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A NEW RANGE STATISTIC

FOR COMPARISONS OF

SEVERAL EXPONENTIAL LOCATION PARAMETERS*

by

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Given k independent exponential populations with unknown and possibly unequal location parameters, with a common but unknown scale parameter, a complete random sample of common size n is drawn from each population. Based on these k samples, this paper proposes a new range statistic for comparing these exponential location parameters by a set of simultaneous confidence intervals. Upper percentage points are given, and the power function is also studied. Other applications in multiple range testing and in ranking and selection problems are noted.

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A NEW RANGE STATISTIC FOR COMPARISONS OF SEVERAL EXPONENTIAL LOCATION PARAMETERS

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ABSTRACT

Let there be k independent exponential populations with unknown and possibly unequal location parameters, and with a common but unknown scale parameter. A complete random sample of common size n is drawn from each population. Based on these k samples, this paper is to propose a new range statistic called C-statistic, for comparing these exponential location parameters by a set of simultaneous confidence intervals. The upper percentage points of the C-statistic are given, and the power function of the statistic under the alternative hypothesis is also studied. Other applications in multiple range test and in ranking and selection problems could be expected.

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1. INTRODUCTION

In life testing, there are many occasions when the exponential distributions are appropriate statistical models for the random variables under study. The problem of testing the hypothesis concerning the parameters of these distributions has been extensively studied in the past four decades. Paulson (1941), Epstein and Tsao (1953), Kumar and Patel (1971), Weinman, et al. (1973), and others considered the inference on two exponential location parameters based on likelihood ratio or its equivalence. For testing the equality of several exponential parameters Sukhatme (1936) considered likelihood ratio tests; Hogg and Tanis (1963) proposed an iterated testing procedure by which earlier conclusion may be drawn. These procedures are long well known and only applicable for testing a hypothesis. However, the multiple comparisons between these parameters of interest by either pairwise differences or a linear contrast were not reported. In this paper, we propose a new range statistic, C, which is similar to the Studentized range statistic, for comparing several exponential location parameters by their differences or a linear contrast.

The statistic C proposed in this paper has not been seen in the literature nor in the books by Gumbel (1958), Sarhan and Greenberg (1962) and David (1981). The distribution of C and its moments are therefore obtained in Section 2. Applications of the C statistic to hypothesis testing and multiple comparisons concerning these location parameters are given in Section 3. The power function of the test is considered in Section 4. And the table of upper percentage points is also provided.

2. DISTRIBUTION OF C

Let $X_1, \ldots, X_k$ be independent and identically distributed random variables (i.i.d.r.v.'s) with the standard exponential probability density function (p.d.f.), $f(x) = \exp(-x)$, $x > 0$, zero elsewhere. Let $R = \max(X_i) - \min(X_i)$ be the range of the X's. Then, it is easy to see that the distribution of R has the p.d.f. of the form

$$g(r) = (k-1)\exp(-r)\{1 - \exp(-r)\}^{k-2}, \quad r > 0, \qquad (2.1)$$

zero elsewhere. Let V be a chi-square r.v. with 2ν degrees of freedom (d.f.), independent of the X's. Define the range statistic C to be

$$C = \frac{R}{V/2\nu} \quad . \qquad (2.2)$$

Let Y=R. Then the joint p.d.f. of C and Y, g(c,y), can be easily found as

$$g(c,y) = (k-1)\nu^{\nu}y^{\nu}e^{-y(1+\nu/c)}(1-e^{-y})^{k-2}/(\Gamma(\nu)c^{\nu+1}), \qquad (2.3)$$

$$c > 0,\ y > 0,$$

zero elsewhere. The marginal p.d.f. of C can then be obtained by using the binomial expansion, collecting the exponential terms and integrating out y in g(c,y), we have the p.d.f. of C, h(c), where

$$h(c) = (k-1)\sum_{i=0}^{k-2}(-1)^i\binom{k-2}{i}[1+c(i+1)/\nu]^{-(\nu+1)}, \quad c > 0, \qquad (2.4)$$

zero elsewhere.

