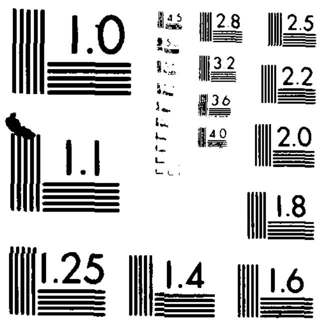
AD-A099 812 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 12/2
RANDOM LINEAR PROGRAMS.(U)
MAR 81 I ADLER, S E BERENGUER N00014-76-C-0134
UNCLASSIFIED ORC-81-4 NL

END
DATE
FILMED
6 81
DTIC



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL II (5)

(14) ORC-81-4

(11)

(6) RANDOM LINEAR PROGRAMS.

by (10)

ILAN ADLER

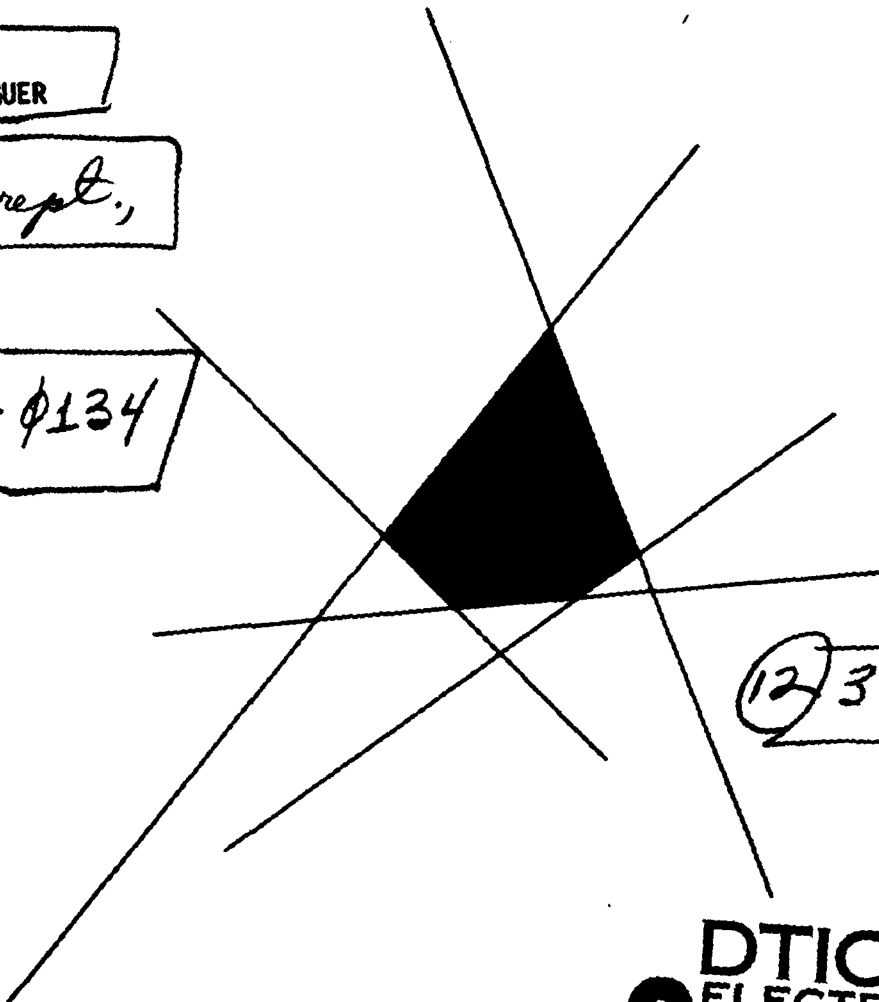
[redacted]

SANCHO E. DE B. BERENGUER

(9) Research rept.,

(16) N00014-76-C-0134

AD A099812



(12) 37

DTIC
ELECTE
JUN 0 8 1981
S D
F

OPERATIONS RESEARCH CENTER

DTIC FILE COPY

UNIVERSITY OF CALIFORNIA · BERKELEY 270 750

81 6 08 069

RANDOM LINEAR PROGRAMS

by

Ilan Adler
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

and

Sancho E. de B. Berenguer
Instituto de Matemática Pura e Aplicada
CNPq
Rio de Janeiro, Brazil

MARCH 1981 ORC ·4

This research was supported by the Office of Naval Research under Contract N00014-76-C-0134 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC 81-4	2. GOVT ACCESSION NO. AD-A099 812	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) RANDOM LINEAR PROGRAMS		5. TYPE OF REPORT & PERIOD COVERED Research Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ilan Adler and Sancho E. de B. Berenguer		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0134
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047 033
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, Virginia 22217		12. REPORT DATE March 1981
		13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear Programming
Random Linear Programs
Polyhedral Sets

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT

In this paper, we study the random generation of a linear program of the type

$$P : \text{Max } cx$$
$$\text{subject to } Ax \leq b .$$

P is randomly generated through the A_{ij}'s , b_i's and c_j's . We assume these random variables to be independent and symmetric around zero and to have continuous distribution functions, therefore, transforming the random generation problem into a distribution free combinatorial problem.

Making use of the theory of d-Arrangements, we compute the probabilities of P being feasible and bounded, and we also calculate the expected number of faces, of all possible dimentions, of the polytope that is the feasibility set of P , given that P is feasible.

INTRODUCTION

The riddle of the gap between the time-proven actual efficiency of the Simplex method for linear programs and its apparent theoretical inefficiency attracted a great deal of research in recent years. One of the more popular approaches is to consider the "average" efficiency of the Simplex method. The idea is to show that even though in rare problems (usually, specifically and cleverly designed to be a bad problem) the Simplex method can take an exponential number of iterations, as a function of its size, it would usually take modest number of steps for almost all randomly generated linear programs.

In order to discuss and investigate intelligently the average efficiency of the Simplex method, one has, of course, first to develop schemes for random generation of linear programs. Thus, several papers were devoted to this subject.

