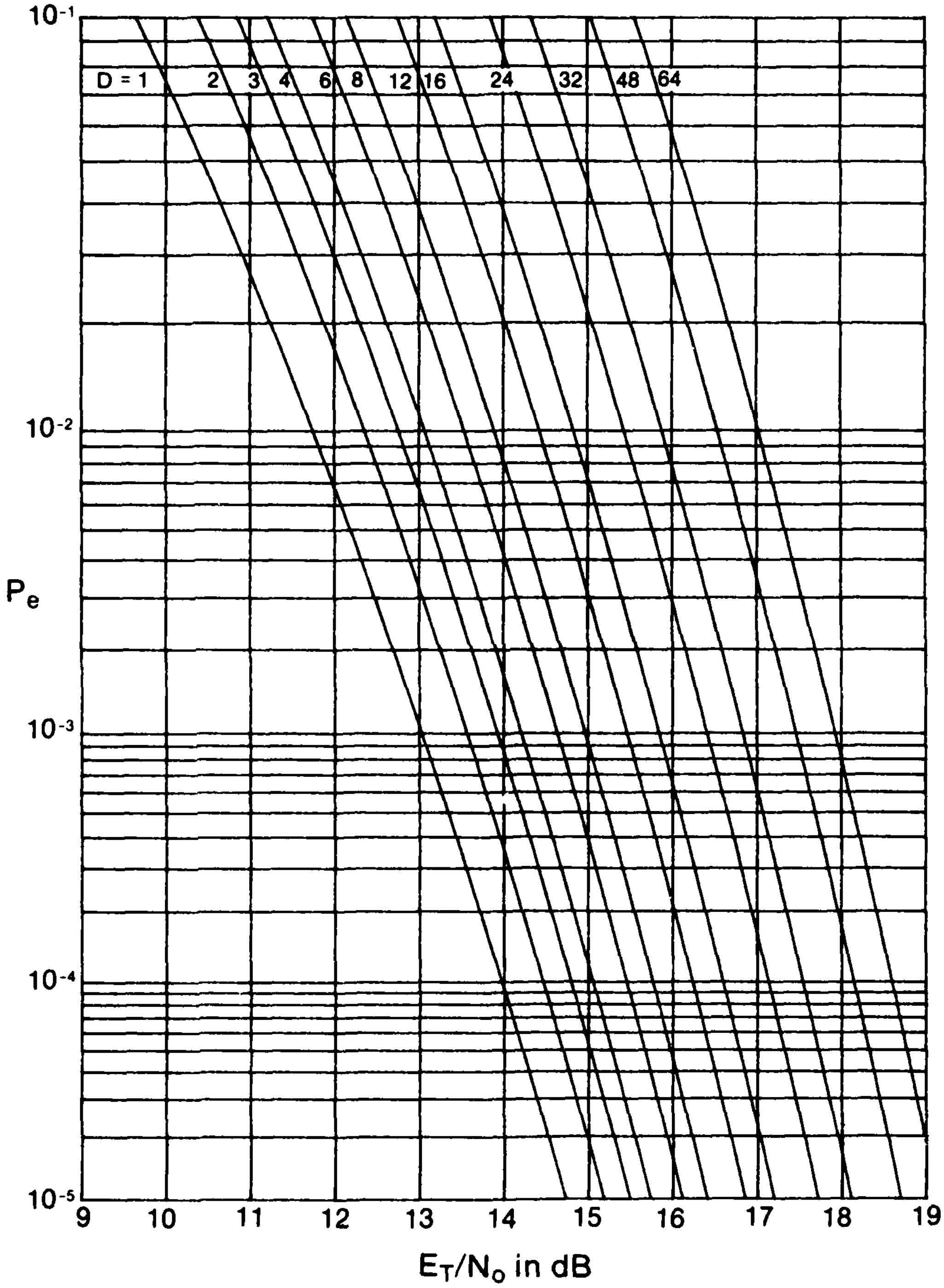


Figure 7. M-ary Character Error Probability for M = 64

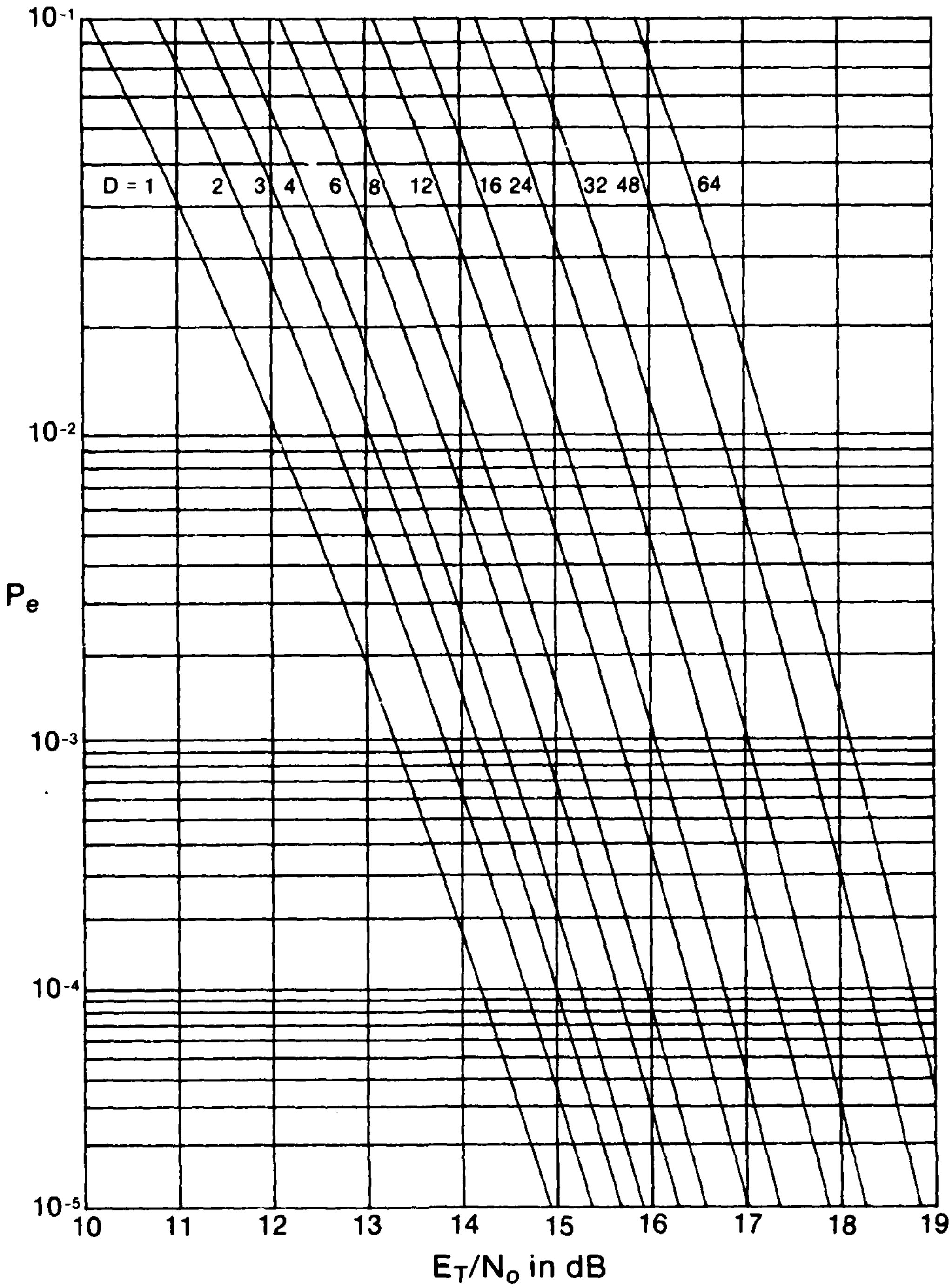


