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For mathematical convenience, military targets are generally represented by simple geometric shapes (circles, elipses and rectangles) in performance models which require the computation of hit probability. A much more realistic representation of target shape would frequently be possible through the use of a polygon. This report derives a simple algorithm for the computation of the probability of hitting a general n-sided polygon.

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THE PROBABILITY OF HITTING A POLYGONAL TARGET

1. INTRODUCTION

Military targets are generally represented by simple geometric shapes in performance models which require the computation of hit probability. Circles and especially rectangles whose sides are parallel to the coordinate axes of the weapon delivery error distribution are the most commonly used shapes. A much more realistic representaion of target shape would frequently be possible through the use of a polygon. However, with such a representation the calculation of hit probability becomes much more difficult. The purpose of this report is to present a simple algorithm for the computation of such probabilities.

2. DISCUSSION

The following paragraphs state the problem, the assumptions made, and develop the computational algorithm.

2.1 Assumptions.

The target representation will be assumed to be a general n-sided planar polygon. The weapon delivery error will be assumed to be bivariate in the plane of the target. The general algorithm to be presented does not require that the delivery errors be normally distributed, but the FORTRAN subroutine and numerical examples included will be based on that further assumption.

2.2 Problem Statement.

The problem is to derive an algorithm which computes P_H, the probability that a random shot from the assumed bivariate delivery error distribution hits the polygonal target.

2.3 Method of Solution.

The basic building block for this algorithm is the integral of the probability density function of delivery errors over a region consisting of the entire target plane "below" a specified line segment. Specifically, given a line segment connecting point i whose coordinates are (x_i,y_i) with point j whose coordinates are (x_j,y_j), we will need to integrate the delivery error distribution over the region R defined by

$$R: \begin{cases} x \text{ between } x_i \text{ and } x_j \\ y \text{ less than } y_i + \left(\dfrac{y_j-y_i}{x_j-x_i}\right)(x-x_i) \; . \end{cases} \qquad (1)$$

Such a region is shown as the shaded area in Figure 1. If $f(x,y)$ denotes the probability density function for the bivariate distribution of weapon delivery error, this integral is given by

$$F_{i,j} = \int_{x=x_i}^{x_j} \int_{y=-\infty}^{y_i + \left(\frac{y_j - y_i}{x_j - x_i}\right)(x - x_i)} f(x,y)\, dy\, dx \qquad (2)$$

This integral will be positive when $x_j > x_i$, and in that case will be the probability of hitting the region R. However, in the present application, this inequality will not always hold; so we will not, in general, refer to this integral as a probability.

Applying the rule for interchanging the upper and lower limits of integration, it follows that

$$F_{i,j} = -F_{j,i} \qquad (3)$$

This relationship will be important in simplifying subsequent results.

The next step is to determine how these $F_{i,j}$'s can be used to determine the probability of hitting a triangular target. The result for the triangle will be used in a mathematical induction development to extend the result to a general n-sided polygon.

There are two possible types of triangles that must be considered. In the first of these (Figure 2a), the "upper boundary" consists of two sides, and the "lower" boundary consists of one side. In the second type (Figure 2b), the "upper" boundary consists of one side and the "lower" boundary consists of two sides. The probabilities of hitting these two types of triangles will be derived in terms of the F function, and it will be shown that the same formal expression can be used for each.

For triangular targets, the hit probability is the probability of hitting between the "upper" boundary and the "lower" boundary. This can be expressed as the probability of hitting below the "upper" boundary, minus the probability of hitting below the "lower" boundary. In the first type of triangle this probability is:

$$P_H(\text{Type a}) = F_{1,2} + F_{2,3} - F_{1,3}\,, \qquad (4)$$

where $F_{1,2} + F_{2,3}$ is the probability of hitting below the "upper" boundary, and $F_{1,3}$ is the probability of hitting below the "lower" boundary, with the vertices labeled consecutively in the clockwise direction around the triangle. Similarly, for the second type of triangle,