Obviously, before studying the difficult problem of explaining the actual behavior of the Simplex method, some studies were devoted to the study of simple problems concerning randomly generated linear programs such as the expected number of extreme points, the probability of having unfeasible or unbounded linear programs, etc.

As it turns out, there are several "reasonable" methods of generating linear programs which unfortunately may give different results. This disturbing fact was our main indication in initiating the current research. We tried somehow to obtain the most natural and robust way of generating linear programs.

Thus, in this paper, we present a method of randomly generating linear programs which is based on randomly generating coefficients of the problem (objective function, matrix of coefficients and right-hand side). We preferred that method over geometrically oriented methods, because this is the way linear program are usually perceived by users. Based on some mild assumptions (independence of all random variables, symmetry around zero and continuity of the density function), we obtain (in Section 2) several results concerning the expected number of vertices and the probabilities of feasibility or unboundedness for any size of linear program. We also extended some of the results to limiting cases (e.g., where the number of constraints or variables approaches infinity). The main feature (we believe) of our results is that under the general assumptions mentioned above, the results are independent of the actual distribution function of the randomly generated coefficients. This feature is the outcome of viewing the problem through an application of the theory of arrangements hyperplanes in a d-dimensional space. This transformation (presented in Section 1) allows us to consider the several questions involving expected values and probabilities as simple counting problems, which are independent of the actual distribution of the coefficients. We should also note (as discussed in Section 3) that our results apply to any form of linear programs.

NOTATION AND DEFINITIONS

Most of the notation and definitions in this paper are derived from Grünbaum [5].

We denote the Euclidean space of dimension d by E^d .

We call a polyhedral set the intersection of a finite number of half spaces. A nonempty polyhedral set is called a polyhedron and a polytope is a bounded polyhedron. If a polyhedron (polytope) is of dimension d , we call it a d-polyhedron (d-polytope). Consider a d-polyhedron P ; a k-dimensional face of P is referred to as a k-face; $F_k(P)$, $0 \leq k \leq d$, denote the set of all k-faces of P , a member of $F_k(P)$ is called a vertex, an edge or a facet of P if $k = 0$, $k = 1$ or $k = d - 1$.

Given an $n \times d$ matrix A and sequences $I \subseteq \{1, \ldots, n\}$ and $J \subseteq \{1, \ldots, d\}$, we denote by $A_{I.}$ the submatrix of A associated with the rows in I ; by $A_{.J}$ the submatrix of A associated with the columns in J . We denote by $A_{i.}$ the i^{th} row of A and by $A_{.j}$ the j^{th} column of A .

Consider the linear program:

$$P : \text{Max } cx$$
$$\text{subject to } \; x \in X$$

where X is the polyhedral set $\{x \in E^d \mid Ax \leq b\}$. X is called the feasibility set or constraint set of P . The hyperplanes $\{x \mid A_{i.} = b_i\}$, for all rows of A are called the supporting hyperplanes of the half spaces of X .

Finally, the symbol □ will be placed at the end of a proof, and the symbol ▮ will be placed at the end of a theorem or lemma, which will be presented without proof.

1. THE RANDOM GENERATION OF LINEAR PROGRAMS

1.0 Introduction

In this section, we define a random linear program and introduce the process of generating such a program.

We define a random linear program as a linear program for which we have a random objective function and a random constraint set. More specifically: we consider random programs of the type:

$$P : \text{Max } cx$$
$$\text{subject to } Ax \leq b$$

where A , c , b are randomly generated so that P is generated through the random variables A_{ij}'s , b_i's and c_j's .

We assume the A_{ij}'s , b_i's and c_j's to be independent random variables and symmetric around zero and to have continuous distribution functions. These assumptions are crucial for obtaining the main results of the paper.

In the following, we shall present the process of randomly generating linear programs and present some theorems which are applied in the development of our results.

1.1 Random Half Spaces

Consider the following linear program, to be randomly generated:

$$P : \text{Max } cx$$
$$\text{subject to } Ax \leq b \text{ ,}$$

where A is $n \times d$, b is $n \times 1$, and c is $d \times 1$, $n \times d$.

(For the application of our results to linear programs in other forms, see the remarks in Section 3.)

The following assumptions are crucial for the results obtained in the next chapters and will be often referred to as the assumptions of Section 1. We will now formalize them.

Assumption 1:

All the A_{ij}'s , b_i's and c_j's of P are independent random variables.

Assumption 2:

The random variables A_{ij}'s , b_i's and c_j's have continuous distribution functions.

Assumption 3:

The random variables A_{ij}'s , b_i's and c_j's are symmetric around zero.

We can now state the following theorem about the random half spaces of P which is the basis for the results about the constraint sets of random linear programs in Section 2.

Let $a \in E^d$ and $\alpha \in E^1$; denote

$$\hat{H}(a) = \{x \mid x = \alpha a \quad \text{for any} \quad \alpha \neq 0\}$$

$$\hat{H}^+(a) = \{x \mid x = \alpha a \quad \text{for any} \quad \alpha > 0\}$$

$$\hat{H}^-(a) = \{x \mid x = \alpha a \quad \text{for any} \quad \alpha < 0\} .$$

Theorem 1.1.1

Let $\tilde{A}_{i.}$, $\tilde{b}_i$ be realized values for the random variables $A_{i.}$, b_i (for some $1 \leq i \leq n$).

Suppose assumptions 1-3 above are satisfied, then if $A_{i.}$, b_i are randomly generated such that $(A_{i.},b_i) \in \hat{H}(A_{i.},b_i)$, then

$$\text{Prob}\left[(A_{i.},b_i) \in \hat{H}^{+}(\tilde{A}_{i.},\tilde{b}_i)\right] = \text{Prob}\left[(A_{i.},b_i) \in \hat{H}^{-}(\tilde{A}_{i.},\tilde{b}_i)\right] = 1/2 .$$

Note that since $\left\{x \in \mathbb{E}^d \mid A_{i.}x \leq b_i\right\} = \left\{x \in \mathbb{E}^d \mid -A_{i.}x \geq -b_i\right\}$ it follows from Theorem 1.1.1 that if $\left\{x \in \mathbb{E}^d \mid A_{i.}x = b_i\right\}$ is a supporting hyperplane of the i^{th} randomly generated constraint, then it is equally likely that the i^{th} constraint is either $\left\{x \in \mathbb{E}^d \mid A_{i.}x \leq b_i\right\}$ or $\left\{x \in \mathbb{E}^d \mid A_{i.}x \geq b_i\right\}$.