Figure 8. M-ary Character Error Probability for M = 128

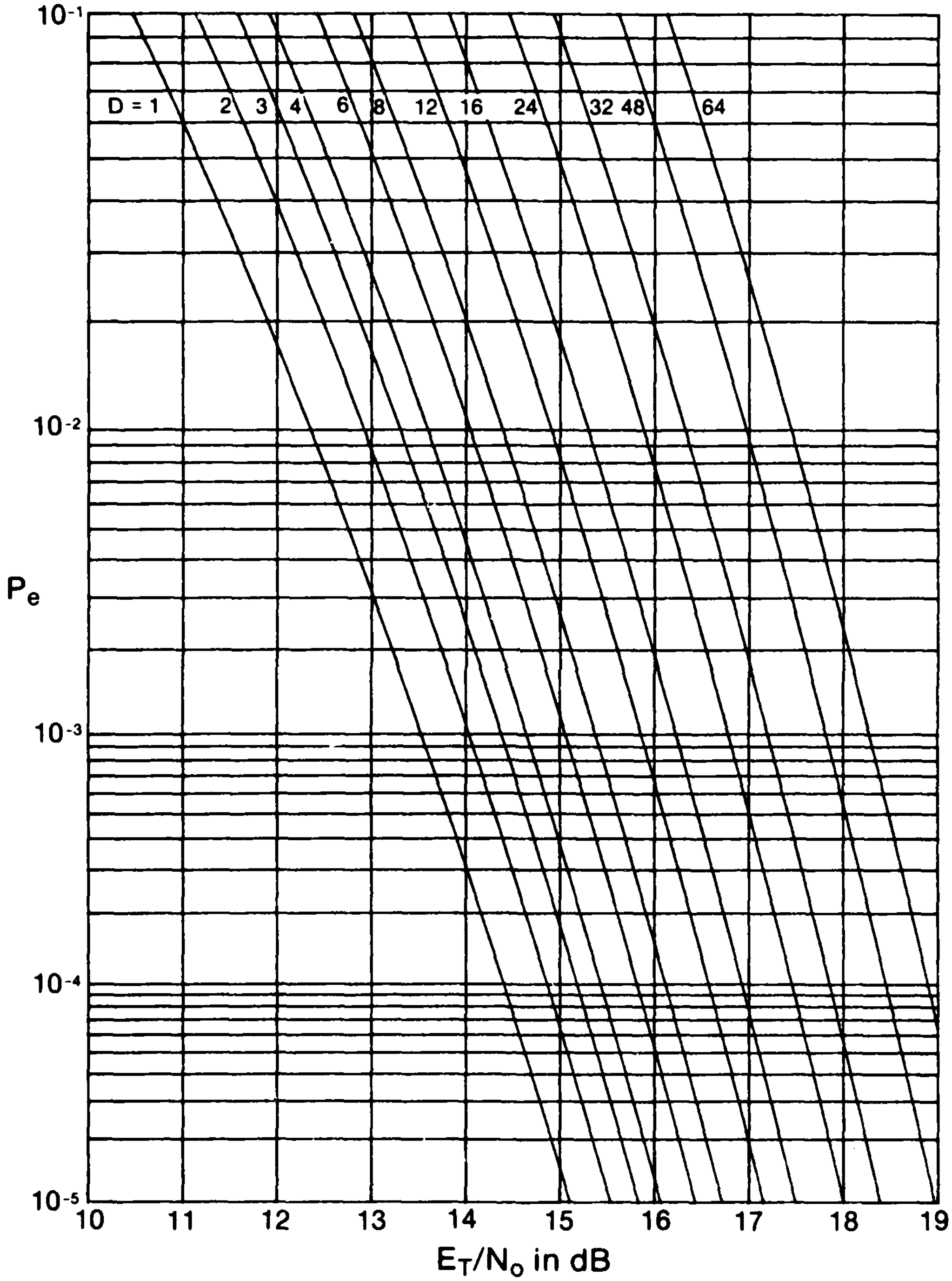
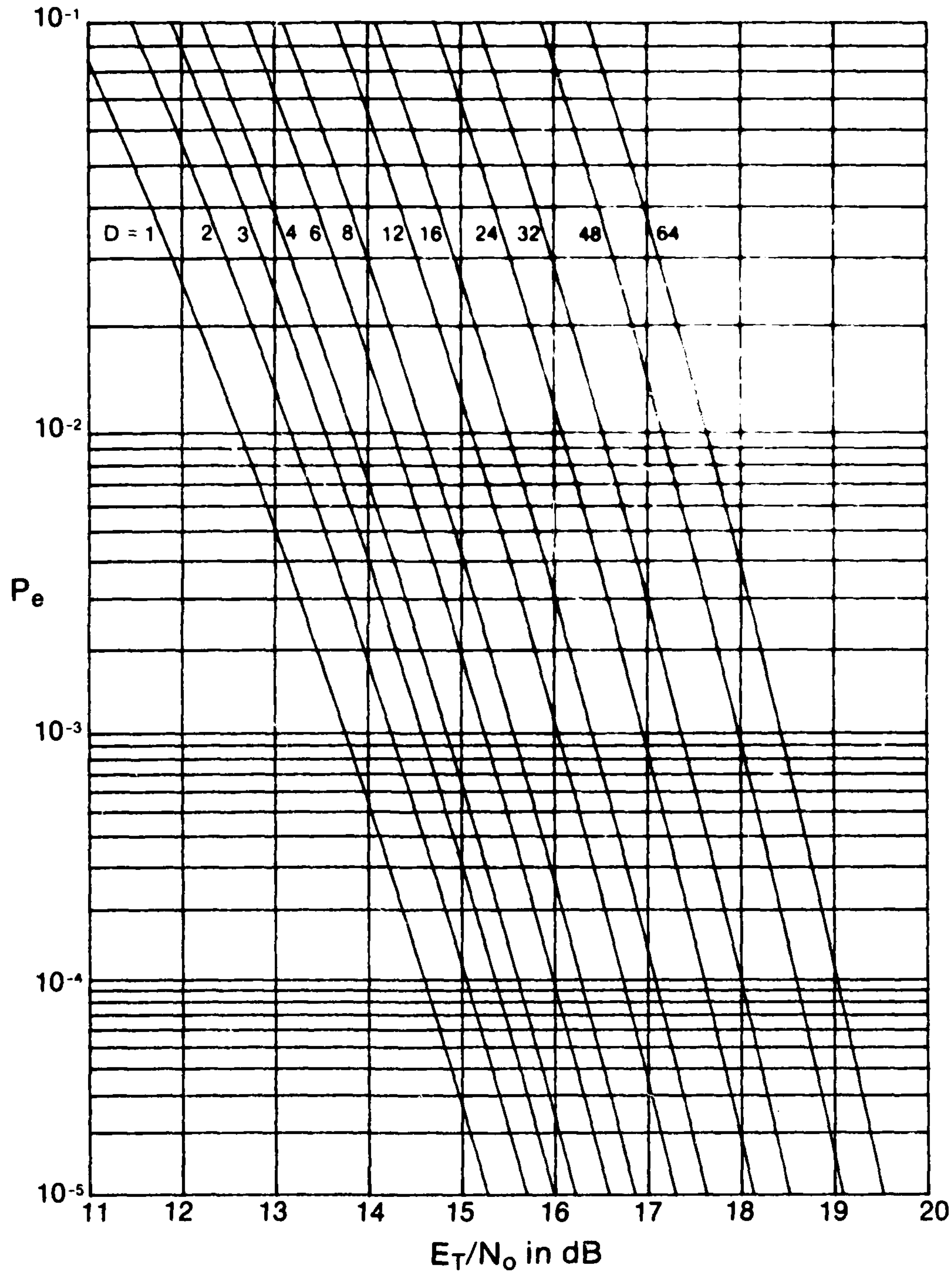


