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ON POLYNOMIAL INTERPOLATION IN THE POINTS OF A GEOMETRIC PROGRESSION, STIRLING, SCHELLBACH, RUNGE AND ROMBERG

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ABSTRACT

It is very well known that Newton's interpolation series

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots$$

simplifies considerably in the case that the points $x_n = a + nh$ form an arithmetic progression. Indeed, in this case

$$f(a,a+h,\cdots,a+nh) = \frac{1}{n!h^n} \Delta_h^n f(a) .$$

It seems much less known that a similar simplification occurs in the case when the points of interpolation form a geometric progression. This paper deals with this interpolation problem and its main contribution is to call attention to the references [6], [5], [3] to the work of Stirling (1730), Schellbach (1864), and Runge (1891), which seems now practically forgotten. This work is here described and also its close connection with the elegant algorithm of Romberg (See [1]). We illustrate these connections with numerical examples.

AMS(MOS) Subject Classification: 65B05, 65D05

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SIGNIFICANCE AND EXPLANATION

It is very well known that Newton's interpolation series with divided differences simplifies considerably in the case that we interpolate in the points of an arithmetic progression. It seems much less known that a similar simplification occurs in the case when the points of interpolation form a geometric progression. We describe here the practically forgotten work of Stirling (1730), Schellbach (1864), and Runge (1891), and its connection with the elegant and more recent algorithm of Romberg (1955).

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON POLYNOMIAL INTERPOLATION IN THE POINTS OF A GEOMETRIC PROGRESSION, STIRLING, SCHELLBACH, RUNGE AND ROMBERG

I. J. Schoenberg

1. Introduction. It is very well known that Newton's interpolation series

(1.1) $f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots$

simplifies considerably in the case that the points $x_n = a + nh$ form an arithmetic progression. Indeed, in this case we have

$$f(a,a+h,\cdots,a+nh) = \frac{1}{n!h^n} \Delta_h^n f(a) \quad (1.2)$$

It seems much less known that a similar simplification occurs in the case when the points of interpolation form a geometric progression, at least this is not mentioned in the standard trestises on this subject.

The main contribution of the present paper is to call attention to the references [6], [5], [3], to the work of Stirling, Schellbach and Runge, which seem now to be practically forgotten. These were known to the author since 1943. Stirling expands the function $F(z)$ in the form

$$F(z) = a_0 + a_1 r^z + a_2 r^{2z} + \cdots \qquad (r > 1) ,$$

determining the coefficients $a_0, a_1, a_2, \cdots$ by the interpolation at $z = 0,1,2,\cdots$ and then extrapolates at $z = -\infty$. Schellbach retains Stirling's approach casting the method in an elegant algorithmic form. The obvious change of variable $x = r^z$ transforms the problem into our polynomial interpolation in the points of a geometric progression (Theorem 1 below).

Quite recently I noticed the close connection with the elegant algorithm of Romberg (Theorem 2) for which algorithm we refer to the important paper [1] of Bauer, Rutishauser and Stiefel. Also recently I noticed that Runge [3]

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also solves the same interpolation problem without stating this fact explicitly (Theorem 3). Rather he applies the idea of the "Richardson deferred approach to the limit" 20 years before Richardson, working out error estimates. We illustrate these connections with numerical examples.

I. Stirling and Schellbach

2. The Stirling-Schellbach algorithm. Let

$$(2.1)\qquad x_k = aq^k\ ,\ (k = 0,1,\cdots),\ a \neq 0,\quad 0 < |q| < 1$$

be a geometric progression. Let $f(x)$ be a function which is analytic and regular at the origin $x = 0$, the problem being to describe explicitly Newton's series (2.1). Particular interest for applications is to obtain approximations to the value of $f(0)$ by polynomial extrapolation at $x = 0$ from the $n + 1$ data

$$(2.2)\qquad u_k = f(aq^k)\ ,\quad (k = 0,1,\cdots,n).$$

Following Schellbach [5, §157, p 280] we define the q-differences $\mathcal{D}^m u_k$ recursively by

$$(2.3)\qquad \mathcal{D}^m u_k = \mathcal{D}^{m-1}u_{k+1} - q^{m-1}\mathcal{D}^{m-1}u_k\ ,\quad (m = 1,2,\cdots),$$

