LEVEL II ②

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LOCAL EXISTENCE FOR THE CAUCHY PROBLEM
OF A REACTION-DIFFUSION SYSTEM WITH
DISCONTINUOUS NONLINEARITY

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ABSTRACT

The pure initial value problem for the system of equations

$$v_t = v_{xx} + f(v) - w$$

$$w_t = \epsilon(v - \gamma w)$$

is considered. Here ϵ and γ are positive constants, and

$f(v) = v - H(v - a)$ where H is the Heaviside step function and

$a \in (0, \frac{1}{2})$. Because of the discontinuity in f one cannot expect the

solution of this system to be very smooth. Sufficient conditions on the

initial data are given which guarantee the existence of a classical solution

in $R \times (0, T)$ for some positive time T.

AMS(MOS) Subject Classification: 35K65

Key words: Reaction Diffusion Equations, Classical Solution

Work Unit Number 1 - Applied Analysis

SIGNIFICANCE AND EXPLANATION

The most famous model for nerve conduction is due to Hodgkin and
Huxley. However, a mathematical analysis of their model has proven very
difficult. The complexity of the Hodgkin and Huxley model has led a number of
other authors to introduce simpler models. In this report we consider one
such simplification.

It has been demonstrated (experimentally) that impulses in the nerve axon
travel with constant shape and velocity. Mathematically, this corresponds to
traveling wave solutions. A number of authors have proven that the mathe-
matical equations considered here do possess traveling wave solutions.
Another property of impulses in the nerve axon is the existence of a threshold
phenomenon. This corresponds to the biological fact that a minimum stimulus
is needed to trigger an impulse. Here we prove some preliminary results which
will be used in a later report when it is demonstrated that the equations
under study do indeed exhibit a threshold phenomenon.

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LOCAL EXISTENCE FOR THE CAUCHY PROBLEM OF A

REACTION-DIFFUSION SYSTEM WITH DISCONTINUOUS NONLINEARITY

David Terman

1. INTRODUCTION

In this paper we consider the pure initial value problem for the FitzHugh-Nagumo
equations

$$(1.1) \qquad \qquad v_t = v_{xx} + f(v) - w$$
$$w_t = \epsilon(v - \gamma w) \ ,$$

the initial data being $(v(x,0), w(x,0)) = (\varphi(x),0)$. Here ϵ and γ are positive
constants. These equations were introduced as a qualitative model for nerve conduction
[2,5,7]. We follow McKean [4] and assume that $f(v)$ is given by $f(v) = v - H(v - a)$
where H is the Heaviside step function and $a \in (0, \frac{1}{2})$.

Note that because $f(v)$ is discontinuous we cannot expect the solution, (v,w), to
be very smooth. By a classical solution of System (1.1) we mean the following:

Definition: Let $S_T = R \times (0,T)$ and $G_T = \{(x,t) \in S_T: v(x,t) \neq a\}$. Then
$(v(x,t),w(x,t))$ is said to be a classical solution of the Cauchy problem (1.1) in S_T if:

(a) (v,w) along with (v_x,w_x) are bounded continuous functions in S_T.

(b)   in G_T, v_{xx}, v_t and w_t are continuous functions which satisfy the system
        of Equations (1.1),

(c) $\lim\limits_{t \downarrow 0} v(x,t) = \varphi(x)$ and $\lim\limits_{t \downarrow 0} w(x,t) = 0$ for each $x \in R$.

Throughout this paper we assume that $\varphi(x) = v(x,0)$ satisfies the following
conditions:

(a) $\varphi(x) \in C^1(\mathbb{R})$

(b) $\varphi(x) = \varphi(-x)$ in $\mathbb{R}$

(c) $\varphi(x_0) = a$ for some $x_0 > 0$

(1.2) (d) $\varphi(x) > a$ if and only if $|x| < x_0$

(e) $\varphi'(x_0) < 0$

(f) $\varphi''(x)$ is a bounded continuous function except possibly at

$x = x_0$.

This last condition is needed in order to obtain sufficient a priori bounds on the derivatives of the solution of System (1.1).

In this paper we prove that if $\varphi(x)$ satisfies (1.2) then there exists a classical solution of the Cauchy problem (1.1) in S_T for some $T > 0$. Here we give an outline of the proof.

From Assumption (1.2) we expect there to exist a positive constant T and a smooth function $s(t)$, defined in $[0,T]$, such that $v > a$ for $|x| < s(t)$ and $v < a$ for $|x| > s(t)$. Suppose that this is the case. We then set $G = \{(x,t) : |x| < s(t), 0 < t < T\}$ and let χ_G be the characteristic step function of the region G. It follows that if $|x| \neq s(t)$, then (v,w) is a solution of the system of equations

$$v_t = v_{xx} - v + \chi_G - w$$

(1.3) $$w_t = \varepsilon(v - \gamma w) \qquad \text{in } S_T ,$$

$$(v(x,0), w(x,0)) = (\varphi(x), 0) \qquad \text{in } \mathbb{R} .$$

Note that the first equation in (1.3) is similar to a nonhomogeneous heat equation while the second is just an ordinary differential equation. Formally, the solution of (1.3) can be written as:

$$v(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(x-\xi,t-\tau)d\xi$$

(1.4) $$- \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)w(\xi,\tau)d\xi$$

$$w(x,t) = \varepsilon e^{-\varepsilon\gamma t} \int_0^t e^{\varepsilon\gamma\eta} v(x,\eta)d\eta .$$

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Here $K(x,t) = \dfrac{e^{-t}}{2\pi^{1/2}t^{1/2}}\, e^{-x^2/4t}$ is the fundamental solution of the linear differential equation

(1.5) $$\psi_t = \psi_{xx} - \psi .$$

Setting $x = s(t)$ in (1.4) we find that, formally, $s(t)$ must satisfy the integral equation

$$a = \int_{-\infty}^{\infty} K(s(t)-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(s(t)-\xi,t-\tau)d\xi$$

(1.6)

$$- \int_0^t d\tau \int_{-\infty}^{\infty} K(s(t)-\xi,t-\tau)w(\xi,\tau)d\xi .$$

Using an iteration procedure, we prove the existence of functions $v(x,t), w(x,t)$, and $s(t)$ which satisfy the Equations (1.4) and (1.6). We then show that (v,w) is the desired classical solution of the Cauchy problem (1.1) in S_T .