Figure 9. M-ary Character Error Probability for M = 256



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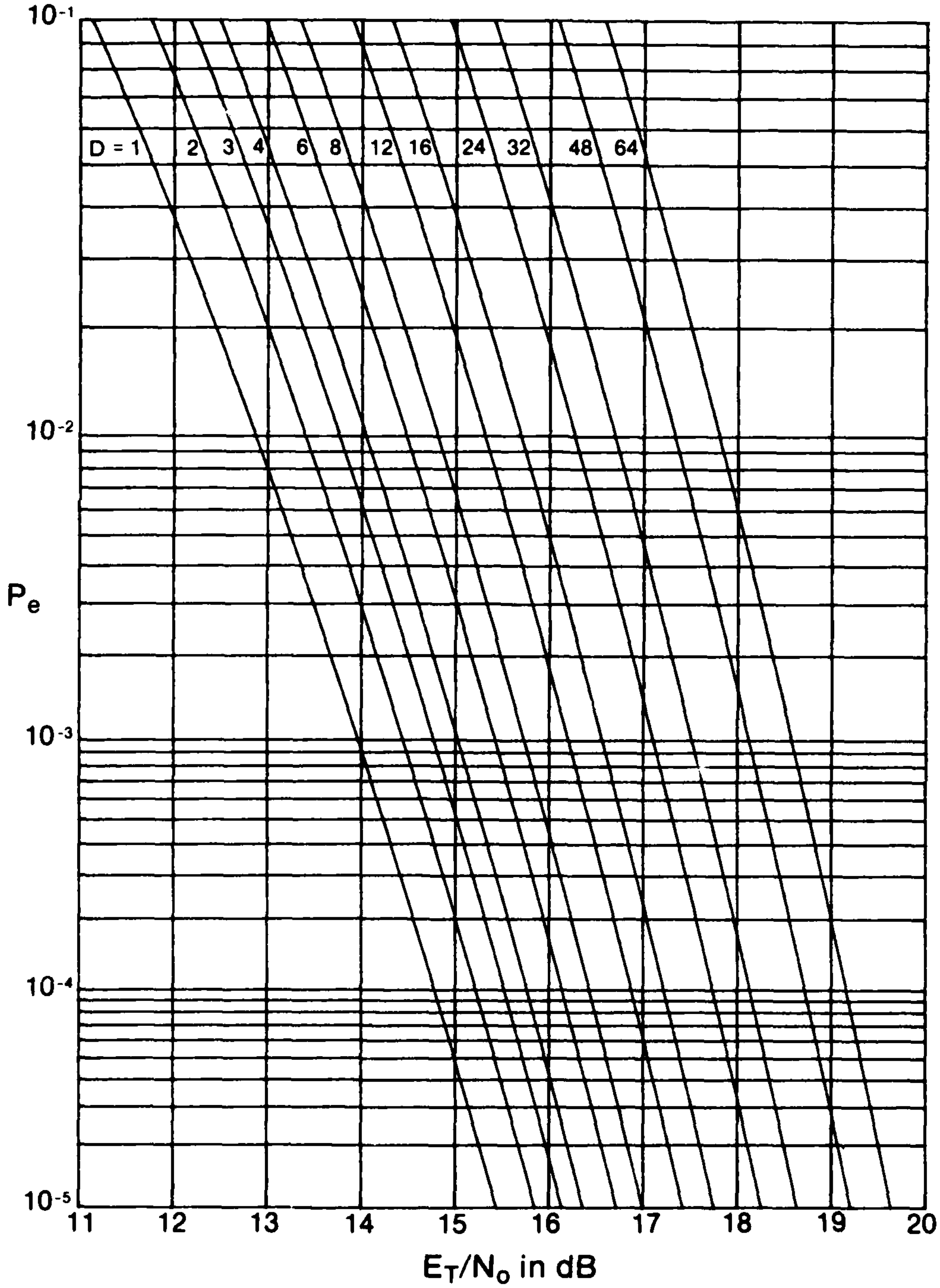

Figure 11. M-ary Character Error Probability for M = 1024

Correct Decision with a Threshold

The processing system above presumed that one of the M signal alternatives was always transmitted. A more general situation occurs when the additional case of no signal present at all is allowed. Then the largest of the M decision variables is compared with a threshold. There are three types of probabilities of interest for signal present (reference 3, pages 2-5); they are the probability of a missed decision, the probability of an incorrect decision, and the probability of a correct decision. The last is given by a slight generalization of (1) to (reference 3, equation 5)

$$P_{CD} = \frac{\exp\left(-d_T^2/2\right)}{d_T^{D-1}} \int_V^{\infty} dy\ y^D\ e^{-y^2/2}\ I_{D-1}\left(d_T y\right)\left[B_{D-1}\left(y^2/2\right)\right]^{M-1} , \tag{6}$$

where V is the threshold value.

For signal absent, we then have false alarm probability (reference 3, equation 2)

$$P_{FA} = 1 - \text{Prob}\left\{\max\left(y_1, \ \ldots \ , y_M\right) < V \text{ for } d_T = 0\right\}$$

$$= 1 - \left[\int_0^V dy\ y\ \frac{\left(y^2/2\right)^{D-1} \exp\left(-y^2/2\right)}{(D-1)!}\right]^M$$

$$= 1 - \left[B_{D-1}\left(V^2/2\right)\right]^M . \tag{7}$$

Here we used the limit of the integrand of (6) as $d_T \to 0$ and (3). Result (7) agrees with reference 3, equation 25, as it must, since both receivers are processing identical noise-only channel outputs. No numerical results on this more general situation, (6) and (7), are presented here.

Extension to Fading Signal

All the earlier results in this report have pertained to constant amplitude signal components on the D diversity branches. Now we presume that there is slow signal fading, meaning that the total received signal energy E_T is a random variable. In particular, we let energy-density ratio (signal-to-noise ratio)

$$R_T = \frac{d_T^2}{2} = \frac{E_T}{N_o} \tag{8}$$

have probability density function

$$p\left(R_T\right) = \frac{R_T^{\nu} \exp\left(-R_T/\mu\right)}{\mu^{\nu+1}\ \Gamma(\nu + 1)} \quad \text{for } R_T > 0 \tag{9}$$

with

$$\mu = \frac{\overline{R_T}}{\nu + 1} = \frac{\overline{E_T}/N_o}{\nu + 1} \quad , \; \nu > -1 \quad . \tag{10}$$

Here $\overline{E}_T$ is the average received total signal energy on the D branches, and ν is a constant, not necessarily an integer.

This very general model of fading was presented and utilized in reference 4, pages 11-12, in the investigation of a maximum likelihood detector. For example, the four signal fading cases considered by Swerling (reference 5) are subsumed by (9) for particular choices of ν; see table 1.

