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(6) MULTIVARIATE SPLINES

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UNIVERSITY OF WISCONSIN - MADISON
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MULTIVARIATE SPLINES

Klaus Höllig

Technical Summary Report #2188
March 1981

ABSTRACT

It is shown that the span of a collection of multivariate B-splines, $S_k := \operatorname{span}_{j \in J} M(\cdot|x^{j_0},\cdots,x^{j_{k+m}})$, contains polynomials of total degree $\leq k$ if the index set $\{j = (j_0,\cdots,j_{k+m})\}_{j\in J}$ has a certain combinatorial structure, in particular the knots $\{x^\nu\} \subset \mathbb{R}^m$ can be chosen essentially arbitrarily. Under mild assumptions on the distribution of the knots, a dual basis $\{\lambda_j\}_{j\in J}$ for S_k is constructed which has local support, i.e., $\operatorname{supp} \lambda_j \subset \operatorname{supp} M_j$, and the condition number of the B-spline basis is estimated. This leads to good local approximation schemes.

AMS(MOS) Subject Classification: 41A15, 41A63

Key Words: Multivariate, B-splines, spline functions, dual basis, condition number

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SIGNIFICANCE AND EXPLANATION

Spline approximation provides good approximation methods in one variable. This is to a great deal due to the properties of the B-spline basis. Recently, multivariate B-splines $M(\cdot|x^0,\cdots,x^n)$ have been defined and their basic recurrence relations derived. However, there was no satisfactory way to choose for a given set of knots $\{x^\nu\} \subset R^m$ a span of B-splines that contains polynomials; this property is essential for good approximation properties.

In this report we give a sufficient condition in terms of the index set $\{j = (j_0,\cdots,j_{k+m})\}_{j\in J}$ for $S_k := \operatorname{span}_{j\in J} M(\cdot|x^{j_0},\cdots,x^{j_{k+m}})$ to contain locally polynomials of total degree $\leq k$. Since the knots may be chosen almost arbitrarily, the resulting space of multivariate splines S_k has the local flexibility familar from the univariate theory. We show that under mild assumptions on the distribution of the knots the L_p-norm of a spline $s \in S_k$ is equivalent to the ℓ_p-norm of its normalized B-spline coefficients. As an application, we indicate how local approximation schemes can be constructed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MULTIVARIATE SPLINES

Klaus Höllig

0. Introduction

The purpose of this report is to introduce a class of multivariate non tensor product spline functions and study its basic properties. The basis for our approach are recent results of C. A. Micchelli and W. Dahmen [8-12, 21, 22] on multivariate B-splines $M(\cdot|x^0,\cdots,x^n)$ introduced by C. de Boor in [3]. We consider a space of spline functions $S_k = \operatorname{span}_{j\in J} M(\cdot|x^{j_0},\cdots,x^{j_n})$, where the index set J has a certain combinatorial structure and the knots $x^\nu \in \mathbb{R}^m$ may be chosen almost arbitrarily. It is shown that S_k contains polynomials of total degree $k = n-m$ which is the basis for good approximation properties. Under mild assumptions on the distribution of the knots we construct a dual basis λ_j, $j \in J$, for S_k with local support, i.e. $\operatorname{supp} \lambda_j \subset \operatorname{supp} M_j$. Using extensions of the functionals λ_j to L_p the condition number of the B-spline basis is estimated which turns out to be essentially independent of the distribution of the knots. As an application we obtain the basic error estimate for local approximation schemes.

Since the spaces S_k have the local flexibility familiar from the univariate theory one might expect applications to finite element methods, smoothing of data and surface fitting.

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1. B-Splines

The geometric interpretation of the univariate B-spline by H. B. Curry and I. J. Schoenberg [7] gave rise to the following generalization.

Definition 1. C. de Boor [3]. Let $x^0,\cdots,x^n$ be any points in $\mathbb{R}^m$, $n > m$, which span a proper convex set and choose any simplex $\sigma((x^0,\tilde{x}^0),\cdots,(x^n,\tilde{x}^n))$ with vertices $(x^\nu,\tilde{x}^\nu) \in \mathbb{R}^n$. The B-spline $M(\cdot|x^0,\cdots,x^n):\mathbb{R}^m \to \mathbb{R}$ corresponding to the "knots" x^ν is defined by

$$(1) \qquad M(x|x^0,\cdots,x^n) = \frac{\mathrm{vol}_{n-m}\{\tilde{x} \in \mathbb{R}^{n-m} | (x,\tilde{x}) \in \sigma((x^0,\tilde{x}^0),\cdots,(x^n,\tilde{x}^n))\}}{\mathrm{vol}_n \sigma((x^0,\tilde{x}^0),\cdots,(x^n,\tilde{x}^n))}$$

where for $n=m$ $\mathrm{vol}_0(A) := \begin{cases} 0\,, & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases}$.

Denote by $\Delta_n = \{(\lambda_0,\cdots,\lambda_n) | \sum_{\nu=0}^n \lambda_\nu = 1,\ \lambda_\nu > 0,\ \nu = 0,\cdots,n\}$ the standard n-dimensional simplex. A simple calculation shows [22, p. 3,4] that for $f \in C(\mathbb{R}^m)$

$$(2) \qquad n! \int_{\Delta_n} f(\sum_{\nu=0}^n \lambda_\nu x^\nu)\,d\lambda_1\cdots d\lambda_n = \int_{\mathbb{R}^m} M(x|x^0,\cdots,x^n) f(x)\,dx$$

a relation used by C. A. Micchelli to define the multivariate B-splines [21]. Equation (2) defines M even for $n < m$ as a positive measure supported in the convex hull of the points $x^0,\cdots,x^n$. From (2) the Fourier transform of M can be computed via the Hermite Genocchi formula for divided differences [23, p. 16]

$$(3) \qquad [t_0,\cdots,t_n]f = \int_{\Delta_n} f^{(n)}(\sum_{\nu=0}^n \lambda_\nu t_\nu)\,d\lambda_1\cdots d\lambda_n\,.$$

Applying Parseval's identity $\int f\hat{g} = \int \hat{f}g$, $\hat{h}(y) = \int h(x)e^{-ixy}dx$, to the equation (2) we have

$$\int_{\mathbb{R}^m} \hat{M}(x|x^0,\cdots,x^n)g(x)\,dx = \int_{\mathbb{R}^m} M(x|x^0,\cdots,x^n)\hat{g}(x)\,dx =$$

$$n! \int_{\Delta_n} \hat{g}\Big(\sum_{\nu=0}^{n} \lambda_\nu x^\nu\Big) d\lambda_1 \cdots d\lambda_n = n! \int_{\Delta_n} \int_{\mathbb{R}^m} \exp\Big(\sum_{\nu=0}^{n} \lambda_\nu(-ixx^\nu)\Big) g(x)\,dx\, d\lambda_1 \cdots d\lambda_n =$$

$$n! \int_{\mathbb{R}^m} [-ixx^0,\cdots,-ixx^n]_t\, e^t\, g(x)\,dx$$

which implies that the Fourier transform of the B-spline is given by [21]

