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MINIMIZATION PROBLEMS IN L1(R3).(U)

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MINIMIZATION PROBLEMS IN $L^1(R^3)$

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ABSTRACT

In this paper, minimization problems in $L^1(\mathbb{R}^3)$ are considered. These problems arise in astrophysics for the determination of equilibrium configurations of axially symmetric rotating fluids (rotating stars). Under nearly optimal assumptions a minimizer is proved to exist by a direct variational method, which uses heavily the symmetry of the problem in order to get some compactness. Finally, by looking directly at the Euler equation, we give some existence results (of solutions of the Euler equation) even if the infimum is not finite.

AMS (MOS) Subject Classifications: 35J60, 49H05, 76C99.

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SIGNIFICANCE AND EXPLANATION

The determination of equilibrium configurations of axially symmetric rotating fluids (rotating stars) reduces to the following variational problem: one has to minimize a functional (called the energy) depending on the density of the fluid subject to the constraint that the total mass is prescribed. This problem is thus a minimization problem in $L^1(\mathbb{R}^3)$ which, together with the fact that the domain $(\mathbb{R}^3)$ is unbounded, creates difficulties (lack of compactness) which are overcome by new compactness results using heavily the axial symmetry of the problem. An application of this method is given concerning other minimization problems in L^1 arising in Thomas-Fermi theory in Quantum Mechanics. Finally, looking directly at the Euler equation associated with the minimization problem, we obtain solutions (of the Euler equation) even if the infimum (of the energy) is not finite.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MINIMIZATION PROBLEMS IN $L^1(\mathbb{R}^3)$

P.L. LIONS

I - INTRODUCTION

We study in this paper a class of minimization problems of the following type : find ρ in $L^1(\mathbb{R}^3)$ minimizing

$$(1) \qquad \inf_{\substack{\rho \geq 0 \\ \int_{\mathbb{R}^3} \rho(x)\,dx = M}} \left\{ \int_{\mathbb{R}^3} j(\rho(x))\,dx + \int_{\mathbb{R}^3} V(x)\rho(x)\,dx - \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dx\,dy \right\} ;$$

where j is some given positive convex function, V is a given axially symmetric function and where M is a prescribed positive constant.

Problems of this type arise in many situations : in celestial mechanics, as a model to study the geometry of stars and planets (see [18], [24], [34], [38] or [26] for the classical theory concerning these problems and their origins) ; or in quantum mechanics as Thomas-Fermi type problems. Some particular solutions may be found in [26], [33], and [39], but the first general results are given in [6], [4] and [5] (see also [7] for a physical interpretation, and [17], [19], [20], [21] for qualitative properties of the solutions).

The variational method given in [6] to solve (1) is to solve first an approximate problem : find ρ_R in $W_R = \{\rho \in L^1(\mathbb{R}^3),$ $0 < \rho < R, \int \rho = M, \rho \equiv 0$ in $\mathbb{R}^3 - \{|\xi| < R\}\}$ such that $\mathcal{E}(\rho_R) = \min_{W_R} \mathcal{E}(\rho)$.

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Then one has to obtain estimates on ρ_R : namely to prove there exists R_o such that for $R > R_o$ $\rho_R \in W_{R_o}$. And finally this provides a solution of (1). This somewhat complicated and undirect method is used in order to avoid the difficulty due to the term

$$- \frac{1}{2} \iint_{R^3 \times R^3} \frac{\rho(x)\rho(y)}{|x-y|} \, dxdy$$

in the functional.

Indeed if all other terms are " lower semi-continuous " (in a vague sense) this term is concave in ρ and thus there seems to be a difficulty to pass to the limit on a minimizing sequence.

We present here a method to pass to the limit in this term (which is related to minimization techniques introduced in [10], [11], [12] or [29]) and therefore we give a direct (and simple) minimization approach to (1). This enables us to generalize the results of [6], [4] and to treat problems like (1), arising in Thomas-Fermi theory. This method is based on new compactness results using heavily the axial symmetry of the problem. In addition, this method works in some situations for which it is not clear that the method of [6] may be applied (since it is not clear that the solutions we find have compact support).

In section II, we give our main results concerning (1) (the compressible case for axisymmetric rotating fluids) ; while in section III we apply our techniques to the incompressible case. In section IV we study some variational problems of Thomas-Fermi type. Finally in section V, we look directly at the Euler equations associated with (1).

Let us conclude this introduction remarking that another class of minimization problems is treated in [28], [8] and [15] : the main difference between these problems is in the presence of

$+\frac{1}{2}\iint \rho(x)\rho(y)|x-y|^{-1}dxdy$ instead of $-\frac{1}{2}\iint \rho(x)\rho(y)|x-y|^{-1}dxdy$.

This difference also makes the Euler equations simpler in the case of [8] since these equations (even nonlinear) satisfy the maximum principle (see [8]).

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II - AXISYMMETRIC ROTATING FLUIDS : THE COMPRESSIBLE CASE

II.1. Angular velocity prescribed :

Let us give a few notations : $D(\mathcal{E}) = \{\rho \in L^1 \cap L^{6/5}(\mathbb{R}^3), j(|\rho|) \in L^1(\mathbb{R}^3)\}$;

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^3} j(\rho)dx + \int_{\mathbb{R}^3} V(x)\rho(x)dx - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dxdy$$

for ρ in $D(\mathcal{E})$, where we assume

(2) j is a nonnegative continuous convex function on $\mathbb{R}_+$ such that $j(0) = 0$,

(3) $V \in L^\infty(\mathbb{R}^3)$ and $V(x) = V((\sqrt{x_1^2 + x_2^2}, 0, 0))$ for all $x = (x_1, x_2, x_3)$;

(this last assumption means that V depends only in r if we use cylindrical coordinates $x = (r,\theta,z)$).

Remark II.1. From well-known results we have for all ρ in $L^{6/5}$:

(4) $$\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dxdy\right| \leq C_0 \|\rho\|^2_{L^{6/5}} \quad ;$$

(if $\rho \in L^{6/5}$, $\int_{\mathbb{R}^3} \rho(y)\frac{1}{|x-y|} dy \in L^6$) .

Furthermore an easy argument gives : $\left(\frac{C_0}{4\pi}\right)^{1/2} = C$ where C is the best constant in the following Sobolev inequality :

$$\|u\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^3)} \text{ , for all } u \text{ in } H^1(\mathbb{R}^3) .$$

Now, in view of [35], [3] or [37], $C = \frac{1}{\sqrt{3}}(\frac{\pi^2}{4})^{-1/3}$; thus we have

(5) $$C_o = (\frac{4}{\pi})^{1/3} \times \frac{1}{3} .$$

In conclusion for ρ in $L^{6/5}$, the following functional E is well defined

$$E(\rho) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} \, dxdy \quad ;$$

and $E(\rho)$ is bounded as in (4) (where C_o is given by (5)).

In addition, we have by Hölder inequality, for all ρ in $L^1 \cap L^{4/3}$:

(5') $$\left| \iint_{R^3 \times R^3} \rho(x)\rho(y)|x-y|^{-1} dxdy \right| \leq C_1 \|\rho\|^{2/3}_{L^1} \|\rho\|^{4/3}_{L^{4/3}} .$$

and let C_1 be the best constant satisfying (5') for all ρ in $L^1 \cap L^{4/3}$.

Now , we give our main result

<u>THEOREM 1</u>. *Let $M > 0$ be fixed. We assume (2), (3) and*

(6) $$\lim_{t \to \infty} \frac{j(t)}{t^{4/3}} = K > 0 \text{ , with } \frac{C_1}{2} M^{2/3} < K < +\infty \quad ;$$

(7) $\mathcal{E}(\rho) < 0$, *for some* ρ *in* $\tilde{D}(\mathcal{E})$ *such that* $\rho \geq 0$, $\int \rho \leq M$;

(8) $V \geq 0$ a.e. *in* $\mathbb{R}^3$.

Then, there exists ρ *in* $D(\mathcal{E})$ *such that* $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho = M$,

ρ *depends only of* r *and* z, ρ *is even and is nonincreasing in* z ,

and $$\mathcal{E}(\rho) = \min_{\substack{\int\tilde{\rho} = M \\ \tilde{\rho} \geq 0, \tilde{\rho} \in \tilde{D}(\mathcal{E})}} \mathcal{E}(\tilde{\rho}) = \min_{\substack{\int\tilde{\rho} \leq M \\ \tilde{\rho} \geq 0, \tilde{\rho} \in \tilde{D}(\mathcal{E})}} \mathcal{E}(\tilde{\rho}) < 0 \quad ;$$

where $\tilde{D}(\mathcal{E}) = D(\mathcal{E}) \cap \{\tilde{\rho} = \tilde{\rho}(r,z)$ *with* $x = (r,\theta,z)\}$.

<u>Remark II.2</u>. This result contains the corresponding result of [6] where some extra assumptions are made upon j . Let us also remark that in [6] (and this is the physical problem) V is replaced by $-\tilde{V}$ where $\tilde{V}$ satisfies (3) and

(9) $\tilde{V}$ is nondecreasing in $r = (x_1^2+x_2^2)^{1/2}$, $\tilde{V} \geq 0$ a.e. ,

$$\lim_{r\to\infty} r\,[\tilde{V}(r) - \sup_{s\geq 0} \tilde{V}] = 0 .$$

Thus if we set $V = \sup_{r\geq 0} \tilde{V} - \tilde{V}$, we have

$$\min_{\substack{\int\tilde{\rho} = M \\ \tilde{\rho}\geq 0}} \int j(\rho) - \int \tilde{V}\,\rho + \frac{1}{2}E(\rho) = \left\{ \min_{\substack{\int\tilde{\rho} = M \\ \tilde{\rho}\geq 0}} \mathcal{E}(\rho) \right\} - M \sup_{r\geq 0} \tilde{V} .$$

And by this simple observation, we see that the physical problem is included in our general framework.

We will see below (in Corollary II.1) some sufficient conditions for (7) to hold. Let us also point out that regularity results are proved in [6] and can be easily adapted to the situation treated here.

Remark II.3. The fact that $\frac{4}{3}$ is critical power and $M_o = (\frac{2K}{C_1})^{3/2}$ is a critical mass is proved in [6] . Indeed, if we consider $j(t) = \alpha t^{\beta}$ with $0 < \alpha$, $0 < \beta < \frac{4}{3}$ then a straightforward argument shows that

$$\min_{\substack{\int\rho = M \\ \rho\geq 0}} \int j(\rho) - E(\rho) = -\infty ,$$

and this implies : $\min_{\substack{\int\rho = M \\ \rho\geq 0}} \mathcal{E}(\rho) = -\infty$,

We will come back on this point later on (in Section V).

Remark II.4. The case of other potentials V (i.e. not satisfying (3) and (8)) is considered in Remark II.7.

COROLLARY II.1. *Under assumptions (2), (3), (6) and (8) and if we assume either*

$$(10)\quad \begin{cases} \lim\limits_{t \to 0} j(t)t^{-4/3} = 0 \\ \\ \lim\limits_{r \to \infty} rV(r) = 0 \end{cases}$$

or

$$(11)\quad \begin{cases} \lim\limits_{t \to \infty} j(t)t^{-4/3} = \infty\ , \\ \\ M > M_o' > 0\ , \text{ for some constant } M_o' \text{ large enough,} \end{cases}$$

then (7) is satisfied and thus the conclusion of Theorem II.2 holds.

Proof of Corollary II.1. If we assume (10), (7) follows from Lemma 6 in [6] : we reproduce the proof .

We choose σ such that $\int_{\mathbb{R}^3} \sigma = M$, $\sigma \in \mathcal{D}_+(\mathbb{R}^3)$ and $\sigma = 0$ if $|x| > 1$ or if $r < \frac{1}{2}$. Next, we consider $\sigma_R = \frac{1}{R^3}\sigma(\frac{x}{R})$ and we compute

$$\begin{cases} \int_{\mathbb{R}^3} j(\sigma_R)\,dx = o(1) \int_{\mathbb{R}^3} \sigma_R^{4/3} = o(\frac{1}{R}) \quad (\text{as } R \to \infty\,) \\ \int_{\mathbb{R}^3} V(r)\sigma_R(x)\,dx = \int_{\mathbb{R}^3} V(Rr)\sigma(x)\,dx = \int_{\frac{1}{2} < r \leq 1} V(Rr)\sigma(x)\,dx = o(\frac{1}{R})\ , \end{cases}$$

on the other hand

$$\iint \frac{\sigma_R(x)\sigma_R(y)}{|x-y|}\,dxdy = \frac{1}{R} \iint \frac{\sigma(x)\sigma(y)}{|x-y|}\,dxdy\ .$$

And this implies that for R large enough we have $\mathcal{E}(\sigma_R) < 0$.