and arrange these in a triangular array

$$(2.4)\qquad \begin{array}{llll}
u_0 & & & \\
 & \mathcal{D}u_0 = u_1 - u_0 & & \\
u_1 & & \mathcal{D}^2u_0 = \mathcal{D}u_1 - q\mathcal{D}u_0 & \\
 & \mathcal{D}u_1 = u_2 - u_1 & & \mathcal{D}^3u_0 = \mathcal{D}^2u_1 - q^2\mathcal{D}^2u_0. \\
u_2 & & \mathcal{D}^2u_1 = \mathcal{D}u_2 - q\mathcal{D}u_1 & \\
 & \mathcal{D}u_2 = u_3 - u_2 & & \\
u_3 & & & \\
\vdots & & &
\end{array}$$

In terms of these differences we may state

Lemma 1. For the interpolation points (2.1) the divided differences are

$$(2.5)\qquad f(a,aq,\cdots,aq^n) = \frac{\mathcal{D}_0{}^n u}{a^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}.$$

Proof. Indeed, for $n = 1$

$$f(a,aq) = \frac{f(aq) - f(a)}{aq - a} = \frac{u_1 - u_0}{a(q-1)} = \frac{\mathcal{D}u_0}{a(q-1)} .$$

Assuming (2.5) correct for $n-1$ we have

$$f(a,aq,\cdots,aq^{n-1}) = \frac{\mathcal{D}^{n-1}u_0}{a^{n-1}(q^{n-1}-1)(q^{n-1}-q)\cdots(q^{n-1}-q^{n-2})}$$

and therefore also

$$f(aq,aq^2,\cdots,aq^n) = \frac{\mathcal{D}^{n-1}u_1}{a^{n-1}q^{n-1}(q^{n-1}-1)\cdots(q^{n-1}-q^{n-2})} .$$

These imply that

$$f(a,aq^2,\cdots,aq^n) = \frac{f(aq,\cdots,aq^n)-f(a,\cdots,aq^{n-1})}{aq^n - a}$$

$$= \frac{1}{a(q^n-1)} \left\{\frac{\mathcal{D}^{n-1}u_1}{q^{n-1}} - \mathcal{D}^{n-1}u_0\right\} \frac{1}{a^{n-1}(q^{n-1}-1)\cdots(q^{n-1}-q^{n-2})}$$

$$= \frac{\mathcal{D}^n u_0}{a^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}$$

which proves (2.5) by induction.

Using Newton's expansion (1.1) we immediately obtain

Theorem 1. (Stirling-Schellbach). *For the interpolation points* (2.1) *Newton's series* (1.1) *becomes*

(2.6) $$f(x) = \sum_{n=0}^{\infty} P_n(x)\ \mathcal{D}^n u_0 ,$$

where

$$P_n(x) = \frac{(x-a)(x-aq)\cdots(x-aq^{n-1})}{a^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} . \tag{2.7}$$

In particular, for $x = 0$ we obtain the expansion

$$f(0) = u_0 + \frac{\mathcal{D}u_0}{1-q} + \frac{\mathcal{D}^2u_0}{(1-q)(1-q^2)} + \cdots + \frac{\mathcal{D}^n u_0}{(1-q)\cdots(1-q^n)} + \cdots . \tag{2.8}$$

We propose to call Stirling-Schellbach algorithm the sequence to sequence transformation

$$(u_n) \to (v_n) \tag{2.9}$$

which transforms the sequence (u_n) into the sequence (v_n) of partial sums of the series (2.8), hence

$$v_n = u_0 + \frac{\mathcal{D}u_0}{1-q} + \cdots + \frac{\mathcal{D}^n u_0}{(1-q)\cdots(1-q^n)} , \quad (n = 0,1,\cdots). \tag{2.10}$$

It is not difficult to show that (2.9) is limit preserving in the sense that if $u_n \to \ell$, then also $v_n \to \ell$. However, the converse is also true: If $v_n \to \ell$, then also $u_n \to \ell$. This shows that (2.9) can not be used as a limiting method that changes some divergent sequences (u_n) into convergent ones. Rather the importance of (2.9) lies in another direction: It speeds up the convergence of some slowly convergent sequences.

A dramatic example of acceleration of convergence was given by Stirling himself.