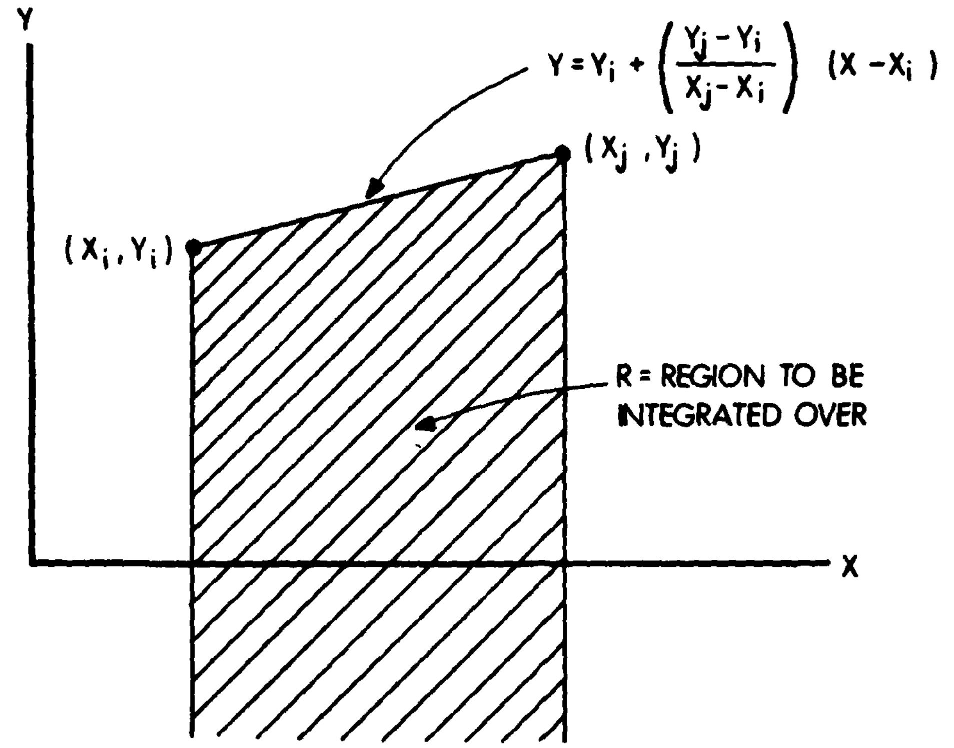

Figure 1. Description of Basic Probability Building Block.

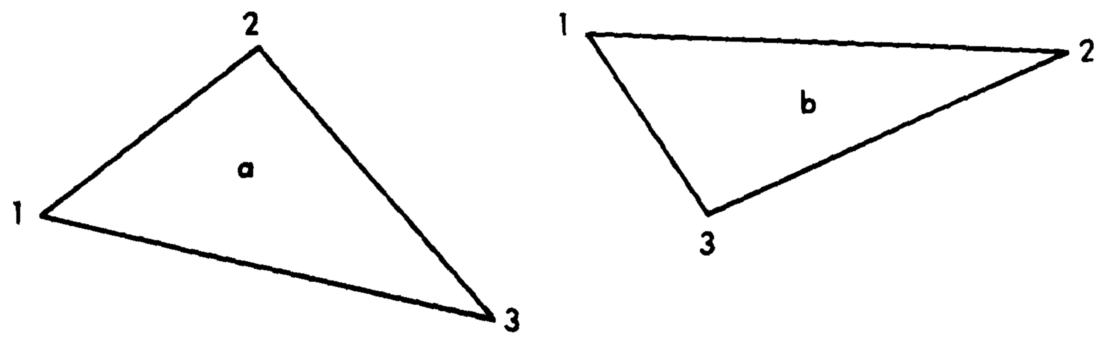

Figure 2. The Two Possible Types of Triangles.