The proof of Theorems 1.1.1 and 1.2.1 follows directly from the assumptions above, standard manipulations and the use of the theory of transformation of random vectors, that can be found in Bickel and Doksum [2]; the details of these proofs can be found in Berenguer [1].

1.2 The Objective Function

We generate the objective function cx by randomly generating the vector $c = (c_1, \ldots, c_d)$.

Obviously, for every objective function cx , there corresponds a set $\left\{(kc_1, \ldots, kc_d) \in \mathbb{E}^d \mid k > 0\right\}$ which represent the same objective function. Thus, we shall refer to a given objective function as a family of hyperplanes in which any member of this family represents the same objective function.

<u>Theorem 1.2.1</u>

Let $\tilde{c}$ be a realizable objective function vector for P , then if c is a randomly generated objective function for P , such that $c \in \hat{H}^-(\tilde{c})$, then

$$\text{Prob}\ [c \in \hat{H}^-(\tilde{c})] = \text{Prob}\ [c \in \hat{H}^+(\tilde{c})] = 1/2 \ . \qquad \blacksquare$$

Note that since max cx is equivalent to -min (-cx) , Theorem 1.1.2 implies that given cx as the objective hyperplane it is equally likely that the objective is either maximizing or minimizing.

2. d-ARRANGEMENTS AND THE RANDOM GENERATION OF LINEAR PROGRAMS

2.0 Introduction

Using the assumptions of Section 1 and the theory of d-Arrangements, we will transform our linear program random generation problem into a combinatorial problem.

In this section we will compute the expected value for the number of faces of every possible dimension of the random linear program. Probabilities of being feasible and of being bounded for the random linear program will also be calculated.

2.1 d-Arrangements

Definition:

A finite family A of $n > d$ hyperplanes forms a d-Arrangement of hyperplanes in $\mathbb{E}^d$, provided that no point in $\mathbb{E}^d$ belongs to all elements of A. The n hyperplanes partition $\mathbb{E}^d$ into a finite number of d-polyhedral sets: the facets of these d-polyhedral sets are formed by those hyperplanes and no point in the interior of any of the d-polyhedral sets belongs to any of the hyperplanes. These polyhedral sets are also called the d-faces of the d-Arrangement. The k-faces of the d-Arrangement, for $0 \leq k \leq d$, are the k-faces of the d-faces of the d-Arrangement.

Definition:

A set of n hyperplanes have the *general intersection property* if the intersection of every set of K of those hyperplanes is a $(d-k)$-dimensional linear affine space (if $d-k$ is negative, then the intersection is considered void).

Let $F_d(k,n)$ be the total number of k-faces of all d-polytopes of the d-Arrangement formed by a family of n hyperplanes. Also, define $F_0(0,n) = 1$.

We can now state the following lemmas, due to Buck [3].

Lemma 2.1.1:

The number of k-faces of a d-Arrangement formed by a family A of n hyperplanes, having the general intersection property in $\mathbb{E}^d$, is given by

$$F_d(k,n) = \begin{bmatrix} n \\ d-k \end{bmatrix} F_k(k,n+k-d)$$

$$= \begin{bmatrix} n \\ d-k \end{bmatrix} \sum_{i=0}^{k} \begin{bmatrix} n+k-d \\ i \end{bmatrix} .$$

Corollary:

The number of d-polyhedral sets in A is given by

$$F_d(d,n) = \sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix} .$$

Lemma 2.1.2:

The number of bounded k-faces of A , given the general intersection property is:

$$F_d^*(k,n) = \frac{d+1}{n+k-d} \begin{bmatrix} d \\ k \end{bmatrix} \begin{bmatrix} n \\ d+1 \end{bmatrix} .$$

Corollary:

The number of d-polytopes of A is given by

$$F_d^*(d,n) = \begin{bmatrix} n-1 \\ d \end{bmatrix} .$$

2.2 d-Arrangements and the Constraint Set of a Randomly Generated Linear Program

We will now relate the results of Section 2.1 about d-Arrangements to the constraint sets of random linear programs by making use of the results obtained in Section 1. Recall that we assume the random linear program P to have the following representation:

$$P : \text{Max } cx$$
$$\text{subject to } \; x \in X = \{x \mid Ax \leq b\} ,$$

where A is $n \times d$, b is $n \times 1$, c is $d \times 1$, for $2 \leq d \leq n$ and $n \leq d$.

Theorem 2.2.1:

Given the assumptions of Section 1, the supporting hyperplanes of the half spaces of X , $\{x \mid A_{i.} = b_i\}$, $i = 1, \ldots, n$, have the general intersection property with probability one.

Proof:

Berenguer in [1] shows that the hyperplane $H_i = \{x \mid A_{i.}x = b_i\}$ can be fully characterized by $x_{iH} = \left(x^i_{1H}, \ldots, x^i_{dH}\right)$, the point where the line perpendicular to H_i and passing through the origin meets the hyperplane; so, H_i can be determined by x^i_H and the density $f_H(x_{1H}, \ldots, x_{dH})$ characterizes the random hyperplane. It is also shown that, given the assumptions of Section 1, x_{iH} has a continuous distribution function.

We can show that the probability of any two hyperplanes, say H_1 and H_2 , characterized by the points x^1_H and x^2_H with densities f_1 and f_2 being parallel to each other is equal to zero. Let $P[H_2 \parallel H_1]$ be this probability. Then:

$$P[H_2 \parallel H_1] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1\left(x_{1H}^1, \ldots, x_{dH}^1\right) P[H_2 \parallel H_1 \mid H_1] dx_{1H}^1, \ldots, dx_{dH}^1 .$$

But

$$P[H_2 \parallel H_1 \mid H_1] = \int_{L_1} f_2\left(x_{1H}^2, \ldots, x_{dH}^2\right) dx_{1H}^2, \ldots, dx_{dH}^2 ,$$

where L_1 is the line that passes through the origin and x_H^1 and L_1 characterizes all the hyperplanes that are parallel to H_1 .

