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(6) L_∞-LOWER BOUND OF L_2-PROJECTIONS

ONTO SPLINES ON A GEOMETRIC MESH

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L_∞-Lower Bound of L_2-Projections onto Splines on a Geometric Mesh

Y. Y. Feng* and J. Kozak**

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ABSTRACT

In [1], we gave another proof of the boundedness of L_2-projections onto splines on a geometric mesh. In this paper, we obtain the sharp lower bound for the inverse of the corresponding B-spline Gram matrix. I.e.

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)}\right| > 2k-1, \quad \text{for} \quad r=k,\ k-1.$$

AMS (MOS) Subject Classification: 41A15

Key Words: Euler-Frobenius polynomial, splines, sharp lower bound, Least-squares approximation

Work Unit Number 3 - Numerical Analysis and Computer Science

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SIGNIFICANCE AND EXPLANATION

Least-squares approximation by polynomial splines is a very effective means of approximation, particularly when the knots are appropriately nonuniformly spaced to adapt to the particular behaviour of the function being approximated. Unfortunately, the stability of this process has been established only for nearly uniform knot sequences. The stability can be linked to the norm of the inverse of the Gram matrix of a (appropriately scaled) B-spline basis. In an earlier report [2], we studied an important special case, that of a geometric knot sequence and there showed the norm of the inverse of that Gramian to be bounded independent of the mesh ratio.

In the present report, we continue these investigations and show, in particular, the surprising fact that the norm of the inverse of the Gramian is least (i.e., the stability is greatest) when the mesh is most nonuniform.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

L_∞-Lower Bound of L_2-Projections onto Splines on a Geometric Mesh

Y. Y. Feng* and J. Kozak**

1. Introduction

We begin with the explanation of some notations.

$\Pi_n(\lambda;q) := \frac{1}{n!t^n}\sum_{i=0}^{n}(-)^{n-i}\binom{n}{i}\prod_{\substack{j=0\\ j\neq i}}^{n}(q^j-\lambda)$, the generalized Euler-Frobenius polynomial of order n. $t := \ell n\, q$

$\binom{n}{r} := \frac{n!}{r!(n-r)!}$, a binomial coefficient.

$a_{n,i}(q)$ $(i = 0,1,\ldots n-1)$:= the coefficients of the polynomial defined by

$$\sum_{i=0}^{n-1} a_{n,i}(q)\lambda^i := \frac{1}{\gamma_n(q-1)^n}\Pi_n(\lambda;q).\quad \gamma_n := \frac{1}{n!t^n}$$

$a_{n,i}^{(j)}$ $(i = 0,1,\ldots n-1,\ j = 0,\ldots i(n-1-i))$: = the coefficients of polynomial defined by

$$a_{n,i}(q) =: q^{(n-i)(n-1-i)/2}\sum_{j=0}^{i(n-1-i)} a_{n,i}^{(j)} q^j .$$

Given a biinfinite geometric knot sequence $t := (q^i)_{-\infty}^{+\infty}$ for some $q \in (0,\infty)$ with

$$t_{\pm\infty} := \lim_{i\to\pm\infty} t_i,\ I := (t_{-\infty},t_{+\infty}) .$$

we denote by

$$N_{n,i} := (t_{i+n}-t_i)\,[t_i,t_{i+1},\ldots t_{i+n}]\,(\cdot-x)_+^{n-1}$$

the corresponding B-splines normalized so that

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$$\sum_i N_{n,i}(x) = 1$$

and by $S_{n,t} := \mathrm{span}\{N_{n,i}\}$, the space of splines of degree $n-1$ with knots t.

We can consider the projectors $P_{k,r} : C(I) \to S_{2k-r,t}$ defined by the condition that

$$P_{k,r}f = \sum_i a_i\,(f)N_{2k-r,i}$$

$$\sum_j (N_{r,i}, N_{2k-r,j})a_j(f) = (N_{r,i}, f)$$

with $(f,g) := \int_a^b f(x)g(x)dx$.

Then $P_{k,0}$ is the interpolation projector and $P_{k,k}$ the usual L_2-projector onto $S_{k,t}$.