Table 1. Particular Cases of General Fading

ν	0	D-1	1	2D-1
Swerling Case	1	2	3	4

Cases 1 and 3 correspond to common signal fading on all D branches, whereas cases 2 and 4 correspond to independent and identically distributed signal fading on each of the D branches. In particular, case 2, $\nu = D - 1$, corresponds to exponential probability density function

$$p_1(R) = \frac{1}{\overline{R_1}} \exp\left(-\frac{R}{\overline{R_1}}\right) \text{ for } R > 0 \; , \quad \overline{R_1} = \frac{\overline{R_T}}{D} \; , \tag{11}$$

for the received energy-density ratio, E_1/N_o, on one branch; this is Rayleigh fading of the received signal voltage amplitude on each branch. Equation (9) is a D-fold convolution of (11) when $\nu = D - 1$. Case 4, $\nu = 2D - 1$, corresponds to probability density function

$$p_1(R) = \frac{4}{\overline{R_1}^2} R \exp\left(-\frac{2}{\overline{R_1}} R\right) \text{ for } R > 0 \; , \quad \overline{R_1} = \frac{\overline{R_T}}{D} \; , \tag{12}$$

for the received energy-density ratio, E_1/N_o, on one branch. Equation (9) is a D-fold convolution of (12) when $\nu = 2D - 1$.

The average character error probability is evaluated in appendix D; it is given by

$$\overline{P_e} = 1 - \left[(D-1)! \; (\mu + 1)^{\nu+1}\right]^{-1}$$

$$\cdot \int_0^\infty dt \; t^{D-1} \, e^{-t} \left[B_{D-1}(t)\right]^{M-1} \; {}_1F_1\left(\nu + 1; \; D; \; \frac{\mu}{\mu + 1} t\right) \quad . \tag{13}$$

This single integral can be evaluated numerically for any ν of interest; of course, M, D, and $\overline{E}_T/N_o$ need to be specified also for this numerical procedure.

As a special instance of fading (9), consider case 2 in more detail:

$$\nu = D - 1,$$

$$\mu = \frac{\overline{R_T}}{D} = \frac{\overline{E_T}/N_o}{D} = \text{average energy-density ratio per branch.} \quad (14)$$

Then, if we use reference 2, equation 13.6.12, and (3) above, (13) specializes to

$$\overline{P_e} = 1 - \int_0^\infty dt\ q_1(t) \left[\int_0^t dx\ q_o(x)\right]^{M-1} , \quad (15)$$

where

$$q_1(t) = \frac{t^{D-1} \exp\left(-\frac{t}{\mu + 1}\right)}{(D - 1)!\ (\mu + 1)^D} \text{ for } t > 0 \ ,$$

$$q_o(x) = \frac{x^{D-1} \exp(-x)}{(D - 1)!} \text{ for } x > 0 \ , \quad (16)$$

are the probability density functions of the decision variables formed by the sum of D envelope-squared filter outputs for signal-present and noise-only, respectively. Result (15) is identical to reference 1, equation C-1. It is what we would have gotten had we averaged the signal probability density function in (1) — the function multiplying the bracket — with respect to the signal strength, before expressing P_e in integral form. In fact, a more general integral than (15) has already been encountered and evaluated in reference 3, equation 26, where a thresholding operation was included in the receiver processor for the fading signal.

For M = 2, general fading result (13) can be evaluated in closed form; these special cases are presented in appendix D, especially (D-5) and (D-8).

Discussion and Summary

This report has addressed the problem of numerically assessing the additional signal-to-noise ratio (i.e., energy-density ratio) required when a nonfading signal is broken into D (unequal) components and combined incoherently. Some related studies into the cost of imperfections of receivers, or the cost of lack of knowledge of the received signal structure, are presented and evaluated in references 6 and 7.

Since the performance of the processor considered here for the nonfading signal depends only on the total received signal energy, regardless of how fractionalized it may be among the diversity channels, these results also apply to a situation where uncertainty in the time of arrival or the doppler shift of the received signal causes the receiver to search over several channels, even though the particular received signal alternative occupies only one channel. The D-fold summation for each M-ary alternative leads to a loss in performance; stated alternatively, the cost of the uncertainty is additional received signal energy required in order to maintain the same quality of performance, P_e.

For a large signal-to-noise ratio, the M-ary character error probability is given approximately by (5) in terms of the binary error probability, P_{e2}. The usual approach is then to use (A-3) with M = 2, or to use (B-4), to evaluate P_{e2}. However, the situation frequently arises where the characteristic functions of the decision variables can be evaluated fairly easily, whereas the corresponding probability density functions and/or cumulative distribution functions cannot. In that case, an alternative representation of P_{e2} directly in terms of the pertinent characteristic functions would be useful. This problem is addressed in appendix E, with the following results for the binary error probability:

$$P_{e2} = \frac{1}{i2\pi} \int_{C_-} \frac{d\xi}{\xi} f_o(\xi) f_1(-\xi)$$

$$= \frac{-1}{i2\pi} \int_{C_+} \frac{d\xi}{\xi} f_o(-\xi) f_1(\xi) \quad .$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{d\xi}{\xi} \operatorname{Im}\{f_o(\xi) f_1(-\xi)\} \quad . \tag{17}$$

Here $f_o(\xi)$ and $f_1(\xi)$ are the characteristic functions of the noise-only and signal-plus-noise decision variables, respectively. C_- is a contour in the complex ξ-plane along the real axis, with a small downward identation at $\xi = 0$; C_+ is a similar contour indented upward at $\xi = 0$. The last form in (17) can be particularly useful for numerical evaluation of P_{e2} for some characteristic functions.

Another alternative to evaluation of M-ary error probabilities is to use bounds which are simpler to compute than the exact result. This is attractive if the bounds are tight, at least for the range of signal-to-noise ratios and error probabilities of interest. Some general bounds on error probability (which were not used here) are presented in appendix F for a situation that includes interference as well as noise.

Appendix A

Derivation of M-ary Character Error Probability

To avoid duplication of effort, we will rely heavily on reference 8, pages 2, 3, 8, 9 and appendixes A and B, especially (B-7)-(B-12). We obtain directly the probability density function of the signal decision variable (sum of D filter outputs) as*

$$p_1(x) = \frac{1}{2}\left(\frac{x}{d_T^2}\right)^{\frac{D-1}{2}} \exp\left(-\frac{x + d_T^2}{2}\right) I_{D-1}\left(d_T\, x^{1/2}\right) \text{ for } x > 0 \quad , \tag{A-1}$$

and that for any one of the M-1 noise-only decision variables as

$$p_o(y) = \frac{1}{2}\,\frac{(y/2)^{D-1}}{(D-1)!}\exp\left(-\frac{y}{2}\right) \text{ for } y > 0 \quad . \tag{A-2}$$

A correct decision is yielded only if x is greater than all M-1 independent noise variables; the probability of this event is

$$P_c = \int_0^\infty dx\; p_1(x)\left[\int_0^x dy\; p_o(y)\right]^{M-1} \quad . \tag{A-3}$$

But the integral on y in (A-3) and (A-2) is given by (3) as $B_{D-1}(x/2)$. Then by use of (A-1) and the substitution $x = y^2$, (A-3) becomes

$$P_c = \frac{\exp\left(-d_T^2/2\right)}{d_T^{D-1}} \int_0^\infty dy\; y^D\, e^{-y^2/2}\, I_{D-1}\left(d_T y\right)\left[B_{D-1}\left(y^2/2\right)\right]^{M-1} \quad . \tag{A-4}$$

The M-ary character error probability is $P_e = 1 - P_c$.