3. Stirling's computation of π. We interpolate the entire function

$$f(x) = \frac{2}{\sqrt{x}} \sin\frac{\pi\sqrt{x}}{2} = \pi - \frac{1}{24}\pi^3 x + \cdots \tag{3.1}$$

at the points

$$x_k = q^k , \quad (k = 0,1,\cdots,n), \quad \text{where} \quad q = \frac{1}{4} . \tag{3.2}$$

Observing that the area $A(m)$ of a regular polygon of m sides inscribed in the unit circle is given by

$$A(m) = m \sin \frac{\pi}{m} \cos \frac{\pi}{m}$$

we find that

$$u_k = f\left(\frac{1}{4^k}\right) = 2^{k+1} \sin \frac{\pi}{2^{k+1}} = 2^{k+2} \sin \frac{\pi}{2^{k+2}} \cos \frac{\pi}{2^{k+2}} \tag{3.3}$$

and therefore

$$u_k = f\left(\frac{1}{4^k}\right) = A(2^{k+2}). \tag{3.4}$$

Stirling computes the areas

$$A(4),\ A(8),\ A(16),\ A(32),\ A(64),$$

each requiring a square root extraction to derive it from the previous one. We use the Texas Instruments SR-50 and working with 8 decimal we find that (2.4) becomes

(3.5)

$u_0 = A(4)$	$= 2.$				
		.8284 2712			
$u_1 = A(8)$	$= 2.8284\ 2712$		.0259 3356		
		.2330 4034		.0000 9675	
$u_2 = A(16)$	$= 3.0614\ 6746$		.00171760		.0000 0006
		.0599 7769		.0000 0157	
$u_3 = A(32)$	$= 3.1214\ 4515$		.0001 0892		
		.0151 0334			
$u_4 = A(64)$	$= 3.1365\ 4849$				

We find the relevant q-differences $\mathcal{D}^k u_0$, $(k = 0,1,2,3,4)$ in the top diagonal of this array. Forming the partial sums (2.10) we find the approximations

$$
(3.6) \qquad \begin{aligned}
v_0 &= 2 \\
v_1 &= 3.1045\ 6949 \\
v_2 &= 3.1414\ 5277 \\
v_3 &= 3.1415\ 9256 \\
v_4 &= 3.1415\ 9265
\end{aligned}
$$

Notice the rapid convergence of the v_n to π, the value of v_4 having all its decimals correct.

<u>Remarks.</u> 1. Stirling [6, page 133], and also Schellbach who reproduces Stirling's computations [5, page 286], uses also the next area $u_5 = A(128)$. For some reason they use in their algorithm these values in their reverse order $A(128), A(64), \cdots, A(4)$. This reversal requires to replace $q = 1/4$ by $q = 4$. (Extrapolation at $+\infty$!) How does Schellbach justify the choice of $q = 4$? He argues as follows: "Since each difference is nearly 4 times as large as the previous difference, the choice of $q = 4$ will result in small values of the higher differences and lead to rapid convergence". They compute with 14 decimals and obtain π with 14 correct decimals. Arranging the algorithm in the natural order of (3.5), and extending it to include the next area $u_5 = A(128)$, a fairly easy estimate will show the error to be

$$|\pi - v_5| < 1.068 \times 10^{-16} .$$

Schellbach devotes an entire chapter [5, 275-294] to Stirling's interpolation series and its applications and concludes the chapter by saying "... this series, which seems to have escaped so far the attention of mathematicians, appears to be of exceptionally high practical importance".

2. We want to estimate the error $f(0) - v_n$ of polynomial interpolation of $f(x)$ at $x = 0$. From (1.1) we know the error at x to be

$$f(x_0,\cdots,x_n,x)(x-x_0)\cdots(x-x_n) = (x-x_0)\cdots(x-x_n)\frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{(z-x_0)\cdots(z-x_n)(z-x)},$$

where Γ is a closed curve containing x and the interpolation points. In particular, for $x = 0$, we obtain an estimate

$$(3.7) \qquad |f(0) - v_n| < C \cdot K^n |q|^{\frac{n(n+1)}{2}},$$

valid for all n. Here C and K depend only on $f(x)$.