This paper is a continuation of [1]. In [1] the uniform boundedness of

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)}\right|$$

for $q \in (0,\infty)$ with $r = k,\ k-1$ was proved. Here, G_r^{-1} is the inverse of corresponding B-spline Gram matrix. In this paper we obtain the sharp lower bound for $\|G_r^{-1}\|_\infty$. I.e. we prove that for any $q \in (0,\infty)$ and $r = k,\ k-1$, the inequality

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)}\right| \geq 2k - 1$$

holds.

In order to prove this, we need some properties of $\Pi_n(\lambda;q)$ which were studied in [1] and [2]. For the reader's convenience we copy some of them as follows.

__Proposition 1__ [2] $\Pi_n(\lambda;q)$ satisfies a "difference-delay" equation

$$\Pi_0(\lambda;q) := 1$$

$$\Pi_{n+1}(\lambda;q) = \frac{1}{(n+1)t}\left((1-\lambda)q^n \Pi_n(q^{-1}\lambda;q) - (q^{n+1}-\lambda)\Pi_n(\lambda;q)\right), \quad n = 0,1,\ldots \ .$$

__Proposition 2__ [1] The polynomial $\Pi_n(\lambda;q)$ satisfies

$$\Pi_n(\lambda;q) = \lambda^{n-1} q^{-n(n-1)/2} \Pi_n(q^n \lambda^{-1};q) \ . \qquad (1.1)$$

The coefficients $a_{n,i}(q)$ can be computed recursively by

$$a_{n+1,i}(q) = (q-1)^{-1}\left((q^{n+1}-q^{n-i})a_{n,i}(q) + (q^{n+1-i}-1)a_{n,i-1}(q)\right) , \qquad (1.2)$$

where

$$a_{n,0}(q) := 1, \ a_{n,-1}(q) = a_{n,n}(q) := 0 \ .$$

__Proposition 3__ [1] The coefficients $a_{n,i}(q)$ satisfy

$$a_{n,i}(q) = q^{n(n-2i-1)/2} a_{n,n-1-i}(q) \ . \qquad (1.3)$$

and for $n > 2$ the integer coefficients $a_{n,i}^{(j)}$ are symmetric

$$a_{n,i}^{(j)} = a_{n,i}^{(i(n-1-i)-j)}, \quad \text{all } j \ . \qquad (1.4)$$

In particular

$$a_{n,i}^{(0)} = \binom{n-1}{i} \quad , \tag{1.5}$$

$$a_{n,i}^{(1)} = (n-2)\binom{n-1}{i} - \binom{n-2}{i+1} - \binom{n-2}{i-2} \quad . \tag{1.6}$$

2. The sharp lower bound for $\|G_r^{-1}\|_\infty$

Before proving the theorem we need to do some preparation.

<u>Lemma 2.1</u> The following equalities hold

$$\frac{\Pi_{2k-1}(-q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = (-)^{k-1} q^{\frac{3}{2}k(k-1)} \sum_{j=0}^{(k-1)^2} (\alpha_{2k-1,j} - \beta_{2k-1,j}) q^j$$

with

$$\begin{aligned} \alpha_{2k-1,j} &:= a_{2k-1,k-1}^{(j)} + \sum_{i=1}^{\infty} \left(a_{2k-1,k-1-2i}^{(j-i(2i-1))} + a_{2k-1,k-1-2i}^{(j-i(2i+1))}\right) \\ \beta_{2k-1,j} &:= a_{2k-1,k-2}^{(j)} + \sum_{i=1}^{\infty} \left(a_{2k-1,k-2i}^{(j-i(2i-1))} + a_{2k-1,k-2-2i}^{(j-i(2i+1))}\right) \quad . \end{aligned} \tag{2.1}$$

For convenience, here and below we use

$$a_{n,i}^{(r)} := 0, \quad \text{if} \quad r < 0 \quad \text{or} \quad r > i(n-1-i) \quad \text{as well as} \quad i < 0 \quad . \tag{2.2}$$