We note that, if k=2, the C has an F distribution with 2 and 2ν d.f. and it has an L-shaped density. When $k > 2$, the distribution of C has an unimodel occurring at $c = c_o > 0$ and the shape of the density is skewed to the right.

The upper αth percentage point $c = c_{k,\nu}^{\alpha}$ is obtained according to the equation

$$\int_c^\infty h(c)dc = \sum_{i=0}^{k-2} (-1)^i \binom{k-1}{i+1} \{1+c(i+1)/\nu\}^{-\nu} = \alpha. \tag{2.5}$$

Since the range statistic will be used in certain applied problems associated with samples of size n, we let $\nu = k(n-1)$ and then compute the table of the percentage point c for $k = 2(1)10(2)20, 25, 30, 40, 50$, $n = 2(1)10, 12, 16, 20, 30, 60, 120, \infty$; $\alpha = .10, .05, .01$ which is given in Table 1.

The mth moment of C about origin may be obtained by the changes of variable $y = 1/(1+c(i+1)/\nu)$. Thus, we have

$$\begin{aligned} E(C^m) &= \int_0^\infty c^m h(c)dc \\ &= (k-1) \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} \left(\frac{\nu}{i+1}\right)^{m+1} \int_0^1 \sum_{j=0}^{m} \binom{m}{j} (-1)^j y^{\nu-m-1+j} dy \\ &= (k-1) \sum_{i=0}^{k-2} \binom{k-2}{i} \left(\frac{\nu}{i+1}\right)^{m+1} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{j} / (\nu-m+j) \end{aligned}$$

provided $\nu > m$.

It is easy to show that, as n or ν goes to infinity, the distribution of C in (2.4) converges to g(r). Thus, the critical value of c can be obtained by the equation

$$c = -\ln(1-(1-\nu)^{1/(k-1)}) \ . \tag{2.6}$$

For large n, the approximation is fairly good as compared with the tabled values.

3. MULTIPLE COMPARISONS

Let $\pi_1, \pi_2, \ldots, \pi_k$ denote $k(\geq 2)$ populations so that n independent observations $X_{i1}, X_{i2}, \ldots, X_{in}$ taken from population π_i are exponentially distributed with the probability density function (p.d.f.)

$$f(x;\alpha_i,\theta) = (1/\theta)\exp\{-(x-\alpha_i)/\theta\}, \ \alpha_i < x < \infty, \ \theta > 0, \tag{3.1}$$

zero elsewhere, where α_i is an unknown location parameter and θ is a common but unknown scale parameter for $i = 1, 2, \ldots, k$. It is known that the location parameter is also referred to as the guaranteed life span and the scale parameter the standard deviation.

Let Y_i denote the first order statistic of the sample of size n from population π_i. Let $Y_{[1]} \leq \cdots \leq Y_{[k]}$ denote the ordered values of the k first order statistics $Y_1, \ldots, Y_k$ and let the well known minimum variance unbiased estimator of θ be given by

$$\hat{\theta} = \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - Y_i)/(k(n-1)).$$

Define the range statistic C to be

$$C = n(Y_{[k]} - Y_{[1]})/\hat{\theta}. \tag{3.2}$$

Under the null hypothesis $H_0 : \alpha_1 = \ldots = \alpha_k$, C has the same distribution as of (2.4). This is because C is the ratio of $R = n(Y_{[k]} - Y_{[1]})/\theta$ to $V = 2\nu\hat{\theta}/\theta$, where R and V are independent.

The C range statistic provides a quick test for H_0; one rejects H_0 at α level of significance if the computed value of C is larger than $c_{k,\nu}^{\alpha}$ where $P(C > c_{k,\nu}^{\alpha} \mid H_0) = \alpha$.

Since the C statistic is based on the range, it will not be as powerful as the F-test considered by Sukhatme (1936) when a test of hypothesis is concerned. However, if the multiple comparison among α's is interested, the Sukhatme's test is not applicable. Like Tukey's Studentized range statistic for comparing normal means, the major role of the C range statistic will rest on its extensive use in pairwise multiple comparisons and simultaneous inferences on the location parameters.