$$P_H\ (\text{Type b}) = F_{1,2} - (F_{1,3} + F_{3,2}) = F_{1,2} - F_{1,3} - F_{3,2} \qquad (5)$$

where $F_{1,2}$ is the probability of hitting below the "upper" boundary and $F_{1,3} + F_{3,2}$ is the probability of hitting below the "lower" boundary. Applying the relation $F_{i,j} = -F_{j,i}$ (Equation 3) to each of the "lower" boundary terms in these two expressions,

$$P_H\ (\text{Type a}) = F_{1,2} + F_{2,3} + F_{3,1} \qquad (6)$$

and

$$P_H\ (\text{Type b}) = F_{1,2} + F_{2,3} + F_{3,1}\ . \qquad (7)$$

Thus, the two types of triangles can be treated the same formally: the summation of the F's proceeds in a cyclic fashion around the triangle, taking the sides in turn and integrating in each case from the first value of x to the next value in the cyclic order. The relative magnitudes of the x's themselves will automatically determine the proper sign for the associated value of F, so that the F's can always be added to obtain the desired result. Therefore, for any triangle target with the vertices ordered consecutively in a clockwise order, the hit probability is

$$P_H\ (3) = F_{1,2} + F_{2,3} + F_{3,1} \qquad (8)$$

This result can be rewritten as

$$P_H\ (3) = \sum_{i=1}^{3} F_{i,\text{imod}3+1} \qquad (9)$$

where imod 3 denotes the remainder when i is divided by 3.

Actually desired is the extension of this result to a simple n-sided polygon, that is, one whose sides never cross each other. Since such a polygon also has n vertices, the desired extension is:

$$P_H(n) = \sum_{i=1}^{n} F_{i,\text{imod}n+1} \qquad (10)$$

This result will be established using mathematical induction. In particular, it must be shown that:

I: The result holds for n=3, the polygon with the least possible number of sides, and

II: Assuming the result true for a polygon of (k-1) sides, show that this leads to its truth for a polygon of k sides.

Part I has already been demonstrated, leaving only Part II. At this point two additional propositions are required. These are

- Any k-sided polygon can be partitioned into a triangle and a polygon of (k-1) sides, and

- Given two polygons that share a common side, a traversing of the perimeter of each polygon in the clockwise direction results in the common side being traversed in one direction as one polygon is being traversed, and in the opposite direction as the other polygon is being traversed.

In "Analytic Function Theory, Vol. 1" by Einar Hille, Blaisdell Publishing Company, 1959, there is a proof (page 286) that every simple closed polygon can be triangulated, that is, partitioned into triangles. This result establishes proposition (1). Figure 3 shows a number of polygons, some more odd shaped than others, with each partitioned into a triangle and a polygon of one less side. Figure 4 illustrates the idea of the side common to a pair of polygons being traversed in opposite directions as each polygon is traversed in a clockwise motion. A proof of proposition (2) has not been found, but its truth is almost obvious, and will be assumed.

The proof of Part II of the mathematical induction proceeds as follows: Partition the k-sided polygon into a triangle and a polygon of (k-1) sides. This is possible by proposition (1). This is accomplished by connecting a pair of vertices of the k-sided polygon which are separated by a single vertex. Denote the vertices connected in this way as vertices m and m+2.

By the assumption of Part II of the induction, the probability of hitting the (k-1) sided polygon is

$$P_H(k-1) = F_{1,2}+F_{2,3}+\cdots+F_{m-1,m}+F_{m,m+2}+F_{m+2,m+3}+\cdots+F_{k,1} \cdot \qquad (11)$$

By the result of Part I of the induction, the probability of hitting the triangle is

$$P_H(3) = F_{m,m+1} + F_{m+1,m+2} + F_{m+2,m} \cdot \qquad (12)$$

Proposition (2) insures that the side common to the (k-1)-sided polygon and the triangle (the side connecting vertices m and m+2) will be traversed in opposite directions in the development of $P_H(k-1)$ and $P_H(3)$. Thus, when these two probabilities are added to obtain $P_H(k)$, Equation (3) insures that $F_{m,m+2}$ and $F_{m+2,m}$ cancel each other, establishing the desired result, namely, that

$$P_H(k) = F_{1,2}+F_{2,3}+\cdots+F_{m-1,m}+F_{m,m+1}+F_{m+1,m+2}+\cdots\cdots+F_{k,1} \cdot \qquad (13)$$

$$= \sum_{i=1}^{k} F_{i,i\bmod k+1}$$

This completes the proof that establishes Equation (10). This result depends only on the coordinates of the vertices of the polygonal