In particular

$$\alpha_{2k-1,0} = \binom{2k-2}{k-1}$$

$$\beta_{2k-1,0} = \binom{2k-2}{k-2} \quad . \tag{2.3}$$

Similarly

$$\frac{\Pi_{2k-2}(-q^{k-1};q)}{\gamma_{2k-2}(q-1)^{2k-2}} = (-)^{k-1} q^{\frac{1}{2}(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} (\alpha_{2k-2,j} - \beta_{2k-2,j}) q^j \tag{2.4}$$

with

$$\alpha_{2k-2,j} = \beta_{2k-2,j} := a^{(j)}_{2k-2,k-2} + \sum_{i=1}^{\infty} \left(a^{(j-i(2i-1))}_{2k-2,k-1-2i} + a^{(j-i(2i+1))}_{2k-2,k-2-2i} \right)$$

$$= \sum_{i=0}^{\infty} a^{(j-\frac{1}{2}i(i+1))}_{2k-2,k-2-i} \quad .$$

__Proof:__ By the definition of $a_{n,i}(q)$ and $a^{(j)}_{n,i}$

$$\frac{\Pi_{2k-1}(-q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = \sum_{i=0}^{2k-2} a_{2k-1,i}(q)(-q^k)^i$$

$$= \sum_{i=0}^{2k-2} \sum_{j=0}^{i(2k-2-i)} (-)^i q^{\Phi_i + j} a^{(j)}_{2k-1,i}$$

with

$$\Phi_i := \frac{1}{2}(2k-1-i)(2k-2-i) + ik$$

$$\min_{0 \le i \le 2k-2} \Phi_i = q^{\frac{3}{2}k(k-1)}$$

Let

$$\varphi_i \; := \Phi_i - \frac{3}{2}\,k(k-1) = \frac{1}{2}\,(k-1-i)(k-2-i)$$

then

$$\frac{\Pi_{2k-1}(-q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = q^{\frac{3}{2}\,k(k-1)} \sum_{i=0}^{2k-2} \; \sum_{j=0}^{i(2k-2-i)} (-)^i\, a^{(j)}_{2k-1,i}\, q^{\varphi_i + j}$$

$$= q^{\frac{3}{2}\,k(k-1)} \left[\sum_{i=0}^{k-1} \; \sum_{j=0}^{(k-1)^2-i^2} (-)^{k-1-i} q^{\frac{1}{2}\,i(i-1)+j}\, a^{(j)}_{2k-1,k-1-i} \right.$$

$$\left. + \sum_{i=1}^{k-1} \; \sum_{j=0}^{(k-1)^2-i^2} (-)^{k-1+i} q^{\frac{1}{2}\,i(i+1)+j}\, a^{(j)}_{2k-1,k+i-1} \right]$$

$$= (-)^{k-1} q^{\frac{3}{2}\,k(k-1)} \left[\sum_{j=0}^{(k-1)^2} a^{(j)}_{2k-1,k-1} q^j + \sum_{i=1}^{k-1} \; \sum_{j=0}^{(k-1)^2-i^2} \right.$$

$$\left. (-)^i \left(q^{\frac{1}{2}\,i(i-1)+j}\, a^{(j)}_{2k-1,k-1-i} + q^{\frac{1}{2}\,i(i+1)+j}\, a^{(j)}_{2k-1,k-1-i} \right) \right]$$

$$= (-)^{k-1} q^{\frac{3}{2}\,k(k-1)} \sum_{j=0}^{(k-1)^2} \left[(a^{(j)}_{2k-1,k-1} - a^{(j)}_{2k-1,k-2}) + \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,k-1-2i} \right.$$

$$\left. + a^{(j-i(2i+1))}_{2k-1,k-1-2i}) - \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,k-2i} + a^{(j-i(2i+1))}_{2k-1,k-2-2i}) \right] q^j \quad ,$$

and (2.1) follows.

By (2.1) and (1.5) we get

$$\alpha_{2k-1,0} = a^{(0)}_{2k-1,k-1} = \binom{2k-2}{k-1}$$

$$\beta_{2k-1,0} = a^{(0)}_{2k-1,k-2} = \binom{2k-2}{k-2} \quad .$$

The same kind of argument proves (2.4).