Similar to Tukey's pairwise multiple comparison for normal means, we can construct a set of simultaneous confidence intervals for all of the differences $\alpha_i - \alpha_j$ by the following probability statements:

$$P\{|(Y_i-Y_j) - (\alpha_i-\alpha_j)| \leq c^{\alpha}_{k,k(n-1)} \cdot \frac{\hat{\theta}}{n},\ i,j = 1, \ldots, k,\ i \neq j\} = 1 - \alpha.$$

This is because the inequalities

$$\frac{(Y_i-Y_j) - (\alpha_i-\alpha_j)}{\hat{\theta}/n} \leq c \tag{3.3}$$

hold true for all $i \neq j$ if and only if the inequality

$$\frac{\frac{n}{\theta} \max\limits_{i,j} \{|(Y_i-\alpha_i) - (Y_j-\alpha_j)|\}}{\hat{\theta}/\theta} \leq c \tag{3.4}$$

holds true. The numerator in the left member of the inequality (3.4) is the range of k independent standard exponential r.v.'s and the denominator is the chi-square r.v. (with 2ν d.f.) divided by 2ν. Their ratio defines C as of (2.2). Therefore, a set of $(1-\alpha)100\%$ simultaneous confidence intervals for the difference $\alpha_i - \alpha_j$ is given by

$$\alpha_i - \alpha_j \in (Y_i - Y_j) \pm c^{\alpha}_{k,k(n-1)} \cdot \hat{\theta}/n \tag{3.5}$$

for all $i,j = 1, 2, \ldots, k$, $i \neq j$. In the case where the sample sizes are not all equal, we suggest to replace $1/n$ in (3.5) by $(1/n_i + 1/n_j)/2$ and $k(n-1)$ by

$\nu = \sum_{1}^{k} (n_i-1)$, in which a conservative set of confidence intervals is conjectured.

Furthermore, a generalization of the above pairwise comparisons of α's can be made by the linear contrasts of α's. The probability statement for the family of all contrasts is given by

$$P\{|\sum_{i=1}^{k} c_i(Y_i - \alpha_i)| \leq c^{\alpha}_{k,k(n-1)} \frac{\hat{\theta}}{n} \sum_{i=1}^{k} \frac{|c_i|}{2}\} = 1 - \alpha,$$

where $\sum_{i=1}^{k} c_i = 0$. This is because inequalities (3.3) hold true for all $i \neq j$ if and only if the inequality

$$|\sum_{i=1}^{k} c_i(Y_i - \alpha_i)| \leq c^{\alpha}_{k,k(n-1)} \frac{\hat{\theta}}{n} \sum_{i}^{k} \frac{|c_i|}{2}$$

is satisfied for all linear contrasts $\sum_{1}^{k} c_i = 0$. Thus, the $(1-\alpha)100\%$ simultaneous confidence intervals for all linear contrasts of the location parameters is given by

$$\sum_{i=1}^{k} c_i\alpha_i \in \sum_{i=1}^{k} c_iY_i \pm c^{\alpha}_{k,k(n-1)} \frac{\hat{\theta}}{n} \sum_{1}^{k} \frac{|c_i|}{2}.$$

These comparisons by linear contrasts include pairwise comparisons and also the comparisons of the differences between weighted averages among α's.

Other applications of the C-distribution in multiple range test and in ranking and selection problem could be expected.

4. POWER FUNCTION

The cumulative distribution function (c.d.f.) of the statistic C under the alternative hypothesis $H_1 : \alpha_i \neq \alpha_j$ may be derived in the following way:

$$P(C \leq c|H_1) = P(n(Y_{[k]} - Y_{[1]})/\hat{\theta} \leq c|H_1)$$

$$= \sum_{j=1}^{k} P(nY_i \leq c\hat{\theta} + nY_j,\ Y_j = Y_{[1]},\ \text{for all } i,\ i \neq j|H_1)$$

$$= \sum_{j=1}^{k} P(\max(Z_j + \delta_{ji}, 0) < Z_i < cX + Z_j + \delta_{ji},\ \text{for all } i,\ i \neq j),$$

$$= \sum_{j=1}^{k} \int_0^\infty \int_0^\infty \prod_{i=1}^{k} \{F(cx + z_j + \delta_{ji}) - F(\max(z_j + \delta_{ji}, 0))\} \cdot dF(z_j)dG(x) \qquad (4.1)$$

where

$$Z_i = n(Y_i - \alpha_i)/\theta,\ \delta_{ji} = n(\alpha_j - \alpha_i)/\theta\ ,\ \text{for } i = 1, 2, \ldots, k,\ i \neq j,$$