II. Romberg

4. The Romberg algorithm. We use with slight modifications the notations of the beautiful paper [1] of Bauer, Rutishauser and Stiefel. Let r be a constant such that

$$|r| > 1. \tag{4.1}$$

Starting from the column of values $R_0^{(m)}$ we form the Romberg triangular array

(4.2)

$$
\begin{array}{llll}
R_0^{(0)} & & & \\
 & R_1^{(0)} = \dfrac{rR_0^{(1)} - R_0^{(0)}}{r-1} & & \\
R_0^{(1)} & & R_2^{(0)} = \dfrac{r^2R_1^{(1)}-R_1^{(0)}}{r^2-1} & \\
 & R_1^{(1)} = \dfrac{rR_0^{(2)}-R_0^{(1)}}{r-1} & & R_3^{(0)} = \dfrac{r^3R_2^{(1)}-R_2^{(0)}}{r^3-1} \\
R_0^{(2)} & & R_2^{(1)} = \dfrac{r^2R_1^{(2)}-R_1^{(1)}}{r^2-1} & \quad \cdots \\
 & R_1^{(2)} = \dfrac{rR_0^{(3)}-R_0^{(2)}}{r-1} & & \\
R_0^{(3)} & & & \\
\vdots & & &
\end{array}
$$

the general definition being

$$R_m^{(k)} = \frac{r^m R_{m-1}^{(k+1)} - R_{m-1}^{(k)}}{r^m - 1} \quad \text{or} \quad R_m^{(k)} = \frac{R_{m-1}^{(k+1)} - r^{-m}R_{m-1}^{(k)}}{1 - r^{-m}}. \tag{4.3}$$

Our main result is

Theorem 2. The Stirling-Schellbach algorithm (2.10) is equivalent with the Romberg algorithm (4.2) such that

$$r = \frac{1}{q} \; . \tag{4.4}$$

This means the following: If we identify the first columns of the arrays (2.4) and (4.2), so that

$$u_m = R_0^{(m)} \; , \; (m = 0, 1, \cdots), \tag{4.5}$$

then the elements of the leading diagonal of (4.2) are respectively equal to the partial sums of the series (2.8), i.e.

$$v_m = R_m^{(0)} \; , \quad (m = 0, 1, \cdots). \tag{4.6}$$

We give two proofs.

First proof. My colleague C. de Boor pointed out to me the following *remark:* If we apply the Neville algorithm for a geometric progression $x_m = aq^m$ and for interpolation at $x = 0$, then Neville's fractions simplify and become identical with the elements of Romberg's algorithm. This connection was already mentioned in [4, 301-302].

Second proof. This proof is more direct but longer. The equation (4.6) is true by definition if $m = 0$, because $v_0 = u_0 = R_0^{(0)}$. To establish the equation

$$u_0 + \frac{\mathcal{D}u_0}{1-q} + \cdots + \frac{\mathcal{D}^m u_0}{(1-q)\cdots(1-q^m)} = R_m^{(0)}$$

for all m, we assume that it holds for $m - 1$, and we are to show that

$$\frac{\mathcal{D}^m u_0}{(1-q)\cdots(1-q^m)} = R_m^{(0)} - R_{m-1}^{(0)} \; , \quad \text{with} \quad R_{-1}^{(0)} = 0 \; . \tag{4.7}$$

Let us prove this by induction. For $m = 0$ this reduces to $u_0 = R_0^{(0)}$. Assuming (4.7) true for $m - 1$ we have

$$(4.8)\qquad \frac{\mathcal{D}^{m-1}u_0}{(1-q)\cdots(1-q^{m-1})} = R_{m-1}^{(0)} - R_{m-2}^{(0)} ,$$

and we are to derive from it that also (4.7) holds, i.e.,

$$(4.9)\qquad \frac{\mathcal{D}^{m-1}u_1 - q^{m-1}\mathcal{D}^{m-1}u_0}{(1-q)\cdots(1-q^{m})} = R_{m}^{(0)} - R_{m-1}^{(0)} .$$

However, the assumption (4.8) also implies that

$$\frac{\mathcal{D}^{m-1}u_k}{(1-q)\cdots(1-q^{m-1})} = R_{m-1}^{(k)} - R_{m-2}^{(k)}$$

and in particular, for $k = 1$, that

$$(4.10)\qquad \frac{\mathcal{D}^{m-1}u_1}{(1-q)\cdots(1-q^{m-1})} = R_{m-1}^{(1)} - R_{m-2}^{(1)} .$$