An analogous argument gives

$$\frac{\Pi_{2k-1}(q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = q^{\frac{3}{2}k(k-1)} \sum_{j=0}^{(k-1)^2} (\alpha_{2k-1,j} + \beta_{2k-1,j})q^j \quad , \tag{2.5}$$

$$\frac{\Pi_{2k-2}(q^{k-1};q)}{\gamma_{2k-2}(q-1)^{2k-2}} = q^{\frac{1}{2}(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} (\alpha_{2k-2,j} + \beta_{2k-2,j})q^j \quad . \tag{2.6}$$

A straightforward calculation starting with the formula defining $\Pi_n(\lambda;q)$ leads to the expressions

$$\frac{\Pi_{2k-1}(q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = \binom{2k-1}{k} q^{\frac{3}{2}k(k-1)} \prod_{i=1}^{k-2} (1+q+\cdots+q^i)^2 (1+q+\cdots+q^{k-1})$$

$$=: \binom{2k-1}{k} q^{\frac{3}{2}k(k-1)} \sum_{j=0}^{(k-1)^2} d_{2k-1,j}q^j \tag{2.7}$$

$$\frac{\Pi_{2k-2}(q^{k-1},q)}{\gamma_{2k-2}(q-1)^{2k-2}} = \binom{2k-2}{k-1} q^{\frac{1}{2}(k-1)(3k-4)} \prod_{i=1}^{k-2} (1+q+\cdots+q^i)^2$$

$$=: \binom{2k-2}{k-1} q^{\frac{1}{2}(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} d_{2k-2,j}q^j \quad . \tag{2.8}$$

From (2.5), (2.6), (2.7), (2.8) it is easy to find the following relations

$$\binom{2k-1}{k} d_{2k-1,i} = \alpha_{2k-1,i} + \beta_{2k-1,i} \tag{2.9}$$

$$\binom{2k-2}{k-1} d_{2k-2,i} = \alpha_{2k-2,i} + \beta_{2k-2,i} = 2\alpha_{2k-2,i} \tag{2.10}$$

$$d_{2k-1,i} = \sum_{j=0}^{k-1} d_{2k-2,i-j} \tag{2.11}$$

$$d_{2k-2,i} = \sum_{j=0}^{k-2} d_{2k-3,i-j} \tag{2.12}$$

and

$$d_{2k-1,i} = d_{2k-1,(k-1)^2-i} \tag{2.13}$$

$$d_{2k-2,i} = d_{2k-2,(k-1)(k-2)-i} \tag{2.14}$$

where

$$d_{n,i} := \alpha_{n,i} := 0 \quad \text{if} \quad i < 0 \quad .$$

<u>Lemma 2.2</u> The following equality holds

$$a^{(\ell)}_{n,i} = \sum_{r=\ell-i}^{\ell} a^{(r)}_{n-1,i} + \sum_{r=\ell+i-n+1}^{\ell} a^{(r)}_{n-1,i-1} \quad . \tag{2.15}$$

(i)

<u>Proof</u> By (1.2) and the definitin of $a^{(i)}_{n,i}$

$$a_{n,i}(q) = q^{(n-i)(n-1-i)/2} \sum_{\ell=0}^{i(n-1-i)} a^{(\ell)}_{n,i}\, q^{\ell} \tag{*}$$

and

$$a_{n,i}(q) = q^{n-1-i}(1 + q+\cdots+q^{i})a_{n-1,i}(q) + (1 + q+\cdots+q^{n-i-1})\ a_{n-1,i-1}(q) \tag{2.16}$$

$$= q^{(n-i)(n-1-i)/2} \sum_{j=0}^{i(n-1-i)} \Big(\sum_{r=j-i}^{j} a_{n-1,i}^{(r)} + \sum_{r=j+i-n+1}^{j} a_{n-1,i-1}^{(r)}\Big) q^{j} \quad .$$

Comparing with (*), (2.15) follows.

<u>Corollary 2.1</u> *The following inequalities*

$$a_{n,i}^{(\ell)} > a_{n,i}^{(\ell-1)} \quad \text{for} \quad \ell < [\tfrac{1}{2}\ i(n-i-1)],\ 0 < i < n-1 \tag{2.17}$$

and

$$a_{n,i}^{(\ell)} > a_{n,i-1}^{(\ell)} \quad \text{for} \quad 0 < \ell < i\,(n-1-i),\ i < [\tfrac{n-1}{2}] \tag{2.18}$$

hold.

