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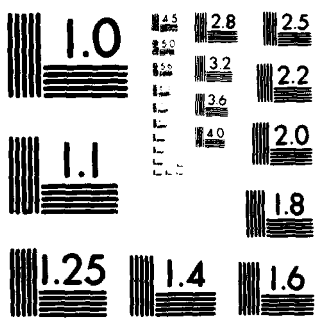

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CONDITIONS UNDER WHICH $E\{N_1\} = \infty$ FOR TONG'S ADAPTIVE SOLUTION TO RANKING AND SELECTION PROBLEMS

by

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Abstract

Tong [9] proposed an adaptive approach as an alternative to the classical indifference-zone formulation of the problems of ranking and selection. With a fixed pre-selected γ^* $(1/k < \gamma^* < 1)$ his procedure calls for the termination of vector-at-a-time sampling when the estimated probability of a correct selection exceeds γ^* for the first time. The purpose of this note is to show that for the case of two normal populations with common known variance, the expected number of vector-observations required by Tong's procedure to terminate sampling approaches infinity as the two population means approach equality for $\gamma^* \geq \sim 0.8413$. This phenomenon presumably also persists if the two largest of $k \geq 3$ population means approach equality. Since in the typical ranking and selection setting it usually is assumed that the experimenter has no knowledge concerning the differences between the population means, the experimenter who uses Tong's procedure clearly does so at his own risk.

1. Introduction. We retain the notation of Tong [9] and consider the important case of Section 5 of his paper, namely that of the normal family. Here $\{X_{ij}\}_{j=1}^{\infty}$ are i.i.d. normal r.v.'s with means θ_i $(1 \leq i \leq k)$ and common known variance σ^2. For every n $(n = 1,2,\ldots)$ let $\overline{X}_i^{(n)} = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ $(1 \leq i \leq k)$, and denote the ordered sample means by $\overline{X}_{[1]}^{(n)} <\ldots< \overline{X}_{[k]}^{(n)}$. Define $\hat{\delta}_i = \hat{\delta}_i^{(n)} = \overline{X}_{[k]}^{(n)} - \overline{X}_{[i]}^{(n)}$ $(1 \leq i \leq k-1)$, and

$$\hat{\gamma}_1^{(n)} = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi(y + n^{1/2}\hat{\delta}_i/\sigma)d\Phi(y), \tag{1}$$

the latter being an estimate of the probability of a correct selection (PCS) at stage n. (See Gibbons, Olkin, and Sobel [5], Section 2.3.4.) Here $\Phi(\cdot)$ is the standard normal cdf. Tong's primary procedure, R_1, stops at the first stage that $\hat{\gamma}_1^{(n)} \geq \gamma^*$ where γ^* $(1/k < \gamma^* < 1)$ is a pre-selected constant, analogous to P^* in the classical indifference-zone formulation of Bechhofer [1]. Thus the stopping stage for R_1 is $N_1 = \inf\{n: \hat{\gamma}_1^{(n)} \geq \gamma^*\}$ where the infimum of the empty set is defined to be $+\infty$.

Denoting the ordered values of the θ_i by $\theta_{[1]} \leq\cdots\leq \theta_{[k]}$, Tong [9] proved for arbitrary $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)$ with $\theta_{[k-1]} < \theta_{[k]}$ that asymptotically $(\gamma^* \to 1)$ we have $E_{\underset{\sim}{\theta}}\{N_1\} \sim n_1^*$ where $n_1^* = n_1^*(\underset{\sim}{\theta})$ is the smallest single-stage sample size that achieves a PCS of γ^* for that $\underset{\sim}{\theta}$. If $\underset{\sim}{\theta}'$ is more (less) favorable to the experimenter than $\underset{\sim}{\theta}$, and n_1^* was chosen under the assumption that $\underset{\sim}{\theta}$ is the true state of nature, then one would anticipate that $E_{\underset{\sim}{\theta}'}\{N_1\} < n_1^* (> n_1^*)$. In particular, when $\theta'_{[k-1]} = \theta'_{[k]}$ one might anticipate that $E_{\underset{\sim}{\theta}'}\{N_1\} >> n_1^*$. In Section 2 we prove for $k = 2$ that $E_{\underset{\sim}{\theta}}\{N_1\}$ is actually infinite when $\theta_{[1]} = \theta_{[2]}$ and γ^* is moderately large.