Using (4.8) and (4.10), (4.9) becomes

$$(4.11)\qquad \frac{R_{m-1}^{(1)} - R_{m-2}^{(1)}}{1-q^m} - q^{m-1}\,\frac{R_{m-1}^{(0)} - R_{m-2}^{(0)}}{1-q^m} = R_{m}^{(0)} - R_{m-1}^{(0)} .$$

However, the second equation (4.3), with $r^{-1} = q$, shows that

$$R_{m-1}^{(1)} = (1-q^m)R_m^{(0)} + q^m R_{m-1}^{(0)}$$

$$R_{m-2}^{(1)} = (1-q^{m-1})R_{m-1}^{(0)} + q^{m-1}R_{m-2}^{(0)} .$$

Substituting into (4.11) we obtain

$$\frac{(1-q^m)R_m^{(0)} + q^m R_{m-1}^{(0)} - (1-q^{m-1})R_{m-1}^{(0)} - q^{m-1}R_{m-2}^{(0)}}{1-q^m} - q^{m-1}\,\frac{R_{m-1}^{(0)} - R_{m-2}^{(0)}}{1-q^m} =$$

$$= R_m^{(0)} - R_{m-1}^{(0)}$$

which simplifies to

$$\frac{(1-q^m)R_m^{(0)} + q^m R_{m-1}^{(0)} - R_{m-1}^{(0)}}{1-q^m} = R_m^{(0)} - R_{m-1}^{(0)}$$

which is visibly an identity.

Remarks. 1. Even though the two algorithms (2.10) and (4.2), with $u_m = R_0^{(m)}$ and $r = q^{-1}$, solve the same interpolation problem, it is clear that the elegant Romberg algorithm is much to be preferred. We illustrate this by returning to

2. The computation of π. With $r = q^{-1} = 4$ and for the u_k of (3.5) Romberg's triangular array becomes

	$u_0 = 2$				
		3.1045 6949			
	$u_1 = 2.8284\ 2712$		3.1414 5277		
		3.1391 4757		3.1415 9257	
(4.12)	$u_2 = 3.0614\ 6746$		3.1415 9039		3.1415 9265
		3.1414 3771		3.1415 9265	
	$u_3 = 3.1214\ 4515$		3.1415 9262		
		3.1415 8294			
	$u_4 = 3.1365\ 4849$.				

We recognize in the top-diagonal of (4.12) the values (3.6), except for some rounding errors, as guaranteed by Theorem 2.

III. Runge

5. Runge's first problem. Without knowledge of the work of Stirling and Schellbach, Runge considers in [3] the following problems. Let

(5.1) $$f(x) = a_0 + a_1 x + \cdots + a_1 x^k + \cdots , \quad (|x| < k)$$

be regular in the circle $|x| < r$. Let q and a be constants such that

(5.2) $$0 < |q| < 1 , \quad 0 < |a| < r .$$

Runge's Problem 1 is to approximate to $a_0 = f(0)$ in terms of the $n+1$ values

(5.3) $$u_k = f(aq^k) , \quad (k = 0,1,\cdots,n) .$$

This problem was solved by Theorem 1 by the Stirling-Schellbach algorithm (2.10), and also by Theorem 2 by the Romberg algorithm with

(5.4) $$r = \frac{1}{q} .$$

However, nowhere does Runge mention this polynomial extrapolation approach. Rather he proceeds directly as follows, considering only the case when $q = 1/2$. We write the (5.3) as

(5.5) $$f(aq^{n-k}) = a_0 + a_1 a q^n r^k + a_2 a^2 q^{2n} r^{2k} + \cdots + a_n a^n q^{n^2} r^{nk} + a_{n+1} a^{n+1} q^{(n+1)n} r^{(n+1)k} + \cdots , \quad (k = 0,1,\cdots,n).$$

He multiplies this equation with C_k and sums over all k, the objective being to so choose the C_k as to anihilate the n terms in a^s $(s = 1,2,\cdots,n)$. This is seen to be achieved, provided the C_k are the coefficients of the polynomial

(5.6) $$\varphi_n(x) = C_0 + C_1 x + \cdots + C_n x^n = \frac{(r-x)(r^2-x)\cdots(r^n-x)}{(r-1)(r^2-1)\cdots(r^n-1)}$$

defined by the conditions

(5.7) $$\varphi_n(r) = \varphi_n(r^2) = \cdots = \varphi_n(r^n) = 0 , \quad \varphi_n(1) = 1 .$$

In this way Runge obtains from (5.5) the equation

(5.8) $$\mathfrak{R}_n = \sum_0^n C_k f(aq^{n-k}) = a_0 + a_{n+1} a^{n+1} q^{(n+1)n} \varphi_n(r^{n+1}) + \cdots .$$

Since $\varphi_n(r^{n+1}) = (-1)^n r^{1+2+\cdots+n} = (-1)^n q^{-n(n+1)/2}$ we see that Runge's approximation $\mathcal{R}_n$ to a_0 gives the same order of approximation (3.7) as the Stirling-Schellbach v_n. The following result should therefore come as no great surprise.