(4) $$\hat{M}(x|x^0,\cdots,x^n) = n!\, [-ixx^0,\cdots,-ixx^n]_t\, e^t .$$

Expanding the divided difference on the right hand side of (4) we obtain the Fourier transform of W. Dahmen's truncated power representation of multivariate B-splines [8]

(5) $$\hat{M}(x|x^0,\cdots,x^n) = n! \sum_{\nu=0}^{n} e^{-ixx^\nu} \prod_{\substack{\mu=0\\ \mu\neq\nu}}^{n} [i(x^\mu - x^\nu)x]^{-1} .$$

Identity (5) corresponds to the usually used definition of the univariate B-spline [1]

(6) $$M(x|t_0,\cdots,t_n) = n \sum_{\nu=0}^{n} (t_\nu - x)_+^{n-1} \prod_{\substack{\mu=0\\ \mu\neq\nu}}^{n} (t_\nu - t_\mu)^{-1}$$

$$= n\, [t_0,\cdots,t_n]_t\, (t-x)_+^{n-1} .$$

Using the recurrence relation for divided differences

(7) $$(t_\nu - t_\mu)[t_0,\cdots,t_n] = [t_0,\cdots,t_{\mu-1},t_{\mu+1},\cdots,t_n] - [t_0,\cdots,t_{\nu-1},t_{\nu+1},\cdots,t_n]$$

we can derive from (4) a formula for the directional derivative of M. We set $t_\nu = -ixx^\nu$, apply (7) to the function e^t and take the inverse Fourier transform. It follows that

(8) $$D_{x^\nu - x^\mu} M(x|x^0,\cdots,x^n) = n[M(x|x^0,\cdots,x^{\nu-1},x^{\nu+1}\cdots,x^n) - M(x|x^0,\cdots,x^{\mu-1},x^{\mu+1},\cdots,x^n)]$$

where $D_z := z \cdot \nabla$. This generalizes to

<u>Theorem 1 [8, 21, 22].</u> <u>If</u> $y = \sum_{\nu=0}^{n} \lambda_\nu x^\nu$, $\sum_{\nu=0}^{n} \lambda_\nu = 0$, <u>then we have</u>

(9) $$D_y M(x|x^0,\cdots,x^n) = n \sum_{\nu=0}^{n} \lambda_\nu M(x|x^0,\cdots,x^{\nu-1},x^{\nu+1},\cdots,x^n)$$

<u>where this identity should be interpreted in the sense of distributions if some of the B-splines are supported on sets of zero measure.</u>

A repeated application of (9) shows that the distributional derivatives $D^{\alpha}M(\cdot|x^0,\cdots,x^n)$, $|\alpha| = n-m+1$, being linear combinations of the B-splines $M(\cdot|x^{i_1},\cdots,x^{i_m})$, are supported in convex subsets of hyperplanes. Hence M agrees with a polynomial of total degree $\leq n-m$ on every subset of $\mathbb{R}^m$ not cut by one of the convex sets spanned by $x^{i_1},\cdots,x^{i_m}$, $0 \leq i_1 < \cdots < i_m \leq n$ [21, 22]. The global smoothness of M depends on the geometric configuration of the points x^{ν}. If every subset of ℓ points of $\{x^0,\cdots,x^n\}$ forms a proper convex set it follows again from (9) that $D^{\alpha}M \in L_{\infty}$, $|\alpha| = n-\ell+1$. Therefore,

(10) $$M(\cdot|x^0,\cdots,x^n) \in W_{\infty}^{n-\ell+1}(\mathbb{R}^m) \subset C^{n-\ell}(\mathbb{R}^m) .$$

Perhaps the most striking identity for multivariate B-splines is the recurrence relation, discovered by C. A. Micchelli [21] and also proved by W. Dahmen [8] under more restrictive assumptions. See also [16] for a proof via Fourier analysis.

Theorem 2 [8,21,22]. *Assume that the points* $x^0,\cdots,x^n \in \mathbb{R}^m$, $n > m$ *form a proper convex set and*

$$x = \sum_{\nu=0}^{n} \lambda_{\nu}x^{\nu} , \quad \sum_{\nu=0}^{n} \lambda_{\nu} = 1 ,$$

then we have

(11) $$M(x|x^0,\cdots,x^n) = \frac{n}{n-m} \sum_{\nu=0}^{n} \lambda_{\nu} M(x|x^0,\cdots,x^{\nu-1},x^{\nu+1},\cdots,x^n)$$

if all B-splines occurring in this equation are continuous at x.

Formula (11) expresses M recursively as a convex combination of positive quantities and thus provides a stable way for computing multivariate B-splines [22]. So far we have only discussed some basic results on multivariate B-splines and we refer to the literature [8-12, 21, 22] for many interesting recent developments.

2. Some topological preliminaries

We need some basic facts about triangulations [14]. Since there seems to be no convenient reference for a reader with little background in topology we sketch the proofs.

Let (X,I) denote a collection of nondegenerate closed simplices $\sigma_i(X) = \sigma(x^{i_0},\cdots,x^{i_m})$, $i = (i_0,\cdots,i_m) \in I \subset \mathbb{Z}^{m+1}$ with vertices $x^\nu \in X \subset \mathbb{R}^m$. (X,I) is called a triangulation of a set $\Omega \subset \mathbb{R}^m$ if $\Omega = \bigcup_{i\in I} \sigma_i(X)$ and the intersection of two different simplices is either empty or a common lower dimensional face. Moreover we assume that every compact subset of Ω is intersected by finitely many simplices only.

Definition 2. Denote by $B_i := \omega_i + \rho_i B$ the inscribed balls with center ω_i and radius ρ_i for the simplices $\sigma_i(X)$. We call a triangulation regular, if there exists a constant γ such that

$$\text{diam}\ \sigma_i(X) \leq \gamma\, \rho_i\ , \quad i \in I\ . \tag{12}$$

Lemma 1. Let (X,I) be a triangulation of an open set $\Omega \subset \mathbb{R}^m$. Then for any finite set of vertices $\{x^\nu\}_{\nu\in J} \subset X$ there is an $\varepsilon > 0$ such that $(\tilde{X},I)$ with $\tilde{x}^\nu = x^\nu$, $\nu \notin J$ is also a triangulation of Ω if $|\tilde{x}^\nu - x^\nu| < \varepsilon$, $\nu \in J$.