$F(x) = 1 - e^{-x}$ and $G(x)$ is the c.d.f. of $X = \hat{\theta}/\theta$. Note that for any given i, j, the probability $F(cx + z_j + \delta_{ji}) - F(\max(z_j + \delta_{ji}, 0))$ is positive if $cx + z_j + \delta_{ji} > 0$, zero otherwise.

The c.d.f. in (4.1) which is also termed the probability of type II error depends on the $\binom{k-1}{2}$ differences of the α's and it cannot be simplified further. Thus, the power function can only be expressed as $\beta(\underline{\alpha}) = 1 - p(C \leq c|H_1)$, for which a numerical integration or computer simulation should be used under H_1. It is clear that the power function under H_0 reduces to (2.5) as a special case. It can also be examined that for a particular configuration of α's, $\alpha_1 = \ldots = \alpha_{k-1}$, $\alpha_k = \alpha_{k-1} + \delta$, $\delta > 0$, expression (4.1) can be reduced to

$$P(C \le c|H_1) = e^{-(k-1)d} \sum_{\ell=0}^{k-1} (-1)^{\ell} \binom{k-1}{\ell} \{(1 + \ell c/\nu)^{-\nu}$$

$$- (k-1) \frac{\nu^{\nu}}{\Gamma(\nu)} \sum_{i=0}^{\nu-1} i! \binom{\nu-1}{i} \left(\frac{d}{c}\right)^{\nu-i-1} (\nu+\ell c)^{-i-1}$$

$$\sum_{j=0}^{\nu-i-1} (-1)^j \binom{\nu-i-1}{j} \sum_{m=0}^{j} m! \binom{j}{m} d^{-m} \left(k - \frac{\nu+\ell c}{c}\right)^{-m-1}\},$$

where $\nu = k(n-1)$, $d = n\delta/\theta$, $c = c^{\alpha}_{k,k(n-1)}$, which converges to $1-\alpha$ as $\delta \to 0$ and to 0 as $\delta \to \infty$.

TABLE 1. Upper Percentage Points of the Range Statistic C.

$\alpha = .10$

n \ k	2	3	4	5	6	7	8	9	10
2	4.32	4.81	4.98	5.08	5.16	5.23	5.30	5.36	5.41
3	3.11	3.77	4.09	4.31	4.48	4.61	4.73	4.83	4.92
4	2.81	3.48	3.83	4.08	4.27	4.42	4.55	4.66	4.76
5	2.67	3.34	3.71	3.97	4.16	4.32	4.46	4.58	4.68
6	2.59	3.26	3.64	3.90	4.10	4.27	4.41	4.53	4.64
7	2.54	3.21	3.59	3.86	4.06	4.23	4.37	4.50	4.60
8	2.50	3.18	3.56	3.83	4.04	4.21	4.35	4.47	4.58
9	2.48	3.15	3.53	3.81	4.02	4.19	4.33	4.46	4.57
10	2.46	3.13	3.52	3.79	4.00	4.17	4.32	4.44	4.55
12	2.43	3.10	3.49	3.76	3.98	4.15	4.30	4.42	4.54
16	2.39	3.06	3.46	3.73	3.95	4.12	4.27	4.40	4.51
20	2.37	3.04	3.44	3.71	3.93	4.11	4.26	4.39	4.50
30	2.35	3.02	3.41	3.69	3.91	4.09	4.24	4.37	4.48
60	2.33	2.99	3.39	3.67	3.89	4.07	4.22	4.35	4.47
120	2.31	2.98	3.38	3.66	3.88	4.06	4.21	4.34	4.46
∞	2.30	2.97	3.37	3.65	3.87	4.05	4.20	4.34	4.45