Since L_1 has d-dimensional Lebesgue measure (see Halmos [6], Chapter VII) equal to zero $(d \geq 2)$, and by Assumption 2, x_H^2 has continuous distribution function, we have $P[H_2 \parallel H_1 \mid H_1] = 0$. Hence $P[H_2 \parallel H_2 \mid H_1] = 0$.

Also, since no two hyperplanes are parallel, we have that k hyperplanes, $2 \leq k \leq d$, will necessarily meet in some $(d-k)$-dimensional plane in $\mathbb{E}^d$. Clearly, those $(d-k)$-dimensional planes have d-dimensional Lebesgue measure zero in $\mathbb{E}^d$ and because of the continuity of the distribution functions of the x_H^i's , we can easily show, in a similar way of the nonparallelism proof above, that no other hyperplane will meet those k hyperplanes in the same $(d-k)$-dimensional plane. $\square$

<u>Corollary 1</u>:

The supporting hyperplanes of the half spaces of $X = \{x \mid A_{i \cdot} x = b_i\}$, $i = 1, \ldots, n$ form a d-Arrangement.

Proof:

According to Theorem 2.2.1, those supporting hyperplanes have the general intersection property, and therefore, no point belongs to all $n > d$ hyperplanes since that would contradict the fact that $d + 1$ hyperplanes do not intersect at the same point. □

Corollary 2:

The supporting hyperplanes $\{x \mid A_{i.}x = b_i\}$, $i = 1, \ldots, n$ of the half spaces of X partition $\mathbb{E}^d$ into $F_d(d,n) = \sum_{i=0}^{d} \begin{bmatrix} n \\ d \end{bmatrix}$ different d-polytopes.

Proof:

Directly follows from the corollary of Lemma 2.1.1, Theorem 2.2.1 and Corollary 1 above. □

Note that Theorem 2.2.1 allows us to apply combinatorial results of d-Arrangements in calculating probabilities relating to randomly generated linear programs. Note that a randomly generated polyhedron might be of dimension less than d. However, the assumptions of Section 1 guarantees that the randomly generated polyhedron (if nonempty) will be of dimension d with probability one.

Theorem 2.2.2:

Let the assumptions of Section 1 be satisfied for the random generation of P. Let $\tilde{A}$ and $\tilde{b}$ be possible values that can occur for the random matrix A and the random vector b. So, given that $\{\tilde{A}_{i.}x = \tilde{b}_i\}$, $i = 1, \ldots, n$ occur as supporting hyperplanes of the half spaces of X, forming a d-Arrangement in $\mathbb{E}^d$, then all the $F_d(d,n)$ different d-polytopes that are the d-faces of that d-Arrangement have equal probability of occurring as the constraint set of the random linear program P, namely $1/2^n$.

Proof:

Given that the hyperplanes $\{x \mid \tilde{A}_{i.}x = \tilde{b}_i\}$, $i = 1, \ldots, n$, occur, it follows from Theorem 2.2.1 that, with probability one, they have the general intersection property and form a d-Arrangement. Then, each of the d-polytopes of this d-Arrangement corresponds to a unique combination of inequality signs, $\geq$ or $\leq$, for the half spaces of the constraint set of P. Since we have n hyperplanes, we have 2^n possible combinations for the inequality signs, $\geq$ or $\leq$. We will now show that each combination has probability $1/2^n$ of occurring. By Theorem 1.1.1 we have that, given the hyperplane $\{x \mid \tilde{A}_{i.}x = \tilde{b}_i\}$, $1 \leq i \leq n$, each of the inequality signs $\geq$ and $\leq$ occur with probability $1/2$, i.e., the half spaces $\{x \mid \tilde{A}_{i.}x \leq \tilde{b}_i\}$ and $\{x \mid \tilde{A}_{i.}x \geq \tilde{b}_i\}$ occur with probability $1/2$. So, given the hyperplanes, all the inequality signs occur with equal probability and each combination of them has equal probability $1/2^n$ of corresponding to the constraint set of P. □

Corollary 1:

Given the assumptions of Section 1, the probability of the random linear program P having a feasible constraint set is equal to

$$P_F(n,d) = \frac{F_d(d,n)}{2^n} = \frac{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} .$$

Proof:

The n hyperplanes $\{x \mid A_{i.}x = b_i\}$ of P will, with probability one, form a d-Arrangement in E^d, with $F_d(d,n)$ d-polytopes, that will correspond to possible feasible constraint sets for P. It easily follows from Theorem 2.2.2, that the probability of a feasible constraint set is equal to

$$P_F(n,d) = \frac{F_d(d,n)}{2^n} = \frac{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} . \qquad \square$$

Proposition 2.2.1:

$$\lim_{\substack{n\to\infty \\ d \text{ constant}}} P_F = 0 \quad \text{and} \quad \lim_{\substack{n,d\to\infty \\ n-d=m \text{ constant}}} P_F(n,d) = 1 .$$

Corollary 2:

Given the assumptions of Chapter 1, the probability of the random linear program P having a feasible bounded constraint set is equal to

$$P_F^*(n,d) = \frac{F_d^*(d,n)}{2^n} = \frac{\begin{bmatrix} n-1 \\ d \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} .$$

For the random linear program P, we had shown that, with probability one, the supporting hyperplanes of the half spaces of X will form a d-Arrangement, independent of the distributions of A, b and c, as long as the assumptions of Section 1 are satisfied. We also calculated the probabilities for feasible and bounded constraint sets. For this we used the almost sure d-Arrangement structure of the hyperplanes that allowed us to apply combinatorial type arguments to the problem.