<u>Thereom 3.</u> <u>Runge's approximation</u> $\mathcal{R}_n$ <u>of</u> $F(0) = a_0$, <u>defined by</u> (5.8), (5.6), (5.7), <u>is identical with the result of polynomial extrapolation, hence</u>

(5.9) $$\mathcal{R}_n = v_n = R_0^{(n)}.$$

A proof of this is fairly easy if we define the interpolated value v_n by Lagrange's interpolation formula, and also use the Gaussian identity

$$(x-r)(x-r^2)\cdots(x-r^n) = \sum_0^n (-1)^\nu \left[{n \atop \nu} \right] r^{\frac{\nu(\nu+1)}{2}} x^{n-\nu}$$

where

$$\left[{n \atop \nu} \right] = \frac{(r^n-1)(r^{n-1}-1)\cdots(r^{n-\nu+1}-1)}{(r^\nu-1)(r^{\nu-1}-1)\cdots(r-1)} .$$

We omit the details.

<u>Remarks.</u> Besides the Stirling computation of π pf §3, further attractive examples are provided by the following functions.

1. The function

(5.10) $$f(x) = (1+x)^{\frac{1}{x}} = e + a_1 x + \cdots .$$

To determine $f(0) = e$ we can use Romberg's algorithm choosing e.g. $a = 1/8$, $q = 1/16$, hence $r = 16$. Notice that $x_k = aq^k = 2^{-3-4k}$ and therefore the computation of $u_k = f(aq^k)$ from (5.10), requires only successive squaring.

2. The entire function

(5.11) $$f(x) = \frac{1}{x}(2^x-1) = \log 2 + a_1 x + \cdots$$

will give approximations to log 2. With $a = 1$ and $q = 1/2$, the computation of $u_k = f(aq^k) = f(1/2^k)$ requires only successive square root extractions. In (5.11) we may replace 2 by any integer N .

3. *The case when* $f(x)$ *is an even function of* x . If

(5.12) $$f(x) = a_0 + a_2x^2 + a_4x^4 + \cdots$$

we define

(5.13) $$g(x) = f(\sqrt{x}) = a_0 + a_2x + a_2x^2 + \cdots$$

and observe that

(5.14) $$g(a^2q^{2k}) = f(aq^k) , \quad (k = 0,1,\cdots).$$

Since $g(x)$ is also regular at $x = 0$, we obtain

Theorem 4. If the function (5.1) *is even, then we may apply to the data* (5.3) *the Romberg algorithm with* q *replaced by* q^2 . *This means that in Romberg's algorithm* (4.2) *we keep the first columns*

$$R_0^{(m)} = u_m = f(aq^m)$$

fixed and pass from $r = q^{-1}$ *to* $r^2 = q^{-2}$.

Replacing q by q^2 will clearly accelerate the convergence of the v_n to $f(0) = a_0$. This important device was already used in §3: Instead of interpolating the even function

$$\frac{2}{x}\sin\frac{\pi x}{2} \quad \text{at} \quad x = 1, \frac{1}{2}, \frac{1}{2^2}, \cdots$$

we interpolated the entire function (3.1) at $x = 1, 1/4, 1/4^2, \cdots$. In §6 we shall again use this device to good advantage.

6. Runge's second problem: Computing the inverse function. Let

(6.1) $$y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (|x| < r)$$

be such that

(6.2) we can compute the value of $f(xq)$ in terms of $f(x)$,

and therefore, successively,

(6.3) we can compute $f(xq^k)$ $(k = 0,1,\cdots)$ in terms of $y = f(x)$.