Lemma 2. Let (X,I) be a triangulation of a simply connected set Ω and assume that a locally finite collection of simplices $(\tilde{X},I)$ satisfies $\Omega = \bigcup_{i\in I} \sigma_i(\tilde{X})$. Then $(\tilde{X},I)$ is also a triangulation of Ω.

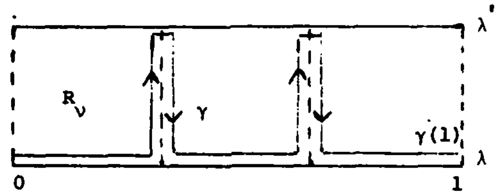
Proof. The mapping $f : \Omega \to \Omega$ defined by

$$f(x^\nu) = \tilde{x}^\nu\ ,$$

$$f_{|\sigma_i(X)} \text{ is affine linear,}$$

is a local homeomorphism which is by assumption surjective. Since Ω is simply connected the monodromy theorem implies that f is bijective and hence $(\tilde{X},I)$ is a triangulation of Ω. Roughly speaking, the argument is as follows. Suppose there exist $z_0, z_1 \in \Omega$, $z_0 \neq z_1$, such that $y = f(z_0) = f(z_1)$ and let C_0 be the image of a curve joining z_0

and z_1. Denote by $C(t,\lambda)$: $[0,1]^2 \to \Omega$ a continuous deformation of $C_0 = C(\cdot,0)$ into the point y, i.e. $y \equiv C(\cdot,1)$. Since we have assumed that only finitely many simplices $\tilde{\sigma}_i$ intersect a given compact set every point $x \in \Omega$ has a neighborhood U_x such that $f^{-1}U_x$ consists of finitely many disjoint sets $V_{x,1},\cdots,V_{x,j}$ and $f|_{V_{x,\nu}}$ is a homeomorphism. Hence the inverse of f along each curve $C_\lambda = C(\cdot,\lambda)$, $f^{-1}(C(\cdot,\lambda))$, with initial value $z_0 = f^{-1}(C(0,\lambda))$ is well defined. We shall show that $z_1 = f^{-1}(C(1,\lambda))$ for all $\lambda \in [0,1]$ contradicting $f^{-1}(C(\cdot,1)) \equiv z_0$. For a sufficiently fine subdivision of $[0,1]^2$ into squares R_ν each of the sets $C(R_\nu)$ is contained in some neighborhood U_x and the following picture shows that inverting f along adjacent curves leads to the same value



$$f^{-1}(C(1,\lambda)) = f^{-1}(C(\gamma(1))) = f^{-1}(C(1,\lambda')).$$

The combinatorial product of the sets $i = \{i_0,\cdots,i_m\}$ and $\{0,\cdots,k\}$ is defined as the collection of all ordered subsets of $\{i_0,\cdots,i_m\} \times \{0,\cdots,k\}$ with $n+1 := m+k+1$ distinct elements, i.e.

$$i\Delta k := \{(j,r) = \begin{pmatrix} j_0,\cdots,j_n \\ r_0,\cdots,r_n \end{pmatrix} \in [\{i_0,\cdots,i_m\} \times \{0,\cdots,k\}]^{n+1} \mid \tag{13}$$

$$(j_{\nu+1} = j_\nu \text{ and } r_{\nu+1} = r_\nu+1) \text{ or } (j_{\nu+1} = j_\nu+1 \text{ and } r_{\nu+1} = r_\nu) \text{ for } \nu = 0,\cdots,n\}.$$

Note that $(j,r) \in i\Delta k$ is already uniquely determined by $j = (j_0,\cdots,j_n)$ so that we may drop r when referring to elements of $i\Delta k$.

<u>Lemma 3 .</u> Denote by $\Delta_k = \sigma(e_0,\cdots,e_k)$, with $e_0 = (0,\cdots,0)$, $e_1 = (1,0,\cdots,0),\cdots,$ the standard k-dimensional simplex and let (X,I) be a triangulation of $\Omega \subset R^m$. Then the simplices

$$\sigma((x^{j_0}, e_{r_0}), \cdots, (x^{j_n}, e_{r_n})), \quad n := m+k,$$

(14)

$$(j,r) \in I\Delta k := \bigcup_{i \in I} i\Delta k$$

form a triangulation of $\Omega \times \Delta_k$.

<u>Proof.</u> Consider the intersection of two simplices $V = \sigma_{j^0} \cap \sigma_{j^1}$, $\sigma_{j^\rho} :=$

$\sigma((x^{j_0^\rho}, e_{r_0^\rho}), \cdots, (x^{j_n^\rho}, e_{r_n^\rho}))$, and the intersection of their projections onto $\mathbb{R}^m$

$W = \sigma_{i^0} \cap \sigma_{i^1}$. Since (X,I) is a triangulation we have $W = \sigma(y_0, \cdots, y_s)$ where $y_0, \cdots, y_s$ are the common vertices of $\sigma_{i^0}, \sigma_{i^1}$ and clearly $V = W \times \Delta_k \cap V$.

Denote by $(y_\nu, e_{t^\rho_{\nu\mu}})$ $\nu = 0, \cdots, s$, $\mu = 0, \cdots, \ell^\rho_\nu$, $\rho = 0,1$, the vertices of $\sigma_{j^0}, \sigma_{j^1}$

contained in the set $W \times \Delta_k$ and let

(*)
$$x = \sum_{\nu=0}^{s} \sum_{\mu=0}^{\ell^\rho_\nu} \lambda^\rho_{\nu\mu} (y_\nu, e_{t^\rho_{\nu\mu}}) \in \sigma_{j^\rho} , \quad \rho = 0,1,$$

$$\sum_{\nu\mu} \lambda^\rho_{\nu\mu} = 1$$

be a point in V . Since the projection of x onto $\mathbb{R}^m$ is a unique convex combination of $y_0, \cdots, y_s$ we have the additional equations

(**)
$$\sum_{\mu=0}^{\ell^0_\nu} \lambda^0_{\nu\mu} = \sum_{\mu=0}^{\ell^1_\nu} \lambda^1_{\nu\mu}, \quad \nu = 0, \cdots, s.$$

By the definition of the index set j we have $t^\rho_{\nu\mu} < t^\rho_{\nu(\mu+1)}$, $t^\rho_{\nu\mu} < t^\rho_{(\nu+1)\mu}$. Using this fact we can successively solve the equations (*), (**) for $\lambda^\rho_{\lambda\mu}$, $\nu = s, \cdots, 0$, and obtain

$$\lambda^0_{\nu\mu} = \lambda^1_{\nu\mu}, \quad \text{iff} \quad e_{t^0_{\nu\mu}} = e_{t^1_{\nu\mu}},$$

$$\lambda^\rho_{\nu\mu} = 0 \qquad \text{otherwise.}$$

This implies that V is the simplex spanned by the common vertices of $\sigma_{j^0}, \sigma_{j^1}$.