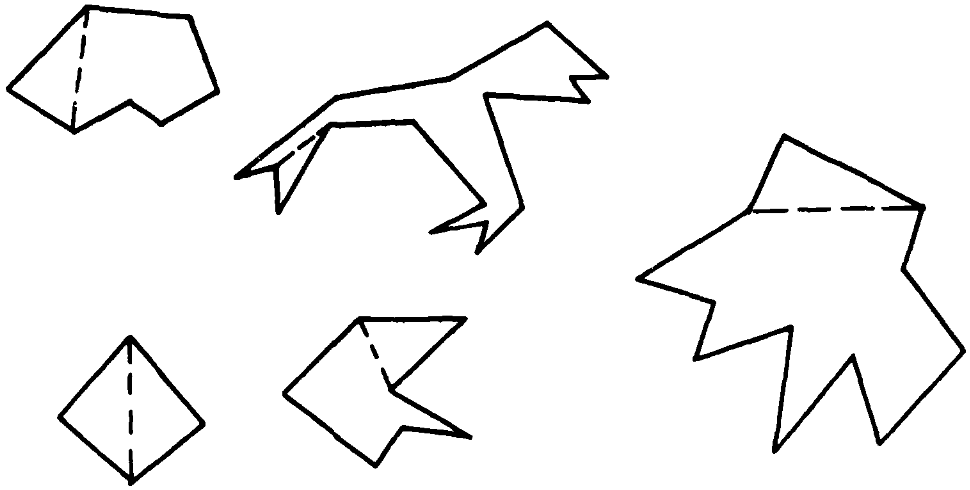

Figure 3. Example of Partitioning Polygons.

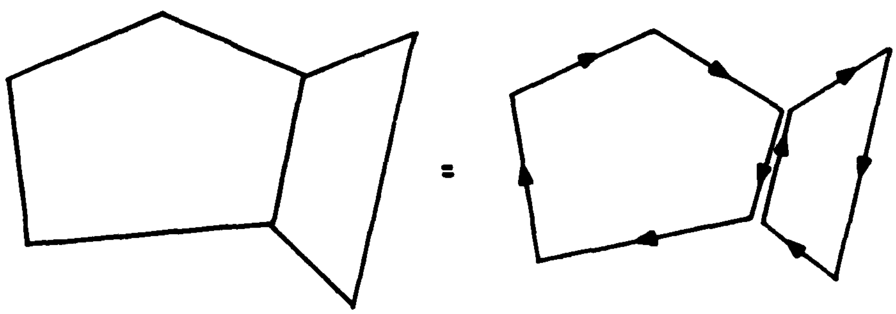

Figure 4. Illustration of Opposite Motion Along Common Side of Two Adjacent Polygons

target. It becomes extremely easy to apply, requiring only a means for calculating the function $F_{i,j}$ defined in Equation (2). When presented with a general polygonal target, the vertices are numbered consecutively in a clockwise rotation around the polygon, starting at an arbitrarily chosen vertex. The values of the $F_{i,j}$ function for the sides, taken in order, are then summed to obtain the hit probability.