Throughout this section we assume that

(6.4) $$q = \tfrac{1}{2} .$$

The formulae

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}} = \sqrt{\frac{1 - \sqrt{1 - \sin^2 x}}{2}}$$

show that $\sin x$ satisfies (6.3) with $q = \tfrac{1}{2}$.

Problem 2. To compute x if $y = f(x)$ is prescribed.

Solution. Runge reduces this problem to Problem 1 of §5 as follows. Writing

$$f(xt) = a_0 + a_1 xt + a_2 x^2 t^2 + a_3 x^3 t^3 + \cdots , \qquad (a_1 \neq 0) ,$$

we define the new function

(6.5) $$g(t) = \frac{f(xt) - a_0}{a_1 t} = x + \frac{a_2}{a_1} x^2 t + \frac{a_3}{a_1} x^3 t^2 + \cdots$$

which depends also on x , and for which

$$u_k = g(q^k) = \frac{f(xq^k) - a_0}{a_1 q^k} = x + \frac{a_2}{a_1} x^2 q^k + \frac{a_3}{a_1} x^3 q^{2k} + \cdots , \qquad (k = 0,1,\cdots).$$

By our assumption (6.3) all these values can be computed, and (6.5) shows that

(6.7) $$x = g(0)$$

can be obtained as the solution of Problem 1 for the data u_k .

Following Runge [3, 222-223] and also Schellbach [5, 88-90] we illustrate this procedure by

The computation of an incomplete elliptic integral of the first kind.

Let

$$x = \int_0^y \frac{d\theta}{\sqrt{1 - c^2\sin^2\theta}}, \quad (0 < c < 1, \quad 0 < y \leq \pi/2), \tag{6.8}$$

where y is prescribed and we are to compute x . A result of Legendre [2, vol. 1, §21, 25-26] is as follows: If we determine acute angles γ and y_1 from the equations

$$\sin\gamma = c \sin y, \quad \sin y_1 = \sin \frac{y}{2} / \cos \frac{\gamma}{2}, \tag{6.9}$$

then

$$\frac{x}{2} = \int_0^{y_1} \frac{d\theta}{\sqrt{1 - c^2\sin^2\theta}}. \tag{6.10}$$

This shows how the value of the integral can be halved. This operation can be repeated: We determine successively angles γ_{n-1} and y_n from the equations

$$\begin{gathered} \sin \gamma_{n-1} = c \sin y_{n-1}, \ \sin y_n = \sin \frac{y_{n-1}}{2} / \cos \frac{\gamma_{n-1}}{2}, \\ (n = 1,2,\cdots,k ; \quad \gamma_0 = \gamma, \ y_0 = y), \end{gathered} \tag{6.11}$$

to obtain y_k such that

$$\frac{x}{2^k} = \int_0^{y_k} \frac{d\theta}{\sqrt{1 - c^2\sin^2\theta}}. \tag{6.12}$$

We now invert the relationship (6.8) to obtain

$$y = f(x) = x + a_3x^3 + a_5x^5 + \cdots , \tag{6.13}$$

observing that it is an odd function which is usually denoted by $f(x) = \mathrm{sn}\ x$. Clearly we may rewrite (6.12) as

(6.14) $$y_k = f(\frac{x}{2^k}) .$$

In terms of (6.13) our function (6.5) becomes

(6.15) $$g(t) = \frac{f(xt)}{t} = x + a_3x^3t^2 + a_5x^5t^4 + \cdots .$$

and in particular (6.6) may be written as

(6.16)
$$\begin{aligned} u_k &= g(\frac{1}{2^k}) = 2^k f(\frac{x}{2^k}) = 2^k y_k \\ &= x + a_3x^3 2^{-2k} + a_5x^5 2^{-4k} + \cdots , \quad (k = 0,1,\cdots) . \end{aligned}$$

This equation shows that $u_k \to x$ as $k \to \infty$, and u_k is Legendre's approximation to x. However, this approximation can be much improved by our extrapolation procedure.

A numerical example. Let us evaluate the incomplete elliptic integral

(6.17) $$x = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{1 - \frac{1}{4}\sin^2\theta}} \quad \text{where} \quad c = \tfrac{1}{2} \quad \text{and} \quad y = \frac{\pi}{4} .$$

From Legendre's Table VIII [2, vol. 3, page 339] we obtain the 12 place value

(6.18) $$x = .80436\ 61012\ 32.$$

The following computations were made in double precision by Fred W. Sauer, of the MRC Computing Staff.