If we assume (11), we now choose σ such that $\int_{\mathbb{R}^3} \sigma = 1$, $\sigma \in \mathcal{D}_+(\mathbb{R}^3)$. If we denote by $\sigma_R(x) = \sigma(\frac{x}{R})$, we have

$$\int j(\sigma_R)(x)\;dx = R^3 \int j(\sigma)\,dx \quad ;$$

$$\int V(r)\sigma_R(x)\,dx = R^3 \int V(Rr)\sigma(x)\,dx \leq CR^3 \quad ;$$

$$\iint \frac{\sigma_R(x)\sigma_R(y)}{|x-y|}\,dxdy = R^5 \iint \frac{\sigma(x)\sigma(y)}{|x-y|}\,dxdy \;.$$

Thus, if $R > R_o$, we have $\mathcal{E}(\sigma_R) < 0$, on the other hand we have

$$\int \sigma_R \;dx = R^3 \quad ;$$

And choosing $M_o = R_o^3$, we conclude .

We would like to point out that, in the case where we assume (10), then by [6] all solutions ρ of the minimization problem have compact support (it is, by the way, a necessary condition for the method of [6] to be applied). Under the general assumptions of Theorem II.1 , we do not know (in general) that all solutions ρ have compact support.

II.2. Proof of Theorem II.1

Let us give the outline of the proof : we first want to solve :

$$\mathcal{E}(\rho) = \min_{\substack{\int \tilde{\rho} \leq M \\ \tilde{\rho} \geq 0 \\ \tilde{\rho} \in \tilde{D}(\mathcal{E})}} \{\mathcal{E}(\tilde{\rho}) \;;\; \int \rho \leq M \,,\, \rho \geq 0 \,,\, \rho \in \tilde{D}(\mathcal{E})\} \;. \tag{12}$$

(Remark that the minimizing set is convex and $\mathcal{E} + E$ is convex). Thus considering a minimizing sequence ρ_n we obtain first 1) bounds on ρ_n ; next 2) we choose a " good " minimizing sequence and 3) we pass to the limit and find a solution ρ of (12).

When this is done, we have to prove in a fourth step 4) that we have actually : $\int \rho = M$. This will be achieved in this section by a simple scaling argument.

Remark II.5. This scheme of proof is somewhat standard in minimization problems and for related problems and techniques the reader is referred to [11], [12], [29] for example. Here, the difficulty is essentially concentrated in steps 3) and 4).

Step 1) Let ρ_n be a minimizing sequence, that is, such that :

$$\rho_n \geq 0 \ , \ \rho_n \in \tilde{D}(\&) \ , \ \int \rho_n \leq M \quad \text{and} \quad \&(\rho_n) \downarrow I \geq -\infty$$

where $I = \inf\limits_{\substack{\tilde{\rho} \geq 0 \ , \ \int \tilde{\rho} \leq M \\ \tilde{\rho} \in \tilde{D}(\&)}} \&(\tilde{\rho})$

Because of (7), $I < 0$ and we may assume $\&(\rho_n) \leq -\lambda < 0$.

In view of (6) and (8), we have : for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\&(\rho_n) \geq (K-\varepsilon) \int \rho_n^{4/3} - C_\varepsilon M - \frac{C_1}{2} M^{2/3} \int \rho_n^{4/3}$$

(indeed, $j(t) \geq (K-\varepsilon) t^{4/3} - C_\varepsilon t$).

And because of (6), we conclude : for any minimizing sequence ρ_n we have

(13) $\|\rho_n\|_{L^{4/3}} \leq \text{Const.} \ ; \ \int j(\rho_n) \leq \text{cont.} \ ; \ \int V(x)\rho_n(x)\,dx \leq \text{Const.}$

In particular this proves that $I > -\infty$.

Step 2) Let ρ_n be a minimizing sequence, we introduce

$$\tilde{\rho}_n = \rho_n^\star$$

where $\rho^\star$ denotes the Steiner symmetrisation of ρ with respect to the plane $x_3 = 0$.

We recall (see [14]) that, because of the properties of the Steiner symmetrisation, we have

$$\int j(\tilde{\rho}_n) = \int j(\rho_n)$$

(since ρ_n and $\tilde{\rho}_n$ have the same distribution function) ;

$\int V(r)\tilde{\rho}_n = \int V(r)\rho_n$ (remark that $\tilde{\rho}_n(r,z)$ and $\rho_n(r,z)$ for any r have the same distribution function in z) ;

$$E(\tilde{\rho}_n) \geq E(\rho_n) .$$

This implies $$\mathcal{E}(\tilde{\rho}_n) \leq \mathcal{E}(\rho_n) .$$

Thus, we may assume that the minimizing sequence ρ_n satisfies :

(14) $$\rho_n = \rho_n^\star$$

(if not, take $\tilde{\rho}_n = \rho_n^\star$) , that is, ρ_n is axially symmetric, even and nonincreasing in z .

Step 3) Let ρ_n be a minimizing sequence satisfying (14) : by step 1) ρ_n satisfies (13). In addition, if we introduce

$$u_n(x) = \int \rho_n(y) \frac{1}{|x-y|} dy ,$$

we have easily : ∇u_n is bounded in $L^q(\mathbb{R}^3)$ for $\frac{3}{2} < q \leq \frac{12}{5}$ and u_n is bounded in $L^p(R^3)$ for $3 < p \leq 12$. In addition, $u_n = u_n^\star$.

Lemma II.1 will be proved in II.3.

Now, we extract (if necessary) a subsequence of ρ_n , we still denote by ρ_n , satisfying :

$$(15)\quad \begin{cases} \rho_n \xrightarrow[L^p(\mathbb{R}^3)]{} \rho \text{ weakly for } 1 < p \leq \frac{4}{3} \\ u_n \xrightarrow[L^q(\mathbb{R}^3)]{} u \text{ weakly for } 3 < q < 12 \text{ and } u_n \to u \text{ a.e.} \\ u_n \xrightarrow[L^q_{loc}(\mathbb{R}^3)]{} u \text{ strongly for } q < 12 \text{ and } u = \int \rho(y) \frac{1}{|x-y|}\, dy . \end{cases}$$

Indeed, since u_n satisfies : $-\Delta u_n = 4\pi\rho_n$ in $\mathcal{D}'(\mathbb{R}^3)$, we have by the well-known L^p estimates (see [1] for example) $\| u_n \|_{W^{2,4/3}_{loc}(\mathbb{R}^3)} \leq$ Const.

and by the Sobolev imbeddings u_n remains in a compact set of $L^q(B_R)$ for any ball B_R and for any $q < 12$.

We now want to prove that ρ is a solution of (12). We first remark that, for all $R > 0$,

$$\int_{B_R} \rho\, dx = \lim_{n \to \infty} \int_{B_R} \rho_n\, dx \leq M$$

thus $\rho \in L^1_+(\mathbb{R}^3)$ and $\int \rho \leq M$.

In the same way, using assumption (8), we prove that

$$0 \leq \int_{\mathbb{R}^3} V(x)\rho(x)\, dx \leq \underline{\lim}_n \int_{\mathbb{R}^3} V(x)\rho_n(x)\, dx .$$

Since j is convex and nonnegative, we prove now

$$\int_{\mathbb{R}^3} j(\rho)\, dx \leq \underline{\lim}_n \int_{\mathbb{R}^3} j(\rho_n)\, dx .$$

Indeed it is enough to prove that J defined on $D(J) = \{\rho \in L^{4/3}(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} j(|\rho|)\, dx < +\infty\}$ by $J(\rho) = \int j(|\rho|)\, dx$ is lower semi-continuous on $L^{4/3}(\mathbb{R}^3)$ (for the topology of the norm) and this follows obviously from Fatou's lemma.

To conclude, we just have to prove that

$$E(\rho_n) \underset{n \to \infty}{\to} E(\rho)$$

But $E(\rho_n) = \int_{\mathbb{R}^3} \rho_n u_n \, dx$ and ρ_n converges weakly in $L^{6/5}(\mathbb{R}^3)$ towards ρ .

Thus we need to prove some compactness on u_n : this will be achieved by the following two lemmas (proved in section II.3) :

<u>*LEMMA II.1*</u> *If* $\nabla u \in L^2(\mathbb{R}^3)$, $u \in L^p(\mathbb{R}^3)$ $(1 \leq p \leq \infty)$ *and if* u *is axially symmetric and* $u = u^\star$ *, then we have*

(16) $$|u(r,z)| \leq C \{\|\nabla u\|_{L^2}^{2/(p+2)} \|u\|_{L^p}^{p/(p+2)}\} |z|^{-2/(p+2)} r^{-\frac{2}{p+2}} ;$$

for some C *independent of* u .

<u>*LEMMA II.2*</u> *If* (∇u_n) *is bounded in* $L^2(\mathbb{R}^3)$, (u_n) *is bounded in* $L^p(\mathbb{R}^3)$ *for* $3 < p < 1$. *and if* u_n *is axially symmetric and* $u_n = u_n^\star$, *then* (u_n) *is relatively compact in* $L^q(\mathbb{R}^3)$ *for* $3 < q < 12$.

In particular u_n (or a subsequence) converges strongly to u in $L^6(\mathbb{R}^3)$ and we are able to conclude.

<u>Step 4)</u> We argue by contradiction. Suppose $\int \rho < M$, where ρ is a solution of (12). Define ρ_σ by :

$$\rho_\sigma(r,z) = \rho(r, \tfrac{z}{\sigma}) .$$

We have : $$\int \rho_\sigma = \sigma \int \rho \quad \text{and}$$

$$\mathcal{E}(\rho_\sigma) = \sigma\{\int j(\rho) + \int V\rho\} - \frac{1}{2} E(\rho_\sigma) .$$

But $E(\rho_\sigma) = \sigma^2 \iint \rho(x)\rho(y)[(x_1-y_1)^2 + (x_2-y_2)^2 + \sigma^2(x_3-y_3)^2]^{-1/2} \, dxdy$.

We want to compute (formally) $\frac{d}{d\sigma} \mathcal{E}(\rho_\sigma)|_{\sigma = 1}$ and to prove this quantity is negative. Suppose we have done so. In this case, for σ near 1 and $\sigma > 1$, we have $\int \rho_\sigma \leq M$ and $\mathcal{E}(\rho_\sigma) < \mathcal{E}(\rho)$.

This contradicts the choice of ρ and this proves that all solutions ρ of (12) satisfy

$$\int \rho = M .$$

Thus, we compute now $\frac{d}{d\sigma}\mathcal{E}(\rho_\sigma)\big|_{\sigma=+1}$ (this can be made rigourous in a straightforward way) :

$$\frac{d}{d\sigma}\mathcal{E}(\rho_\sigma)\Big|_\sigma = \int j(\rho) + \int V\rho - E(\rho) + \frac{1}{2}\iint \rho(x)\rho(y)\frac{|x_3-y_3|^2}{|x-y|^3}\,dxdy$$

$$\leq \mathcal{E}(\rho) - \frac{1}{2}E(\rho) + \frac{1}{2}\iint \rho(x)\rho(y)\frac{1}{|x-y|}\,dxdy = \mathcal{E}(\rho)\ .$$

But $\mathcal{E}(\rho) = I < 0$ and we conclude.

(17) $$\frac{d}{d\sigma}\mathcal{E}(\rho_\sigma)_{\sigma=1} < 0\ .$$

<u>Remark 2.6</u>. We have used in the proof above symmetries of the problem : first, when we choose a good minimizing sequence which enables us to obtain some compactness via Lemma II.2 and arguments similar to [36] ; second, in the scaling argument of step 4).

<u>Remark 2.7</u>. Let us indicate how the above proof may be modified to treat the case of potentials V which do not satisfy (3) or (8).

<u>Case 1</u>. We assume

(3') $V \in L^\infty(\mathbb{R}^3) \cap W^{1,\infty}_{loc}(\mathbb{R}^3)$, $V(x) = V(r,z)$, V is nondecreasing in z

(8') $$\frac{\partial V}{\partial z} z \leq V \quad \text{a.e.}$$

(18) $\text{meas}(V^- > \varepsilon) < \infty$, for every $\varepsilon > 0$.

Then, the conclusion of Theorem II.1 holds.

The proof is exactly as above : just with a few modifications, we use (3') in step 2) since (3') implies

$$\int V\rho^\star \leq \int V\rho\ .$$

In step 3), we pass to the limit in the term $\int V\rho_n$ in the following way :

$$\int V\rho_n = \int V^+\rho_n - \int V^-\rho_n\ .$$

As above $\underline{\lim} \int V^+\rho_n \geq \int V^+\rho$

On the other hand

$$\int V^-\rho_n = \int_{(V^- \leq \varepsilon)} V^-\rho_n + \int_{(V^- > \varepsilon)} V^-\rho_n$$

and $\left|\int_{(V^- \leq \varepsilon)} V^-\rho_n\right| \leq \varepsilon \int \rho_n \leq \varepsilon M$,

$$\int_{(V^- > \varepsilon)} V^-\rho_n \xrightarrow[n \to \infty]{} \int_{(V^- > \varepsilon)} V^-\rho \quad \text{(because of (18))}.$$

We may replace (8) by

(19) $$\frac{\partial V}{\partial r} r + \frac{\partial V}{\partial z} z \leq 2V \text{ a.e.}$$

or by

(20) $$j'(t)t \leq 2t\ , \quad \text{for} \quad t > 0\ .$$

If (19) holds, in step 4) we set $\rho_\sigma(x) = \rho(\frac{x}{\sigma})$ and compute $\frac{d}{d\sigma} \mathcal{E}(\rho_\sigma)\big|_{\sigma=+1}$

If (20) holds, in step 4) we compute $\frac{d}{d\theta} \mathcal{E}(\theta\rho)\big|_{\theta=+1}$.