3. COMPUTATIONAL FORM

The basis for this hit probability formulation is the function $F_{i,j}$ defined in Equation (2), and repeated here as Equation (14).

$$F_{i,j} = \int_{x=x_i}^{x_j} \int_{y=-\infty}^{y_i + \left(\frac{y_j - y_i}{x_j - x_i}\right)(x - x_i)} f(x,y)\, dy\, dx \tag{14}$$

Weapon delivery error is generally assumed to be distributed according to an uncorrelated bivariate normal distribution. Under this assumption,

$$f(x,y) = \frac{1}{\sigma_x \sigma_y} \phi\left(\frac{x-A}{\sigma_x}\right) \phi\left(\frac{y-B}{\sigma_y}\right) \tag{15}$$

where (A,B) is the mean impact point, σ_x and σ_y are the standard deviations of the delivery error in the x and y directions respectively, and

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}$$

for all t, the unit normal probability density function. Thus

$$F_{i,j} = \int_{x=x_i}^{x_j} \int_{y=-\infty}^{y_i + \left(\frac{y_j - y_i}{x_j - x_i}\right)(x - x_i)} \frac{1}{\sigma_x \sigma_y} \phi\left(\frac{x-A}{\sigma_x}\right) \phi\left(\frac{y-B}{\sigma_y}\right) dy\, dx \tag{16}$$

Make the changes of variable $u = \frac{x-A}{\sigma_x}$ and $v = \frac{y-B}{\sigma_y}$, and let

$m_{ij} = \frac{y_j - y_i}{x_j - x_i}$, $S_i = \frac{x_i - A}{\sigma_x}$, $S_j = \frac{x_j - A}{\sigma_x}$ and $T_i = \frac{y_i - B}{\sigma_y}$; then this integral can be written as

$$F_{i,j} = \int_{u=S_i}^{S_j} \phi(u)\,\Phi\left(T_i + \frac{\sigma_x}{\sigma_y} m_{ij}[u - S_i]\right)du$$

where $\Phi(Z) = \int_{t=-\infty}^{Z} \phi(t)dt$, *for all Z, the cumulative normal probability* function.

In general, this integration must be performed numerically. However, there are two special cases where this is not necessary. In the first, identified by $x_i = x_j$ (or $S_i = S_j$), the upper and lower limits of integration are the same and $F_{i,j} = 0$. This is consistent with the notion that $F_{i,j}$ is the integral of the probability density function "under" the line segment connecting (x_i, y_i) and (x_j, y_j). In this special case, the connecting segment is vertical, and thus has no measurable area "under" it. In the second special case, identified by $y_i = y_j$ (or $m_{ij} = 0$), the integral reduces to

$$F_{i,j} = \int_{u=S_i}^{S_j} \phi(u)\Phi(T_i)du = \Phi(T_i)[\Phi(S_j) - \Phi(S_i)] \qquad (18)$$

The function Φ is available on most computers. In the general case requiring numerical integration to evaluate $F_{i,j}$, one further simplification is useful. The function $\phi(u)$ is essentially zero-valued for $|u| > 4$. Thus, the numerical integration can be made more efficient by adjusting the values of S_i and S_j so that no integration steps outside the interval $(-4 < u < 4)$ are included.

A FORTRAN subroutine for calculating the probability of hitting a polygonal target using this method, along with instructions as to how to use this subroutine in a FORTRAN program, are included in the Appendix.

4. SAMPLE CALCULATIONS

The polygonal target shown in Figure 5 was used as a basis for checking this procedure and the associated FORTRAN subroutine. The probability of hitting this polygon was easily hand-calculated by partitioning it into rectangles each of whose sides are parallel to the coordinate axes.