2. The case of $k = 2$ normal populations

Our result is summarized in the following theorem:

<u>Theorem</u>: For $k = 2$ and $\theta_{[1]} = \theta_{[2]}$, the r.v. $N_1 = \inf\{n: \hat{\gamma}_1^{(n)} \geq \gamma^*\}$ has $E_{\underset{\sim}{\theta}}\{N_1\} = \infty$ when $\Phi^{-1}(\gamma^*) \geq 1$.

<u>Remark 2.1</u>: $\Phi^{-1}(\gamma^*) \geq 1$ for $\gamma^* \geq \Phi(1) \sim 0.8413$. Hence, for $\gamma^* \geq \Phi(1)$ we have $E_{\underset{\sim}{\theta}}\{N_1\} \to \infty$ as $\theta_{[2]} - \theta_{[1]} \to 0$.

<u>Proof</u>: For $k = 2$ normal populations we can write (1) as

$$\begin{aligned} \hat{\gamma}_1^{(n)} &= \int_{-\infty}^{\infty} \Phi(y + n^{1/2}\hat{\delta}_1/\sigma)d\Phi(y) \\ &= \Phi\left(\sqrt{\frac{n}{2}}\,\frac{\hat{\delta}_1}{\sigma}\right). \end{aligned} \tag{2}$$

Then $N_1 = \inf\{n: \hat{\delta}_1 \geq \sqrt{\frac{2\sigma^2}{n}}\ \Phi^{-1}(\gamma^*)\}$. Since $\hat{\delta}_1 = |\overline{X}_1^{(n)} - \overline{X}_2^{(n)}|$ we see that

$$\begin{aligned} N_1 &= \inf\{n: |\sum_{j=1}^{n} (X_{1j} - X_{2j})/\sqrt{2}\sigma| \geq n^{1/2}\Phi^{-1}(\gamma^*)\} \\ &= \inf\{n: |S_n| \geq n^{1/2}\Phi^{-1}(\gamma^*)\} \end{aligned}$$

where for $\theta_1 = \theta_2$ we have that S_n is the sum of n i.i.d. $N(0,1)$ r.v.'s. By Corollory 1 of Chow, Robbins and Teicher [3], p. 792, we see that $E\{N_1\}$ is infinite for $\Phi^{-1}(\gamma^*) \geq 1$, i.e., for $\gamma^* \geq \Phi(1) \sim 0.8413$, and thus the theorem is proved.

<u>Remark 2.2</u>: The same result holds for Tong's procedure R_2 since R_2 stops after R_1.

3. Comparisons of $E\{N_1\}$ with the corresponding quantity for other competing procedures

In the setting of Section 1 with $k \geq 2$, the expected number of stages required to terminate experimentation for three other sequential procedures are compared with the corresponding quantity for Tong's R_1. The procedures are the closed sequential procedure (with elimination) of Paulson [6] (with the improvement of Fabian [4]), the open sequential procedure (without elimination) of Bechhofer, Kiefer, and Sobel [2], p. 264, and the two-stage procedure (with elimination) of Tamhane and Bechhofer [7], [8]; all of these procedures are known to guarantee the indifference-zone probability requirement for all $k \geq 2$ and specified $\{\delta^*, P^*\}$ $(0 < \delta^* < \infty,\ 1/k < P^* < 1)$. For Paulson's procedure the number of stages is bounded. For the Bechhofer-Kiefer-Sobel procedure the expected number of stages for $k = 2$ has been proven to be bounded ([2], p. 224); for this procedure with $k \geq 2$ and $\theta_{[1]} = \theta_{[k]}$, an excellent approximation $(P^* \to 1)$ supported by Monte Carlo sampling results is available for the expected number of stages ([2], pp. 296-7). For the Tamhane-Bechhofer procedure the number of vector-observations is bounded (and the expected total number of observations required to terminate experimentation is uniformly in $\underset{\sim}{\theta}$ less than the total number required by the corresponding single-stage procedure of Bechhofer [1]).

In view of the desirable properties of the above procedures there appears to be little justification for using Tong's procedure.

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