We now introduce some notation.

Let $\psi(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi$. Note that $\psi(x,t)$ is the solution of the linear equation (1.5) with initial datum $\psi(x,0) = \varphi(x)$.

Suppose that $\alpha(t)$ is a positive, continuous function defined in $[0,T_1]$ for some $T_1 > 0$. Let $z(x,t)$ be a continuous function defined in $\mathbb{R} \times [0,T_1)$. Let

$$\phi(\alpha)(t) = \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t) - \xi, t-\tau)d\xi \quad \text{in} \quad [0,T_1]$$

and

$$\Gamma(z)(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)z(\xi,\tau)d\xi \quad \text{in} \quad \mathbb{R} \times [0,T_1] .$$

Note that $s(t)$ is a solution of (1.6) if and only if

(1.7) $$\psi(s(t),t) = a - \phi(s)(t) + \Gamma(w)(s(t),t).$$

In Section 2 we prove the properties of ψ and the operators ϕ and Γ which are needed in the proof of the local existence of a classical solution of System (1.1). The proof of local existence is given in Section 3.

2. The Operators Φ and Γ.

In this section we prove the properties of ψ and the operators Φ and Γ which are needed in the proof of the local existence of a classical solution of System (1.1).

Lemma 2.1. $\psi(x,t) \in C^{\infty}(\mathbb{R} \times \mathbb{R}^{+})$. Furthermore, there exist positive constants $\delta_1, \delta_2, \delta_3$ and λ such that $-\delta_1 < \psi_x(x,t) < -\delta_2$ and $|\psi_t(x,t)| < \delta_3$ in the rectangle $P = (x_0 - \lambda, x_0 + \lambda) \times (0, \lambda)$.

Proof: The first assertion is a standard result about solutions of Equation (1.5). The other assertions follow from the Assumptions (1.2e,f). (See Friedman [3], page 65).//

Lemma 2.2. Assume that $\alpha(t) \in C^1(0,T)$. Then $\Phi(\alpha)(t) \in C^1(0,T)$ and

$$\Phi(\alpha)'(t) = \int_{-x_0}^{x_0} K(\alpha(t)-\xi,t)d\xi + \int_0^t K(\alpha(t)+\alpha(\tau),t-\tau)[\alpha'(\tau) + \alpha'(t)]d\tau$$

(2.1)

$$+ \int_0^t K(\alpha(t)-\alpha(\tau),t-\tau)[\alpha'(\tau)-\alpha'(t)]d\tau \ .$$

Proof: Note that

$$\Phi(\alpha)'(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{\Phi(\alpha)(t+\varepsilon) - \Phi(\alpha)(t)\}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_0^{t+\varepsilon} d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} (\alpha(t+\varepsilon)-\xi,t+\varepsilon-\tau)d\xi \right.$$

$$\left. - \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t)-\xi,t-\tau)d\xi \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_{\varepsilon}^{0} d\tau \int_{-\alpha(\tau+\varepsilon)+\alpha(t)-\alpha(t+\varepsilon)}^{\alpha(\tau+\varepsilon)+\alpha(t)-\alpha(\tau+\varepsilon)} K(\alpha(t)-\xi,t-\tau)d\xi \right.$$

$$+ \int_0^t d\tau \int_{-\alpha(\tau+\varepsilon)+\alpha(t)-\alpha(t+\varepsilon)}^{-\alpha(\tau)} K(\alpha(t)-\xi,t-\tau)d\xi$$

$$\left. - \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau+\varepsilon)+\alpha(t)-\alpha(t+\varepsilon)} K(\alpha(t)-\xi,t-\tau)d\xi \right]$$

Passing to the limit we obtain (2.1). //

Lemma 2.3: Suppose that $\alpha(t) \in C^1(0,T)$ and $|\alpha'(t)| < M$ in $(0,T)$. Then

$$|\phi(\alpha)'(t)| < 1 + 4MT^{1/2} \quad \text{in} \quad (0,T) .$$

Proof: From (2.1) it follows that for $t \in (0,T)$,

$$|\phi(\alpha)'(t)| < 1 + 4M \int_0^t K(\alpha(t) - \alpha(\tau), t-\tau) d\tau$$

$$< 1 + 4M \int_0^t \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} d\tau$$

$$< 1 + 4MT^{1/2} . \quad //$$

Lemma 2.4: Let $\alpha(t)$ be as in the previous Lemma. Suppose that for some $\rho \in (0,T)$ there exists a constant M_1 such that

$$|\alpha'(t_1) - \alpha'(t_0)| < M_1 |t_1 - t_0|^{1/2}$$

for each $t_0, t_1 \in (\rho,T)$. Then there exist positive constants K_1 and K_2, which depend only on ρ and M, such that

$$|\phi(\alpha)'(t_1) - \phi(\alpha)'(t_0)| < (K_1 + K_2 M_1 T^{1/2})|t_1 - t_0|^{1/2}$$

for each $t_0, t_1 \in (\rho,T)$.