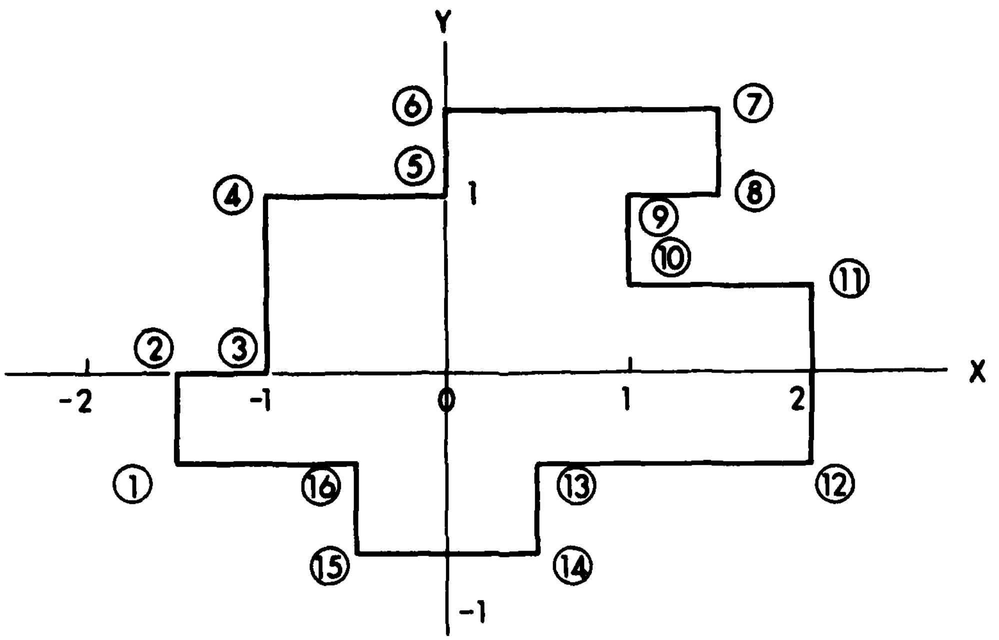

Figure 5. Polygon for Sample Computation.