$\alpha = .05$

2	6.94	6.90	6.75	6.67	6.62	6.59	6.59	6.59	6.60
3	4.46	5.00	5.23	5.39	5.51	5.61	5.70	5.77	5.83
4	3.89	4.50	4.81	5.02	5.18	5.31	5.43	5.52	5.61
5	3.63	4.28	4.62	4.85	5.02	5.17	5.29	5.40	5.50
6	3.49	4.15	4.50	4.75	4.93	5.09	5.22	5.33	5.43
7	3.40	4.06	4.43	4.68	4.87	5.03	5.17	5.28	5.39
8	3.34	4.01	4.38	4.63	4.83	4.99	5.13	5.25	5.36
9	3.29	3.96	4.34	4.60	4.80	4.96	5.10	5.22	5.33
10	3.26	3.93	4.31	4.57	4.78	4.94	5.08	5.21	5.31
12	3.21	3.88	4.26	4.53	4.74	4.91	5.04	5.18	5.29
16	3.15	3.83	4.21	4.49	4.70	4.87	5.02	5.14	5.26
20	3.12	3.79	4.18	4.46	4.67	4.85	5.00	5.12	5.24
30	3.07	3.75	4.15	4.43	4.64	4.82	4.97	5.10	5.21
60	3.03	3.71	4.11	4.39	4.61	4.79	4.94	5.08	5.19
120	3.01	3.69	4.09	4.38	4.60	4.78	4.93	5.06	5.18
∞	3.00	3.68	4.08	4.36	4.58	4.77	4.92	5.05	5.17

$\alpha = .01$

2	18.00	14.05	12.23	11.24	10.64	10.24	9.96	9.76	9.61
3	8.65	8.43	8.25	8.14	8.09	8.06	8.05	8.06	8.07
4	6.93	7.18	7.27	7.33	7.39	7.45	7.51	7.56	7.62
5	6.23	6.64	6.83	6.96	7.07	7.17	7.25	7.33	7.40
6	5.85	6.34	6.58	6.76	6.89	7.01	7.11	7.19	7.28
7	5.61	6.15	6.43	6.62	6.77	6.90	7.01	7.11	7.19
8	5.45	6.02	6.32	6.52	6.69	6.82	6.94	7.04	7.13
9	5.34	5.92	6.25	6.45	6.63	6.77	6.89	7.00	7.09
10	5.25	5.85	6.17	6.40	6.58	6.72	6.85	6.96	7.06
12	5.12	5.74	6.08	6.32	6.51	6.66	6.79	6.91	7.01
16	4.98	5.62	5.98	6.23	6.43	6.59	6.73	6.85	6.95
20	4.90	5.55	5.92	6.18	6.38	6.55	6.69	6.81	6.92
30	4.79	5.46	5.84	6.11	6.32	6.49	6.64	6.77	6.88
60	4.70	5.38	5.77	6.05	6.26	6.44	6.59	6.72	6.84
120	4.65	5.34	5.73	6.02	6.24	6.42	6.57	6.70	6.82
∞	4.61	5.30	5.70	5.99	6.21	6.39	6.55	6.68	6.80

TABLE 1. (Continued)

α = .10

n \ k	12	14	16	18	20	25	30	40	50	100
2	5.51	5.60	5.68	5.75	5.82	5.97	6.10	6.31	6.48	7.05
3	5.07	5.20	5.31	5.41	5.50	5.70	5.86	6.11	6.31	6.95
4	4.93	5.07	5.20	5.30	5.40	5.61	5.78	6.05	6.26	6.92
5	4.86	5.01	5.14	5.25	5.35	5.56	5.74	6.01	6.23	6.90
6	4.82	4.97	5.10	5.22	5.32	5.54	5.71	5.99	6.21	6.89
7	4.79	4.94	5.08	5.19	5.30	5.52	5.70	5.98	6.20	6.88
8	4.77	4.93	5.06	5.18	5.28	5.51	5.69	5.97	6.19	6.88
9	4.76	4.91	5.05	5.17	5.27	5.50	5.68	5.96	6.19	6.87
10	4.74	4.90	5.04	5.16	5.27	5.49	5.67	5.96	6.18	6.87
12	4.73	4.89	5.03	5.15	5.25	5.48	5.66	5.95	6.17	6.86
16	4.71	4.87	5.01	5.13	5.24	5.47	5.65	5.94	6.17	6.86
20	4.70	4.86	5.00	5.12	5.23	5.46	5.64	5.94	6.16	6.86
30	4.68	4.85	4.99	5.11	5.22	5.45	5.64	5.93	6.15	6.85
60	4.67	4.83	4.97	5.10	5.21	5.44	5.63	5.92	6.15	6.85
120	4.66	4.83	4.97	5.09	5.20	5.44	5.62	5.92	6.15	6.85
∞	4.65	4.82	4.96	5.09	5.20	5.43	5.62	5.92	6.14	6.84