Proof: Fix $t_0, t_1 \in (\rho,T)$. Then

$$\phi(\alpha)'(t_1) - \phi(\alpha)'(t_0) = \int_{-x_0}^{x_0} [K(\alpha(t_1)-\xi, t_1) - K(\alpha(t_0)-\xi, t_0)] d\xi$$

$$+ \left[\int_0^{t_1} K(\alpha(t_1)-\alpha(\tau), t_1-\tau)[\alpha'(\tau)-\alpha'(t_1)] d\tau \right.$$

$$\left. - \int_0^{t_0} K(\alpha(t_0)-\alpha(\tau), t_0-\tau)[\alpha'(\tau)-\alpha'(t_0)] d\tau \right]$$

$$+ \left[\int_0^{t_1} K(\alpha(t_1)+\alpha(\tau), t_1-\tau)[\alpha'(\tau)+\alpha'(t_1)] d\tau \right.$$

$$\left. - \int_0^{t_0} K(\alpha(t_0)+\alpha(\tau), t_0-\tau)[\alpha'(\tau)+\alpha'(t_0)] d\tau \right]$$

$$= [A] + [B] + [C].$$

Since $K(x,t)$ is an infinitely differentiable function of (x,t) for $t > 0$, there exists a positive constant D_1 such that $|[A]| < D_1 |t_1 - t_0|^{1/2}$. Note that D_1 depends only on ρ and M.

Next consider [B] which we rewrite as

$$[B] = \int_{t_0 - t_1}^{t_0} K(\alpha(t_1) - \alpha(\tau + t_1 - t_0), t_0 - \tau)(\alpha'(\tau + t_1 - t_0) - \alpha'(t_1)) d\tau$$

$$- \int_0^{t_0} K(\alpha(t_0) - \alpha(\tau), t_0 - \tau)(\alpha'(\tau) - \alpha'(t_0)) d\tau .$$

Note that,

$$\alpha'(\tau + t_0 - t_1) - \alpha'(t_1) = [\alpha'(\tau + t_1 - t_0) - \alpha'(\tau)]$$

$$+ [\alpha'(\tau) - \alpha'(t_0)] + [\alpha'(t_0) - \alpha'(t_1)]$$

$$< [\alpha'(\tau) - \alpha'(t_0)] + 2M_1 |t_1 - t_0|^{1/2} .$$

Therefore,

$$|[B]| < \left[\int_{t_0 - t_1}^0 K(\alpha(t_1) - \alpha(\tau + t_1 - t_0), t_0 - \tau) [\alpha'(\tau + t_1 - t_0) - \alpha'(t_1)] d\tau \right]$$

$$+ \left[\int_0^{t_0} [K(\alpha(t_1) - \alpha(\tau + t_0 - t_1), t_0 - \tau) - K(\alpha(t_0) - \alpha(\tau), t_0 - \tau)] [\alpha'(\tau) - \alpha'(t_0)] d\tau \right]$$

$$+ \left[\int_0^{t_0} K(\alpha(t_1) - \alpha(\tau + t_1 - t_0), t_0 - \tau) 2M_1 |t_1 - t_0|^{1/2} d\tau \right]$$

$$= [B_1] + [B_2] + [B_3] .$$

Now,

$$[B_1] < 2M \int_{t_0 - t_1}^0 \frac{1}{2\pi^{1/2} (t_0 - \tau)^{1/2}} d\tau$$

$$= \frac{2M}{\pi^{1/2}} |t_1^{1/2} - t_0^{1/2}|$$

$$< D_2 |t_1 - t_0|^{1/2}$$

for some constant D_2 which depends only on M and ρ. We also have that

$$|[B_3]| < 2M_1|t_1-t_0|^{1/2} \int_0^{t_0} K(\alpha(t_1) - \alpha(\tau+t_1-t_0), t_0-\tau) d\tau$$

$$< 2M_1|t_1-t_0|^{1/2} \int_0^{t_0} \frac{1}{2\pi^{1/2}(t_0-\tau)^{1/2}} d\tau$$

$$< 2M_1 T^{1/2} |t_1-t_0|^{1/2}.$$

Now consider $[B_2]$. Note that

$$|[B_2]| < 2M \int_0^{t_0} |K(\alpha(t_1)-\alpha(\tau+t_1-t_0), t_0-\tau) - K(\alpha(t_0)-\alpha(\tau), t_0-\tau)| d\tau$$

$$= 2M \int_0^{t_0} \frac{1}{2\pi^{1/2}(t_0-\tau)^{1/2}} |\gamma(t_1,\tau) - \gamma(t_0,\tau)| d\tau$$

where

$$\gamma(t,\tau) = e^{-\frac{[\alpha(t)-\alpha(\tau+t-t_0)]^2}{4(t_0-\tau)}}.$$

Assume that $\tau \in (0,t_0)$. Then, by the Mean Value Theorem,

$$|\gamma(t_1,\tau) - \gamma(t_0,\tau)| < |\frac{\partial}{\partial t} \gamma(\eta,\tau)||t_1-t_0|$$

for some $\eta \in (t_0,t_1)$. (We assume, without loss of generality, that $t_0 < t_1$). Note that

$$|\frac{\partial}{\partial t} \gamma(\eta,\tau)| = \left[\frac{2|\alpha(\eta)-\alpha(\tau+\eta-t_0)|}{|t_0-\tau|} \right] |\alpha'(\eta) - \alpha'(\tau+\eta-t_0)| \cdot e^{-\frac{[\alpha(\eta)-\alpha(\tau+\eta-t_0)]^2}{4(t_0-\tau)}}$$

$$< \left[\frac{2M|t_0-\tau|}{|t_0-\tau|} \right] 2M \cdot 1$$

$$= 4M^2.$$

Therefore,

$$|[B_2]| \leq 8M^3 \int_0^{t_0} \frac{|t_1 - t_0|}{2\pi^{1/2} (t_0 - \tau)^{1/2}} d\tau$$

$$\leq 8M^3 T^{1/2} |t_1 - t_0|$$

$$\leq D_3 |t_1 - t_0|^{1/2}$$

for some positive constant D_3 which depends only on M. (Since we will eventually choose T to be small we assume throughout that $T < 1$.)

We have shown that

$$|[B]| \leq |[B_1]| + |[B_2]| + |[B_3]| \leq D_4 |t_1 - t_0|^{1/2} + 2M_1 T^{1/2} |t_1 - t_0|^{1/2}$$

where $D_4 \equiv D_2 + D_3$ depends only on M and ρ.