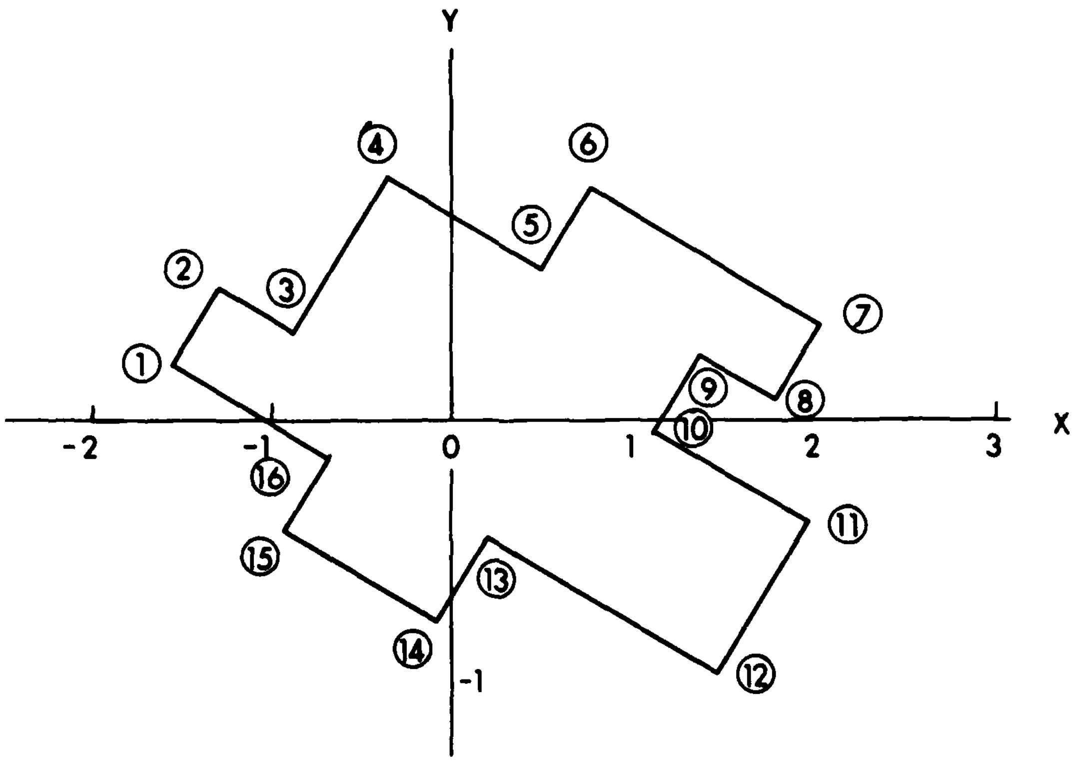

Figure 6. Polygon Rotated 30°

This result was then compared to the result from the approach of this report. As the polygon appears in Figure 5, the two special cases of the method apply - for the vertical sides (such as the side connecting vertex 11 and vertex 12) the function $F_{i,j}=0$ and for the horizontal sides (such as the side connecting vertex 10 and vertex 11) Equation (18) applies. The agreement between the hand computation and the subroutine result was established. Then the polygon was rotated 30° about the origin, as shown in Figure 6. As long as $\sigma_x=\sigma_y$ and A=B=0, the probability of hitting the rotated polygon should be the same as that for the unrotated polygon. In this case, however, the general expression for $F_{i,j}$ applies for all sides since none of them are parallel to the coordinate axes. The resulting probability was the same as that for the unrotated polygon. Figure 7 shows the probability of hitting this rotated polygon for A=B=0 and as a function of $\sigma_x=\sigma_y$.

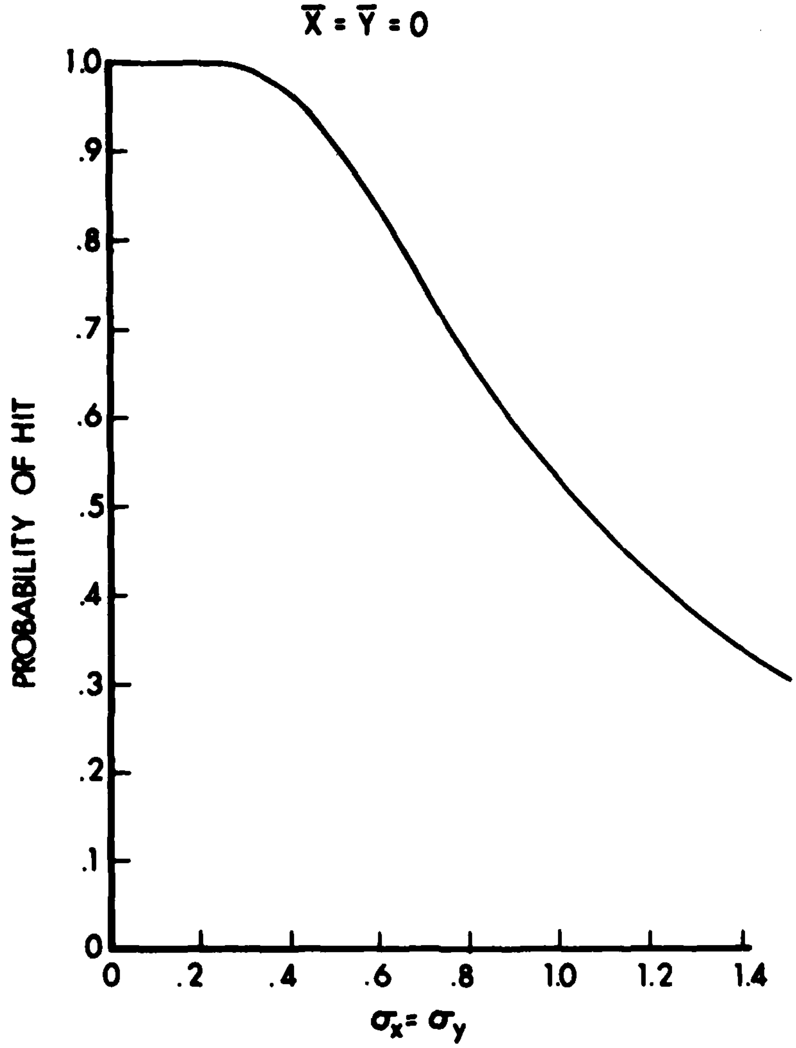

Figure 7. Probability of Hitting Rotated Polygon.