A simple calculation shows that $\text{vol } \sigma_j = \frac{m!}{n!} \text{vol } \sigma_i$, $j \in i\Delta k$. Since $\sigma_j \subset \sigma_i \times \Delta_k$ and $\# \, i\Delta k = \binom{m+k}{m}$ this implies $\bigcup_{j \in i\Delta k} \sigma_j = \sigma_i \times \Delta_k$.

3. Splines as linear combinations of B-splines

Denote by P_k the polynomials of total degree $\leq k$, i.e.

$$P_k = \{f \mid D^\alpha f = 0,\ |\alpha| = k+1\}, \text{ where } \alpha = (\alpha_1,\cdots,\alpha_m),\ |\alpha| = \sum_{\nu=1}^m \alpha_\nu,\ D^\alpha = \prod_{\nu=1}^m \left(\frac{\partial}{\partial y_\nu}\right)^{\alpha_\nu}.$$

We call S_k a space of splines of total degree k defined on a set $\Omega \subset \mathbb{R}^m$ if

(A) S_k is the linear span of a collection of B-splines and

(B) $P_k(\Omega) \subset S_{k|\Omega}$.

Moreover we shall assume that

(C) every compact subset of Ω is intersected by the supports of finitely many B-splines only.

In the univariate case we may associate with any increasing sequence of knots $\{t_\nu\}$, $t_\nu < t_{\nu+k+1}$, a space of splines

$$S_k = \operatorname*{span}_\nu M(\cdot|t_\nu,\cdots,t_{\nu+k+1}). \tag{15}$$

In several variables any triangulation $((X,\tilde{X}),I)$ *of a product* $\Omega \times \tilde{\Omega} \subset \mathbb{R}^m \times \mathbb{R}^k$, $\operatorname{vol} \tilde{\Omega} < \infty$, corresponds by Definition 1 to a family of B-splines which forms a partition of unity over Ω [3], i.e.

$$1 \equiv (\operatorname{vol}\tilde{\Omega})^{-1} \sum_{i\in I} \operatorname{vol} \sigma((x^{i_0},\tilde{x}^{i_0}),\cdots,(x^{i_n},\tilde{x}^{i_n}))\, M(x|x^{i_0},\cdots,x^{i_n}), \tag{16}$$

$$x \in \Omega, \quad n = m+k,$$

and it was shown in [9] that for $\Omega \times \tilde{\Omega} = [0,1]^n$ $P_k \subset \operatorname*{span}_{i\in I} M(\cdot|x^{i_0},\cdots,x^{i_n})$, i.e. condition (B) is satisfied.

To require the existence of an associated triangulation is, however, too restrictive and not quite satisfactory, since the B-splines do only depend on the knots $x^\nu \in X \subset \mathbb{R}^m$. As in the univariate case, one would like to associate with any distribution of knots a corresponding space of splines. This can be accomplished by a multivariate analogon of the process of "pulling apart" knots which we shall now describe.

Definition 3. Let (X,I), $X = \{x^\nu\}_{\nu\in\mathbb{Z}}$, be a triangulation of $\mathbb{R}^m$ and

$$\mathbf{X} = \{x_\mu^\nu\}_{\nu\in\mathbb{Z},\mu=0,\cdots,k}\ ,\ x_\mu^\nu \in \mathbb{R}^m,$$

a collection of points, not necessarily related to (X,I). We define the space of splines of total degree k with respect to the index set I and the knots $\mathbf{X}$ by (c.f. (13), (14))

(17) $$S_{k,\mathbf{X},I} = \operatorname*{span}_{j\in I\Delta k} M_j(\cdot|\mathbf{X})$$

where $M_j(\cdot|\mathbf{X}) := M(\cdot|x_{r_0}^{j_0},\cdots,x_{r_n}^{j_n})$, $n = m+k$. In addition we assume that condition (C) is satisfied.

The rest of this chapter is devoted to the justification of this Definition, i.e. to prove (B). To simplify notation we shall drop most of the subscripts in the sequel, e.g. $S_k := S_{k,\mathbf{X},I}$, $M_j := M_j(\cdot|\mathbf{X})$, $\sigma_i := \sigma_i(X)$, etc.

Lemma 4. If we make for the knots $\mathbf{X}$ the particular choice $x_\mu^\nu = x^\nu$, $\nu \in \mathbb{Z}$, $\mu = 0,\cdots,k$, then $S_{k,\mathbf{X},I}$ coincides with the space of piecewise polynomials of total degree k with respect to the triangulation (X,I), i.e.

(18) $$S_{k,\mathbf{X},I} = P_{k,X,I} := \{f|f_{|\sigma_i(X)} \in P_k\ ,\ \ i \in I\}\ .$$

Proof. For the special choice $x_\mu^\nu = x^\nu$ we have

$$S_{k,\mathbf{X},I|\sigma_i} = \operatorname*{span}_{j\in i\Delta k} M(\cdot|x^{j_0},\cdots,x^{j_n})$$

and $\{x^{j_0},\cdots,x^{j_n}\} = \{x^{i_0},\cdots,x^{i_m}\}$. We could now use the formula for B-splines with multiple knots [22, p. 16] to complete the proof. But we may also apply (9) to conclude that the supports of the distributional derivatives $D^\alpha M$, $|\alpha| = k+1$, are contained in the boundary of the simplex σ_i. This implies $\operatorname*{span}_{j\in i\Delta k} M_j \subset P_k(\sigma_i)$. The reverse inclusion is a special case of Theorem 3 below. Moreover, since $\#i\Delta k = \binom{k+m}{k} = \dim P_k$, the B-splines M_j, $j \in i\Delta k$, form a basis for $P_k(\sigma_i)$.

Remark. Lemma 4 shows that we may view the spaces S_k as perturbations of piecewise polynomials obtained by pulling apart the multiple knots $x^\nu \in X$

$$x^\nu \to x_\mu^\nu , \quad \mu = 0,\cdots,k .$$

One may e.g. choose $a_0,\cdots,a_k \in \mathbb{R}^m$ and define $x_\mu^\nu = x^\nu + a_\mu$, $\mu = 0,\cdots,k$. This very special perturbation corresponds to an affine transformation of the triangulation (14) and is only an appropriate choice if the triangulation (X,I) is quasiuniform and one is interested in a uniform distribution of the knots $\mathbf{X}$.