A similar computation shows that there exist constants D_5 and D_6, which depends only on ρ and M, such that

$$|[c]| \leq D_5 |t_1 - t_0|^{1/2} + D_6 M_1 T^{1/2} |t_1 - t_0|^{1/2}.$$

In fact, this computation is much easier since $K(\alpha(t_1) + \alpha(\tau), t_1 - \tau)$ and $K(\alpha(t_0) + \alpha(\tau), t_0 - \tau)$ are smooth functions of τ.

Setting $K_1 = D_1 + D_4 + D_6$ and $K_2 = 2 + D_6$. The result follows. //

We now consider the operator $\Gamma(z)(x,t)$. In what follows we assume that T is some positive constant and $S_{T_1} = R \times (0,T)$. We also assume that $\alpha(t), M, \rho$, and M_1 are as in the previous two lemmas, and set $h(t) = \Gamma(z)(\alpha(t), t)$.

<u>Lemma 2.5:</u> Assume that $z(x,t) \in C^{1,1}(S_T)$ with $|z|_{C^{1,1}} = Z$. Then,

i) $h(t) \in C^1(0,T)$,

ii)   there exist a constant K_3, which depends only on Z, such that

$$|h'(t)| \leq K_3 + ZMT^{1/2} \text{ for } t \in (0,T),$$

iii)   there exist constants K_4 and K_5, which depend only on $\rho, M,$ and Z such that

$$|h'(t_1) - h'(t_0)| \leq (K_4 + K_5 M_1 T^{1/2}) |t_1 - t_0|^{1/2}$$

for each $t_0, t_1 \in (\rho, T)$.

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<u>Proof:</u> Set $g(x,t) = \Gamma(z)(x,t)$. Then $g(x,t)$ is the solution of the inhomogeneous differential equation

$$u_t = u_{xx} - u + z$$

$$u(x,0) = 0 .$$

Since $z \in C^{1,1}(S_T)$ it follows from the Schauder estimates (see [3], page 65) that $g \in C^{2,\frac{1}{2}}(S_T)$ where $|g|_{C^{2,\frac{1}{2}}(S_T)}$ depends only on z. We set $K_6 = |g|_{C^{2,\frac{1}{2}}(S_T)}$. Furthermore, there exists a constant K_3, which depends only on z, such that $|g_t(x,t)| < K_3$ in S_T. Note that in S_T,

$$|g_x(x,t)| = |\int_0^t d\tau \int_{-\infty}^\infty K_x(x-\xi,t-\tau)z(\xi,\tau)d\xi|$$

$$= |\int_0^t d\tau \int_{-\infty}^\infty K_\xi(x-\xi,t-\tau)z(\xi,\tau)d\xi|$$

$$= |\int_0^t d\tau \int_{-\infty}^\infty K(x-\xi,t-\tau)z_\xi(\xi,\tau)d\xi|$$

$$< z \int_0^t d\tau \int_{-\infty}^\infty K(x-\xi,t-\tau)d\xi$$

$$< z \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau$$

$$< z T^{\frac{1}{2}} .$$

Now (i) follows because $h(t) = g(\alpha(t),t)$ where g and α are both smooth functions. (ii) is true because

(2.2) $\qquad h'(t) = g_x(\alpha(t),t)\alpha'(t) + g_t(\alpha(t),t) \qquad$ in $(0,T)$

and, therefore,

$$|h'(t)| < z MT^{\frac{1}{2}} + K_3 .$$

Finally, it follows from (2.2) that for $t_0, t_1 \in (\rho, T)$,

$$|h'(t_1) - h'(t_0)| \leq |g_x(\alpha(t_1), t_1) - g_x(\alpha(t_0), t_0)||\alpha'(t_1)|$$

$$+ |g_x(\alpha(t_0), t_0)||\alpha'(t_1) - \alpha'(t_0)| + |g_t(\alpha(t_1), t_1) - g_t(\alpha(t_0), t_0)|$$

$$\leq K_6 M|t_1 - t_0|^{1/2} + 2T^{1/2} M_1 |t_1 - t_0|^{1/2} + K_6|t_1 - t_0|^{1/2}.$$

(iii) now follows if we set $K_4 = K_6(M + 1)$ and $K_5 = 2$. //

3. Local Existence

We are now ready to prove the existence of a classical solution of the Cauchy problem (1.1) in S_T for some positive T . The idea of the proof is as follows.

Let $s_0(t) = x_0$ in $\mathbb{R}^+$ and suppose that for some time $T_1 > 0$ we have defined smooth functions $s_k(t)$ for $t \in [0,T_1]$, $k = 0,1,\cdots,n$. We then let $(v_n(x,t),w_n(x,t))$ be the solution of the integral equations

$$v_n(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s_n(\tau)}^{s_n(\tau)} K(x-\xi,t-\tau)d\xi$$

(3.1a)

$$- \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)w_n(\xi,\tau)d\xi$$

(3.1b) $$w_n(x,t) = \varepsilon e^{-\varepsilon\gamma t} \int_0^{\infty} e^{\varepsilon\gamma\eta} v_n(x,\eta)d\eta .$$

That such a solution exists is proved in Lemma (3.1). We then use the Implicit Function Theorem to define $s_{n+1}(t)$ as the solution of the equation

$$\psi(s_{n+1}(t),t) = a - \phi(s_n)(t) + \Gamma(w_n)(s_n(t),t) ,$$

$$s_n(0) = x_0 .$$

We show that the sequences of functions $\{s_n(t)\}$, $\{v_n(x,t)\}$, and $\{w_n(x,t)\}$ converge to functions $s(t)$, $v(x,t)$, and $w(x,t)$. These functions are shown to be solutions of the Equations (1.4) and (1.6). It is then shown that (v,w) is a classical solution of the Cauchy problem (1.1).