Now, we will continue our study to obtain the main result of this section which is the computation of expected values for the number of faces of the d-polyhedron that constitutes the constraint set of the random linear program. The next lemmas are the first steps towards this objective. Before that we need one more definition:

Definition:

n hyperplanes in $\mathbb{E}^d$, $n \geq d$, are said to have the *general position property* if the normal vectors of the elements of every d-subset of the n hyperplanes are linearly independent. We also say that the n *hyperplanes are in general position* or that the *normal vectors are in general position.*

Lemma 2.2.1:

Let n $(n > d)$ hyperplanes in $\mathbb{E}^d$ have the general intersection property. Then they also have the general position property. ∎

The proof of 2.2.1 is easily obtained from the definitions of these two properties.

The next lemma is due to Schäfli in [8].

Lemma 2.2.2:

n hyperplanes in $\mathbb{E}^d$, $n \geq d$, all passing through a given point x^o and having the general position property partition $\mathbb{E}^d$ into $2\sum_{i=0}^{d-1}\begin{bmatrix} n-1 \\ i \end{bmatrix}$ cones with x^o as an apex, such that no point in the interior of any of the cones belongs to any of the hyperplanes. ∎

Corollary:

d hyperplanes having the general intersection property partition $\mathbb{E}^d$ into 2^d cones.

Proof:

From Lemma 2.2.1, we know that d hyperplanes in $\mathbb{E}^d$ having the general intersection property, also have the general position property. Hence, they all meet in a given point and as a straightforward application of Lemma 2.2.2, it follows that those d hyperplanes partition $\mathbb{E}^d$ into

$$2 \sum_{i=0}^{d-1} \begin{bmatrix} d-1 \\ i \end{bmatrix} = 2^d \text{ cones.} \qquad \square$$

Lemma 2.2.3:

For a d-Arrangement formed by n hyperplanes, having the general intersection property in $\mathbb{E}^d$, a k-face is contained in exactly 2^{d-k} d-faces, for $0 \le k \le d$.

Proof:

We will consider 3 possible cases:

(i) $k = 0$; since the hyperplanes have the general intersection property, we know that every d-set of them must intersect in a 0-face. By the corollary of Lemma 2.2.2, we know that these d hyperplanes form 2^d cones with that 0-faces as apex. It easily follows that every 0-face is contained in exactly 2^d d-faces.

(ii) $0 < k < d$; the proof is based on a proof of a similar theorem found in Cover and Efron [4]; consider a 0-face Q of the d-Arrangement and let $H_1, \ldots, H_d$ be the d different hyperplanes that intersect at Q and $a_1, \ldots, a_d$ normal vectors of $H_1, \ldots, H_d$. We will prove that every

> k-face containing Q is itself contained in 2^{d-k} of the 2^d different cones with apex Q and formed by the hyperplanes $H_1, \ldots, H_d$.

Without loss of generality, choose any $d-k$ hyperplanes of $H_1, \ldots, H_d$, say $H_1, \ldots, H_{d-k}$ and let $H = \bigcap_{i=1}^{d-k} H_i$ be the k-dimensional linear subspace orthogonal to the vectors $a_1, \ldots, a_{d-k}$. The remaining k hyperplanes $H_{d-k+1}, \ldots, H_d$ partition H into $\sum_{i=0}^{k-1} \begin{bmatrix} k-1 \\ i \end{bmatrix} = 2^{k-1}$ convex cones C_ℓ , $\ell = 1, \ldots, 2^{k-1}$ (this can be verified by noting that the projections of $a_{d-k+1}, \ldots, a_d$ into H are in general position in that subspace and that the intersection of H_i with H , for $i = d-k+1, \ldots, d$ is the $(k-1)$-dimensional subspace of H that is orthogonal to the projection of a_i). The interior of each *of the cones* C_ℓ *can be characterized as the set of solution vectors* w to the simultaneous relations

$$\operatorname{sgn}(a_i w) = \delta_i^* \qquad i = 1, \ldots, d$$

$$\text{where} \quad \delta_i = 0 \quad \text{for} \quad i = 1, \ldots, d-k$$

$$\text{and} \quad \delta_i = \pm 1 \quad \text{for} \quad i = d-k+1, \ldots, d .$$

Where sgn is the sign function defined on $\mathbb{R}$:

$$\operatorname{sgn}(y) = 1 , \; y > 0$$

$$\operatorname{sgn}(y) = 0 , \; y = 0$$

$$\operatorname{sgn}(y) = -1 , \; y < 0 .$$

Let $\delta = (\delta_1, \ldots, \delta_d)$ be a vector of ± 1's such that $\delta_i = \delta_i^*$ for $i \geq d-k+1$. It follows by continuity that every δ represents a nonempty solution cone having C_ℓ as a k-boundary and that those 2^{d-k} solution cones are the only ones having this property. It follows that every k-face is contained in one of those k-boundaries. Since each d-face to which Q belongs can be associated to one of those 2^d cones and none of the k-faces that contains Q can be contained in any other d-face than those 2^d ones (this could imply that Q would belong to more than 2^d d-faces), then the proof is complete.

(iii) $k = d$; the result is obvious, for this case. □

We can now state and prove the main theorem for this section:

<u>Theorem 2.2.3</u>:

Under the assumptions of Section 1, the expected number of k-faces, $0 \le k \le d$, of the d-polytope that is the constraint set of the random linear program, assuming feasibility, is given by:

$$e_d(k,n) = 2^{d-k} \frac{F_d(k,n)}{F_d(d,n)} = \frac{2^{d-k} \begin{bmatrix} n \\ d-k \end{bmatrix} \sum_{i=0}^{k} \begin{bmatrix} n-d+k \\ i \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} , \; k = 0, \ldots, d .$$

<u>Proof</u>:

From Theorem 2.2.1 we know that the n supporting hyperplanes of the constraint set of P have the general intersection property and, with probability one, will form a d-Arrangement in E^d . By Theorem 2.2.2, it follows that all the $F_d(d,n)$ different d-polytopes of the d-Arrangement have equal probability of occurring as the constraint set of P . Therefore, the expected number of k-face is equal to the total number of k-faces, for all the $F_d(d,n)$ d-polytopes, divided by the number of d-polytopes in the d-Arrangement. It is left to show that $2^{d-k}F_d(k,n)$ is the total number of faces for all the d-polytope; this result easily follows if we recall that we have $F_d(k,n)$ k-faces in the d-Arrangement and that, according to Lemma 2.2.3, each of those k-faces belongs to 2^{d-k} of the d-polytopes. □

It should be noted that the expected numbers of k-faces, for $0 \le k \le d$, do not necessarily combine to form a d-polytope, since it is easy to show that the $e_d(k,n)$'s do not necessarily satisfy Euler's relation for the number of faces of a d-polytope and clearly are not necessarily integers.