Several checks on (A-4) are available. First, for the trivial case M = 1, (A-4) yields $P_c = 1$, which agrees with the fact that there are then no noise-only channels to worry about at all. Second, as $d_T \to 0$, (A-4) reduces to $P_c = 1/M$; this agrees with the observation that we have a completely random selection with zero signal-to-noise ratio. Third, for D = 1, (A-4) agrees with reference 1, equation 9A (when the latter is corrected for the typographical omission of the factor $\exp(-E_1/N_0)$).

A program for the evaluation of $P_e = 1 - P_c$ via (A-4) is given in appendix C.

*A "deflection" interpretation of d_T^2 is available from reference 8, equations (B-10), (B-7), (B-4), (A-4), and (B-5).

Appendix B

Binary Error Probability

This appendix will deal exclusively with the special case of M = 2, a binary decision. Then (1) becomes, with the identifications in (3) and (4),

$$P_{e2} = \frac{\exp\left(-\,d_T^2/2\right)}{d_T^{D-1}} \int_0^{\infty} dy\; y^D\, e^{-y^2}\; I_{D-1}\left(d_T y\right)\; e_{D-1}\left(y^2/2\right)$$

$$= \frac{\exp\left(-\,d_T^2/2\right)}{d_T^{D-1}} \sum_{k=0}^{D-1} \frac{1}{k!\,2^k} \int_0^{\infty} dy\; y^{D+2k}\, e^{-y^2}\; I_{D-1}\left(d_T y\right)$$

$$= \frac{\exp\left(-\,d_T^2/4\right)}{2^D} \sum_{k=0}^{D-1} \frac{(D)_k}{k!\,2^k}\; {}_1F_1\left(-k;\; D;\; -d_T^2/4\right) \quad . \tag{B-1}$$

To evaluate the integral, we used reference 9, equation 6.631 1; the final transformation to (B-1) employed reference 2, equation 13.1.27. An alternative expression to (B-1) is available via reference 2, equation 13.6.9:

$$P_{e2} = \frac{\exp\left(-d_T^2/4\right)}{2^D} \cdot \sum_{k=0}^{D-1} \frac{1}{2^k}\, L_k^{(D-1)}\left(-d_T^2/4\right) \quad . \tag{B-2}$$

If we expand the polynomial in $d_T^2/4$ that occurs in (B-1) and (B-2), and interchange sums, we obtain

$$P_{e2} = \frac{\exp\left(-d_T^2/4\right)}{2^D} \sum_{n=0}^{D-1} \frac{\left(d_T^2/4\right)^n}{2^n\; n!} \sum_{j=0}^{D-1-n} \frac{(D\,+\,n)_j}{2^j\; j!} \quad . \tag{B-3}$$

All three expressions above for P_{e2} are finite sums of positive quantities and are reasonable for computer evaluation, even for fairly large D.

Another expression for $P_{c2} = 1 - P_{e2}$ is available by recalling (A-1) and (A-2) and changing (A-3) to

$$P_{c2} = \int_0^{\infty} dy\; p_0(y) \int_y^{\infty} dx\; p_1(x) \quad . \tag{B-4}$$

But from (A-1), upon making the substitution x = t² and using reference 10, equation 1, we find for the inner integral in (B-4),

$$\int_y^\infty dx\ p_1(x) = \int_{y^{1/2}}^\infty dt\ t\left(\frac{t}{d_T}\right)^{D-1} \exp\left(-\frac{t^2 + d_T^2}{2}\right) I_{D-1}\left(d_T t\right)$$

$$= Q_D\left(d_T, y^{1/2}\right) . \tag{B-5}$$

Then (B-4) yields

$$P_{c2} = \int_0^\infty dy\ \frac{y^{D-1}\ \exp(-y/2)}{2^D (D - 1)!}\ Q_D\left(d_T, y^{1/2}\right)$$

$$= \left[2^{D-1}\ (D - 1)!\right]^{-1} \int_0^\infty du\ u^{2D-1}\ \exp\left(-u^2/2\right)\ Q_D\left(d_T, u\right) . \tag{B-6}$$

At this point, we will use the new integral result

$$\int_0^\infty dx\ x^{2N-1}\ \exp\left(-p^2 x^2/2\right)\ Q_M(a, bx)$$

$$= \frac{2^{N-1}(N - 1)!}{p^{2N}} \left[1 - \left(\frac{b^2}{p^2 + b^2}\right)^{M+N-1} e^{-A} \sum_{n=0}^{N-1} \binom{M+N-1}{n} \left(\frac{p^2}{b^2}\right)^n e_{N-1-n}\ (A)\right] , \tag{B-7}$$

where

$$A \equiv \frac{a^2}{2}\ \frac{p^2}{p^2 + b^2} . \tag{B-8}$$

This integral and several other new integrals of Q_M functions, along with their derivations, will be presented fairly soon by this author in another NUSC technical report.

The use of (B-7) and (B-8) in (B-6) yields

$$P_{e2} = \frac{\exp\left(-d_T^2/4\right)}{2^{2D-1}} \sum_{n=0}^{D-1} \binom{2D-1}{n}\ e_{D-1-n}\ \left(d_T^2/4\right) . \tag{B-9}$$

Expanding the partial exponential expansion in powers of $d_T^2/4$ according to (4), and interchanging summations, we obtain for the binary error probability

$$P_{e2} = \frac{\exp\left(-d_T^2/4\right)}{2^{2D-1}} \sum_{k=0}^{D-1} \frac{\left(d_T^2/4\right)^k}{k!} \sum_{n=0}^{D-1-k} \binom{2D-1}{n} . \tag{B-10}$$

This result is very similar to (B-3), and of course yields identical numerical values; however, a direct verification of the transformation that would take the inner sum in (B-10) to the inner sum in (B-3), or vice versa, has not been discovered.

(B-10) is a very efficient form for numerical evaluation, because the summand of the inner sum (on n) does not depend on the index (k) of the outer sum; this advantageous property is not true of (B-3). Thus for a given D, we can compute the partial sums on n in (B-10) by starting at k = D - 1 and merely adding one additional term to the inner sum for each downward unit step in k; these D partial sums are all stored and then used for the outer sum starting at k = 0. Thus, what appears to be a double sum in (B-10) actually can be computed by means of two single sums of D positive terms. Furthermore, the inner sum on n in (B-10) can be done once for a value of D of interest, and then used repeatedly for different values of d_T^2. A program incorporating these observations is listed at the end of this appendix. All of the results thus far apply to arbitrary values of energy-density ratio E_T/N_o.

One reason for dwelling on the evaluation of the binary error probability is that for large signal-to-noise ratio, i.e., large energy-density ratio E_T/N_o, the M-ary character error probability is given approximately by

$$P_e \approx (M - 1)\, P_{e2} \quad \text{for small } P_e \quad . \tag{B-11}$$

This result may be seen to be valid when it is realized that the probability of any particular noise-alone decision variable exceeding the signal decision variable is a rare event when P_e is small. Thus the probability of two or more noise variables exceeding the signal variable is very rare indeed, and we need only account for the single noise variable case. Since these events are disjoint and there are M-1 such variables, (B-11) follows immediately.