We have defined the spline spaces S_k on $\mathbb{R}^m$ to avoid a separate discussion of various types of "boundary" conditions in the sequel. This is no loss of generality since we include all kinds of finite dimensional spline spaces by simply restricting the domain of definition. The global smoothness of S_k depends on the smoothness of the particular B-splines (c.f. (10)). But, although the set of knots yielding the highest possible smoothness $S_k \subset C^{k-1}$ is dense, there is no canonical way to choose such knots as in the univariate case.

Example 1. Let $\{t_\nu\}$, $t_\nu < t_{\nu+k+1}$, be an increasing sequence of knots. To obtain the canonical B-spline basis (15) we set (c.f. Definition 3) $X = \{t_{(k+1)\nu}\}_{\nu\in\mathbb{Z}}$, $I = \{(\nu,\nu+1)\}_{\nu\in\mathbb{Z}}$, $x_\mu^\nu = t_{(k+1)\nu-\mu}$, $\nu \in \mathbb{Z}$, $\mu = 0,\cdots,k$, which yields

$$(\nu,\nu+1)\Delta k = \left\{\binom{\nu\cdots\nu(\nu+1)\cdots(\nu+1)}{0\cdots\rho\ \ \rho\ \ \cdots\ \ k},\ \rho = 0,\cdots,k\right\},$$

$$\operatorname*{span}_{j\in(\nu,\nu+1)\Delta k} M_j = \operatorname*{span}_{\rho=0,\cdots,k} M(\cdot|t_{(k+1)\nu-\rho},\cdots,t_{(k+1)(\nu+1)-\rho}).$$

The following example shows that Definition 3 is slightly more general than the usual univariate definition, since we do not require that a B-spline does only correspond to adjacent knots.

Example 2. Let $X = \mathbb{Z}$, i.e. $x^\nu = \nu$, $I = \{(\nu,\nu+1)\}_{\nu\in\mathbb{Z}}$ and consider the knots $x_0^1 = 2$, $x_1^1 = 2.5$, $x_0^2 = 1$, $x_1^2 = 0.5$ and $x_\mu^\nu = \nu$ for $\nu \neq 1,2$. Restricting the domain to the interval $[0,3]$ we have

$$S_{1,\mathbf{X},I|[0,3]} = \operatorname{span}\{M(\cdot|0,0,2.5),\ M(\cdot|0,2,2.5),\ M(\cdot|0.5,2,2.5),\ M(\cdot|0.5,1,2),$$

$$M(\cdot|0.5,1,3),\ \ M(\cdot|1,3,3)\}$$

which is a space of continuous piecewise linear splines with an unusual basis.

Theorem 3. The spaces $S_{k,\mathbf{X},I}$ contain polynomials of total degree k. In particular we have for all $x,y \in R^m$

$$(1+xy)^k = \sum_{j\in I\Delta k} C_j(y|\mathbf{X})\, M_j(x|\mathbf{X}) \tag{19}$$

where (c.f. (13), (14), (17))

$$C_j(y|\mathbf{X}) := \varepsilon_j \frac{k!}{n!} \det \begin{vmatrix} (1 + x_{r_0}^{j_0} y) e_{r_0} & \cdots & (1 + x_{r_n}^{j_n}) e_{r_n} \\ x_{r_0}^{j_0} & \cdots & x_{r_n}^{j_n} \\ 1 & \cdots & 1 \end{vmatrix}$$

and $\varepsilon_j \in \{-1,1\}$ is chosen so that

$$C_j(X) := \varepsilon_j \frac{k!}{n!} \det \begin{vmatrix} e_{r_0} & \cdots & e_{r_n} \\ x^{j_0} & \cdots & x^{j_n} \\ 1 & \cdots & 1 \end{vmatrix} > 0 .$$

(Recall that x^ν are the vertices of the triangulation corresponding to the index set I).

For the standard univariate B-spline basis formula (19) is known as Marsden's identity [19]. The basic idea of its multivariate generalization is due to W. Dahmen [9]. For two variables (19) was proved by T. N. T. Goodman and S. L. Lee [15] under the restrictive assumption that the B-splines correspond to a triangulation of $\mathbf{R}^2 \times \Delta_k$. In both cases there is a more explicit formula for the coefficients $C_j(y|\mathbf{X})$ (c.f. Corollaries 1 and 2). Besides the generalization to an arbitrary number of variables we show that identity

(19) is valid for any collection of knots as long as we keep the combinatorial structure determined by the index set I .

Remarks. (i) Identity (19) holds for any finite collection of B-splines M_j, $j \in J \subset I\Delta k$, if we restrict x to the domain $\mathbb{R}^m \setminus \bigcup_{j \notin J} \operatorname{supp} M_j$.

(ii) For $y = 0$, (19) is a generalization of (16) since the coefficients $C_j := C(0|X)$ need not to be positive. The coefficients corresponding to the B-splines in Example 2, e.g., are 1.25, 1, -0.8, -0.4, 1.25, 1. One can show, using Lemma 2, that the coefficients C_j, $j \in I\Delta k$, are positive if and only if the simplices

(20) $$\sigma_j(\mathbb{X}) := \sigma((x_{r_0}^{j_0}, e_{r_0}), \cdots, (x_{r_n}^{j_n}, e_{r_n})), \quad j \in I\Delta k ,$$

form a triangulation of $\mathbb{R}^m \times \Delta_k$.

(iii) Comparing the coefficients of y^α in identity (19) we obtain an explicit representation for the monomials x^α, $|\alpha| \le k$,

(21) $$x^\alpha = \sum_{j \in I\Delta k} C_j^\alpha(\mathbb{X}) \, M_j(x)$$

where $C_j^\alpha(\mathbb{X}) := (\alpha!)^{-1} \binom{k}{|\alpha|}^{-1} D^\alpha C_j(y|X)_{|y=0}$, $\alpha! = \prod_{\nu=1}^{m} \alpha_\nu!$.