We will now present some limiting properties of $e_d(k,n)$.

Corollary 1:

For a given d, and given that P is feasible, we have

(i) $\lim_{n\to\infty} e_d(k,n) = 2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}$, $0 \le k \le d$. That is, when n goes to ∞ and d remains constant the expected number of all dimensional faces of constraint set of P is the same as that of a d-hypercube.

(ii) $2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}$ (the number of k faces of the d-hypercube) is an upper bound for $e_d(k,n)$, for $0 \le k \le d$, for every n. It is also clear that $\lim_{n\to\infty} e_d(k,n) = 2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}$ from below.

Proof:

$$\text{(i)} \quad \lim_{n\to\infty} \frac{2^{d-k}\begin{bmatrix} n \\ d-k \end{bmatrix} \sum_{i=0}^{k} \begin{bmatrix} n-d+k \\ i \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} = \frac{2^{d-k}\begin{bmatrix} n \\ d-k \end{bmatrix}\begin{bmatrix} n-d+k \\ k \end{bmatrix}}{\begin{bmatrix} n \\ d \end{bmatrix}} = 2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}.$$

(ii) We have to show that, for every n , $e_d(k,n) \leq 2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}$, $0 \leq k \leq d$. This is obviously true for $k = d$.

We know that

$$e_d(k,n) = \frac{2^{d-k}\begin{bmatrix} n \\ d-k \end{bmatrix} \sum_{i=0}^{k} \begin{bmatrix} n-d+k \\ i \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}$$

$$= \frac{2^{d-k}\begin{bmatrix} n \\ d-k \end{bmatrix} L(k,n-d+k)}{L(d,n)} ,$$

where $L(d,n) = \sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}$. We can then write

$$e_d(k,n) = \frac{2^{d-k} \frac{n!}{(d-k)!(n-d+k)!} L(k,n-d+k)}{L(d,n)}$$

$$= \frac{2^{d-k} \frac{d!}{k!(d-k)!} \cdot \frac{k!}{d!} \cdot \frac{n!}{(n-d+k)!} L(k,n-d+k)}{L(d,n)}$$

$$= 2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix} \frac{k!}{d!} \frac{n!}{(n-d+k)!} \frac{L(k,n-d+k)}{L(d,n)} .$$

However,

$$\frac{k!}{d!} \frac{n!}{(n-d+k)!} L(k,n-d+k)$$

$$= \frac{k!}{d!} \frac{n!}{(n-d+k)!} \sum_{i=0}^{k} \begin{bmatrix} n-d+k \\ i \end{bmatrix}$$

$$= \frac{k!}{d!} \frac{n!}{(n-d+k)!} \sum_{i=0}^{k} \frac{(n-d+k)!}{(n-d+k-i)!i!}$$

$$= \frac{k!}{d!} \sum_{i=0}^{k} \frac{n!}{(n-d+k-i)!i!}$$

$$= \frac{1}{\prod_{\ell=k+1}^{d} \ell} \sum_{j=d-k}^{d} \frac{n!}{(n-j)!(j-d+k)!} \, , \quad 0 \le k \le d-1$$

$$= \sum_{j=d-k}^{d} \frac{n!}{(n-j)!\left((j-d+k)! \prod_{\ell=k+1}^{d} \ell\right)} < \sum_{i=0}^{d-k-1} \frac{n!}{i!(n-i)!}$$

$$+ \sum_{i=d-k}^{d} \frac{n!}{i!(n-i)!}$$

$$= \sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix} = L(d,n) \; .$$

Hence,

$$e_d(k,n) < 2^{d-k} \begin{bmatrix} d \\ k \end{bmatrix} \frac{L(d,n)}{L(d,n)} = 2^{d-k} \begin{bmatrix} d \\ k \end{bmatrix} \, , \quad 0 \le k \le d-1 \; . \qquad \square$$

As n goes to ∞ and d remains constant, the probability of feasibility, $P_F(d,n)$ goes to zero, but it should be mentioned that, given feasibility, $2^{d-k}\begin{bmatrix} d \\ k \end{bmatrix}$ is an upper bound for $e_d(k,n)$, for $0 \le k \le d$. However, this is not a sharp bound, for $0 \le k \le (d-1)$. We will show it for $k = 0$ and $k = (d-1)$ (extreme points and facets).

Corollary 2:

(i) For $k = 0$, $e_d(0,n) \le \begin{bmatrix} n - [(d+1)/2] \\ n-d \end{bmatrix} + \begin{bmatrix} n - [(d+2)/2] \\ n-d \end{bmatrix} < 2^d$,

for $n < d + [(d+1)/2]$.

(ii) For $k = d-1$, $e_d(d-1,n) < \min(n,2d)$.

<u>Proof</u>:

(i) For a given m , $\begin{bmatrix} d \\ m \end{bmatrix}$, where $d > m$ is an increasing function of d . We can write

$$2^d = \sum_{i=0}^{d} \begin{bmatrix} d \\ i \end{bmatrix} = S \ .$$

We know that

$$n - [(d+2)/2] < n - [(d+1)/2] < d \ .$$

Therefore,

$$n - d < [(d+1)/2] < d \ .$$

Hence,

$$\begin{bmatrix} d \\ n-d \end{bmatrix} > \begin{bmatrix} n - [(d+1)/2] \\ n-d \end{bmatrix} .$$

Also,

$$\begin{bmatrix} d \\ d-(n-d) \end{bmatrix} = \begin{bmatrix} d \\ 2d-n \end{bmatrix} = \begin{bmatrix} d \\ n-d \end{bmatrix} > \begin{bmatrix} n - [(d+2)/2] \\ n-2 \end{bmatrix} .$$