For large $d_T^2/2 = E_T/N_o$, an asymptotic form for P_{e2} can be developed from (B-10); express

$$P_{e2} = \frac{\exp\left(-d_T^2/4\right)}{2^{2D-1}} \frac{\left(d_T^2/4\right)^{D-1}}{(D-1)!}$$

$$* \left[1 + \frac{2D(D-1)}{d_T^2/4} + \frac{(D-1)\,(D-2)\,(2D^2 - D + 1)}{\left(d_T^2/4\right)^2} + \ldots\right] \quad . \tag{B-12}$$

Therefore, asymptotically,

$$P_{e2} \sim \frac{\exp\left(-d_T^2/4\right)}{2^{2D-1}} \frac{\left(d_T^2/4\right)^{D-1}}{(D-1)!} \quad \text{as } d_T \to \infty \quad . \tag{B-13}$$

But reference to (B-12) reveals that in order for (B-13) to be a good approximation to P_{e2}, rather than just an asymptotic result, the requirement $d_T >> D$ must be satisfied. This requirement, however, is typically too large a signal-to-noise ratio requirement to be practically useful. We have found it necessary to resort to (B-10) for the binary error probability for the range of values of interest here.

A program for the binary error probability P_{e2} in BASIC for the Hewlett-Packard 9845 is presented below. It utilizes (B-10) et seq., as discussed above. The variables are self-explanatory and consistent with the earlier text.

```
1 ! Binary Pe for D-fold diversity, but no fading.  TR 6473
10     D=12        ! D>0       NUMBER OF DIVERSITY CHANNELS
20     Etn0=20     ! Etn0>0    TOTAL SIGNAL ENERGY/SINGLE-SIDED NOISE DENSITY
30     D1=D-1
40     D2=D*2
50     A=Etn0/2
60     DIM S(100)
70     REDIM S(D1)
80     S(D1)=T=1
90     FOR J=1 TO D1
100    T=T*(D2-J)/J
110    S(D1-J)=S(D-J)+T
120    NEXT J
130    S=S(0)
140    T=1
150    FOR K=1 TO D1
160    T=T*A/K
170    S=S+T*S(K)
180    NEXT K
190    Pe=EXP(-A)*S/2^(D2-1)
200    PRINT "D =";D,"Etn0 =";Etn0,"Pe =";Pe
210    END
```

```
D = 12                Etn0 = 20              Pe = 3.45181939137E-03
```

Appendix C

Program for M-ary Character Error Probability

The expression for P_e is given by (1). Since we cannot integrate to ∞, we terminate the integral at $y = L$. The error incurred in doing this is given by

$$\frac{\exp\left(-d_T^2/2\right)}{d_T^{D-1}} \int_L^\infty dy\; y^D\, e^{-y^2/2}\, I_{D-1}\left(d_T y\right) \left[B_{D-1}\left(y^2/2\right)\right]^{M-1}$$

$$\leq \frac{\exp\left(-d_T^2/2\right)}{d_T^{D-1}} \int_L^\infty dy\; y^D\, e^{-y^2/2}\, I_{D-1}\left(d_T y\right)$$

$$\approx \frac{1}{d_T^{D-1}} \int_L^\infty dy\; y^D \frac{\exp\left[-\frac{1}{2}\left(y - d_T\right)^2\right]}{\left(2\pi\, d_T\, y\right)^{1/2}}$$

$$\approx \left(\frac{L}{d_T}\right)^{D-\frac{1}{2}} \frac{1}{(2\pi)^{1/2}} \int_L^\infty dy\; \exp\left[-\frac{1}{2}\left(y - d_T\right)^2\right]$$

$$\approx \left(\frac{L}{d_T}\right)^{D-\frac{1}{2}} \frac{\exp\left[-\frac{1}{2}\left(L - d_T\right)^2\right]}{(2\pi)^{1/2}\left(L - d_T\right)} . \qquad \text{(C-1)}$$

We used: the fact that $B_n(x)$ is upper-bounded by 1, as may be seen from (3); the asymptotic behavior of $I_n(x)$ for large x as given by reference 2, equation 9.7.1; and an integration by parts to yield the final form. For given values of D and d_T (M has dropped out of this error bound), we step L up by units of 1 from the value $d_T + 1$ until (C-1) is sufficiently small.

We chose error 10^{-12} in the enclosed program. The reason that the integral in (1) must be evaluated very accurately is that we must subtract the integral result from 1 in order to get the error probability. If we want to evaluate P_e accurately in the range of 10^{-5}, then we need the integral to 8 or 9 decimal places to counteract the effect of subtraction from 1 required by form (1).

We numerically integrate (1) for y in the interval (0, L) via the Trapezoidal rule, with automatic halving of the interval, until a stable result occurs. It is shown in reference 11, pages 55 and 58, that the Trapezoidal rule is excellent for integrands whose odd derivatives at the end points of integration are equal. For the application here, the integrand of (1) may be shown to behave as y^{2DM} as $y \rightarrow 0+$. Thus a high number of odd derivatives are zero at $y = 0+$ and virtually zero at $y = L$ for large DM, meaning that the correction terms to the Trapezoidal rule vanish. In fact, (1) was also integrated via Simpson's rule and via Simpson's rule with end correction (reference 12, pages 414-418); neither performed as well as the Trapezoidal rule, which is incorporated in the program for P_e below. Because of the very small values for the exponentials and the very large values for the Bessel function, it was found necessary to first evaluate the logarithm of the integrand of (1) and then exponentiate it prior to its use in the Trapezoidal rule.

```
1 ! M-ary Pe for D-fold diversity, but no fading.  TR 6473
10    M=8         ! M>1       NUMBER OF SIGNAL ALTERNATIVES
20    D=12        ! D>0       NUMBER OF DIVERSITY CHANNELS
30    Etn0=20     ! Etn0>0    TOTAL SIGNAL ENERGY/SINGLE-SIDED NOISE DENSITY
40    Dt=SQR(2*Etn0)       !  TOTAL "DEFLECTION" STATISTIC
50    PRINT "M =";M,"D =";D,"Etn0 =";Etn0,"Dt =";Dt
60    COM D,D1,M1,Dt,Etn0
70    M1=M-1
80    D1=D-1
90    S1=0
100   S2=Dt
110   A=D-.5
120   B=.4/Dt^A
130   S2=S2+1
140   C=S2-Dt
150   E=B*S2^A*EXP(-.5*C*C)/C
160   IF E>1E-12 THEN 130
170   S3=(S2-S1)*.5
180   S4=2
190   S5=(FNS(S1)+FNS(S2))*.5
200   S6=0
210   FOR S7=1 TO S4-1 STEP 2
220   S6=S6+FNS(S1+S3*S7)
230   NEXT S7
240   PRINT USING "M.12DE,7D";1-(S5+S6)*S3,S4
250   S3=S3*.5
260   S4=S4*2
270   S5=S5+S6
280   GOTO 200
290   !
```

```
300   DEF FNS(Y)                           ! INTEGRAND
310   COM D,D1,M1,Dt,Etn0
320   T=.5*Y*Y
330   B=1-EXP(-T)*FNE(T,D1)
340   IF B<=0 THEN RETURN 0
350   S=D*LOG(Y/Dt)-T+FNInxlog(Dt*Y,D1)+M1*LOG(B)-Etn0
360   RETURN EXP(S)*Dt
370   FNEND
380   !
390   DEF FNE(X,N)        ! SUM OF X^K/K! FROM K = 0 TO N
400   S=T=1
410   FOR K=1 TO N
420   T=T*X/K
430   S=S+T
440   NEXT K
450   RETURN S
460   FNEND
470   !
480   DEF FNInxlog(X,N)                 ! LOG OF In(x)
490   IF X>0 THEN 520
500   IF N=0 THEN RETURN 1
510   IF N>0 THEN RETURN 0
520   A=.5*X
530   F=1
540   FOR I=2 TO N
550   F=F*I
560   NEXT I
570   F=N*LOG(A)-LOG(F)
580   A=A*A
590   S=T=1E-87
600   FOR I=1 TO 500
610   T=T*A/(I*(N+I))
620   S=S+T
630   IF T*2E11<=S THEN 660
640   NEXT I
650   OUTPUT 0;"500 TERMS AT ";X;N
660   Inxlog=F+LOG(S)+200.32490309
670   RETURN Inxlog
680   FNEND
```

```
M = 8                  D = 12                 Etn0 = 20              Dt = 6.32455532033
-.223330444766E+01       2
-.620160241930E+00       4
 .423327566000E-02       8
 .184637510900E-01      16
 .184216702500E-01      32
 .184216610500E-01      64
 .184216611700E-01     128
```