In what follows we let $s_0(t) = x_0$ in $\mathbb{R}^+$ and assume that smooth functions $s_k(t)$, $k = 0,1,\cdots,n$, have been defined in $[0,T_1]$ for some $T_1 > 0$. Restrictions on T_1 will be given later. We assume that $M_k = \sup_{t \in (0,T_1)} |s_k'(t)| < \infty$, for $k = 0,1,2,\cdots,n$. For each $\rho \in (0,T_1)$ we assume that there exist constants C_k such that

$$|s_k'(t_1) - s_k'(t_0)| < C_k |t_1 - t_0|^{1/2} \text{ for each } k \text{ and } t_0,t_1 \in (\rho,T_1).$$

Lemma 3.1: There exist bounded, continuous functions $(v_n(x,t),w_n(x,t))$ which satisfy the Equations (3.1) in S_{T_1}.

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Proof: The proof follows Evans and Shenk [1]. Let $v_{n_0}(x,t) = \varphi(x)$ and $w_{n_0}(x,t) = w(x,0) \equiv 0$ in S_{T_1}. Assuming that $v_{n_j}(x,t)$ and $w_{n_j}(x,t)$ have been defined for $j \geq 0$, we let $v_{n_{j+1}}(x,0) = \varphi(x)$ and $w_{n_{j+1}}(x,0) = w(x,0)$, and, for $(x,t) \in S_{T_1}$,

$$v_{n_{j+1}}(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s_n(\tau)}^{s_n(\tau)} K(x-\xi,t-\tau)d\xi$$

(3.2a)

$$- \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)w_{n_j}(\xi,\tau)d\xi$$

(3.2b)
$$w_{n_{j+1}}(x,t) = \varepsilon e^{-\varepsilon\gamma t}\int_0^t e^{\varepsilon\gamma\eta}v_{n_j}(x,\eta)d\eta.$$

The resulting sequences of functions, $\{v_{n_j}(x,t)\}$ and $\{w_{n_j}(x,t)\}$, are defined and continuous in S_{T_1}. We show that these sequences converge uniformly to a solution of the Equations (3.1). Note that since $\varphi(x)$ is bounded, it follows from induction that each of the functions v_{n_j} and w_{n_j} are bounded.

Let

$$\rho_j(t) = \sup_{(x,t)\in S_{T_1}} \{|v_{n_j}(x,t) - v_{n_{j-1}}(x,t)| + |w_{n_j}(x,t) - w_{n_{j-1}}(x,t)|\}.$$

From Equations (3.2) it follows that, for $(x,t) \in S_{T_1}$,

(3.3a) $\quad |v_{n_{j+1}}(x,t) - v_{n_j}(x,t)| \leq \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)|w_{n_j}(\xi,\tau) - w_{n_{j-1}}(\xi,\tau)|d\xi$

(3.3b) $\quad |w_{n_{j+1}}(x,t) - w_{n_j}(x,t)| \leq \varepsilon\int_0^t |v_{n_j}(x,\eta) - v_{n_{j-1}}(x,\eta)|d\eta.$

Adding (3.3a) and (3.3b) we obtain

$$\rho_{j+1}(t) \leq (1 + \varepsilon) \int_0^t \rho_j(\tau)d\tau \quad \text{for } t \in (0,T_1), \quad j = 1,2,\cdots.$$

If K is a bound on $\rho_1(t)$ for $0 < t < T_1$ we have

$$\rho_1(t) \leq K, \quad \rho_2(t) \leq (1 + \varepsilon)Kt, \cdots, \rho_{j+1}(t) \leq \frac{K[(1+\varepsilon)t]^j}{j!}, \cdots.$$

Thus, $\displaystyle\sum_{j=1}^{\infty} \rho_j(t) \leq \sum_{j=1}^{\infty} \frac{K[(1+\varepsilon)t]^{j+1}}{(j+1)!} \leq Ke^{(1+\varepsilon)T_1}$ and the sequences $\{v_{n_j}(x,t)\}$ and

$\{w_{n_j}(x,t)\}$ converge uniformly in S_{T_1} to limit functions $v_n(x,t)$ and $w_n(x,t)$.

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Moreover, passing to the limit in Equations (3.2) we find that $v_n(x,t)$ and $w_n(x,t)$ satisfy the Equations (3.1) in S_{T_1}. //

From the proof of the preceding Lemma it follows that there exist constants V and W such that $|v_n(x,t)| < V$ and $|w_n(x,t)| < W$ in S_{T_1}. Note that V and W can be chosen independent of the curve $s_n(t)$. From (3.1) it follows that $(v_n)_x(x,t)$ and $(w_n)_x(x,t)$ both exist in S_{T_1}, except possibly at $|x| = s_n(t)$, $0 < t < T_1$.

__Lemma 3.2.__ There exist constants V_1 and W_1, independent of the curve $s_n(t)$, such that $|(v_n)_x(x,t)| < V_1$ and $|(w_n)_x(x,t)| < W_1$ in S_{T_1}, except possibly at $|x| = s_n(t)$.

__Proof:__ Suppose that $x \neq s_n(t)$. We differentiate both sides of (3.1a) to obtain

$$(v_n)_x(x,t) = \int_{-\infty}^{\infty} K_x(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s_n(\tau)}^{s_n(\tau)} K_x(x-\xi,t-\tau)d\xi$$

$$- \int_0^t d\tau \int_{-\infty}^{\infty} K_x(x-\xi,t-\tau)w_n(\xi,\tau)d\xi .$$

Integrating by parts in the first integral yields

$$|(v_n)_x(x,t)| < |\varphi'(x)|_{\infty} + (1 + W) \int_0^t d\tau \int_{-\infty}^{\infty} |K_x(x-\xi,t-\tau)|d\xi .$$

Note that

$$\int_0^t d\tau \int_{-\infty}^{\infty} |K_x(x-\xi,t-\tau)|d\xi = 2 \int_0^t d\tau \int_0^{\infty} |K_x(\eta,t-\tau)|d\eta$$

$$= \int_0^t \frac{e^{-(t-\tau)}}{\pi^{1/2}(t-\tau)^{1/2}} d\tau \int_0^{\infty} \frac{\eta}{2(t-\tau)} e^{-\eta^2/4(t-\tau)} d\eta$$

$$= \int_0^t \frac{e^{-(t-\tau)}}{\pi^{1/2}(t-\tau)^{1/2}} d\tau$$

$$< T_1^{1/2} .$$

Therefore,

$$|(v_n)_x(x,t)| < |\varphi'(x)|_{\infty} + (1 + W)T_1^{1/2} \equiv V_1 .$$

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Differentiating both sides of (3.1b) yields

$$|(w_n)_x(x,t)| \leq \epsilon \int_0^t |(v_n)_x(x,\eta)| \, d\eta$$

$$\leq \epsilon \, V_1 \, T_1 \equiv W_1 \, . \, //$$

Lemma 3.3: $(w_n)_t(x,t)$ is a bounded continuous function in S_{T_1} .