Case 2. We assume now that V is spherically symmetric and V satisfies

(3") $$V \in L^\infty(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$$

(8") $$r \frac{\partial V}{\partial r} \leq 2\ V \quad \text{a.e.}$$

and (18) ; (Again (8") may be replaced by (20), and (3"), (8") may be relaxed a bit, but we will not consider such generalisations here).

Then, the conclusion of Theorem II.1 holds provided $\tilde{D}(\mathcal{E})$ is replaced by

$$\hat{D}(\mathcal{E}) = \{\rho \in D(\mathcal{E}), \rho(x) = \rho(|x|)\}\ .$$

The proof is the same but step 2) is suppressed; in step 3), Lemma II.1 is replaced by Proposition II.1 of the following section and in step 4)

$$\frac{d}{d\sigma} \mathcal{E}(\rho_\sigma)\big|_{\sigma=+1}\ ,$$

where $\rho_\sigma(x) = \rho(\frac{x}{\sigma})$.

II.3. Some technical lemmas

In this section we want to prove Lemma II.1 and some related results : we begin by proving a Proposition which generalizes results of [36], [11].

PROPOSITION II.1. *Let* $N \geq 2$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. *There exists some constant* $C > 0$, $C = C(N,p,q)$ *such that for all* u *satisfying :*

$$\nabla u \in L^p(\mathbb{R}^N)\ ,\ u \in L^q(\mathbb{R}^N)\ ,\ u(x) = u(|x|)\ ;$$

we have

$$|u(x)| \leq C\, \{\|\nabla u\|_{L^p}^{\frac{p'}{q+p'}}\ \|u\|_{L^q}^{\frac{q}{q+p'}}\}\, |x|^{-\frac{(N-1)p'}{q+p'}} \tag{21}$$

where p' *is the conjugate exponent of* $p(\frac{1}{p} + \frac{1}{p'} = 1)$.

Remark II.8. If $p^* \leq q$ (where p^* is the sobolev exponent given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$) then (21) may be replaced by

$$|u(x)| \leq C\, \|\nabla u\|_{L^p}\, |x|^{\frac{N-p}{p}} . \tag{21'}$$

Indeed, if $\nabla u \in L^p$ and $u \in L^q$, then by Sobolev imbeddings :

$$u \in L^{p^*} ,$$

thus we may apply (21) with $q = p^*$ and we find (21').

The special case $p = 2$ of (21') is given in [11], while in [36] the case $p = q = 2$ is given and also used in [29].

Proof of Proposition II.1. Let us make a formal proof, which can be easily made rigorous. We still denote $u(r) = u(x)$, where $r = |x|$. We have $\frac{d}{dr}(|u|^\alpha\, r^{N-1}) \geq -\alpha\, |u|^{\alpha-1} |u'|\, r^{N-1}$ for $r > 0$

where $\alpha = \frac{q}{p'} + 1$.

Thus we have

$$|u|^{\alpha} r^{N-1} \leq \alpha \int_r^{\infty} |u|^{\alpha-1} |u'| s^{N-1}\, ds$$

$$\leq \alpha \left[\int_0^{\infty} |u|^{(\alpha-1)p'} s^{N-1} ds\right]^{1/p'} \left[\int_0^{\infty} |u'|^p s^{N-1} ds\right]^{1/p}$$

$$= \alpha \left[\int_0^{\infty} |u|^q\, s^{N-1} ds\right]^{1/p'} \left[\int_0^{\infty} |u'|^p\, s^{N-1}\, ds\right]^{1/p}$$

$$\leq C \, \| u \|^{q/p'}_{L^q(\mathbb{R}^N)} \, \| \nabla u \|_{L^p(\mathbb{R}^N)} \quad ,$$

and this implies (21).

We now turn to the

Proof of Lemma II.1. We introduce for x in $\mathbb{R}^2$ and $r = |x|$:

$$v(x) = v(r) = \int_{\frac{z}{2}}^{z} u(r,t)\,dt \; .$$

Then v is radial, $\nabla v \in L^2(\mathbb{R}^2)$ since $\nabla u \in L^2(\mathbb{R}^3)$, $v \in L^p(\mathbb{R}^2)$ since $u \in L^p(\mathbb{R}^3)$ and we have

$$|\nabla v|_{L^2(\mathbb{R}^2)} \leq |z|^{1/2} \, \| \nabla u \|_{L^2(\mathbb{R}^3)}$$

$$\| v \|_{L^p(\mathbb{R}^2)} \leq |z|^{\frac{p-1}{p}} \, \| u \|_{L^p(\mathbb{R}^3)} \; .$$

Applying Proposition II.1 , we find

$$|v(r)| \leq C \, \| \nabla u \|^{\frac{2}{p+2}}_{L^2(\mathbb{R}^3)} \, \| u \|^{\frac{p}{p+2}}_{L^p(\mathbb{R}^3)} \, |z|^{\frac{1}{p+2}} |z|^{\frac{p-1}{p+2}} r^{-\frac{2}{p+2}}$$

$$\leq C \, \| \nabla u \|^{2/(p+2)}_{L^2(\mathbb{R}^3)} \, \| u \|^{p/(p+2)}_{L^p(\mathbb{R}^3)} |z|^{p/(p+2)} r^{-2/(p+2)} \; .$$

Now, we use the fact that $u = u^*$, thus for $z > 0$ we have :

$v(r) \geq \frac{z}{2} \; u(r,z)$. And this proves the Lemma.

<u>Proof of Lemma II.2.</u> We just need to prove that if $u_n \longrightarrow u$ weakly in L^p for $3 < p < 12$, strongly in L^p_{loc} for $p < 12$ and if $u_n \underset{a.e}{\to} u$, then this, together with the assumptions made on u_n, implies that u_n converges strongly in L^q for $3 < q < 7$.

We first prove that

$$\int_0^1 dz \int_{\mathbb{R}^2} u_n^p(x,z)dx \to \int_0^1 dz \int_{\mathbb{R}^2} u^p(x,z)dx\ , \ \textit{for}\ \ 3 < p < 7\ .$$

Indeed, remark that we have, for $z > 0$ fixed :

$$\int_{\mathbb{R}^2} u_n^\alpha(x,z)\,dx \leq \frac{1}{z}\int_0^z dz \int_{\mathbb{R}^2} u_n^\alpha(x,z)\,dz \leq \frac{C}{z}\ ,\ \text{for } 3 < \alpha < 12$$

thus $u_n(\cdot,z)$ is bounded in $L^\alpha(\mathbb{R}^2)$, for $3 < \alpha < 12$.

In addition, by Lemma II.1, we have :

$$|u_n(\cdot,z)| \leq \varepsilon(|x|)\ ,\ \text{where}\ \ \varepsilon(t) \to 0\ \text{ as }\ t \to +\infty\ .$$

Applying the compactness lemma proved in [36] (see also [12], [11]) we obtain

$$v_n(z) = \int_{\mathbb{R}^2} u_n^p(x,z)\,dx \underset{n\to\infty}{\longrightarrow} v(z) = \int_{\mathbb{R}^2} u^p(x,z)\,dx\ ,\ \text{for } 3 < p < 12$$

Now, obviously, $\| v_n \|_{L^1(0,1)} \leq C$ and

$$\left\| \frac{\partial v_n}{\partial z} \right\|_{L^1(0,1)} \leq C \int_{\mathbb{R}^3} |u_n|^{p-1} |\nabla u_n|\,dx \leq C \left[\int_{\mathbb{R}^3} |u_n|^{2(p-1)}\right]^{1/2}$$

and for $3 < p < 7$, $2(p-1) \in (3,12)$.

Thus $\left\| \frac{\partial v_n}{\partial z} \right\|_{L^1(0,1)} \leq C$ and this implies $\| v_n \|_{L^\infty(0,1)} \leq C$. (Actually, to be rigorous, we need to assume u_n smooth and we then argue by an obvious density argument). Since $v_n \underset{a.e}{\longrightarrow} v$ in $[0,1]$, we deduce :

$$\int_0^1 dz \int_{\mathbb{R}^2} u_n^p(x,z)dx \to \int_0^1 dz \int_{\mathbb{R}^2} u^p(x,z)dx\ ,\ \textit{for}\ \ 3 < p < 7\ .$$

There just remains to prove

$$\int_{|z|\geq 1} u_n^p(x,z)\,dxdz \to \int_{|z|\geq 1} u^p(x,z)\,dxdz$$

But $u_n \underset{a.e}{\to} u$ and for each $\alpha > 0$, the set $\{u_n > \alpha\} \cap \{|z| \geq 1\}$ is contained in the set $\{r|z| \leq C_\alpha\} \cap \{|z| \geq 1\} = I_\alpha$, here we just use Lemma II.1. But the measure of I_α is finite :

$$|I_\alpha| = 2\int_1^\infty dz \int_0^{\frac{1}{C_\alpha z}} r\,dr = \frac{1}{C_\alpha^2} < \infty \quad ;$$

and this implies

$$\int_{I_\alpha} u_n^p(x,z)\,dxdz \to \int_{I_\alpha} u^p(x,z)\,dxdz \quad , \quad \text{for any } \alpha > 0$$

and $p < 12$ (we just use the fact that $u_n \underset{a.e}{\to} u$ and $\|u_n\|_{L^{12}(I_\alpha)} \leq C$).

And finally

$$\int_{(|z|\geq 1)-I_\alpha} (u_n^p + u^p)(x,z)\,dxdz \leq \alpha^\gamma \int_{\mathbb{R}^3} (u_n^{p-\gamma} + u^{p-\gamma})\,dx$$

$$\leq C\alpha^\gamma \text{ if } \gamma \text{ is choosen small enough}$$

and this concludes the proof of Lemma II.2.

II.4. Angular momentum prescribed

We now consider another type of functional, which comes out from another model for rotating stars (the angular momentum is now prescribed). Let us denote by

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^3} j(|\rho|)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \rho(x) L(m_\rho(r(x)))\,\frac{1}{r(x)^2}\,dx$$

$$- \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy$$

with $\mathcal{E}$ defined on $D(\mathcal{E})$: the set of functions ρ lying in $L^1 \cap L^{6/5}(\mathbb{R}^3)$ and such that

$$\int_{\mathbb{R}^3} j(|\rho(x)|)dx < \infty \quad , \quad \int_{\mathbb{R}^3} \rho(x)L(m_\rho(r(x)))\frac{1}{r(x)^2}\,dx < \infty \quad .$$

Here, we define: $m_\rho(r) = \int_{(r(y)\leq r)} \rho(y)dy$ (with $r(x) = \sqrt{x_1^2+x_2^2}$, for all $x = (x_1,x_2,x_3)$); and L satisfies:

(22) L is a nonnegative, continuous, nondecreasing function on $\mathbb{R}_+$.

We will assume that j satisfies (2), as above. Such a functional $\mathcal{E}(\rho)$, as said above, arises in a model of a rotating fluid with prescribed angular momentum: see [6] and [4] for more detailed explanations.

<u>THEOREM II.2</u>. Let $M > 0$. We assume (2), (6), (7), (22) and

(10') $\overline{\lim_{t\to 0_+}}\, j(r)t^{-4/3} < +\infty.$

Then there exists ρ in $\tilde{D}(\mathcal{E})$ such that $\int_{\mathbb{R}^3} \rho(x)dx = M$, ρ is even in z and ρ is nonincreasing in $z \geq 0$, and

$$\mathcal{E}(\rho) = \min_{\substack{\int\tilde{\rho}=M \\ \tilde{\rho}\geq 0,\tilde{\rho}\in\tilde{D}(\mathcal{E})}} \mathcal{E}(\tilde{\rho}) = \min_{\substack{\int\tilde{\rho}\leq M \\ \tilde{\rho}\geq 0,\tilde{\rho}\in\tilde{D}(\mathcal{E})}} \mathcal{E}(\tilde{\rho})$$

where $\tilde{D}(\mathcal{E}) = \{\rho \in D(\mathcal{E}),\ \rho(x) = \rho(r,z)\}$ (where $x = (r,\theta,z)$ in cylindrical coordinates).

<u>REMARK II.9.</u> Again this result contains those of [6], where some additional assumptions are made upon j. We give below i) some sufficient conditions for (7) to be satisfied, ii) other existence results without assuming (10').