<u>Proof</u> We use mathematical induction to prove (2.17), (2.18). Suppose for $n-1$ (2.17), (2.18) hold. Using (2.15),

$$a_{n,i}^{(\ell)} - a_{n,i}^{(\ell-1)} = (a_{n-1,i}^{(\ell)} - a_{n-1,i}^{(\ell-i-1)}) + \big(a_{n-1,i-1}^{(\ell)} - a_{n-1,i-1}^{(\ell+i-n)}\big) \quad .$$

Since (1.4)

$$a_{n,i}^{(\ell)} = a_{n,i}^{(i(n-1-i)-\ell)}$$

and

$$\ell < \frac{1}{2}\ i(n-1-i) \quad \text{as well as} \quad 0 < i < n-1 \quad .$$

We get

$$a_{n-1,i}^{(\ell)} > a_{n-1,i}^{(\ell-i-1)}, \quad \text{if} \quad \ell < \frac{1}{2}\, i(n-2-i)$$

$$a_{n-1,i}^{(\ell)} > a_{n-1,i}^{[\frac{1}{2}\, i(n-2-i)-\frac{i}{2}]} > a_{n-1,i}^{(\ell-i-1)}, \quad \text{if} \quad \frac{1}{2}\, i(n-2-i) < \ell < \frac{1}{2}\, i(n-1-i) \quad .$$

However

$$a_{n-1,i}^{(\ell)} > a_{n-1,i}^{(\ell-i-1)} \quad \text{for} \quad 0 < \ell < \frac{1}{2}\, i(n-1-i) \quad .$$

The same kind of argument shows

$$a_{n-1,i-1}^{(\ell)} > a_{n-1,i-1}^{(\ell+i-n)} \quad .$$

Now, we bring the induction hypothesis to the next level and (2.17) is proved since it obviously holds for $n = 2$.

In order to prove (2.18), it is enough to prove (2.18), only for $0 < \ell < \frac{1}{2}\,(i-1)(n-i)$ because of (2.17) and (1.4). By (2.15)

$$a_{n,i}^{(\ell)} - a_{n,i-1}^{(\ell)} = \sum_{r=\ell-i+1}^{\ell} \left(a_{n-1,i}^{(r)} - a_{n-1,i-1}^{(r)}\right) + \sum_{r=\ell+i-n+1}^{\ell} \left(a_{n-1,i-1}^{(r)} - a_{n-1,i-2}^{(r)}\right)$$
$$+ \left(a_{n-1,i}^{(\ell-i)} - a_{n-1,i-2}^{(\ell+i-n)}\right) \quad .$$

By induction hypthesis and (2.17), we know

$$a_{n-1,i-1}^{(r)} > a_{n-1,i-2}^{(r)}$$

and

$$a_{n-1,i}^{(r)} - a_{n-1,i-1}^{(r)} > 0$$

as well as

$$a_{n-1,i}^{(\ell-i)} > a_{n-1,i-1}^{(\ell-i)} > a_{n-1,i-1}^{(\ell+i-n))} > a_{n-1,i-2}^{(\ell+i-n)} .$$

Therefore (2.18) holds for n and so (2.18) is proved, since it is obviously right for $n = 2$.

Lemma 2.3 The following equalities

$$\alpha_{2k-1,j} = 2 \sum_{r=j-k+1}^{j} a_{2k-2,k-1}^{(r)} + \sum_{i=1}^{\infty} \Big(\sum_{r=j-k+1-i(2i-3)}^{j-i(2i-1)} a_{2k-2,k-1-2i}^{(r)} + \sum_{r=j-k+1-i(2i+1)}^{j-i(2i-1)} a_{2k-2,k-2-2i}^{(r)} + \sum_{r=j-k+1-i(2i-1)}^{j-i(2i+1)} a_{2k-2,k-1-2i}^{(r)} + \sum_{r=j-k+1-i(2i+3)}^{j-i(2i+1)} a_{2k-2,k-2-2i}^{(r)} \Big) \quad (2.19)$$

$$\sum_{i=j-k+1}^{j} \alpha_{2k-2,i} = \sum_{r=j-k+1}^{j} a_{2k-2,k-2}^{(r)} + \sum_{i=1}^{\infty} \Big(\sum_{r=j-k+1-i(2i-1)}^{j-i(2i-1)} a_{2k-2,k-1-2i}^{(r)} + \sum_{r=j-k+1-i(2i+1)}^{j-i(2i+1)} a_{2k-2,k-2-2i}^{(r)} \Big) . \quad (2.20)$$

hold.