α = .05

n \ k	12	14	16	18	20	25	30	40	50	100
2	6.63	6.67	6.71	6.75	6.80	6.90	7.00	7.17	7.32	7.84
3	5.97	6.08	6.18	6.26	6.34	6.52	6.66	6.90	7.09	7.70
4	5.76	5.89	6.01	6.11	6.20	6.39	6.55	6.81	7.01	7.66
5	5.66	5.80	5.92	6.03	6.13	6.33	6.50	6.77	6.97	7.63
6	5.60	5.75	5.87	5.99	6.08	6.29	6.47	6.74	6.95	7.62
7	5.56	5.71	5.84	5.95	6.06	6.27	6.45	6.72	6.94	7.61
8	5.54	5.69	5.82	5.93	6.04	6.25	6.43	6.71	6.93	7.60
9	5.52	5.67	5.80	5.92	6.02	6.24	6.42	6.70	6.92	7.60
10	5.50	5.65	5.79	5.90	6.01	6.23	6.41	6.69	6.91	7.59
12	5.48	5.63	5.77	5.89	5.99	6.21	6.40	6.68	6.90	7.59
16	5.45	5.61	5.74	5.86	5.97	6.20	6.38	6.67	6.89	7.58
20	5.43	5.59	5.73	5.85	5.96	6.19	6.37	6.66	6.89	7.58
30	5.41	5.57	5.71	5.84	5.94	6.17	6.36	6.65	6.88	7.57
60	5.39	5.55	5.70	5.82	5.93	6.16	6.35	6.64	6.87	7.57
120	5.38	5.55	5.69	5.81	5.92	6.16	6.34	6.64	6.87	7.57
∞	5.37	5.54	5.68	5.80	5.92	6.15	6.34	6.63	6.86	7.56

α = .01

n \ k	12	14	16	18	20	25	30	40	50	100
2	9.41	9.28	9.20	9.15	9.12	9.09	9.10	9.16	9.24	9.62
3	8.10	8.15	8.19	8.24	8.29	8.41	8.51	8.70	8.86	9.41
4	7.71	7.80	7.89	7.96	8.03	8.19	8.33	8.55	8.73	9.34
5	7.53	7.64	7.74	7.83	7.91	8.09	8.23	8.48	8.67	9.30
6	7.42	7.54	7.65	7.75	7.83	8.02	8.18	8.43	8.64	9.28
7	7.35	7.48	7.59	7.69	7.78	7.98	8.14	8.41	8.61	9.26
8	7.29	7.43	7.55	7.65	7.75	7.95	8.12	8.39	8.59	9.25
9	7.26	7.40	7.52	7.63	7.72	7.93	8.10	8.37	8.58	9.25
10	7.23	7.37	7.49	7.60	7.70	7.91	8.09	8.36	8.57	9.24
12	7.19	7.33	7.46	7.57	7.67	7.89	8.06	8.34	8.56	9.23
16	7.13	7.29	7.42	7.54	7.64	7.86	8.04	8.32	8.54	9.22
20	7.11	7.26	7.40	7.51	7.62	7.84	8.02	8.31	8.53	9.22
30	7.07	7.23	7.37	7.49	7.59	7.82	8.00	8.29	8.52	9.21
60	7.03	7.20	7.34	7.46	7.57	7.80	7.99	8.28	8.50	9.20
120	7.02	7.18	7.32	7.45	7.56	7.79	7.98	8.27	8.50	9.20
∞	7.00	7.17	7.31	7.43	7.54	7.78	7.97	8.26	8.49	9.19

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