Since $\begin{bmatrix} d \\ n-d \end{bmatrix}$ and $\begin{bmatrix} d \\ 2d-n \end{bmatrix}$ are terms of S , we have $S > \begin{bmatrix} n - [(d+1)/2] \\ n-d \end{bmatrix} + \begin{bmatrix} n - [(d+2)/2] \\ n-d \end{bmatrix}$. But, since $\begin{bmatrix} n - [(d+1)/2] \\ n-d \end{bmatrix} + \begin{bmatrix} n - [(d+2)/2] \\ n-d \end{bmatrix}$ is the maximal number of extreme points for a d-polyhedron, with n facets

(McMullen [7]), we then have

$$e_d(0,n) \leq \begin{bmatrix} n - [(d+1)/2] \\ n-d \end{bmatrix} + \begin{bmatrix} n - [(d+2)/2] \\ n-d \end{bmatrix} < 2^d ,$$

$$\text{for} \quad n < d + [(d+1)/2] . \qquad \square$$

(ii) It suffices to show that for $d > \frac{n}{2}$, $e_d(d-1,n) < n$. Let

$$L(d,n) = \sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} - \sum_{i=d+1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}$$

$$= 2^n - \bar{L}(n,d) .$$

Similarly,

$$L(d-1,n-1) = 2^{n-1} - \bar{L}(n-1,d-1)$$

$$\bar{L}(d,n) = \frac{n!}{(d+1)!(n-d-1)!} + \frac{n!}{(d+2)!(n-d-2)!} + \cdots + \frac{n!}{(n-2)!2!}$$

$$+ \frac{n!}{(n-1)!1!} + \frac{n!}{n!}$$

$$\bar{L}(d-1,n-1) = \frac{(n-1)!}{d!(n-d-1)!} + \frac{(n-1)!}{(d+1)!(n-d-2)!} + \cdots + \frac{(n-1)!}{(n-3)!2!}$$

$$+ \frac{(n-1)!}{(n-2)!1!} + \frac{(n-1)!}{(n-1)!} .$$

Since $d > \frac{n}{2}$, $2 > \frac{n}{d}$ and

$$2\bar{L}(d-1,n-1) > \frac{n!}{d(d!)(n-d-1)!} + \frac{n!}{d(d+1)!(n-d-2)!} + \cdots$$

$$+ \frac{n!}{d(n-3)!2!} + \frac{n!}{d(n-2)!1!} + \frac{n!}{d(n-1)!} > \bar{L}(d,n) .$$

Therefore,

$$2\bar{L}(d-1,n-1) > \bar{L}(d,n) \quad \text{or} \quad 2\bar{L}(d-1,n-1) - \bar{L}(d,n) > 0$$

and

$$L(d,n) - 2L(d-1,n-1)$$

$$= 2^n - \bar{L}(d,n) - 2^n + 2\bar{L}(d-1,n-1) = 2\bar{L}(d-1,n-1) - \bar{L}(d,n) > 0 \ .$$

Hence,

$$L(d,n) > 2L(d-1,n-1)$$

and

$$e_d(d-1,n) = 2n \frac{\sum_{i=0}^{d-1} \begin{bmatrix} n-1 \\ i \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} = \frac{2nL(d-1,n-1)}{L(d,n)} < \frac{2nL(d-1,n-1)}{2L(d-1,n-1)} \ .$$

Therefore, for $d > n/2$, $e_d(d-1,n) < n$. □

In a similar way we can show

Corollary 3:

When $n \to \infty$ and $n-d = m$ remains constant, $e_d(k,n)$ goes to

$$\frac{\begin{bmatrix} n \\ d-k \end{bmatrix} \sum_{i=0}^{k} \begin{bmatrix} m+k \\ i \end{bmatrix}}{\sum_{i=0}^{m+k} \begin{bmatrix} m+k \\ i \end{bmatrix}} \ .$$

Note that the limit of $e_d(d,n)$ (i.e., the expected number of facets), is equal to n . This should be expected since $P_F(n,d) = 1$, i.e., the probability of feasibility in this case is equal to one, and all the supporting hyperplanes of the half spaces of X should, with probability one, form a facet of the d-polyhedron that is the constraint set of P .

2.3 d-Arrangements and the Objective Function of Randomly Generated Linear Programs

Let the family of hyperplanes $\hat{H}$ characterize the objective function cx of P .

Lemma 2.3.1:

Given a nonempty cone C , with some point $Q \in \mathbb{E}^d$ as an apex, and a hyperplane H passing through Q , exactly one of the following two cases can happen (with probability 1):

(i) H is tangent to C

(ii) H partitions C to two nonempty cones, C_1 and C_2 , each with an apex Q . ∎

Using Lemma 2.3.1, one can easily prove

Lemma 2.3.2:

Given 2^d nonempty cones formed by d hyperplanes $H_1, \ldots, H_d$ intersecting at a point Q in $\mathbb{E}^d$, having the general intersection property and another hyperplane H , passing through Q , such that H , $H_1, \ldots, H_d$ are in general position, then H partitions 2^{d-1} of the 2^d nonempty cones to two cones each and H is tangent to exactly two of the 2^d nonempty cones. ∎

Theorem 2.3.1:

Let the assumptions of Section 1 be satisfied. Then, given that the random linear program P is feasible, we have

(i) The probability of a given vertex of the d-Arrangement formed by the supporting hyperplanes $\{x \mid A_i x = b_i\}$, $i = 1, \ldots, n$ of the half spaces of X, being the optimal extreme point for P is equal to $\dfrac{1}{\sum_{i=0}^{d} \begin{bmatrix} n \\ d \end{bmatrix}}$.