Proof of Theorem 3. Note, that by Definition 1 we may associate with the B-splines M_j the simplices σ_j (c.f. (17), (20)). First we assume that the collection of simplices σ_j, $j \in I\Delta k$, is a small perturbation of the triangulation (14), i.e. $S_{k,\mathbb{X},I}$ is close to $P_{k,X,I}$ (c.f. (18)). More precisely, for a finite set $J \subset \mathbb{Z}$ and $\varepsilon > 0$, to be chosen later, we assume

(*) $$|x_\mu^\nu - x^\nu| < \varepsilon, \quad \mu = 0,\cdots,k, \quad \nu \in J$$

(*)

(**) $$x_\mu^\nu = x^\nu , \qquad \mu = 0,\cdots,k, \quad \nu \notin J .$$

By Lemma 1, σ_j , j $I\Delta k$, is a triangulation for small ε (Since $\mathbb{R}^m \times \Delta_k \subset \mathbb{R}^n$ is not an open set. Lemma 1 cannot be applied directly. But, we may consider suitable extensions of the triangulations to $\mathbb{R}^n$). Let K be an arbitrary bounded subset of $\mathbb{R}^m$. Obviously it is sufficient to prove (19) under the assumption $x \in K$ and y small. We define the mapping

$$T : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$$

$$T(x,\tilde{x}) = (x,(1 + xy)\tilde{x}) .$$

Consider the finitely many B-splines M_j, $j \in \tilde{J}$, the supports of which intersect K (c.f. the assumption in Definition 3), and let $\{x_\mu^\nu\}_{\mu=0,\cdots,k,\ \nu\in J}$ be the corresponding set of knots. For small enough y the simplices

$$\sigma_j^T := \sigma(T(x_{r_0}^{j_0},e_{r_0}),\cdots,T(x_{r_n}^{j_n},e_{r_n})), \quad j \in \tilde{J} ,$$

form also a triangulation. Moreover, since T maps the hyperplanes bounding $\mathbb{R}^m \times \Delta_k$ into hyperplanes and leaves projections on $\mathbb{R}^m$ fixed we have

$$[\bigcup_{j\in\tilde{J}} \sigma_j^T] \cap K \times \mathbb{R}^k = T(K \times \Delta_k) .$$

By Definition 1 and (16) it follows that for small y and $x \in K$

$$\mathrm{vol}\{\tilde{x} \in \mathbb{R}^k | (x,\tilde{x}) \in T(K \times \Delta_k)\} = (k!)^{-1}(1 + xy)^k =$$

$$\sum_{j\in\tilde{J}} \mathrm{vol}\{\tilde{x} \in \mathbb{R}^k | (x,\tilde{x}) \in \sigma_j^T\} = \sum_{j\in\tilde{J}} \mathrm{vol}\, \sigma_j^T M_j .$$

For $|x_\mu^\nu - x_\nu|$, $\nu \in J$, small we have $\mathrm{sgn}\, C_j(y|X) = \mathrm{sgn}\, C_j(X) = 1$ which implies $\mathrm{vol}\, \sigma_j^T = (k!)^{-1}C_j(y|X)$ and completes the proof under the assumptions (*), (**).

We now drop the assumption (*). Let M_j, $j \in \tilde{J}$, be the finitely many B-splines having at least one of the knots $\{x_\mu^\nu\}_{\mu=0,\cdots,k,\ \nu \in J}$ and consider the equation

$$\int_{\mathbb{R}^m} (1+xy)^k\hat{\phi}(x)dx - \int_{\mathbb{R}^m} (\sum_{\substack{j\in I\Delta k\\ j\notin\tilde{J}}} C_j(y|X)M_j(x))\hat{\phi}(x)dx =$$

(+)

$$\int_{\mathbb{R}^m} \sum_{j\in\tilde{J}} C_j(y|X)M_j(x)\hat{\phi}(x)dx$$

where $\hat{\phi}$, $\phi \in \mathcal{D}$ is the Fourier transform of a test function with compact support. By (4) and Parseval's identity, the right hand side is equal to

$$n! \sum_{j\in\tilde{J}} C_j(y|X) \int_{\mathbb{R}^m}[-ix_{r_0}^{j_0}x,\cdots,-ix_{r_n}^{j_n}x]_t\, e^t\, \phi(x)dx .$$

The coefficients $C_j(y|X)$ and the divided difference of the exponential function are also defined for complex values $x_\mu^\nu \in \mathbb{C}^m$, and it is easily seen that the right hand side of (+) is an entire function of the arguments $\{x_\mu^\nu\}_{\mu=0,\cdots,k,\ \nu \in J} \in (\mathbb{C}^m)^{\#J\cdot(k+1)}$. The left hand side of (+) is constant with respect to these arguments. Therefore, since by the first part of the proof the identity is valid in a real neighborhood of the knots x^ν, $\nu \in J$, it follows globally by the identity theorem for power series [17]. To obtain (19) from the identity (+) we note that $\{\hat{\phi} | \phi \in \mathcal{D}\}$ is dense in $\mathcal{D}$ with respect to the norm $\|\psi\| := \sup_{x \in \mathbb{R}^m} (1 + |x|)^\ell |\psi(x)|$ for all $\ell \in \mathbb{N}$.

The general case follows now easily. Consider an arbitrary perturbation of the initial knots

$$x^\nu + x_\mu^\nu, \quad \mu = 0, \cdots, k$$

such that any bounded subset $K \subset \mathbb{R}^m$ is intersected by the supports of finitely many B-splines only. Restricting x to K, only finitely many knots are relevant for the identity (19) and the previous arguments apply.

We now examine in more detail the coefficients $C_j(y|\mathbb{Z})$. Indicating terms of the form $(1 + x_\mu^\nu y)$ by " $*$'s", the determinant in their definition has the following structure

(22)
$$\begin{array}{l}
\left.\begin{array}{ccccccc}
\overbrace{0 \cdots 0}^{\rho_0+1} & * & * & & & & 0 \\
 & & & * \cdots * & & & \\
 & 0 & & & & * & \cdots * \\
\end{array}\right\} k \\
\left.\begin{array}{ccc}
x_{r_0}^{j_0} & \cdots & x_{r_n}^{j_n}
\end{array}\right\} m \\
\begin{array}{ccc}
1 & \cdots & 1
\end{array}
\end{array}$$

where the position of the nonzero entries in the first k rows depends on the index set $r = (r_0, \cdots, r_n)$. If $\rho_\nu + 1$, $\nu = 1, \cdots, k$, denotes the number of $*$'s in the ν's row we have $\rho_0 + \cdots + \rho_k = m$. For small m this leads to considerable simplification.

<u>Corollary 1 [19].</u> For any increasing sequence of knots $\{t_\nu\}_{\nu \in \mathbb{Z}} \subset \mathbb{R}$, $t_\nu < t_{\nu+k+1}$, we have

(23) $$(y-x)^k = (k+1)^{-1} \sum_\nu (t_{\nu+k+1} - t_\nu) \prod_{\mu=1}^{k} (y - t_{\nu+\mu}) \, M(x | t_\nu, \cdots, t_{\nu+k+1}) .$$

Corollary 2 [15]. Let $S_{k,X,I}$ be defined on $\mathbb{R}^2$. Then we have for all $x, y \in \mathbb{R}^2$

$$(1+xy)^k = \sum_{j \in I\Delta k} c_j \prod_{\nu=1}^{k} (1+z_j^\nu y)\, M_j(x) \tag{24}$$

where $z_j^\nu := z_j^\nu(X)$ can be determined by the knots of the B-spline M_j and the index set j. For details we refer to [15].