Proof From (2.1), (2.4) and Lemma 2.2, by straightforward calculations, (2.19) and (2.20) can be proved.

■

Lemma 2.4 For $j < [\frac{1}{2}(k-1)^2]$, the inequality

$$\alpha_{2k-1,j} < \alpha_{2k-1,0} \cdot d_{2k-1,j}$$

holds.

Proof Since

$$\alpha_{2k-1,0}\cdot\alpha_{2k-1,j}=\binom{2k-2}{k-1}\sum_{\ell=j-k+1}^{j}\alpha_{2k-2,\ell}$$
$$=2\sum_{\ell=j-k+1}^{j}\alpha_{2k-2,\ell}\ . \tag{2.21}$$

In order to prove Lemma 2.4 it is enough to show

$$\alpha_{2k-1,j}<2\sum_{\ell=j-k+1}^{j}\alpha_{2k-2,\ell}\ . \tag{2.22}$$

From (2.19) and (2.20)

$$2\sum_{\ell=j-k+1}^{j}\alpha_{2k-2,\ell}-\alpha_{2k-1,j}$$
$$=\sum_{i=1}^{\infty}\Big(\sum_{r=j-k+1-i(2i-1)}^{j-k-i(2i-3)}a^{(r)}_{2k-2,k-1-2i}+\sum_{r=j+1-i(2i+1)}^{j-i(2i-1)}a^{(r)}_{2k-2,k-1-2i}$$
$$-\sum_{r=j+1-i(2i+1)}^{j-i(2i-1)}a^{(r)}_{2k-2,k-2-2i}-\sum_{r=j-k+1-i(2i+3)}^{j-k-i(2i+1)}a^{(r)}_{2k-2,k-2-2i}\Big)$$
$$=\sum_{i=1}^{\infty}\Big(\sum_{r=j+1-i(2i+1)}^{j-i(2i-1)}\big(a^{(r)}_{2k-2,k-1-2i}-a^{(r)}_{2k-2,k-2-2i}\big)$$
$$+\sum_{r=j-k+1-i(2i+1)}^{j-k-i(2i-3)}\big(a^{(r)}_{2k-2,k-1-2i}-a^{(r-4i)}_{2k-2,k-2-2i}\big)\Big)\ .$$

Because of (2.17), (2.18) and (1.4), Lemma 2.4 is verified.

∎

After the preceding preparations now it is time to prove the following theorem.

Theorem For any $q\in(0,\infty)$ and $r=k-1,k,$ the inequality

$$|G_r^{-1}|=\left|\frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)}\right|>2k-1 \tag{3.0}$$

holds, and $\lim\limits_{q \to \infty} \| G_r^{-1} \| = 2k-1.$

<u>Proof</u> Because of the symmetry of $\Pi_n(\lambda;q)$, we can restrict our discussin to the case $q \in [1,\infty)$.

Since (2.1), (2.3), (2.7), Lemma 2.4 and (1.1), we know

$$\left|\frac{\Pi_{2k-1}(q^{k-1};q)}{\Pi_{2k-1}(-q^{k-1};q)}\right| = \left|\frac{\Pi_{2k-1}(q^{k};q)}{\Pi_{2k-1}(-q^{k};q)}\right|$$

$$= \frac{\binom{2k-1}{k} \sum\limits_{i=0}^{(k-1)^2} d_{2k-1,i} q^i}{\sum\limits_{i=0}^{(k-1)^2} (2\alpha_{2k-1,i} - \binom{2k-1}{k} d_{2k-1,i}) q^i}$$

$$> \frac{\binom{2k-1}{k} \sum\limits_{i=0}^{(k-1)^2} d_{2k-1,i} q^i}{\sum\limits_{i=0}^{(k-1)^2} (2\alpha_{2k-1,0} \cdot d_{2k-1,i} - \binom{2k-1}{k} d_{2k-1,i}) q^i}$$

$$= 2k-1 \quad ,$$

and refer to [1] for equality.