We now claim, that, except maybe for (10'), the assumptions are optimal: indeed, as in II.1 (Remark II.3) we just indicate that $\frac{4}{3}$

is a critical exponent and we refer to [6] for the justification of this exponent (see also section V below). Now, concerning (7), we prove it is, in general, necessary: take $j(r) = \alpha t^{\beta}$ $(\alpha > 0,\ \beta \geq \frac{4}{3})$ and assume ρ *is a minimum of* $\mathcal{E}$ over $\rho \geq 0$ satisfying the constraint $\int \rho = M$, then necessarily

$$\frac{d}{d\sigma}\, \mathcal{E}\left(\frac{1}{\sigma^3}\, \rho\left(\frac{x}{\sigma}\right)\right)\Big|_{\sigma=+1} = 0 \quad ,$$

but
$$\mathcal{E}\left(\frac{1}{\sigma^3}\, \rho\left(\frac{x}{\sigma}\right)\right) = \frac{1}{\sigma^{3(\beta-1)}} \int j(\rho) + \frac{1}{\sigma^2}\frac{1}{2} \int \rho(x) L(m_\rho(r))\, \frac{1}{r^2}\, dx$$
$$- \frac{1}{\sigma}\frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy \quad ;$$

therefore we have: $3(\beta-1) \int j(\rho) + \int \rho(x) L(m_\rho(r))\, \frac{1}{r^2} - \frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy = 0$ and this implies $\mathcal{E}(\rho) \leq -\frac{1}{2} \int \rho(x) L(m_\rho(r))\, \frac{1}{r^2}\, dx < 0$ (as soon as $\rho \not\equiv 0$, $L \not\equiv 0$).

Proof of Theorem II.2. The proof follows exactly the one of Theorem II.1. The only real modification is in step 4, where, having obtained a minimizer $\rho (\not\equiv 0)$ of $\mathcal{E}$ among all ρ satisfying $\rho \geq 0$, $\int \rho \leq M$, we want to prove that $\int \rho = M$. The argument we are going to use is inspired from a technique due to E. H. Lieb and B. Simon [28]: it is the only part of the proof where we use assumption (10').

Let us first remark that (10') implies that j is differentiable at 0 and $j'(0) = 0$. In addition, it is easily seen that it implies that j is Lipschitz continuous on every interval $[0,T]$ (for all $T > 0$) and that we have

$$\overline{\lim_{t \to 0_+}}\ j'(r)\, t^{-1/3} < \infty \quad .$$

For simplicity of notations and of the presentation of the argument, we will assume that j is C^1 and j' is strictly increasing.

If we argue by contradiction and if we assume: $\int \rho < M$. Then using for example [6], we see that ρ satisfies the following Euler equation:

$$\min(j'(\rho),\ j'(\rho) + f(r) - u) = 0 \quad \text{a.e. in } \mathbb{R}^3 \quad ,$$

where $u(x) = B\rho = \int \rho(y)\ |x-y|^{-1}dy$ and $f(r) = \int_r^\infty \frac{L(m_\rho(r))}{s^3}\, ds$.

Of course this may be rewritten: $\rho = (j')^{-1}((u-f(r))^+)$ and we finally obtain:

$$-\Delta u = \beta((u-f(r))^+),\ \nabla u \in L^2(\mathbb{R}^3),\ \ u \in L^p\ (\mathbb{R}^3)\ \ 3 < p \leq 12 \quad ;$$

where $\beta = 4\pi(j')^{-1}$.

Since we have $-\Delta u \geq 0$, $-\Delta u \not\equiv 0$, $u > 0$ in $\mathbb{R}^3$, we deduce by classical results:

$$u(x) \geq \frac{\alpha}{|x|} \quad \text{for} \quad |x| \geq 1 \quad \text{and for some} \quad \alpha > 0 \quad .$$

On the other hand: $f(r) \leq \frac{L(M)}{2r^2} = \frac{C}{2r^2}$; thus we finally obtain

$$\rho \geq (j')^{-1}((\frac{\alpha}{|x|} - \frac{C}{2r^2})^+) \quad .$$

In particular, we have

$$\int_A \rho(x) \geq \int_A (j')^{-1}(\frac{\alpha}{2|x|})dx$$

where $A = \{|z| \leq \frac{1}{C} r^2,\ r \geq C\}$, for some constant $C > 0$. Since we have, because of (10'): $\underline{\lim}_{t\to 0_+} (j')^{-1}(t)t^{-3} > 0$ and since $\int_A \frac{1}{|x|^3} dx = +\infty$, we obtain a contradiction. This proves $\int \rho = M$ and we conclude.

Now, we give some conditions under which (7) is satisfied:

COROLLARY II.2: Under assumptions (2), (6) and (22); and if we assume either

$$(10'') \qquad \lim_{t\to 0_+} j(t)t^{-4/3} = 0$$

or (10') and

$$(11) \qquad \lim_{t\to+\infty} j(t)t^{-4/3} = +\infty$$

$M > M_0' > 0$, *for some constant* M_0' *large enough;*

then (7) is satisfied and thus the conclusion of Theorem II.2 holds.

The proof of this result is identical to the proof of Corollary II.1 and we will skip it. Let us give another existence result where (10') is no longer assumed:

COROLLARY II.3. Under assumptions (2), (6), (7), (22) and either

$L(t)t^{-4/3}$ *is nonincreasing for* $t > 0$

or $j(t)t^{-3/2}$ *is nonincreasing for* $t > 0$, $L(t)t^{-1}$ *is nondecreasing for* $t > 0$;

then the conclusion of Theorem II.2 holds.

Proof of Corollary II.3:

Again the only argument to be changes is the one corresponding to step 4 of the proof of Theorem II.1. With the same notations, we assume $\int\rho < M$. For the sake of simplicity let us assume j and L are of class C^1: then

$$\mathcal{E}(\rho(\tfrac{x}{\sigma})) = \sigma^3 \int j(\rho) + \sigma\frac{1}{2}\int \frac{L(\sigma^3 m_\rho(r))}{r^2}\rho(x)dx - \sigma^5\frac{1}{2}\iint\frac{\rho(x)\rho(y)}{|x-y|}dxdy \quad .$$

Since we must have: $\frac{d}{d6}\mathcal{E}(\rho_\sigma)\Big|_{\sigma=+1} = 0$, we deduce

$$3\int j(\rho) + \frac{1}{2}\quad\frac{L(m_\rho(r))}{r^2}\rho(x)dx + \frac{1}{2}\int\frac{3L'(m_\rho(r))m_\rho(r)}{r^2}\rho(x)dx$$

$$- 5\,\frac{1}{2}\iint\frac{\rho(x)\rho(y)}{|x-y|}dxdy = 0 \quad .$$

Now, if we assume $L(t)t^{-4/3}$ is nonincreasing, we have: $3L'(t)t \leq 4L(t)$ for $t > 0$ and the preceding equality yields a contradiction with $\mathcal{E}(\rho) < 0$.

The case of the other assumption of Corollary II.3 is treated by a similar method, using $\rho(\frac{r}{\sigma},z)$ instead of $\rho(\frac{x}{\sigma})$.

III - AXISYMMETRIC ROTATING FLUIDS : THE INCOMPRESSIBLE CASE

We now consider briefly the incompressible case of an axisymmetric rotating fluid . In this case (see [4] for the justification) we consider the same functional $\mathcal{E}$ as in II.1 or in II.4. but with $j \equiv 0$, that is either

$$\mathcal{E}_1(\rho) = \int_{\mathbb{R}^3} V(x)\rho(x)\,dx - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \rho(x)\rho(y)\frac{1}{|x-y|}\,dxdy$$

or

$$\mathcal{E}_2(\rho) = \frac{1}{2}\int_{\mathbb{R}^3} \rho(x)L(m_\rho(r))\frac{1}{r^2}\,dx - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy\ .$$

And we want now to minimize $\mathcal{E}_i$ over all functions ρ (in $\tilde{D}(\mathcal{E})$) such that :

$$0 \leqslant \rho \leqslant 1 \quad \text{a.e. and} \quad \int_{\mathbb{R}^3}\rho(x)\,dx = M\ .$$

Like before the first functional corresponds to the case when the angular velocity is prescribed, while the second gives the case when the angular momentum is prescribed.

For the sake of simplicity, we just give a result concerning the minimization of $\mathcal{E}_2$ since the results and methods of this section are very similar of those of the preceding one.

THEOREM III.1. *Let* $M > 0$ *and let* L *satisfy (22). Then there exists* ρ *in* $\tilde{D}(\mathcal{E}_2)$ *such that* $\tilde{\rho}$ *is even, nonincreasing in* z *and*

$$\mathcal{E}_2(\rho) = \min_{\substack{\tilde{\rho}\in\tilde{D}(\mathcal{E}_2)\\ \tilde{0}\leqslant\tilde{\rho}\leqslant 1\\ \int\tilde{\rho}=M}} \mathcal{E}_2(\tilde{\rho}) = \min_{\substack{\tilde{\rho}\in\tilde{D}(\mathcal{E}_2)\\ 0\leqslant\tilde{\rho}\leqslant 1\\ \int\tilde{\rho}\leqslant M}} \mathcal{E}_2(\tilde{\rho}) < 0\ ,$$

where $\tilde{D}(\mathcal{E}_2) = \{\tilde{\rho} \in L^1 \cap L^{6/5}(\mathbb{R}^3),\ \rho(x) = \tilde{\rho}(r,z)$ *for all*

$x = (r,\theta,z)$, *and* $\int_{R^3} \tilde{\rho}(x) \frac{1}{r^2} L(m_{\tilde{\rho}}(r)\,dx < +\infty\}$.

Remark III.1. This result is essentially the same than the existence result in [4] (in [4], L is required in addition to satisfy $L(0) = 0$) but proved in a different way (Remark also that a third approach, consisting in solving directly the Euler equation by a fixed point argument is given in [5]).

Remark III.2. Some qualitative properties of the solutions of the above minimization problems are given in [4] in particular any solution ρ satisfies

$$\rho = 1_G$$

where G is some bounded, axially symmetric, measurable set. Further qualitative properties of G may be found in [26], [17] .

Proof of Theorem III.1. The proof is exactly the same than the proof of Theorems II.1 and II.2 . We just remark that as in Corollary II.1 or II.2 , (7) is satisfied. That is : $\exists \tilde{\rho}$ in $\tilde{D}(\mathcal{E}_2)$ such that $0 \leqslant \tilde{\rho} \leqslant 1$ and $\int_{R^3} \tilde{\rho}(x)\,dx \leqslant M$, satisfying

$$\mathcal{E}_2(\tilde{\rho}) < 0 \ .$$

IV - SOME VARIATIONAL PROBLEMS OF THOMAS-FERMI TYPE

We are now interested by finding ρ minimizing

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^N} j(\rho(x))\,dx - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho(x) f(x-y) \rho(y)$$

over all ρ satisfying : $\rho \geqslant 0$ a.e and $\int \rho = M$.

More precisely $\mathcal{E}(\rho)$ is defined over $D(\mathcal{E})$ where

$$D(\mathcal{E}) = \{\rho \in L^1(R^N) \ , \ j(\rho) \in L^1(R^N) \quad \text{and}$$
$$\iint_{R^N \times R^N} \rho(x)\,|f(x-y)|\,\rho(y) < +\infty\} \ ,$$

and j satisfies (2), while f is some given measurable function satisfying

(23) $\qquad f(x) = f(r)$ where $r = |x|$, for all x in R^N .

Such a problem occurs in quantum mechanics as Thomas-Fermi type problems and we will see below some particular examples of interest for physics (*) . We could also add in $\mathcal{E}(\rho)$ a term like $\int V(x)\rho(x)dx$, but we will not consider this case here which (more or less) can be treated by a combination of the techniques below and of Section II.

We will first investigate (in IV.1) the " extended " problem namely : minimize $\mathcal{E}$ over all ρ in $D(\mathcal{E})$ satisfying $\rho \geqslant 0$ and $\int \rho \leqslant M$. Then, we give some conditions in IV.2 for solutions of the " extended " problem to satisfy actually : $\int \rho = M$. And finally in IV.3 we consider some examples.

IV.1. Resolution of the " extended " minimization problem :

We first consider the case where f is nonnegative and nonincreasing in r :

PROPOSITION IV.1. *Let* f *satisfy (23) and*

(24) $\qquad f \geqslant 0$ *a.e.* , $f(r)$ *is nonincreasing ;*

(*) Those examples were communicated to us by Prof. E.F. Redisch.

(25) $\quad f \in M^p(\mathbb{R}^N) \qquad (1 < p < \infty)$.

Let j *satisfy (2) and*

(6') $\quad \displaystyle\lim_{t\to\infty} \frac{j(t)}{t^q} = K > 0$, *with* $C_p M^{1-1/p} < K \leq +\infty$ *and* $q = 1 + \frac{1}{p}$,

(C_p *is some positive constant defined in Remark IV.1 below) ;*

then there exists ρ *in* $D(\mathcal{E})$ *minimizing* $\mathcal{E}$ *over all* $\tilde{\rho}$ *satisfying :*

$$\tilde{\rho} \in D(\mathcal{E}) \quad , \quad \tilde{\rho} \geq 0 \text{ a.e. and } \int \tilde{\rho} \leq M .$$

In addition if $f \not\equiv 0$ *all such* ρ *solutions of this minimization problem satisfy necessarily(up to a translation) :* ρ *is spherically symmetric, nonnegative and* ρ *is nonincreasing in* $|x|$.