Appendix D

Average Error Probability for Fading Signal

The energy-density ratio variable R_T was defined in (8), and its probability density was given in (9) for a fading signal. From (1), the error probability expressed in terms of R_T is given by

$$P_e\left(R_T\right) = 1 - \frac{\exp\left(-R_T\right)}{\left(2R_T\right)^{\frac{D-1}{2}}} \int_0^\infty dy\ y^D\ e^{-y^2/2}\ I_{D-1}\left(\sqrt{2R_T}\ y\right)\left[B_{D-1}\left(y^2/2\right)\right]^{M-1} \quad . \tag{D-1}$$

The average error probability is

$$\overline{P_e} = \int_0^\infty dR_T\ p\left(R_T\right)\ P_e\left(R_T\right) \quad . \tag{D-2}$$

Substitution of (9) and (D-1) in (D-2) and interchange of integrals lead to

$$\overline{P_e} = 1 - \int_0^\infty dy\ y^D\ e^{-y^2/2}\left[B_{D-1}\left(y^2/2\right)\right]^{M-1}$$

$$* \int_0^\infty dR_T\ \frac{\exp\left(-R_T\right)}{\left(2R_T\right)^{\frac{D-1}{2}}}\ I_{D-1}\left(\sqrt{2R_T}\ y\right)\ \frac{R_T^\nu\ \exp\left(-R_T/\mu\right)}{\mu^{\nu+1}\ \Gamma(\nu+1)} \quad . \tag{D-3}$$

In the inner integral, let $R_T = t^2/2$ and use reference 9, equation 6.631 1. Simplification leads to

$$\overline{P_e} = 1 - \left[2^{D-1}(D-1)!\ (\mu+1)^{\nu+1}\right]^{-1} \int_0^\infty dy\ y^{2D-1}\ e^{-y^2/2}$$

$$* \left[B_{D-1}\left(y^2/2\right)\right]^{M-1}\ {}_1F_1\left(\nu+1;\ D;\ \frac{\mu}{\mu+1}\ \frac{y^2}{2}\right) \quad . \tag{D-4}$$

Use of the variable $t = y^2/2$ in (D-4) yields the result already quoted in (13). Results (D-4) and (13) hold for general ν and M. When ν is chosen as the special value D-1, the resulting average error probability is given by (15) and (16).

Specialization to M = 2

Now we let parameter ν in (9) be general, but we set M = 2. Thus we will develop the average binary error probability for a general signal fading model. Upon use of

(3) and (4), (13) yields for M = 2,

$$\overline{P_{e2}} = \left[(D-1)!\ (\mu+1)^{\nu+1}\right]^{-1}$$

$$\cdot \sum_{k=0}^{D-1} \frac{1}{k!} \int_0^\infty dt\ t^{D-1+k}\ e^{-2t}\ {}_1F_1\left(\nu+1;\ D;\ \frac{\mu}{\mu+1}\, t\right)$$

$$= \frac{(\mu+1)^{D-1-\nu}}{(\mu+2)^D} \sum_{k=0}^{D-1} \binom{D-1+k}{k} \left(\frac{\mu+1}{\mu+2}\right)^k F\left(D-1-\nu,\ D+k;\ D;\ \frac{-\mu}{\mu+2}\right) . \tag{D-5}$$

This finite sum of Gaussian hypergeometric functions is not too useful in general. However, for the special case $\nu = D - 1$ considered in (14)-(16), (D-5) immediately yields

$$\overline{P_{e2}} = \frac{1}{(\mu+2)^D} \sum_{k=0}^{D-1} \binom{D-1+k}{k}\left(\frac{\mu+1}{\mu+2}\right)^k \quad \text{for } \nu = D-1 ; \tag{D-6}$$

this last result can also be derived directly from (15) and (16) for M = 2, by reference to (3) and (4).

A much more appealing result than (D-5) for average binary error probability $\overline{P_{e2}}$ for general ν is attained if we start with the binary no-fading result in (B-9) and use identification (8):

$$P_{e2} = \frac{\exp\left(-R_T/2\right)}{2^{2D-1}} \sum_{n=0}^{D-1} \binom{2D-1}{n} \sum_{k=0}^{D-1-n} \frac{1}{k!}\left(\frac{R_T}{2}\right)^k . \tag{D-7}$$

Then the average binary error probability is, by use of (D-7) and (9),

$$\overline{P_{e2}} = \int_0^\infty dR_T\ p\left(R_T\right)\ P_{e2}$$

$$= \left[2^{2D-1}\ \mu^{\nu+1}\ \Gamma(\nu+1)\right]^{-1} \sum_{n=0}^{D-1} \binom{2D-1}{n} \sum_{k=0}^{D-1-n} \frac{1}{k!\ 2^k}$$

$$\cdot \int_0^\infty dR_T\ R_T^{\nu+k} \exp\left(-\frac{R_T}{\mu} - \frac{R_T}{2}\right)$$

$$= \left[2^{2D-1}\left(1+\frac{\mu}{2}\right)^{\nu+1}\right]^{-1} \sum_{n=0}^{D-1}\binom{2D-1}{n} \sum_{k=0}^{D-1-n} \frac{(\nu+1)_k}{k!}\left(\frac{\mu/2}{1+\mu/2}\right)^k$$

$$= \left[2^{2D-1} \left(1 + \frac{\mu}{2}\right)^{\nu+1} \right]^{-1} \sum_{k=0}^{D-1} \frac{(\nu + 1)_k}{k!} \left(\frac{\mu/2}{1 + \mu/2}\right)^k \sum_{n=0}^{D-1-k} \binom{2D-1}{n} . \tag{D-8}$$

This closed form result is a finite sum of positive quantities and holds for general values of $\nu(>-1)$. The summations are very similar to those encountered in non-fading binary error probability (B-10). A program for the evaluation of (D-8), which takes advantage of the observations made under (B-10), is given below. For $\nu = D - 1$, (D-8) must reduce to (D-6); we have not discovered the transformation that accomplishes this corroboration, but we have confirmed the equality numerically for $\nu = D - 1$, as well as for the alternative, more general result (D-5). No numerical results for (D-8) have been presented here.

```
1 ! Average binary Pe for D-fold diversity with fading.  TR 6473
10    D=12       ! D>0     NUMBER OF DIVERSITY CHANNELS
20    Nu=6.1     ! Nu>-1   MEASURE OF FADING, EQS. 9-10
30    Mu=3.7     ! Mu>=0   MEASURE OF SIGNAL-TO-NOISE RATIO
40    PRINT "D =";D,"Nu =";Nu,"Mu =";Mu
50    D1=D-1
60    D2=D*2
70    A=Mu/(2+Mu)
80    DIM S(100)
90    REDIM S(D1)
100   S(D1)=T=1
110   FOR J=1 TO D1
120   T=T*(D2-J)/J
130   S(D1-J)=S(D-J)+T
140   NEXT J
150   S=S(0)
160   T=1
170   FOR K=1 TO D1
180   T=T*(Nu+K)*A/K
190   S=S+T*S(K)
200   NEXT K
210   Pe=S/(2^(D2-1)*(1+.5*Mu)^(Nu+1))
220   PRINT "Pe =";Pe
230   END


D = 12               Nu = 6.1              Mu = 3.7
Pe = 6.10615516863E-03
```