Proof: This follows because $(w_n)_t = \epsilon(v_n - \gamma w_n)$ in S_{T_1} . We choose W_2 so that

$$|(w_n)_t(x,t)| < W_2 \quad \text{in} \quad S_{T_1} . \, //$$

Let $\overline{W} = W_1 + W_2 + W_3$.

We wish to define $s_{n+1}(t)$ implicitly as the solution of the equation:

(3.4)
$$\psi(s_{n+1}(t),t) = a - \phi(s_n)(t) + \Gamma(w_n)(s_n(t),t) \, ,$$

$$s_{n+1}(0) = x_0 \, .$$

Recall that we are assuming that $s_n(t)$ is a smooth function in $(0,T_1)$,

$M_n = \sup_{t \in (0,T_1)} s_n'(t) < \infty$, and given $\rho \in (0,T_1)$, there exists a constant C_n such that

$|s_n'(t_1) - s_n'(t_0)| < C_n|t_1 - t_0|^{1/2}$ for each $t_0, t_1 \in (\rho, T_1)$. From Lemmas 2.3, 2.4 and 2.5 we conclude the following.

Let $\beta(t)$ equal the right hand side of (3.4). Then,

a) $\beta(t) \in C^1(0,T_1)$,

b)   there exists constants K_7 and K_8 such that

(3.5)
$$|\beta'(t)| < K_7 + K_8 \, M_n T_1^{1/2} \quad \text{in} \quad (0,T_1) \, ,$$

c)   there exist constants K_9 and K_{10} such that

$$|\beta'(t_1) - \beta'(t_0)| < (K_9 + K_{10} \, C_n \, T_1^{1/2}) \, |t_1 - t_0|^{1/2}$$

for each $t_1, t_0 \in (\rho, T_1)$.

Note that the constants K_7 and K_8 depend only on $\overline{W}$, and are, therefore, independent of n . Furthermore, K_9 and K_{10} depend on ρ and the bound on $|\beta'(t)|$ given in (3.5b). Hence, K_9 and K_{10} can be chosen independently of n .

We conclude from Lemma 2.1 and the implicit function theorem that there exists a smooth function $s_{n+1}(t)$, defined for some time, (say $t \in [0,T_2]$), which is a solution of (3.4). We show that as long as $(s_{n+1}(t),t)$ stays in the rectangle P, defined in Lemma 2.1, then $s'_{n+1}(t)$ is bounded, independently of n.

We differentiate Equation 3.4 to obtain

$$\psi_x(s_{n+1}(t),t)s'_{n+1}(t) + \psi_t(s_{n+1}(t),t) = \beta'(t) \ ,$$

or

(3.6)
$$s'_{n+1}(t) = \frac{1}{\psi_x(s_{n+1}(t),t)} [\beta'(t) - \psi_t(s_{n+1}(t),t)] \ .$$

From Lemma 2.1 and (3.5b) it follows that if $(s_{n+1}(t),t) \in P$, then

$$|s'_{n+1}(t)| < \frac{1}{\delta_2} [K_7 + K_8 M_n T_1^{1/2} + \delta_3]$$

$$= K_{11} + K_{12} M_n T_1^{1/2}$$

where $K_{11} = \frac{1}{\delta_2} (K_7 + \delta_3)$ and $K_{12} = K_8/\delta_2$ do not depend on n.

Suppose that $T_1 < \left(\frac{1}{2K_{12}} \right)^2$. Then, as long as $(s_{n+1}(t),t) \in P$,

$$|s'_{n+1}(t)| < K_{11} + \tfrac{1}{2} M_n \ .$$

Hence,

$$M_{n+1} < K_{11} + \tfrac{1}{2} M_n < K_{11} + \tfrac{1}{2} (K_{11} + \tfrac{1}{2} M_{n-1}) < \cdots <$$

$$< K_{11}(1 + \tfrac{1}{2} + \cdots + \frac{1}{2^{n-1}}) + 2^{-n} M_0 < 2K_{11} + M_0 \equiv \tilde{M} \ .$$

Therefore, the sequence $\{s'_n(t)\}$ is uniformly bounded by the constant $\tilde{M}$. It follows that there exists a constant T such that $T < T_1$, and $(s_n(t),t) \in P$ for each $t \in (0,T)$ and each n. Furthermore, there exists a subsequence $\{s_{n_j}(t)\}$ which converges uniformly on $[0,T]$ to a continuous function $s(t)$. We assume, without loss of generality, that $\{s_{n_j}(t)\} = \{s_n(t)\}$.

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<u>Lemma 3.4</u>: Fix $\rho \in (0,T)$. There exist positive constants K_{13} and K_{14} such that

$$|s'_{n+1}(t_1) - s'_{n+1}(t_0)| < (K_{13} + K_{14} C_n T_1^{1/2})|t_1 - t_0|^{1/2}$$

for each n and $t_0, t_1 \in (\rho, T)$. The constants K_{13} and K_{14} can be chosen independently of n.