(ii) The probability of the random linear program P being bounded is equal to

$$P_B = \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} .$$

<u>Proof</u>:

(i) Due to the assumptions of Section 1, the supporting hyperplanes of the half spaces of the constraint set of P, $H_1, \ldots, H_n$, with probability one, form a d-Arrangement in $\mathbb{E}^d$, as shown in Theorem 2.2.1. Let, after the random generation $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ be the values generated for A, b and c. $H_i = \{x \mid \tilde{A}_i x = \tilde{b}_i\}$ $i = 1, \ldots, n$ and let $\hat{H}$ represent the family of hyperplanes that characterizes the objective function $\tilde{c}x$. Then, let Q be a vertex of the d-Arrangement formed by the hyperplanes H_i, such that, without loss of generality, Q is the intersection of the hyperplanes $H_1, \ldots, H_d$. Consider now a hyperplane $H \in \hat{H}$ passing through Q. Due to the assumptions of Chapter 1, we have that the hyperplanes H, $H_1, \ldots, H_d$ are in general position with probability one. From Lemma 2.3.2 we know that H is tangent to two of the d-polyhedra to which Q belongs. So, there exists exactly two polyhedra say X^1 , X^2 in the d-Arrangement for which H is tangent to them at Q . Moreover, it is obvious that Q maximizes the objective function represented by H over one polyhedron (say X^0) and minimizing it over the other (say X^2). Since (by Theorem 1.2.1) there is equal probability that the objective is maximize or minimize and since X^1 and X^2 can be selected with a probability of $1/F_d(n,d)$ we get that Q has probability of $\frac{1}{2} \cdot \frac{1}{F_d(n,d)}$ of being the optimal extreme point of X^1 and similarly for X^2 . Thus Q has a probability of $2 \cdot \frac{1}{2} \frac{1}{F_d(n,d)} = \frac{1}{F_d(n,d)}$ of being the optimal extreme point of the randomly generated P .

(ii) As it follows that for any $H \in \hat{H}$, H , $H_1, \ldots, H_n$ are in general position, then P is bounded if and only if an extreme point is the unique optimal solution. Since we have $F_d(0,n) = \begin{bmatrix} n \\ d \end{bmatrix}$ vertices in the d-Arrangement and each of them has equal probability $\frac{1}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}$ of being the optimal extreme point, it follows that the desired probability is

$$P_B(n,d) = \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} . \qquad \square$$

Corollary:

The probability of the random linear program P being bounded (and feasible), not conditioning on feasibility, is equal to

$$\bar{P}_B(n,d) = \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} .$$

Proof:

Since $P_B(n,d)$ is the desired probability when we condition on feasibility and the probability of feasibility is equal to $P_F(n,d)$, then

$$\bar{P}_B(n,d) = P_B(n,d) \cdot P_F(n,d) = \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} \cdot \frac{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} . \qquad \square$$

Proposition 2.3.1:

(i) $\lim_{n\to\infty} P_B(n,d) = 1$ and

(ii) $\lim_{n\to\infty} P_B(n,d) = 0$.

Proof:

(i) $\lim_{n\to\infty} P_B(n,d) = \lim_{n\to\infty} \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}$. But,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}}{\begin{bmatrix} n \\ d \end{bmatrix}} = \lim_{n\to\infty} \frac{\begin{bmatrix} n \\ 0 \end{bmatrix}}{\begin{bmatrix} n \\ d \end{bmatrix}} + \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ d \end{bmatrix}} + \cdots + \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\begin{bmatrix} n \\ d \end{bmatrix}} = 1 .$$

(ii) $\lim_{n\to\infty} P_B(n,d) = \lim_{n\to\infty} \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{\sum_{i=0}^{d} \begin{bmatrix} n \\ i \end{bmatrix}} = 0$. $\square$

Proposition 2.3.2:

$$\lim_{n\to\infty} \bar{P}_B(n,d) = 0 \quad \text{and} \quad \lim_{n\to\infty} \bar{\bar{P}}_B(n,d) = 0 .$$

Proof:

Since $\bar{\bar{P}}_B(n,d) = \frac{\begin{bmatrix} n \\ d \end{bmatrix}}{2^n}$, the result easily follows. □

3. FINAL REMARKS

Since our main objective in this paper was to present the distribution-independence nature of our method of generating randomly linear programs we left some uncovered related topics.

(1) We assumed a standard form of linear program. Obviously, it is desirable to have the results form free.

(2) Generating a linear program actually resulted in another uniquely defined second linear program, namely the dual. Is the (probabilistic) behavior of these dual programs identical with the results obtained for the primals?

(3) What would happen if the continuity assumption of the density function of the coefficients was dropped? Specifically, what if (as it is believed to the "real" case) *there is a positive probability for a zero* value?

(4) What can be said about the variance of the random variables for which expected values were calculated in Section 2?

(5) How the results could be extented to other characteristics of linear programs, specifically, those which are related to the efficiency of the Simplex method? (such as the expected diameter of the feasibility set, expected number of iterations of some variants of the Simplex method, etc.).

In a subsequent paper, we shall provide the answer to the first three questions and will show that our results are valid to any form of linear programs and for the dual as well.

Calculating the variances of the number of different faces seems to be a much more difficult task, though we believe they are finite and in the same order of magnitude as the expected values. As for the last (and most important) question, we started some preliminary research and hope to report it in some future time.

REFERENCES

[1] Berenguer, S. E., de B., "Randomly Generated Linear Programs," Ph.D. Thesis, University of California, Berkeley, (1978).

[2] Bickel, P. J. and K. A. Doksum, MATHEMATICAL STATISTICS: BASIC IDEAS AND SELECTED TOPICS, Holden Day, San Francisco, 1977.

[3] Buck, R. C., "Partition of Space," American Mathematical Monthly, Vol. 50, pp. 541-544, (1943).

[4] Cover, T. M. and B. Efron, "Geometrical Probability and Random Points on a Hypersphere," Annals of Mathematical Statistics, Vol. 38, pp. 212-220, (1962).

[5] Grünbaum, B., CONVEX POLYTOPES, John Wiley & Sons, Inc., New York, (1967).

[6] Halmos, P. R., MEASURE THEORY, D. Van Nostrand, Princeton, New Jersey, (1950).

[7] McMullen, P., "The Maximum Number of Faces of a Convex Polytope," Mathematika, Vol. 17, pp. 179-184, (1970).

[8] Schläfi, L., GESAMMELTE MATHEMATISCHE ABHANDLUGEN I, Verlag Birkhauser, Basel, (1950).

DAT
ILM