Remark IV.1. If we take $N = 3$ and $f(x) = \frac{1}{|x|}$, then $p = 3$ and $q = \frac{4}{3}$ and (6') reduces to (6). The constant C_p is defined by

$$\left|\iint \rho(x)\rho(y)|f(x-y)|\,dxdy\right| \leq C_p \|\rho\|_{L^1}^{1-1/p} \|\rho\|_{L^q}^{1+1/p}, \text{ for all } \rho$$

in $L^1 \cap L^q$.

Remark IV.2. Of course it is possible to generalize (25) by (25')

(25') $\quad f \in \sum_i M^{p_i}(\mathbb{R}^N) + L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \qquad 1 < p_i < \infty$,

and (6') has to be replaced by the corresponding assumption with $q = 1 + \frac{1}{\wedge_i p_i}$ (and $q = 2$ if in all decompositions of f as in (25') there is a term in $L^1(R^N)$).

Remark IV.3. We may replace (24) by the following :

$$(24') \quad \begin{cases} f = f_1 - f_2 \text{ a.e. , where } f_1 , f_2 \geq 0 \text{ a.e. and } f_1(x) = f_1(r) \\ \text{is nonincreasing while } f_2(x) = f_2(r) \text{ is nondecreasing.} \end{cases}$$

Then (25) has to hold only for f_1 and the Proposition IV.1 (and its proof) are still valid.

<u>Proof of Proposition IV.1.</u> By a classical use of Schwarz symmetrisation process (see [14] , for example, for the definition and properties involved here of the Schwarz symmetrisation) the necessary part of the Theorem is proved. This also proves that we may assume that a minimizing sequence ρ_n satisfies

$$\rho_n(x) = \rho_n(r) \quad \text{and} \quad \rho_n \quad \text{is nonincreasing} .$$

Now, because of (24) , (25) and (6'), exactly as in the proof of Theorem II.1, we get a priori estimates : $\int j(\rho_n)\,dx \leq C$, $\iint \rho_n(x)\, f(x-y)\rho_n(y)\,dxdy \leq C$, (and in particular $\|\rho_n\|_{L^1 \cap L^q} \leq C$).
Now using the fact that ρ_n is nonincreasing and that $\|\rho\|_{L^1} \leq M$, we deduce $0 \leq \rho_n(x) \leq \dfrac{C}{|x|^N}$.

In addition, from a classical result, since ρ_n is a sequence of nonincreasing functions bounded in $L^\infty((\varepsilon,\infty))$ (for every $\varepsilon > 0$) there exists a subsequence n_k such that : $\rho_{n_k} \underset{a.e}{\to} \rho$.
(One could also obverse that ρ_n is of bounded variation, except maybe near 0). In the same way as for Theorem II.1, one concludes since :

$$\iint \rho_{n_k}(x) f(x-y)\rho_{n_k}(y)\,dxdy \xrightarrow[n_k \to \infty]{} \iint \rho(x) f(x-y)\rho(y)\,dxdy$$

Next, we give some results illustrating a method to prove the existence of a solution to the extended problem, we use i) the spherical symmetry, ii) regularity properties of the " potential "

$$u(x) = f * \rho(x) = \int_{\mathbb{R}^N} f(x-y)\rho(y)\,dy$$

PROPOSITION IV.2. *Let* f *satisfy (23) and*

$$(26)\quad \begin{cases} f = f_1 - f_2 \text{ a.e. with } f_i \geq 0 \text{ a.e. } (i = 1,2),\ f_i \text{ is radial } (i=1,2) \\ \text{and } f_1 \in M^p(\mathbb{R}^N) \\ \nabla f_1 \in M^s(\mathbb{R}^N) \text{ with } 1 < s < \infty \text{ and } s < p+1 \text{ ; or } \nabla f_1 \in L^1(\mathbb{R}^N). \end{cases}$$

Let j *satisfy (2), (6') ;*

then there exists ρ *in* $D(\mathcal{E})$, *with* $\int \rho \leq M$, *radial nonnegative minimizing* $\mathcal{E}$ *over all* $\tilde{\rho}$ *satisfying* $\tilde{\rho} \in D(\mathcal{E})$, $\tilde{\rho}$ *is radial,* $\tilde{\rho} \geq 0$ *a.e. and* $\int \tilde{\rho}\, dx \leq M$.

Remark IV.4. As in Remark IV.2 , (26) may be generalized assuming that ∇f_1 belongs to a sum of M^p-spaces or L^p-spaces (and a similar assumption for f_1). In some sense (24) is a particular case of (26) since f being nonincreasing in (24) is essentially of bounded variation.

Remark IV.5. If $f(x) = \frac{1}{|x|}$ and $N = 3$, then $\nabla f \in M^{3/2}(\mathbb{R}^3)$ (and $1 + \frac{1}{p} = \frac{4}{3} < \frac{s}{s-1} = 3$).

Proof of Proposition IV.2. Let ρ_n be a minimizing sequence such that ρ_n is radial. We know by the same argument as in Theorem II.1 (Prop. IV.1) :

$$\int j(\rho_n) \leq C \ , \iint \rho_n(x) f_i(x-y) \rho_n(y) \leq C \ , \|\rho_n\|_{L^1 \cap L^q} \leq C \ ,$$

for some $C > 0$.

Next, we introduce : $u_n(x) = \int_{\mathbb{R}^N} f_1(x-y)\rho_n(y)\,dy$, then $u_n \in L^\alpha(\mathbb{R}^N)$,

(for $p \leq \alpha \leq p(p+1)$).

In addition because of (26) : $\nabla u_n \in L^\beta(\mathbb{R}^N)$ (for $s \leq \beta \leq \frac{s(p+1)}{p+1-s}$).

Since u_n is radial $u_n(x) = u_n(|x|)$, we may apply Proposition II.1 and this enables us to apply the method of proof of Theorem II.1 , and we conclude.

Remark IV.6. It is possible that (26) may be relaxed, assuming only f_1 to be in a fractional Sobolev space, then one would need to extend Prop. II.1. to fractional Sobolev spaces.

Remark IV.7. If instead of (23), we assume

$$f(x) = f(r,z) \text{ where } r = (x_1^2 + x_2^2)^{1/2}, \quad x \in \mathbb{R}^3$$

and $f \geq 0$ a.e., f is nonincreasing in z; then similar results may be proved but we will not consider such a generalization.

IV.2. Saturation of the constraint.

We are now looking to the problem of determining if $\int \rho = M$, where ρ is the solution of the extended problem.

More precisely, in this section, we will assume that $\rho \in D(\mathcal{E})$ and ρ minimizes $\mathcal{E}$ over all $\tilde{\rho}$ in $D(\mathcal{E})$ such that $\tilde{\rho} \geq 0$ a.e. and $\int \tilde{\rho} \leq M$. Of course $D(\mathcal{E})$ may be replaced by $\tilde{D}(\mathcal{E})$ the subspace of $D(\mathcal{E})$ consisting of radial functions.

We give essentially two simple methods to prove that $\int \rho \, dx = M$:

PROPOSITION IV.3. If we assume (2), (23) and $\mathcal{E}(\rho) < 0$ *(or in a equivalent way* $\exists\, \tilde{\rho} \in D(\mathcal{E})$, $\tilde{\rho} \geq 0$, $\int \tilde{\rho} \leq M$ *such that* $\mathcal{E}(\tilde{\rho}) < 0$) *and if either*

(27) $$j'(t)t \leq 2j(t) \text{ a.e. for } t \geq 0$$

or

(28) $$Nf(r) + f'(r)r \geq 0 \text{ a.e. for } r \geq 0,$$

where f *is assumed (for example) to be* C^1 *on* $(0,\infty)$;

then ρ *satisfies :* $\int_{\mathbb{R}^N} \rho(x)dx = M$.

Proof of Proposition IV.3. If we assume (27) and that $\int \rho \, dx < M$ then computing

$$\frac{d}{d\theta}\,\mathcal{E}(\theta\rho)\Big|_{\theta=1} = \int_{\mathbb{R}^N} j'(\rho)\rho\,dx - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)f(x-y)\rho(y)\,dxdy$$

$$\leq 2\mathcal{E}(\rho) < 0$$

we get a contradiction from the definition of ρ .

On the other hand, if we assume (28) and that $\int \rho\,dx < M$, then we have, denoting by $\rho_\sigma(x) = \rho(\frac{x}{\sigma})$

$$\frac{d}{d\sigma}\,\mathcal{E}(\rho_\sigma)\Big|_{\sigma=1} = N\int_{\mathbb{R}^N} j(\rho(x))\,dx - N\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)f(x-y)\rho(y)\,dxdy$$

$$-\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)f'(|x-y|)|x-y|\rho(y)\,dxdy$$

$$\leq \frac{N}{2}\,\mathcal{E}(\rho) < 0 ,$$

and the result is proved.

Remark IV.8. The above proof shows that (28) may be replaced by

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\{f'(x-y)(x-y) + N\,f(x-y)\}\rho(y)\,dxdy \geq 0 ,$$

for all ρ in $\mathcal{D}(\mathbb{R}^N)$ and $\rho \geq 0$ a.e.

Remark IV.9. Obviously (27) is satisfied for $j(t) = t^\gamma$ and $\gamma \leq 2$, while (28) is satisfied for $f(r) = r^{-k}$ and $0 < k \leq N$.

Remark IV.10. We would like to give another method taken from [28] which could be useful in some particular cases. For simplicity, we assume $N = 3$, $f(r) = \frac{1}{r}$: an easy argument gives that one has, if $\int_{\mathbb{R}^3}\rho\,dx < M$,

$$j'(\rho) = \frac{1}{|x|} * \rho \quad \text{a.e. in } \mathbb{R}^3 .$$

If we assume that j' is strictly increasing and $j'(0) = 0$, then denoting by $u = \frac{1}{|x|} * \rho$, we have

$$\rho = \beta(u) \quad \text{a.e. in } \mathbb{R}^3 ,$$

where $\beta = (j')^{-1}$. On the other hand, because of the spherical symmetry :

$$u(r) = \int_{\mathbb{R}^3} \rho(y) \frac{1}{\max(r,|y|)}\, dy \geq \left(\int_{\mathbb{R}^3} \rho(y)\,dy\right)\frac{1}{r} .$$ If

$\mathcal{E}(\rho) < 0$, then $\rho \not\equiv 0$ and $\int_{\mathbb{R}^3} \rho(y)\,dy = m > 0$.

Thus $\rho \geq \beta(\frac{m}{|x|})$ a.e. in $\mathbb{R}^3$; and now, if we assume $\int^{\infty} r^2\beta(\frac{1}{r})dr = +\infty$, we get a contradiction, since $\rho \in L^1$. This contradiction proves : $\int \rho\, dx = M$. Remark that if $j(t)t^{-(1+\gamma)} \xrightarrow[t\to\infty]{} \ell \in (0,\infty)$, then $\lim_{t\to\infty} \beta(t)t^{-1/\gamma}$ exists and is > 0 , and

and $$\int^{\infty} r^2\beta\left(\frac{1}{r}\right) = +\infty \quad \text{as soon as} \quad \gamma \geq \frac{1}{3} .$$

Of course, it is possible to combine Propositions IV.1 , IV.2 on one hand, and IV.3 on the other hand to get general results for the existence of a solution to the minimization problem. But for simplicity we prefer not to give these results but to explain how they apply to some particular examples. In addition we would like to point out that Propositions IV.1-3 are only examples of methods described in the corresponding proofs.

IV.3 Some examples.

Example 1. " non-rotating stars "

This means we consider the case : $N = 3$, $f(x) = \frac{1}{|x|}$. We may apply Proposition IV.1 since $f \in M^3(\mathbb{R}^3)$; Proposition IV.3 also applies since (28) is obviously satisfied and thus we find back Theorem II.1 (in the special case $V = 0$) by the combination of Proposition IV.1 and IV.3.

Example 2. We consider the case $N = 3$, $f(x) = e^{-\mu|x|} \frac{1}{|x|} (\mu > 0)$.

We may obviously apply Proposition IV.1 since $f \in M^3(\mathbb{R}^3)$, thus if j satisfies (2), (6') (with $p=3$) and (27), and if there exists $\tilde{\rho}$ in $D(\varepsilon)$ such that $\mathcal{E}(\tilde{\rho}) < 0$ and $\int \tilde{\rho} \leq M$ then there exists ρ in $D(\varepsilon)$ satisfying

i) ρ is radial, nonincreasing, $\rho \geq 0$ a.e., $\int \rho \, dx = M$.

ii) $$\mathcal{E}(\rho) = \min_{\substack{\tilde{\rho} \in D(\mathcal{E}) \\ \tilde{\rho} \geq 0, \int \tilde{\rho} = M}} \mathcal{E}(\tilde{\rho}) = \min_{\substack{\tilde{\rho} \in D(\mathcal{E}) \\ \tilde{\rho} \geq 0, \int \tilde{\rho} = M}} \mathcal{E}(\tilde{\rho}) .$$

Finally, for the existence of $\tilde{\rho}$ in $D(\mathcal{E})$ such that $\mathcal{E}(\tilde{\rho}) < 0$, and $\int \tilde{\rho} \leq M$ we indicate two simple cases where this can be checked :

1) If j is differentiable at 0, if $j'(0) < 0$, then for every $\rho \geq 0$ in $\mathcal{D}(\mathbb{R}^3)$ $\frac{d}{dt} \mathcal{E}(t\rho)\big|_{t=0} = j'(0) \int_{R^3} \rho \, dx$,
and thus the condition is satisfied for every $M > 0$.