<u>Proof</u>: This follows from (3.5c), (3.6), and Lemma 2.1. //

We now assume that $T < \left(\dfrac{1}{2K_{14}} \right)^2$. Then the previous lemma implies that

$$C_{n+1} < K_{13} + \tfrac{1}{2} C_n < \cdots < K_{13}(1 + \tfrac{1}{2} + \cdots + \dfrac{1}{2^{n-1}}) + 2^{-n} M_0$$

$$< 2K_{13} + C_0 \equiv \tilde{C} .$$

That is, given $\rho \in (0,T)$, there exists a constant $\tilde{C}$ such that

$$|s'_n(t_1) - s'_n(t_0)| < \tilde{C} |t_1 - t_0|^{1/2}$$

for each n and $t_0, t_1 \in (\rho, T)$. It follows that $s'(t)$ is continuously differentiable in $(0,T)$ and a subsequence of $\{s'_n(t)\}$ converges uniformly on compact subsets of $(0,T)$ to $s'(t)$. With loss of generality we assume that $\{s'_n(t)\}$ converges uniformly on compact subsets of $(0,T)$ to $s'(t)$. //

<u>Lemma 3.5</u>: The sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly in S_T to continuous functions v and w which satisfy the Equations (1.4).

<u>Proof</u>: Let $\rho_n(t) = \sup_{x \in \mathbb{R}} \{|v_{n+1}(x,t) - v_n(x,t)| + |w_{n+1}(x,t) - w_n(x,t)|\}$. From (3.1a) it follows that for $(x,t) \in S_T$,

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$$|v_{n+1}(x,t) - v_n(x,t)| \leq |\int_0^t d\tau \int_{-s_{n+1}(\tau)}^{s_{n+1}(\tau)} K(x-\xi,t-\tau)d\xi -$$

$$- \int_0^t d\tau \int_{-s_n(\tau)}^{s_n(\tau)} K(x-\xi,t-\tau)d\xi| + \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)|w_{n+1}(\xi,\tau) - w_n(\xi,\tau)|d\xi$$

$$= |\int_0^t d\tau \int_{-s_{n+1}(\tau)}^{-s_n(\tau)} K(x-\xi,t-\tau)d\xi + \int_0^t d\tau \int_{s_n(\tau)}^{s_{n+1}(\tau)} K(x-\xi,t-\tau)d\xi|$$

$$+ \int_0^t d\tau \int_{-\infty}^{\infty} K(x-\xi,t-\tau)|w_{n+1}(\xi,\tau) - w_n(\xi,\tau)|d\xi$$

$$\leq 2 \sup_{0<\tau<t} |s_{n+1}(\tau) - s_n(\tau)| \int_0^t \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} d\tau +$$

$$+ \int_0^t \sup_{x\in\mathbb{R}} |w_{n+1}(x,\tau) - w_n(x,\tau)|d\tau$$

$$(3.7) \qquad \leq 2T^{1/2} \sup_{0<\tau<t} |s_{n+1}(\tau)-s_n(\tau)| + \int_0^t \sup_{x\in\mathbb{R}} |w_{n+1}(x,\tau) - w_n(x,\tau)|d\tau .$$

From (3.1b) it follows that for $(x,t) \in \mathcal{B}_T$,

$$(3.8) \qquad |w_{n+1}(x,t) - w_n(x,t)| \leq \epsilon \int_0^t |v_{n+1}(x,\tau) - v_n(x,\tau)|d\tau .$$

Let $\delta_n = 2T^{1/2} \sup_{0<\tau<T} |s_{n+1}(\tau) - s_n(\tau)|$. Note that $\delta_n \to 0$ as $n \to \infty$. Adding (3.7) and

(3.8) we find that for $t \in (0,T)$,

$$\rho_n(t) \leq \delta_n + (1+\epsilon) \int_0^t \rho_n(\tau)d\tau .$$

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From Gronwall's inequality it follows that

$$\rho_n(t) < C \ \delta_n$$

for some constant C independent of n. Therefore $\rho_n(t) \to 0$ uniformly as $n \to \infty$, and the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly to limit functions $v(x,t)$ and $w(x,t)$. Passing to the limits in (3.1) we find that $(v(x,t),w(x,t))$ satisfies the Equations (1.4). This implies that v and w are continuous functions in S_T. //

Theorem 3.6: Let $K = \frac{1}{4}\min(\frac{1}{2} - a, a)$ and suppose that $T < \frac{K}{2\varepsilon V}$. Then (v,w) is a classical solution of the Cauchy problem (1.1) in S_T.

Proof: Throughout this proof we assume that $t \in (0,T)$. Recall that $v(x,t), w(x,t)$, and $s(t)$ satisfy the Equations (1.4) and (1.6). Setting $x = s(t)$ in (1.4) and subtracting the resulting equation from (1.6) we find that $v(s(t),t) = a$.

Equation (1.4) implies that for $|x| < s(t)$, (v,w) satisfies the differential equations

(3.9)
$$v_t = v_{xx} - v + 1 - w$$
$$w_t = \varepsilon(v - \gamma w)$$

and, for $|x| > s(t)$, (v,w) satisfies the differential equations

(3.10)
$$v_t = v_{xx} - v - w$$
$$w_t = \varepsilon(v - \gamma w) \ .$$

We show that $v > a$ for $|x| < s(t)$, and $v < a$ for $|x| > s(t)$. This implies that for $x \neq s(t)$, (v,w) satisfies the system of Equations (1.1).

Suppose it were not true that $v > a$ for $|x| < s(t)$, and $v < a$ for $|x| > s(t)$. For example, suppose that $v(x_1,t_1) < a$ where $|x_1| < s(t_1)$. Since $v(x,0) > a$ for $|x| < x_0 = s(0)$, we may assume that $v(x_1,t_1) = a$ and $v(x,t) > a$ in the region

$G = \{(x,t):|x| < s(t), t \in (0,t_1)\}$. We use the maximum principle (see [6], page 159) to show that this is impossible. Note that $v = a$ for $|x| = s(t)$ and $v(x,0) > a$ for $|x| < x_0$. Let L be the operator defined by $Lv \equiv v_t - v_{xx} + v$. Then, in G, $Lv = 1 - w$. From (1.4b) it follows that in $\mathbb{R} \times (0,t_1)$,

$$|w(x,t)| < \varepsilon \int_0^t |v(x,\eta)|\,d\eta$$

(3.11)

$$< \varepsilon \sqrt{T} < K .$$

Therefore, in G, $Lv > 1 - K > a = L(a)$. It now follows from the maximum principle that $v(x_1,t_1) > a$. This is a contradiction. A similar argument shows that it is impossible for $v > a$ for $|x| > s(t)$.