Both Corollaries can be obtained from Theorem 3 by direct computation of the determinant (22). Note that in (23) we have made the substitution $y \to -1/y$. Unfortunately, an identity of the form

$$C_j(y|X) = c_j \prod_{\nu=1}^{k} (1 + z_j^\nu(X)y)$$

is no longer valid in more than two dimensions.

Example 4.

$$\det \begin{vmatrix} 0 & 0 & (1+y_1-y_3) & (1+y_2+y_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+y_1-y_3) & (1+y_2+y_3) \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

cannot be written in the form

$$C(1 + z_1 y)(1 + z_2 y)$$

for some vectors $z_1, z_2 \in \mathbb{R}^3$.

4. The B-spline basis

The standard univariate B-spline basis (15) has the striking property (c.f. C. de Boor [1,3]) that the L_p-norm of a spline is equivalent to the ℓ_p- norm of its coefficients with respect to a normalized B-spline basis independently of the sequence of knots. Under mild assumptions on the distribution of knots we shall establish this result for the multivariate spline spaces $S_{k,\mathbb{X},I}$.

<u>Definition 4.</u> We call $\mathbb{X} = \{x_\mu^\nu\}$ a regular perturbation of a triangulation (X,I) if there exists a constant κ and balls $\Omega_i := x_i + h_i B$ such that for every $i \in I$ (c.f. (13), Definition 3)

(25) $$\Omega_i \subset \bigcap_{j \in i\Delta k} \operatorname{supp} M_j$$

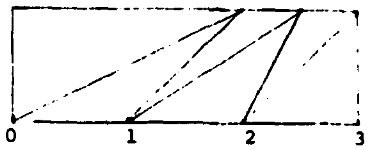
(26) $$\Omega_i \cap \operatorname{supp} M_j = \emptyset\ , \quad j \in i'\Delta k\ , \quad i' \neq i$$

(27) $$\operatorname{diam} \operatorname{supp} M_j \leq \kappa\, h_i\ , \qquad j \in i\Delta k$$

Note, that the conditions (25)-(27) are local, i.e. we do not assume any relationship between h_i, $h_{i'}$, $i \neq i'$.

The following example shows that assumptions (25)-(27) cannot be essentially weakened.

<u>Example 4</u> Consider the following perturbation of the standard triangulation (14) of $\mathbb{R} \times [0,1]$ restricted to $[0,3] \times [0,1]$



Among the corresponding B-splines, $M(\cdot|1,2,2.5)$ occurs twice, a possibility not excluded by Definition 3. Although the B-splines become linearly independent if we move the vertex (2,1) slightly to the left, the basis is not well conditioned.

Lemma 5. With the notation of Definitions 2-4 assume that x_i can be chosen as the center of the inscribed ball of the simplex $\sigma_i(X)$, i.e. $x_i = \omega_i$, $h_i < \rho_i$, and

(28)
$$|x_\mu^\nu - x^\nu| < h_\nu$$

where $h_\nu := \text{Min}\{\rho_i - h_i | x^\nu$ is a vertex of $\sigma_i(X)\}$. Then $\mathbb{X}$ is a regular perturbation of (X,I).

This shows that conditions (25)-(27) allow perturbations of the same magnitude as the perturbation of Example 4.

Proof. The support of each B-spline M_j, $j \in I\Delta k$, is the union of simplices $\tilde{\sigma}_i := \sigma(\tilde{x}^{i_0},\cdots,\tilde{x}^{i_m})$ where $|\tilde{x}^{i_\nu} - x^{i_\nu}| < \rho_i - h_i$. By Lemma 2 the collection of simplices $\{\tilde{\sigma}_i\}_{i\in I}$ is a triangulation. To prove (25), (26) it is sufficient to show that for any such triangulation $\Omega_i \subset \tilde{\sigma}_i$. We define the simplices

$$\sigma_{i,t} = \sigma(t\tilde{x}^{i_0} + (1-t)x^{i_0},\cdots,t\tilde{x}^{i_m} + (1-t)x^{i_m}),\quad t \in [0,1]$$

and observe that none of the (k-1)-dimensional faces of $\sigma_{i,t}$, $t \in [0,1]$, intersects $\overset{0}{\Omega}_i$. This follows because any such face can be separated from $\overset{0}{\Omega}_i$ by the hyperplane parallel to the corresponding face of $\sigma_i(X)$ touching Ω_i. Concerning (27) we note that under the assumptions of the Lemma $\kappa \le \gamma\rho_i+\rho_i-h_i$.

Theorem 4. Let $\mathbb{X}$ be a regular perturbation of a triangulation (X,I). With the notation of Theorem 3 and Definition 4 we define functionals $\lambda : C^k(\mathbb{R}^m) \to \mathbb{R}$ by

(29)
$$\lambda_j(\mathbb{X},\tau)f = \sum_{|\alpha|\le k} (\alpha!)^{-2}\binom{k}{|\alpha|}^{-1} D^\alpha C_j(0|\mathbb{X} - \tau)\, D^\alpha f(\tau),\quad j \in I\Delta k,$$

where $\tau \in \mathbb{R}^m$ and $\mathbb{X} - \tau := \{x_\mu^\nu - \tau\}$ is a translation of the point set $\mathbb{X}$. Then for any set of points $\tau_j \in \Omega_i$, $j \in I\Delta k$, the functionals $\lambda_j := \lambda_j(\mathbb{X},\tau_j)$, $j \in I\Delta k$, are dual to the B-spline basis for $S_{k,\mathbb{X},I}$, i.e.

(30)
$$\lambda_j M_{j'} = \delta_{jj'},\quad j,j' \in I\Delta k.$$

We note the following immediate consequences of this result which for univariate spline functions is due to C. de Boor [2].

(i) The B-splines M_j , $j \in I\Delta k$, are linearly independent over the set $\Omega = \bigcup_{i\in I} \Omega_i$.

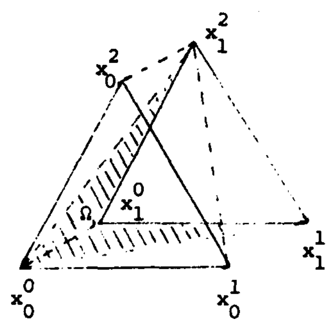
(ii) For any polynomial of total degree $\le k$, $p \in P_k$, $\lambda_j(\mathbb{X},\tau)p$ is constant as a function of $\tau \in \mathbb{R}^m$.

To see this we observe that the polynomial $g(\tau) := \lambda_j(\mathbb{X},\tau)p$ is constant for $\tau \in \Omega_i$, $j \in i\Delta k$. This follows from (30) since by (26) and Theorem 3

(31) $$S_{k,\mathbb{X},I|\Omega_i} = \operatorname*{span}_{j\in i\Delta k} M_{j|\Omega_i} = P_{k|\Omega_i} .$$

Concerning (i) we note, that in contrast to the univariate case in more than one variable the B-spline basis of S_k is not linearly independent over arbitrary open subsets of $\mathbb{R}^m$.