2) If $\lim_{t \to \infty} j(t)/t^2 = 0$, then for M large enough the condition is satisfied indeed $\mathcal{E}(t\rho)$ is negative for t large enough if $\rho \in \mathcal{D}_+(\mathbb{R}^3)$, $\rho \not\equiv 0$. Remark that (27) implies that $j(t)/t^2$ is nonincreasing. We shall see further on (in the next section) that in the case where $j'(0) = 0$ and $\lim_{t \to \infty} j(t)/t^2 = 0$, this condition on M has to be assumed : in otherwords if M is not large enough then the minimization problem does not have any solution.

Example 3. We consider the case where : $N = 3$, $j(t) = \frac{3}{5} \; t^{5/3}$ (*).

Of particular interest are the following functions f :

(*) This case was brought to our attention by Prof. E. F. Redish and seems to arise in quantum mechanics.

(29) $\quad f(r) = + \frac{A}{r} e^{-\mu r} - \frac{A'}{r} e^{-\mu' r}$, with $\mu, \mu', A, A' > 0$.

Before looking at the special case where f is given by (29), let us remark that in order to apply Proposition IV.1-3 , one only needs to assume either (24) and

(25") $\qquad f \in M^p(\mathbb{R}^3)$, for some $\rho > \frac{3}{2}$

(or $\rho = \frac{3}{2}$ and $\|f\|_{M^{3/2}}$ small enough) ;

or (26) (with $p = \frac{3}{2}$).

In particular we have :

<u>*PROPOSITION IV.4.*</u> *Let* f *satisfy (26) (with* $p = \frac{3}{2}$*) and assume there exists* $\tilde{\rho}$ *in* $D(\mathcal{E})$, *such that* $\tilde{\rho}$ *is radial ,* $\tilde{\rho} \geq 0$ *a.e. ,* $\int \tilde{\rho} \leq M$ *and* $\mathcal{E}(\tilde{\rho}) < 0$ *. Then there exists* ρ *in* $D(\mathcal{E})$ *satisfying :*

i) ρ *is radial,* $\rho \geq 0$ a.e., $\int_{\mathbb{R}^3} \rho \, dx = M$.

ii) $\mathcal{E}(\rho) = \min\{\mathcal{E}(\tilde{\rho}) / \tilde{\rho}$ *radial* , $\tilde{\rho} \geq 0$, $\tilde{\rho} \in D(\mathcal{E})$, $\int \tilde{\rho} \leq M\}$.

Finally the existence of $\tilde{\rho}$ may be obtained using the methods of Corollary II.1. : in particular we find that

i) if, for r small enough, we have : $f(r) \geq Cr^{-k}$ for some $C, k > 0$, then for every M , there exists $\tilde{\rho}$ as in Proposition IV.4. (just compute $\mathcal{E}(\frac{1}{\sigma^3} \rho(\frac{x}{\sigma})))$;

ii) if there exists $\tilde{\rho}$ such that $\iint \tilde{\rho}(x) f(x-y) \tilde{\rho}(y) \, dxdy < 0$, $\tilde{\rho} \geq 0$ then for M large enough, the existence of $\tilde{\rho}$ as in Proposition IV.4 is insured (just use the fact that $\mathcal{E}(\theta\tilde{\rho}) \to -\infty$ if $\theta \to +\infty$) . Of course $\iint \rho(x) f(x-y) \rho(y) \, dxdy < 0$ being true for some ρ is necessary for the existence of $\tilde{\rho}$ as in Proposition IV.4. And it is quite obvious that it is satisfied for example if $\int_0^\infty f(r)r^2\,dr < 0$ (eventually $-\infty$).

We turn now to the particular case where f is given by (29) :

$$f(r) = \frac{A}{r}\,e^{-\mu r} - \frac{A'}{r}\,e^{-\mu' r}$$

Obviously (26) is satisfied. Then, a solution of the minimization problem exists as soon as there exists $\tilde{\rho}$ as specified in Proposition IV.4. By the preceding remarks , this is in particular satisfied for every $M > 0$, if $A > A'$ (or if $A = A'$ and $\mu' > \mu$) ; and for $M > M_0$, if $\frac{A}{\mu^2} - \frac{A'}{\mu'^2} < 0$.

In the next section, we consider the Euler equation associated to the special case $N = 3$, $f(x) = \frac{1}{|x|}$ or $f(x) = e^{-\mu|x|}\frac{1}{|x|}$.
Before that, we would like to point out that we deliberately ignored some aspects of these minimization problems :
i) regularity of solutions, ii) properties of compact support of solutions. These two aspects may be studied with the help of techniques due to [6], or by direct examination of the associated Euler equations ; we will not study such problems here.

V - THE EULER EQUATION

First, let us derive formally the Euler equation associated with (1) : a solution ρ of (1) must satisfy for some $\lambda \in \mathbb{R}$:

$$(30) \quad \begin{cases} j'(\rho) + V - B\rho \geq \lambda & \text{if} \quad \rho = 0 \\ j'(\rho) + V - B\rho = \lambda & \text{if} \quad \rho > 0 \end{cases} ,$$

where $B\rho(x) = \int \frac{1}{|x-y|}\rho(y)\,dy$, see [6] for the proof of this assertion (provided natural assumptions hold for j and V).

In all this section we will assume that j satisfies :

(31) j is a C^1 , positive, strictly convex function on $\mathbb{R}_+$ such that $j(0) = j'(0) = 0$. In particular j' is increasing and $\tilde{\beta} = (j')^{-1}$ exists, is continuous on $\mathbb{R}_+$ and $\tilde{\beta}(0) = 0$. Thus, (30) is equivalent to

$$(31') \quad \begin{cases} j'(\rho) - \max(u+\lambda-V,0) = 0 \quad \text{a.e. in } \mathbb{R}^3 , \\ u = B\rho , \end{cases}$$

or

$$(32) \quad \begin{cases} -\Delta u = 4\pi\tilde{\beta}((u+\lambda-V)^+) \quad \text{a.e. in } \mathbb{R}^3 , \\ u = B\rho , \end{cases}$$

since $\rho \in L^1 \cap L^{6/5}(\mathbb{R}^3)$, $u \in L^6(\mathbb{R}^3)$ and in some sense $u = 0$ at infinity. Thus, we are looking for a solution (λ,u) of

$$(33) \quad \begin{cases} -\Delta u = \beta((u+\lambda-V)^+) \quad \text{a.e. in } \mathbb{R}^3 , \; u(\infty) = 0 , \quad u \geq 0 , \\ \int_{\mathbb{R}^3} \beta((u+\lambda-V)^+)\,dx = M , \end{cases}$$

with $\beta(t) = 4\pi\,\tilde{\beta}(t)$.

The goal of this section is to look directly at (33), to see if there may exist a solution (λ,u) of (33) or equivalently a solution ρ

of (30) even when the energy $\mathcal{E}$ is not bounded from below. This phenomena occurs in a somewhat related problem : see [8], [15] for example. We only consider the case where $V \equiv 0$ (but for general V similar arguments can be made) : Section V.1 is devoted to the study of (33), while in section V.2 we consider the case when the potential $\frac{1}{|x|}$ is replaced by $\frac{1}{|x|} e^{-\mu|x|}$. Finally in section V.3 we study the limit case $j(t) = \frac{3}{4} t^{4/3}$.

Remark V.1. If V is given by a Coulomb type potential in $\mathbb{R}^3$:

$$V(x) = -\frac{C}{|x|}$$

for some $C > 0$, then (33) is equivalent to

$$(33') \qquad -\Delta\tilde{u} = \beta((\tilde{u}+\lambda)^+) + 4\pi C\delta_0 \quad \text{in } \mathcal{D}'(R^3),$$

where $\tilde{u} = u + \frac{C}{|x|}$, thus $\tilde{u} \geq 0$ a.e.

If we take β to be a pure power : $\beta(t) = t^{1/\gamma}(\gamma < 1)$, then by the results of [30], there is no solution of (33') if $\gamma \leq \frac{1}{3}$; this implies that (1) does not have any solution for $j(t) = Ct^{1+\gamma}$ and $\gamma \leq \frac{1}{3}$. If $\gamma > \frac{1}{2}$, we may apply our minimization techniques : for example the fact that the constraint is saturated follows from the method given in Remark IV.10. We believe that, combining methods of [30] and of the next section, the remaining case $\frac{1}{3} < \gamma \leq \frac{1}{2}$ can be treated (note that for $\gamma \leq \frac{1}{2}$, the energy $\mathcal{E}$ is not bounded from below) ; and we hope to be able to treat this case in a future study. We now restrict our attention to $V \equiv 0$ (smooth, positive functions V may be treated by similar techniques).

V.1. Solving the Euler equation

We thus consider the problem : find (u,λ) solution of

$$(33)\quad \begin{cases} -\Delta u = \beta((u+\lambda)^+) \text{ a.e. in } \mathbb{R}^N \text{ , } u \geqslant 0 \\ \int_{\mathbb{R}^N} \beta((u+\lambda)^+)dx = M > 0 \text{ ,} \end{cases}$$

where $\lambda \in \mathbb{R}$, M is given and the exact class where we look for u is

$$H = \hat{\mathcal{D}}_r^{1,2}(\mathbb{R}^N) = \{u \text{ , } u \text{ is radial, } \nabla u \in L^2(\mathbb{R}^N) \text{ , } u \in L^{2N/(N-2)}(\mathbb{R}^N)\} \text{ .}$$

In all what follows, we assume $N \geqslant 3$. The fact that we restrict our attention to radial solutions may be justified by the results of [22], [23] . Remark also that λ necessarily is nonpositive : $\lambda \leqslant 0$. In conclusion, we look for (u,λ) in $H \times (-\infty,0]$ solution of (33).

A direct application of the results of [11] gives immediately :

<u>*THEOREM V.1.*</u> *Let* j *satisfy (31) and*

$$(34)\qquad \lim_{t \to \infty} j'(t)t^{-(N-2)/(N+2)} = \infty \text{ ;}$$

then, for every $\lambda < 0$ *, there exists a solution* u *of*

$$(33')\qquad -\Delta u = \beta((u+\lambda)^+) \text{ a.e. in } \mathbb{R}^N \text{ , } u > 0 \text{ , } u \in H \text{ ;}$$

in addition u *is decreasing ,* $u \in C^2(\mathbb{R}^N)$, $\beta((u+\lambda)^+) \in L^1(\mathbb{R}^N)$ *and there exists* $R_o > 0$ *such that*

$$\begin{cases} u(R_o) = -\lambda \quad , \quad u'(R_o) = (N-2)\lambda R_o \\ u(r) = -\lambda\left(\frac{R_o}{r}\right)^{N-2} \quad , \text{ for } r \geqslant R_o \text{ .} \end{cases}$$

In particular, there exists at least one $M > 0$ *such that (33) has a solution* (u,λ) *in* $H \times (-\infty,0)$.

<u>Remark V.2.</u> (34) is equivalent to : $\lim_{t \to \infty} \beta(t)t^{-(N+2)/(N-2)} = 0$ and one knows (see [32], [36] or [12]) that $\frac{N+2}{N-2}$ is the best exponent in order to solve (33'), this proves that (34) is nearly optimal.

The exponent given by Theorem II.1 (or its immediate extension to higher dimensions) is $\frac{N}{N-2}$: we see that looking directly to the Euler equation improves the conditions on j and that there are solutions of the Euler equation even with $\mathcal{E}$ unbounded below.

We will see further on that in general one cannot say more for the existence of one M : indeed we will show that for $j(t) = t^{4/3}$ (in $\mathbb{R}^3$) there is <u>only one M</u> such that (33) has a solution. Nevertheless, by a more precise analysis, we will be able , with suitable restrictions on β(or j') to generalize this conclusion.

<u>Remark V.3</u>. Because of the prior reduction the corresponding density ρ is $\rho = C_N\beta((u+\lambda)^+)$, where C_N is a constant depending only on N . And obviously , $\rho \in C(\mathbb{R}^N)$, is radial, nonincreasing and with a compact support equal to $B_{R_0} = \{|\xi| \leq R_0\}$.

<u>Proof of Theorem V.1</u>. The first part of the Theorem is an immediate consequence of the general existence results of [11] . Now, since u is decreasing there exists a unique R_0 such that $u(R_0) = -\lambda$ and for $r \geq R_0$ we have : $u(r) \leq -\lambda$ and thus $-\Delta u = 0$ if $|x| \geq R_0$. But, this implies $r^{N-1}u'(r) = R_0^{N-1}u'(R_0)$ for $r \geq R_0$ and since $u(r) \downarrow 0$ as $r \uparrow +\infty$, we conclude by a straightforward computation. (Remark that in [31], another proof is given reducing (33) to a problem in a ball and using then the results of [32]).