Appendix E

Binary Error Probability Via Characteristic Functions

Let $p_0(y)$ and $p_1(x)$ be the general probability density functions of a noise-only decision variable and a signal-plus-noise decision variable, respectively. The binary probability of error is then

$$P_{e2} = \text{Prob}(x < y) = \int_{-\infty}^{+\infty} dx\ p_1(x) \int_{x}^{+\infty} dy\ p_0(y) \quad . \tag{E-1}$$

Now by reference 13, equation 6, the exceedance probability in (E-1) is given in terms of the characteristic function $f_0(\xi)$ according to

$$\begin{aligned}\int_{x}^{+\infty} dy\ p_0(y) &= 1 - P_0(x) \\ &= \frac{1}{2} + \frac{1}{i2\pi} ⨍_{-\infty}^{+\infty} \frac{d\xi}{\xi} f_0(\xi) \exp(-i\xi x) \\ &= \frac{1}{i2\pi} \int_{C_-} \frac{d\xi}{\xi} f_0(\xi) \exp(-i\xi x) \quad . \end{aligned} \tag{E-2}$$

The integral with a slash on it is a principal value integral; the contour C_- is a contour on the real axis of the complex ξ-plane except for a small downward identation at $\xi = 0$. Employment of (E-2) in (E-1) yields

$$\begin{aligned} P_{e2} &= \int_{-\infty}^{+\infty} dx\ p_1(x) \frac{1}{i2\pi} \int_{C_-} \frac{d\xi}{\xi} f_0(\xi) \exp(-i\xi x) \\ &= \frac{1}{i2\pi} \int_{C_-} \frac{d\xi}{\xi} f_0(\xi)\ f_1(-\xi) \\ &= \frac{-1}{i2\pi} \int_{C_+} \frac{d\xi}{\xi} f_0(-\xi)\ f_1(\xi) \quad , \end{aligned} \tag{E-3}$$

where C_+ is indented upward at $\xi = 0$. Both contours $C_\pm$ start at $\xi = -\infty$ and go to $\xi = +\infty$.

Expression (E-3) gives binary error probability P_{e2} directly in terms of characteristic functions $f_0(\xi)$ and $f_1(\xi)$, by means of a contour integral. Since characteristic functions satisfy the relation

$$f(-\xi) = f^*(\xi) \quad \text{for real } \xi \quad , \tag{E-4}$$

an alternative real integral form for (E-3) is

$$P_{e2} = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{d\xi}{\xi} \, \mathrm{Im}\{f_o(\xi)\ f_1(-\xi)\} \quad . \qquad \text{(E-5)}$$

Appendix F

Bounds on Error Probability

We consider the situation where there are three different variables upon which a decision as to the transmitted multiple signal alternative must be made. They are, respectively, the signal, noise, and interference variables. In particular, let random decision variable S denote the output of the signal channel; S may itself be the sum of several diversity branch outputs. Let N be the output of the *maximum* of the noise channels outputs, of which there may be many; each noise channel output may itself be the sum of several diversity branch outputs. And let I be the output of the *maximum* of the interference-plus-noise channel outputs, of which there may be many; each interference channel output may itself be the sum of several diversity branch outputs.

For independent signal, noise, and interference channel outputs, the probability of a correct decision is

$$P_c = \text{Prob}(I < S,\ N < S) = \int dx\ p_s(x)\ P_i(x)\ P_n(x) \quad , \tag{F-1}$$

where $p_a(x)$ is the probability density function of random variable a, and $P_a(x)$ is its corresponding cumulative distribution function. We also express

$$P_i(x) = \text{Prob}(I < x) = 1 - Q_i(x) \quad ,$$

$$P_n(x) = \text{Prob}(N < x) = 1 - Q_n(x) \quad , \tag{F-2}$$

where $Q_a(x)$ is the exceedance probability of random variable a.

The probability of character error is

$$\begin{aligned} P_e = 1 - P_c &= \int dx\ p_s(x) \left[1 - \{1 - Q_i(x)\}\{1 - Q_n(x)\}\right] \\ &= \int dx\ p_s(x) \left[Q_i(x) + Q_n(x) - Q_i(x)\ Q_n(x)\right] \\ &= \text{Prob}(I > S) + \text{Prob}(N > S) - \text{Prob}(I > S,\ N > S) \quad . \end{aligned} \tag{F-3}$$

The last quantity in (F-3) is generally quite small for practical cases of interest. But in any event, we have the upper bound

$$P_e \le \text{Prob}(I > S) + \text{Prob}(N > S) \qquad (\text{and } P_e \le 1) \quad . \tag{F-4}$$

In addition, since

$$\int dx\ p_s(x)\ Q_i(x)\ Q_n(x) \le \int dx\ p_s(x)\ \min\{Q_i(x),\ Q_n(x)\}$$

$$\le \min\left\{\int dx\ p_s(x)\ Q_i(x),\ \int dx\ p_s(x)\ Q_n(x)\right\}$$

$$= \min\{\text{Prob}(I > S),\ \text{Prob}(N > S)\} \quad , \tag{F-5}$$

we have the lower bound

$$P_e > \max\{\text{Prob}(I > S),\ \text{Prob}(N > S)\} \quad . \tag{F-6}$$

An alternative to (F-4) is obtained as follows:

$$\begin{aligned} \text{Prob}(I > S,\ N > S) &= \text{Prob}(I > S)\ \text{Prob}(N > S | I > S) \\ &\ge \text{Prob}(I > S)\ \text{Prob}(N > S) \quad . \end{aligned} \tag{F-7}$$

Therefore (F-3) yields upper bound

$$P_e \le \text{Prob}(I > S) + \text{Prob}(N > S) - \text{Prob}(I > S)\ \text{Prob}(N > S) \quad . \tag{F-8}$$

This is a tighter upper bound than (F-4), and is also probably a good approximation in many practical cases. The bound has a maximum value of 1 when either Prob(I>S) = 1 or Prob(N>S) = 1.

If we let

$$\begin{aligned} a &= \min\{\text{Prob}(I > S),\ \text{Prob}(N > S)\} \quad , \\ b &= \max\{\text{Prob}(I > S),\ \text{Prob}(N > S)\} \quad , \end{aligned} \tag{F-9}$$

then the ratio of upper and lower bounds in (F-8) and (F-6) is

$$\frac{\text{upper bound}}{\text{lower bound}} = \frac{(a + b) - ab}{b} = 1 + \frac{a}{b} - a \quad . \tag{F-10}$$

This is largest when $b = a << 1$, in which case

$$\frac{\text{upper bound}}{\text{lower bound}} = 2 - a \approx 2 \quad ; \tag{F-11}$$

thus the ratio of bounds is never greater than 2.

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