We have shown that except for $x \neq s(t)$, (v,w) satisfies the system of Equations (1.1) in S_T. It remains to show that $v_x(x,T)$ exists for $|x| = s(t)$.

Assume that $|x| < s(t)$ and $|\xi| < s(\tau)$. Then $(v(\xi,\tau),w(\xi,\tau))$ satisfies the system of equations

$$v_\tau - v_{\xi\xi} + v = 1 - w$$
$$w_\tau = \varepsilon(v - \gamma w) .$$

Multiply both sides of the first equation by $K(x-\xi,t-\tau)$, integrate by parts, and use the fact that $K_\tau + K_{\xi\xi} - K = 0$ to obtain:

$$(Kv)_\tau - (Kv_\xi)_\xi + (K_\xi v_\xi) = (1 - w)K .$$

We integrate this last equation for $-s(\tau) < \xi < s(\tau)$, $\delta < \tau < t - \delta$, and let $\delta \to 0$ to obtain:

$$v(x,t) - \int_{-x_0}^{x_0} K(x-\xi,t)\varphi(\xi)d\xi - \int_0^t K(x-s(\tau),t-\tau)as'(\tau)d\tau$$

$$- \int_0^t K(x+s(\tau),t-\tau)as'(\tau)d\tau - \int_0^t K(x-s(t),t-\tau)v_\xi(s(\tau)^-,\tau)d\tau$$

(3.12a)
$$+ \int_0^t K(x+s(\tau),t-\tau)v_\xi(-s(\tau)^+,\tau)d\tau + \int_0^t a\, K_\xi(x-s(\tau),t-\tau)d\tau$$

$$- \int_0^t a\, K_\xi(x+s(\tau),t-\tau)d\tau$$

$$= \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} (1-w)K(x-\xi,t-\tau)d\xi \ .$$

Next assume that $\xi > s(\tau)$. Then $v(\xi,\tau)$ satisfies the differential equation

$$v_\tau - v_{\xi\xi} + v = -w \ .$$

Multiply both sides of this equation by $K(x-\xi,t-\tau)$ and integrate by parts to obtain:

$$(Kv)_\xi - (Kv_\xi)_\xi + (K_\xi v)_\xi = -Kw \ .$$

Integrate this last equation for $s(\tau) < \xi < \infty$, $\delta < \tau < t - \delta$ and let $\delta \to 0$ to

obtain:

$$- \int_{x_0}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t K(x-s(\tau),t-\tau)as'(\tau)d\tau$$

(3.12b)
$$+ \int_0^t K(x-s(\tau),t-\tau)v_\xi(s(\tau)^+,\tau)d\tau - \int_0^t a\, K_\xi(x-s(\tau),t-\tau)d\tau$$

$$= \int_0^t d\tau \int_{s(\tau)}^{\infty} K(x-\xi,t-\tau)w(\xi,\tau)d\xi \ .$$

Similarly, for $\xi < s(\tau)$ we obtain:

$$\int_{-\infty}^{-x_0} K(x-\xi,t-\tau)d\xi + \int_0^t K(x+s(\tau),t-\tau)a\, s'(\tau)d\tau$$

(3.12c)
$$- \int_0^t K(x+s(\tau),t-\tau)v_\xi(-s(\tau)^-,\tau)d\tau$$

$$+ \int_0^t a\, K_\xi(x+s(\tau),t-\tau)d\tau = \int_0^t d\tau \int_{-\infty}^{-s(\tau)} K(x-\xi,t-\tau)w(\xi,\tau)d\xi \ .$$

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Adding (3.12a), (3.12b), and (3.12c), and using (1.6) we find that

$$(3.13) \qquad \int_0^t [K(x-s(\tau),t-\tau)[v_\xi(s(\tau)^+,\tau) - v_\xi(s(\tau)^-,\tau)]$$

$$+ K(x+s(\tau),t-\tau)[v_\xi(-s(\tau)^+,\tau) - v_\xi(-s(\tau)^-,\tau)]]d\tau = 0 .$$

Using the assumption that $\varphi(x) = \varphi(-x)$ it follows from (1.4) that $v(x,t) = v(-x,t)$ in S_T. Therefore, (3.13) can be rewritten as

$$\int_0^t [K(x-s(\tau),t-\tau) - K(x+s(\tau),t-\tau)][v_\xi(s(\tau)^+,\tau) - v_\xi(s(\tau)^-,\tau)]d\tau = 0 .$$

Since $[K(x-s(\tau),t-\tau) - K(x+s(\tau),t-\tau)] > 0$ in $(0,T)$ we conclude that $v_x(s(t)^-,t) = v_x(s(t)^+,t)$ in $(0,T)$.

We have shown that $v_x(x,t)$ is a bounded continuous function in S_T. From (1.4b) it follows that $w_x(x,t)$ is also a bounded continuous function in S_T . //

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The pure initial value problem for the system of equations

$$v_t = v_{xx} + f(v) - w$$

$$w_t = \varepsilon(v - \gamma w)$$

is considered. Here ε and γ are positive constants, and

20. ABSTRACT - Cont'd.

$f(v) = v - H(v - a)$ where   H   is the Heaviside step function and
$a \in (0, \frac{1}{2})$.   Because of the discontinuity in   f   one cannot expect the
solution of this system to be very smooth.   Sufficient conditions on the
initial data are given which guarantee the existence of a classical solution
in $\mathbb{R} \times (0,T)$ for some positive time   T.