We now prove the claim made on Remark V.2. concerning the case $j(t) = t^{4/3}$: actually we consider the case of $\beta(t) = t^\alpha$ (corresponding to $j(t) = Ct^{1+1/\alpha}$) , for simplicity we restrict ourselves to N=3.

PROPOSITION V.1. *We assume that* $\beta(t) = t^{\alpha}$ *, with* $0 < \alpha < \infty$ *.* *If* $0 < \alpha < 5$ *and if* $\alpha \neq 3$ *, then for all* $M > 0$ *, there exists a unique* $(u,\lambda) \in H \times (-\infty,0)$ *solution of (33). Moreover* $u \in C^2(\mathbb{R}^N)$ *and* u *is decreasing . If* $\alpha = 3$ *, then there exists a unique* M_0 *such that (33) has a solution in* $H \times (-\infty\ , 0)$ *: in addition for all* λ *,* (33') *has a unique solution* u_λ *(with the same properties as in Theorem V.1) and* $M_0 = \int_{\mathbb{R}^3} \beta((u_\lambda+\lambda)^+)dx$ *is independent of* λ *.*

Remark V.3. It is easy to prove if $\alpha > 5$, then (33) has no solutions in $H \times (-\infty,0)$ for every $M > 0$.

Before going into the proof of Proposition V.1. , let us make some preliminary reductions (independent of the choice of β).

Indeed, instead of looking for (u,λ), we are going to look for (u,R_0) as in Theorem V.1.

More precisely suppose we have a solution v of

$$(35)\quad \begin{cases} -\Delta v = \beta(v) \text{ in } B_R\ , \ v > 0 \text{ in } B_R \\ v = 0 \text{ on } \partial B_R\ , \ v \in W^{2,q}(B_R)\ (\forall\ q < \infty) \\ \int_{B_R} \beta(v)\,dx = M\ , \end{cases}$$

for some $R > 0$, then by [22], v is radial decreasing and if we set

$$\begin{cases} v(r) = -\dfrac{R^{N-1}v'(R)}{r^{N-2}} + Rv'(R)\ , \quad \text{for } r \geqslant R \\ \lambda = Rv'(R) < 0\ , \\ u(x) = v(|x|) - \lambda\ , \text{ for all } x \text{ in } \mathbb{R}^N\ ; \end{cases}$$

then (u,λ) is a solution of (33) in $H\times(-\infty,0)$.

On the other hand if (u,λ) is a solution of (33) in $H \times (-\infty,0)$, then it is easy to see that u is decreasing and if R is the unique solution of $u(r) = -\lambda$, setting $v = (u+\lambda)$, we see that (35) is satisfied for such a v and such a R.

This remark (which we will also use later) being made, we turn now to the proof of Proposition V.1.

<u>Proof of Proposition V.1</u>. (see also [31]).

We consider first the case when $\alpha \neq 1$; then let us denote by v the unique solution of

$$- \Delta v = \beta(v) \text{ in } B_1 , \; v > 0 \text{ in } B_1 ,$$

$$v = 0 \text{ on } \partial B_1 .$$

The existence for $\alpha > 1$ follows from [2] (for instance) while the uniqueness is proved in [22] (the case $\alpha < 1$ is well-known).

Obviously $v_R(x) = R^{2(1-\alpha)} v(\frac{x}{R})$ is the unique solution of

$$- \Delta v_R = \beta(v_R) \text{ in } B_R , \; v_R > 0 \text{ in } B_R , \; v_R = 0 \text{ on } \partial B_R .$$

Now let us compute

$$\lambda = R v_R'(R) = R^{2/(1-\alpha)}$$

and $\int_{B_R} \beta(v_R)\,dx = R^{2\alpha/(1-\alpha)} R^3 \int_{B_1} \beta(0)\,dx = R^{\frac{3-\alpha}{1-\alpha}} \int_{B_1} \beta(0)\,dx$,

and this proves the Proposition (at least for $\alpha \neq 1$).

If $\alpha = 1$, then R in (35) is prescribed by : $\lambda_1(R_o) = +1$ where $\lambda_1(R)$ is the first eigenvalue of $-\Delta$ over $H_o^1(B_R)$. In addition v in (35) is prescribed by $\int_{B_{R_o}} v\,dx = M$ and we conclude.

We now give a few partial results which give a more precise description of the set of M such that (33) has a solution (u,λ) or equivalently such that (35) has a solution (v,R). We insist on the fact that this study is a priori difficult in view of Proposition V.1. and that the results we give are only partial ones. The first result, in some sense, represents the counterpart of Theorem II.1 where the case $\overline{\lim}_{t\to\infty} \beta(t)t^{-3} < \infty$ is considered.

<u>*PROPOSITION V.2.*</u> *Under assumptions (34) and*

(36) $$\lim_{t\to\infty} \beta(t)\, t^{-N(N-2)} = \infty \quad ,$$

(37) $$\overline{\lim_{t\to 0}}\ \beta(t)\, t^{-1} < \infty ,$$

and

(38) $$\overline{\lim_{t\to\infty}}\ \frac{t\beta(t)-\theta\gamma(t)}{t^2\beta(t)^{2/N}} < 0 \quad , \text{ with } \gamma(t) = \int_0^t \beta(s)\,ds \text{ and } 0 < \theta < \frac{2N}{N-2}$$

then there exists $M_o \in (0, +\infty]$ *such that for all* $M \in (0,M_o)$ *there exists a solution* (u,λ) *of (33) in* $H \times (-\infty,0)$.

<u>Remark V.4</u>. Assumptions (34), (36) and (37) are just assumptions on the shape of β which are quite natural. On the other hand (38) is a technical assumption (we believe it is not necessary) which insures that all solutions of

(39) $$-\Delta v = \beta(v) \text{ in } B_R ,\ v = 0 \text{ on } \partial B_R ,\ v > 0 \text{ in } B_R$$

satisfy $\|v\|_{L^\infty(B_R)} \leq C(R_o,R_1)$, if $0 < R_o \leq R \leq R_1 < +\infty$.

This a priori estimate is proved in [13].

<u>Proof of Proposition V.2</u>. We begin with a remark : we only need to prove

1) for every $\varepsilon > 0$ there exists a connected component $\mathcal{C}_\varepsilon$ in $\mathbb{R} \times C_b(\mathbb{R}^N)$ such that :

i) if $(R,v) \in \mathcal{C}_\varepsilon$, then v solves (39) ,

ii) $\{R, \exists v, (R,v) \in \mathcal{C}_\varepsilon\} \supset [\varepsilon, R_o - \varepsilon]$, where R_o is some fixed positive real (eventually infinite then $R_o - \varepsilon$ means $\frac{1}{\varepsilon}$).

2) Let $K_R = \{v$ solutions of (39)$\}$ and $m_R = \sup\limits_{v \in K_R} \int_{B_R} \beta(v)\,dx$, then $m_R \underset{R \to 0}{\to} 0$.

These two claims will be proved in two steps. We first define R_o by

$$\lambda_1(R_o) = \overline{\lim_{t \to 0}} \ \beta(t)t^{-1} \quad \text{(if this limit is } 0 \text{ , then } R_o = \infty).$$

<u>Step 1</u>. We are going to use a topological degree argument and a theorem due to Leray and Schauder [25] : our argument is reminiscent of a similar argument used in [9]. Let us first transform (39) , by a simple rescaling (39) is equivalent to

$$(40) \qquad -\Delta v = R^2\beta(v) \ \text{ in } B_1 \text{ , } v = 0 \ \text{ on } \partial B_1 \text{ , } v > 0 \ \text{ in } B_1 \text{ .}$$

In view of the choice of R_o , we have for all $R < R_o$:

$$R^2 \ \overline{\lim_{t \to 0}} \ \frac{\beta(t)}{t} = R^2\lambda_1(R_o) < R_o^2\lambda_1(R_o) = \lambda_1(1) \text{ .}$$

Let $\varepsilon > 0$ be fixed, because of the preceding inequality, we have

$$\|v\|_{L^\infty(B_1)} > \alpha > 0$$

for every v solution of (40) with $R \leqslant R_o - \varepsilon$ and for some $\alpha > 0$; see [16] or [13] for a proof of that observation.

On the other hand, by the result recalled in Remark V.4. we have

$$\|v\|_{L^\infty(B_1)} < C$$

for every v solution of (40) with $\varepsilon \leq R \leq R_o - \varepsilon$, for some $C > 0$.

Now, we introduce the following compact operator F_R from $C(\overline{B}_1)$ into $C(\overline{B}_1)$: $F_R u = v$ is defined by

$$(41) \quad \begin{cases} -\Delta v = R^2\beta(u) \text{ in } B_1 , \; v = 0 \text{ on } \partial B_1 , \\ v \in W^{2,q}(B_1) , \; \forall\, q < \infty ; \end{cases}$$

where β is defined on $\mathbb{R}$ by $\beta(t) = 0$ if $t \leq 0$.

Suppose we have proved that the topological degree of F_R on the open set $Q = \{v \in C(\overline{B}_1) , \alpha < \| v \|_{L^\infty} < C\}$ is different from 0 and more precisely suppose we have proved

$d(I - F_R, Q, 0) = -1$, for all $R \in [\varepsilon, R_o - \varepsilon]$, then by a fundamental result of Leray and Schauder [25], the first claim is proved (extending functions which are zero on ∂B_R by zero outside B_R) .

Thus, it just remains to compute this degree. First, let us compute $d(I-F_R, Q_1, 0)$ where $Q_1 = \{u \in C(\overline{B}_1), \| u \|_{L^\infty} < \alpha\}$. In view of the estimate recalled above and its proof (see [16]) we see that $d(I-tF_R, Q_1, 0)$ is well defined for $0 \leq t \leq 1$ and is thus independent of t :

$$d(I-F_R, Q_1, 0) = d(I, Q_1, 0) = +1 ,$$

since $0 \in Q_1$.

Now, we want to compute $d(I-F_R, Q_2, 0)$ where $Q_2 = \{ u \in C(\overline{B}_1) , \| u \|_{L^\infty} < C\}$, with C choosen large enough for this degree to be well defined in view of the a priori estimates .

Let $\mu > \lambda_1(B_1)$, it is easy to check that the proof in [13] gives the following estimate : $\| u \|_{L^\infty} < C$ for all u solution of

$$-\Delta u = tR^2\beta(u) + (1-t)(\mu u^+ + 1) \text{ in } B_1 , \; u > 0 \text{ in } B_1 , \; u = 0 \text{ on } \partial B_1 ;$$

where t is any real in $[0,1]$. Thus, if F is the compact operator defined on $C(\overline{B}_1)$ by $Fu = v$ is the solution of

$$-\Delta v = \mu u^+ + 1 \quad \text{in } B_1 \,, \quad v = 0 \quad \text{on } \partial B_1 \;;$$

then, we have :

$$d(I-F_R, Q_2, 0) = d(I-F, Q_2, 0) \,.$$

But, if $Fu = u$ then u by the maximum principle satisfies

$$-\Delta u = \mu u + 1 \quad \text{in } B_1 \,, \; u > 0 \quad \text{in } B_1 \,, \; u = 0 \quad \text{on } \partial B_1 \,.$$

Since we have chosen $\mu > \lambda_1(1)$, this is impossible and F has no fixed point in Q_2, thus : $d(I-F, Q_2, 0) = 0$.

In conclusion, we have, since $Q = Q_2 - \overline{Q}_1$,

$$\begin{aligned} d(I-F_R, Q, 0) &= d(I-F_R, Q_2, 0) - d(I-F_R, Q_1, 0) \\ &= -1 \,. \end{aligned}$$

<u>Step 2.</u> We now prove that $m_R \xrightarrow[R \to 0]{} 0$. Let v_R^1 be a positive eigenfunction of $-\Delta$ corresponding to $\lambda_1(R)$: we normalize it by

$$\int_{B_R} v_R^1 \, dx = +1 \,.$$

Thus, in particular, on $B_{R/2}$ we have : $v_R^1(x) \geq \frac{\alpha}{R^N} > 0$, for some fixed $\alpha > 0$.

Now multiply (39) by v_R^1 and integrate twice by parts, we obtain :

$$\frac{C_1}{R^2} \int_{B_R} v v_R^1 \, dx = \int_{B_R} \beta(v) v_R^1 \, dx \,, \text{ for some } C_1 > 0 \,.$$

Because of (36), for every $K > 0$, there exists t_o ($= t_o(K)$ such that $\beta(t) \geq K t^{N/N-2}$ if $t \geq t_o$.