We solve the equations (6.11) with $k = 4$ and find the following angles in radians

γ = .36136 71239 06708 $\qquad$ y_1 = .39956 34259 31126

γ_1 = .19575 59443 11023 $\qquad$ y_2 = .20075 55982 73493

γ_2 = .09987 08442 57596 $\qquad$ y_3 = .10050 35008 99138

γ_3 = .05018 82616 25625 $\qquad$ y_4 = .05026 75900 94179 .

From (6.16) we get the values

$$u_0 = \quad y = .78539\ 81633\ 97448$$

$$u_1 = 2y_1 = .79912\ 68518\ 62251$$

(6.19) $$u_2 = 4y_2 = .80302\ 23930\ 93970$$

$$u_3 = 8y_3 = .80402\ 80071\ 93103$$

$$u_4 = 16y_4 = .80428\ 14415\ 06865.$$

The last value u_4 is Legengre's 3-place approximation to x . We can now improve this approximation in two different ways:

1. The Romberg algorithm applied to the data (6.19) with

(6.20) $r = 2$ gives the approximation $R_4^{(0)} = .80436\ 56250$

which is seen to be correct to 6 decimal places.

2. Because the $g(t)$ of (6.15) is an even function we may use Theorem 4 and find that the Romberg algorithm with

(6.21) $r = 2^2 = 4$ gives the approximation $\bar{R}_4^{(0)} = .80436\ 61012\ 29163$

which is seen to be correct to nearly 12 decimals.

To conclude it seems worthwhile to recall that Romberg invented his algorithm, also with $r = 4$, for the evaluation of a definite integral (6.17) in terms of its binary trapezoidal sums (See [1]). Writing $F(\theta) = (1 - 1/4 \sin^2\theta)^{-1/2}$, we consider the sums

$$T_{2^k} = \frac{\pi}{4} 2^{-k}\{ \tfrac{1}{2} F(0) + \sum_{s=1}^{2^k-1} F(\tfrac{\pi}{4} s \cdot 2^{-k}) + \tfrac{1}{2} F(\tfrac{\pi}{4})\}, \quad (k = 0,1,\cdots) .$$

Dividing $[0,\frac{\pi}{4}]$ into $2^4 = 16$ equal parts, we can compute the 5 sums

(6.22) $$T_1, T_2, T_4, T_8, T_{16} .$$

If we now apply to the column of the sums (6.22) the Romberg algorithm with $r = 4$ we find with double precision the approximation

(6.23) $$\tilde{R}_4^{(0)} = .80436\ 61012\ 31069.$$

A comparison with Legendre's value (6.18) shows that (6.23) is even slightly more accurate than (6.21).

References

1. F. L. Bauer, H. Rutishauser and E. Stiefel, New aspects of numerical quadrature, Proc. of Symposium on Applied Math, vol. XV Amer. Math. Soc., Providence, R.I., 1963, 199-218.

2. A. M. Legendre, Exercices de Calcul Intégral, 3 volumes, Paris 1816.

3. C. Runge, Über eine numerische Berechnung der Argumente der cyclischen, hyperbolischen und logarithmischen Funktionen, Acta Mathematica, 15 (1891), 221-247.

4. R. Sauer and I. Szabó, Editors, Mathematische Hilfsmittel des Ingenieurs, Springer-Verlag, 1968.

5. K. H. Schellbach, Die Lehre von den elliptischen Integralen und den Theta-Funktionen, Berlin, 1864.

6. James Stirling, The differential method or a Treatise concerning summation and interpolation of infinite series, London, 1749. This is a translation from Latin made by F. Holliday of the original edition of 1730.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is very well known that Newton's interpolation series

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots$$

simplifies considerably in the case that the points $x_n = a + nh$ form an arithmetic progression. Indeed, in this case

$$f(a,a+h,\cdots,a+nh) = \frac{1}{n!h^n} \Delta_h^n f(a).$$

It seems much less known that a similar simplification occurs in the case when the points of interpolation form a geometric progression. This paper deals with

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20. Abstract (continued)

this interpolation problem and its main contribution is to call attention to the references [6], [5], [3] to the work of Stirling (1730), Schellbach (1864), and Runge (1891), which seems now practically forgotten. This work is here described and also its close connection with the elegant algorithm of Romberg (See [1]). We illustrate these connections with numerical examples.