The proof of the theorem relies on Lemma 2.4 mainly. In order to prove the monotonicity of

$$\left|\frac{\Pi_{2k-1}(q^{k};q)}{\Pi_{2k-1}(-q^{k};q)}\right| \quad \text{for} \quad q \in (0,\infty) \quad ,$$

it is sufficient to prove the stronger inequality

$$\frac{\alpha_{2k-1,j}}{\alpha_{2k-1,j+1}} > \frac{d_{2k-1,j}}{d_{2k-1,j+1}} \quad \text{for} \quad 0 < j < [\tfrac{1}{2} (k-1)^2] \quad ,$$

we fail to prove this inequality. But numerical results (see appendix) show the inequality is true at least for $n \leq 9$.

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1. Y. Y. Feng and J. Kozak, On the generalized Euler-Frobenius polynomial, MRC Technical Summary Report #2088.
2. A. Micchelli, Cardinal L-splines, in "Studies in Spline Functions and Approximation Theory", Academic Press, 1976, 203-250.

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3. Appendix. The table of $\alpha_{n,i}$, $\beta_{n,i}$, $d_{n,i}$, $a_{n,i}^{(j)}$ for $n \leq 9$

A. Table of $\alpha_{n,i}$, $\beta_{n,i}$, $d_{n,i}$ for $4 \leq n \leq 9$, $i = 0,1,\cdots [\frac{n-1}{2}] \cdot [\frac{n}{2}]$

n	α, β, d \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	α=β	3	6	3														
	d	1	2	1														
5	α	6	17	22	17	6												
	β	4	13	18	13	4												
	d	1	3	4	3	1												
6	α=β	10	40	80	100	80	40	10										
	d	1	4	8	10	8	4	1										
7	α	20	96	242	422	548	548	422	242	96	20							
	β	15	79	213	383	502	502	383	213	79	15							
	d	1	5	13	23	30	30	23	13	5	1							
8	α=β	35	210	665	1470	2485	3360	3710	3360	2485	1470	665	210	35				
	d	1	6	19	42	71	96	106	96	71	42	19	6	1				
9	α	70	476	1728	4449	9005	15073	21448	26354	28202	26354	21448	15073	9005	4449	1728	476	70
	β	56	406	1548	4119	8509	14411	20636	25432	27238	25432	20636	14411	8509	4119	1548	406	56
	d	1	7	26	68	139	234	334	411	440	411	334	234	139	68	26	7	1

B. The table of $a_{n,i}^{(j)}$ $4 \le n \le 9$, $i = 0,1, \quad [\frac{n-1}{2}]$, $j = 0,1,\cdots \quad i(n-1-i)$

n	I \ J	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	0	1																
	1	3	5	3														
5	0	1																
	1	4	9	9	4													
	2	6	16	22	16	6												
6	0	1																
	1	5	14	19	14	5												
	2	10	35	66	80	66	35	10										
7	0	1																
	1	6	20	34	34	20	6											
	2	15	64	149	233	269	233	149	64	15								
	3	20	90	222	382	494	494	382	222	90	20							
8	0	1																
	1	7	27	55	69	55	27	7										
	2	21	105	288	540	765	855	765	540	288	105	21						
	3	35	189	560	1175	1918	2540	2785	2540	1918	1175	560	189	35				
9	0	1																
	1	8	35	83	125	125	83	35	8									
	2	28	160	503	1091	1806	2400	2632	2400	1806	1091	503	160	28				
	3	56	350	1198	2913	5561	8767	11736	13536	13536	11736	8767	5561	2913	1198	350	56	
	4	70	448	1568	3918	7754	12764	17956	21916	23402	21916	17956	12764	7754	3918	1568	448	70

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In [1], we gave another proof of the boundedness of L_2-projections onto splines on a geometric mesh. In this paper, we obtain the sharp lower bound for the inverse of the corresponding B-spline Gram matrix. I.e.

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)}\right| \geq 2k-1, \quad \text{for} \quad r=k,\ k-1.$$

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