In particular $\beta(t) \geq \frac{2C_1}{R^2} t$ if $t \geq \max(t_o \,, (\frac{2C_1}{K})^{\frac{N-2}{2}} \frac{1}{R^{N-2}})$

This implies

$$\frac{C_1}{R^2}\int_{B_R} vv_R^1\,dx > \int_{B_R\cap\{v>t_1\}} \beta(v)v_R^1\,dx >$$

$$> \frac{2C_1}{R^2} \int_{B_R\cap\{v>t_1\}} vv_R^1\,dx$$

and

$$\frac{1}{R^2}\int_{B_R\cap(v>t_1)} vv_R^1\,dx < \frac{1}{R^2}\int_{B_R\cap(v<t_1)} vv_R^1\,dx < \frac{t_1}{R^2} .$$

Thus,

$$\int_{B_R} \beta(v)v_R^1\,dx < \frac{2C_1t_1}{R^2}$$

and

$$\int_{B_{R/2}} \beta(v)\,dx < \frac{2C_1}{\alpha} R^{N-2}t_1 < \frac{2C_1}{\alpha} R^{N-2}\left(t_o+\left(\frac{2C_1}{K}\right)^{\frac{N-2}{2}} \frac{1}{R^{N-2}}\right)$$

$$< C_2t_oR^{N-2} + C_3K^{-(N-2)/2} .$$

Since all solutions of (39) are radial and decreasing (cf. [22]), we have

$$\int_{B_R} \beta(v)dx = C_N\int_0^R \beta(v(r))r^{N-1}\,dr = C_N\int_0^{R/2} \beta(v(r))r^{N-1}\,dr$$

$$+ C_N\int_{R/2}^R \beta(v(r))r^{N-1}\,dr$$

$$< \int_{B_{\frac{R}{2}}} \beta(v)dx + C_N2^{N-1}\int_{R/2}^R \beta(v(\tfrac{r}{2}))(\tfrac{r}{2})^{N-1}\,dr$$

since β is increasing and v is decreasing, and therefore

$$\int_{B_R} \beta(v)dx < (1+2^N)\int_{B_{R/2}} \beta(v)dx .$$

In conclusion, we have proved :

$$\int_{B_R} \beta(v)dx < C_4\ t_o(K)R^{N-2} + C_5K^{-(N-2)/2} .$$

Choosing first K small and then R small, the claim is proved.

Remark V.5. We use (36) only in Step 2, and to obtain similar results with other types of growth at infinity, one would have to make similar arguments as in step 2 (maybe for $R \to \infty$).

Another example of the same general method is the following result which we will not prove here for the sake of simplicity.

PROPOSITION V.2. We assume that β *satisfies :*

(42) $$\lim_{t \to \infty} \beta(t)t^{-1} = 0 ,$$

(42') $$\lim_{t \to \infty} \beta(t)t^{-1} \text{ exists in } (0, \infty] .$$

then for all $M > 0$ *, there exists* (u,λ) *in* $H \times (-\infty,0)$ *solution of (33).*

Remark V.6. This can also be proved by a simple use of bifurcation results.

V.2. Another type of potential

We now consider the case where in (1), $\frac{1}{|x|}$ is replaced by $e^{-\mu|x|}\frac{1}{|x|}$, then (33) has to be replaced by

$$(44)\quad \begin{cases} -\Delta u + \mu^2 u = \beta((u+\lambda)^+) \text{ a.e. in } \mathbb{R}^3 , \ u > 0 \text{ in } \mathbb{R}^3 , \\ u \in C^2(\mathbb{R}^3) \cap H , \quad \lambda \leq 0 ; \\ \int_{\mathbb{R}^3} \beta((u+\lambda)^+)dx = M . \end{cases}$$

Similar arguments to those developped in the preceding section can be made to prove that under very general assumptions (similar to those encountered in V.1) the set of M such that a solution (u,λ) of (44) exists is of the form (M_o,∞) for some $M_o > 0$. In addition u is decreasing and if $\lambda > 0$, we have that $u(x) = C\, e^{-\mu|x|} \frac{1}{|x|}$ for $|x|$ large enough. In order to restrict the length of the paper, we will not prove here such results but we will just examine some general example proving that for M small enough, there cannot exist a solution of (44) and thus there does not exist a solution of the associated minimization problem (see Example 2 in section IV.3).

PROPOSITION V.4. *If we assume :*

(45) $$\lim_{t \to 0} \frac{\beta(t)}{t} < \mu^2 \quad ,$$

(46) $$\lim_{t \to 0} \beta(t) t^{-\alpha} = 0 \text{ , } \textit{with } \alpha < \frac{N}{N-2} \; ;$$

then there exists M_o *such that , for all* $M \in (0,M_o)$ *, there exist no solution of (44).*

Proof of Proposition V.4. Assume there exist (u_n,λ_n) solution of (14) with $M = \frac{1}{n}$ and let us derive a contradiction . We first prove that u_n converges to 0 in $L^\infty(\mathbb{R}^N)$.

Indeed we have

$$0 \leq - r^{N-1} \frac{du_n}{dr} = \int_0^r s^{N-1} \{\beta((u_n+\lambda_n)^+) - \mu^2 u_n\} ds$$

and therefore

(47) $$\left|\frac{du_n}{dr}\right| \leq \frac{\varepsilon_n}{r^{N-1}} \quad \text{for} \quad r > 0 \text{ ,}$$

where $\varepsilon_n \xrightarrow[n \to \infty]{} 0$.

Since we assume (45), by [36] (see also [11], [12]) we know that

u_n is exponentially small at infinity and we deduce

$$|u_n(r)| \leq \frac{\varepsilon_n}{r^{N-2}} \quad \text{for} \quad r > 0 . \tag{48}$$

Now let $R > 0$ be fixed, on B_R we have

$$\begin{cases} -\Delta u_n + \mu^2 u_n = f_n & \text{in } B_R \\ u_n|_{\partial B_R} \leq \dfrac{\varepsilon_n}{R^{N-2}} , \end{cases}$$

where $f_n \in L^1(B_R)$ and $f_n \xrightarrow[L^1(B_R)]{n\to\infty} 0$

By wellknown regularity results, this implies :

$$u_n \xrightarrow[L^q(B_R)]{n\to\infty} 0 , \quad \text{for all } q < \frac{N}{N-2} .$$

But $0 \leq \beta((u_n + \lambda_n)^+) \leq \beta(u_n)$ and by an easy bootstrap argument (using (46)) we obtain : $u_n \xrightarrow[L^\infty(B_R)]{n\to\infty} 0$.

This, together with (48), implies $u_n \xrightarrow[L^\infty(\mathbb{R}^N)]{n\to\infty} 0$.

Now, from the maximum principle (since $u_n(0) = \| u_n \|_{L^\infty}$) we deduce :

$$\mu^2 u_n(0) \leq \beta((u_n + \lambda_n)^+)(0) \leq \beta(u_n(0)).$$

In view of (45), we have a contradiction ; and that contradiction proves the Proposition.

<u>Remark V.7</u>. This proposition, compared with Theorem II.1 , shows that if we replace $\frac{1}{|x|}$ by $e^{-\mu|x|}\frac{1}{|x|}$, then in general, we have to assume $M > M_0$ in order to solve (1) (or (33)-(44)). The fact that it is enough to assume $M > M_0$ is studied in Section IV.

V.3. The limit case : $j(t) = \frac{3}{4} t^{4/3}$

We have seen (cf V.1. and section II) that $j(t) = \frac{3}{4} t^{4/3}$ is a limiting case. The goal of this section is to explain exactly what happens in that case. First, let us introduce some notations : by Proposition V.1, we know there exists M_o such that, for every $\lambda < 0$, the equation

$$- \Delta u_\lambda = 4\pi (u_\lambda + \lambda)^{+3} \text{ a.e. in } \mathbb{R}^3 , \; u_\lambda \geq 0 \tag{49}$$

has a unique radial solution u_λ in H (having the properties listed in Theorem V.1) and in addition we have for all $\lambda < 0$:

$$\int_{\mathbb{R}^3} (u_\lambda + \lambda)^{+3} dx = M_o ,$$

for some $M_o > 0$, independent of λ.

Then, we have the following result for the corresponding minimization problem :

PROPOSITION V.5. *Let* $M > 0$.

i) if $M < M_o$, *then for every* ρ *in* $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ *such that* $\| \rho \|_{L^1} \leq M$, *we have*

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^3} \frac{3}{4} |\rho|^{4/3} dx - \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dxdy \geq 0$$

and $\mathcal{E}(\rho) = 0$ *implies* $\rho \equiv 0$.

ii) $M = M_o$, *then for every* ρ *in* $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ *such that* $\| \rho \|_{L^1} \leq M$, *we have* $\mathcal{E}(\rho) \geq 0$ *and* $\mathcal{E}(\rho) = 0$ *if and only if* $\rho \equiv 0$ *or* $\rho(x) = (u_\lambda + \lambda)^{+3}(x - x_o)$, *for some* x_o *in* $\mathbb{R}^3$ *and for some* $\lambda < 0$.

iii) if $M > M_o$ *, then* $\mathcal{E}$ *is unbounded below on the set*

$$\{\rho \in L^1 \cap L^{4/3} \ , \ \|\rho\|_{L^1} \leq M\} \ .$$

In other words the minimization problem has no solution for $M \neq M_o$ and for $M = M_o$ its solutions are exactly (up to a translation) $\rho_\lambda(x) = (u_\lambda+\lambda)^{+3}$. We will see in the proof below that $M_o = (\frac{3}{2C_1})^{3/2}$ (C_1 is given by (5')).

Proof of Proposition V.5. Let $M_1 = (\frac{3}{2C_1})^{3/2}$. Obviously if $M \leq M_1$, we have $\mathcal{E}(\rho) \geq 0$, for ρ in $L^1 \cap L^{4/3}$, $\|\rho\|_{L^1} \leq M$.

On the other hand a simple argument proves that for $M > M_1$, $\mathcal{E}$ is unbounded below on $\{\rho \in L^1 \cap L^{4/3} \ , \ \|\rho\|_{L^1} \leq M\}$.

Now let us assume that $\mathcal{E}(\rho) = 0$ for some ρ in $L^1 \cap L^{4/3}$, $\|\rho\|_{L^1} \leq M_1$. Since, by well-known results, $\mathcal{E}(\rho^*) < \mathcal{E}(\rho)$ except if $\rho(x) = \rho^*(x-x_o)$ (for some x_o in $\mathbb{R}^3$) – where as above ρ^* denotes the spherical decreasing rearrangement of ρ – we deduce that (up to a translation) $\rho = \rho^*$. Now, if $\|\rho\|_{L^1} < M_1$, we get (setting $\rho_\sigma(x) = \rho(\frac{x}{\sigma})$)

$$\frac{d}{d\sigma}\mathcal{E}(\rho_\sigma)\big|_{\sigma=+1} = 3\int_{\mathbb{R}^3}\frac{3}{4}\rho^{4/3}\,dx - 5\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{1}{2}\rho(x)\rho(y)|x-y|^{-1}dxdy$$

$$= -\iint_{\mathbb{R}^3\times\mathbb{R}^3}\rho(x)\rho(y)|x-y|^{-1}dxdy$$

Then if $\rho \not\equiv 0$, this would imply $\mathcal{E}(\tilde{\rho}) < 0$, for some $\tilde{\rho}$ in $L^1 \cap L^{4/3}$ with $\|\tilde{\rho}\|_{L^1} < M_1$ and this is impossible.

Next, if $\|\rho\|_{L^1} = M_1$, ρ satisfies the following Euler equation

$$\begin{cases} -\Delta u = 4\pi(u+\lambda)^{+3} \text{ a.e. in } \mathbb{R}^3 \ , \ u \in H \ , \ u = u^* > 0 \ , \ \lambda \in \mathbb{R} \ . \\ \rho = (u+\lambda)^{+3} \in L^1 \cap L^{4/3} \ . \end{cases}$$

This implies easily $\lambda \leq 0$. And applying Pohozaev identity (see [12]) we ,o, in

$$(50) \qquad 0 < \int_{R^3} |\nabla u|^2 dx = 6\pi \int_{R^3} (u+\lambda)^{+4} dx = 4\pi \int_{R^3} u(u+\lambda)^{+3}\, dx \ ;$$

therefore $\lambda < 0$. And this implies : $u = u_\lambda$, $\rho = (u_\lambda+\lambda)^{+3}$.

To conclude the proof of the proposition, we just need to recall the result proved in [6]: for $M = M_1$, there exists ρ satisfying $\mathcal{E}(\rho) = 0$, $\rho \in L^1 \cap L^{4/3}$ and $\|\rho\|_{L^1} = M_1$. Thus $M_1 = M_0$. And it just remains to check that for all $\lambda < 0$.

$$\mathcal{E}(\rho_\lambda) = 0 \quad , \quad \text{where } \rho_\lambda = (u_\lambda+\lambda)^{+3} \ .$$

But in view of (50) , we have

$$6 \int_{R^3} \rho_\lambda^{4/3}\, dx = 4 \int_{R^3} \rho_\lambda(x) u_\lambda(x)\, dx = 4 \iint_{R^3\times R^3} \rho_\lambda(x)\rho_\lambda(y)|x-y|^{-1} dxdy$$

and this gives the desired equality.

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In this paper, minimization problems in $L^1(\mathbb{R}^3)$ are considered. These problems arise in astrophysics for the determination of equilibrium configurations of axially symmetric rotating fluids (rotating stars). Under nearly optimal assumptions a minimizer is proved to exist by a direct variational method, which uses heavily the symmetry of the problem in order to get some compactness. Finally, by looking directly at the Euler equation, we give some existence results (of solutions of the Euler equation) even if the infimum is not finite.

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