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APPENDIX

FORTRAN FUNCTIONS

This appendix presents the following three FORTRAN functions required for the use of this method for computing the probability of hitting a polygonal target. These functions are

1. PHIT (called by user's main program)
2. FIJ (called by PHIT)
3. FUN (called by FIJ)

The user must include all three of these in his main program, but needs only to call PHIT. The arguments in the call statement are

N = number of sides in the polygonal target

X = array name holding X coordinates of vertices of polygon

Y = array name holding Y coordinates of vertices of polygon

XB = X coordinate of mean of hit distribution

YB = Y coordinate of mean of hit distribution

SX = standard deviation of hit distribution in x

SY = standard deviation of hit distribution in y

The size of the x and y arrays (>N) must be established with a DIMENSION statement in the calling program.

The actual numerical integration is accomplished with a Simson integration subroutine (SIMSON), which requires the function FUN to calculate values of the integrand. SIMSON is resident on the CDC computer at Aberdeen Proving Ground, but users of other computers may have to substitute another integration subroutine. In the subroutine used here, a relative error of 0.001 has been specified. This should provide values of $F_{i,j}$ accurate to the third decimal place. The accuracy of the hit probabilities depend on the number of these F's that must be added, but in general should be good to at least two places, and probably more.

```
      FUNCTION PHIT(N,X,Y,XB,YB,SX,SY)
C ***
C     THIS FUNCTION COMPUTES THE PROBABILITY OF HITTING AN
C     N-SIDED POLYGONAL TARGET.  THE COORDINATES OF THE VERTICES
C     OF THE POLYGON ARE STORED IN THE X AND Y ARRAYS IN
C     CONSECUTIVE CLOCKWISE ORDER AROUND THE POLYGON.  THE
C     DELIVERY ERROR DISTRIBUTION IS ASSUMED TO BE BIVARIATE
C     NORMAL WITH MEAN (XB,YB) AND WITH STANDARD DEVIATIONS
C     SX AND SY RESPECTIVELY.
C ***
      DIMENSION X(1),Y(1)
    1 FORMAT(' BOTH SX AND SY MUST BE GREATER THAN ZERO'/
     * '    SX = ',F10.3,'      SY = ',F10.3)
      IF(SX.GT.0. .AND. SY.GT.0.)GOTO 5
        PRINT 1, SX,SY
        STOP
    5 PHIT=0.
        DO 10 I=1,N
        J=MOD(I,N)+1
        PHIT=PHIT+FIJ(X(I),Y(I),X(J),Y(J),XB,YB,SX,SY)
   10   CONTINUE
      RETURN
      END

      FUNCTION FIJ(XI,YI,XJ,YJ,XB,YB,SX,SY)
C ***
C     THIS FUNCTION PERFORMS THE INTEGRATION OF THE BIVARIATE
C     NORMAL DENSITY FUNCTION 'UNDER' THE DIRECTED LINE SEGMENT
C     FROM POINT (XI,YI) TO THE POINT (XJ,YJ).
C ***
      COMMON /ABC/ A,B,SI
      EXTERNAL FUN
      FIJ=0.
      IF (XI.EQ.XJ)RETURN
      SI=(XI-XB)/SX
      SIP=SI
      SIP=AMAX1(SIP,-4.)
      SIP=AMIN1(SIP,4.)
      SJ=(XJ-XB)/SX
      SJP=SJ
      SJP=AMAX1(SJP,-4.)
      SJP=AMIN1(SJP,4.)
      IF(SIP.EQ.SJP)RETURN
      A=(YI-YB)/SY
      B=SX*(YJ-YI)/(SY*(XJ-XI))
      IF(YI.EQ.YJ) FIJ=FND(A)*(FND(SJ)-FND(SI))
      IF(YI.NE.YJ) CALL SIMSON(FUN,FIJ,SIP,SJP,.001,0)
      RETURN
      END

      FUNCTION FUN(S)
      COMMON /ABC/ A,B,C
      FUN=.3989422804*EXP(-.5*S*S)*FND(A+B*(S-C))
      RETURN
      END
```

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