Example 5. Consider the following set of knots in $\mathbb{R}^2$



which may be extended to a regular perturbation of a triangulation of $\mathbb{R}^2$ and the corresponding piecewise linear B-splines $M(\cdot|x_0^0,x_1^0,x_1^1,x_1^2)$, $M(\cdot|x_0^0,x_0^1,x_1^1,x_1^2)$, $M(\cdot|x_0^0,x_0^1,x_0^2,x_1^2)$. Since also supports of other B-splines overlap the set Ω, Ω contains a set of linear dependence $\tilde{\Omega}$, i.e. the B-splines $\{M_j | \text{supp } M_j \cap \tilde{\Omega} \neq \emptyset\}$ are linearly dependent over $\tilde{\Omega}$.

Proof of Theorem 4. Since $\text{supp } \lambda_j \subset \Omega_i$, $j \in i\Delta k$, and $S_{k|\Omega_i} = P_{k|\Omega_i}$ we have to show

$$\lambda_j\Big(\sum_{j'\in i\Delta k} a_{j'} M_{j'}\Big) = a_j$$

for all polynomials $\sum_{j'\in i\Delta k} a_{j'} M_{j'|\Omega_i} \in P_{k|\Omega_i}$. A change of variables in identity (19) yields

(19')
$$(1+(x-\tau)(y-\tau))^k = \sum_{j'\in i\Delta k} C_{j'}(y-\tau|\mathbf{X}-\tau)M_{j'}(x), \quad x \in \Omega_i .$$

We choose $\tau = \tau_j$, apply the functional λ_j to the left hand side and obtain

$$\lambda_j\Big(\sum_{|\beta|<k} \binom{k}{|\beta|}(x-\tau_j)^\beta(y-\tau_j)^\beta\Big) = \sum_{|\alpha|<k} (\alpha!)^{-1}D^\alpha C_j(0|\mathbf{X}-\tau_j)(y-\tau_j)^\alpha .$$

Since the set of polynomials $\{(1+(x-\tau_j)(y-\tau_j))^k\}_{y\in\mathbb{R}^m}$ spans P_k, the proof is complete, if we show

$$\sum_{|\alpha|<k} (\alpha!)^{-1}D^\alpha C_j(0|\mathbf{X}-\tau_j)(y-\tau_j)^\alpha = C_j(y-\tau_j|\mathbf{X}-\tau_j), \quad y \in \mathbb{R}^m .$$

But this follows simply by differentiating with respect to y at $y = \tau_j$.

Formula (29) is only one particular representation of the dual basis for M_j, $j \in I\Delta k$. We now construct bounded extensions $\Lambda_j := \Lambda_j(\mathbf{X})$ of the functionals λ_j to L_p, i.e. $\Lambda_j : L_p \to \mathbb{R}$

(32)
$$\Lambda_j s = \lambda_j s \quad , \quad s \in S_k .$$

If we require $\operatorname{supp}\Lambda_j \subset \Omega_i$, $j \in i\Delta k$, the system (32) reduces to

(33)
$$\Lambda_j p = \lambda_j p \; , \quad p \in P_k .$$

Hence we may determine Λ_j from the identities (19), (21). After a change of variables we have

(21')
$$(x-\tau)^\alpha = \sum_{j\in I\Delta k} (\alpha!)^{-1}\binom{k}{|\alpha|}^{-1} D^\alpha C_j(0|\mathbf{X}-\tau)M_j(x)$$

and clearly, (33) is equivalent to the system

(33')
$$\Lambda_j(\cdot - \tau)^\alpha = (\alpha!)^{-1}\binom{k}{|\alpha|}^{-1}D^\alpha C_j(0|\mathbf{X}-\tau), \; |\alpha| < k .$$

Let L_α be any set of functions dual to the monomials on the unit ball B, i.e.

(34)
$$\int_B L_\alpha(x)\, x^\beta dx = \delta_{\alpha\beta} \; , \; |\alpha|, \; |\beta| < k .$$

We transform the functions L_α to the balls Ω_i (c.f. Definition 4) and normalize them in $L_{p'}$,

(35)
$$\phi_\alpha(x) = h_i^{-m/p'} \; L_\alpha(h_i^{-1}(x-x_i))$$

$$\|\phi_\alpha\|_{p',\Omega_i} \le C(m,k,p),$$

i.e. ϕ_α represents a bounded functional on $L_p(\Omega_i)$. We now set

(36)
$$\Lambda_j = \sum_{|\alpha|<k} a_\alpha \, \phi_\alpha$$

and determine the coefficients from (33'). Choosing $\tau = x_i$, $j \in i\Delta k$, we obtain

$$\Lambda_j(\cdot - x_i)^\alpha = \sum_{|\beta|<k} a_\beta \, h_i^{m/p} \int_B L_\beta(x)(h_i x)^\alpha dx = a_\alpha h_i^{m/p+|\alpha|} \tag{37}$$

which implies

$$a_\alpha = (\alpha!)^{-1}\binom{k}{|\alpha|}^{-1} h_i^{-m/p-|\alpha|} \, D^\alpha C_j(0|\mathbf{X} - x_i) \,. \tag{38}$$

To simplify statement and proof of the theorem below we introduce the following notation. With a space of splines $S_{k,\mathbf{X},I}$ we associate a partition $\dot{\bigcup_\nu} U_\nu = \mathbb{R}^m$ such that for each $j \in I\Delta k$ there exists a set of indices J_j such that

$$\operatorname{supp} M_j = \bigcup_{\nu \in J_j} U_\nu . \tag{39}$$

Moreover we define

$$J_\nu := \{j \in I\Delta k \,|\, \operatorname{supp} M_j \cap U_\nu \neq \phi\}. \tag{40}$$

Note, that in the univariate case (15) we may take $U_\nu = (t_\nu, t_{\nu+1})$ and clearly $\#J_j = \#J_\nu = k+1$.

<u>Theorem 5. Let</u> $\mathbf{X}$ <u>be a regular perturbation of a triangulation</u> (X,I). <u>We define the</u> L_p-<u>normalized B-spline basis for</u> $S_{k,\mathbf{X},I}$ <u>by</u>

$$M_j^{(p)}(\cdot|\mathbf{X}) = \delta_j^{(p)} M_j(\cdot|\mathbf{X}),$$

$$\|M_j^{(p)}\|_p = 1 \tag{41}$$

<u>and assume that</u> (c.f. (39), (40))

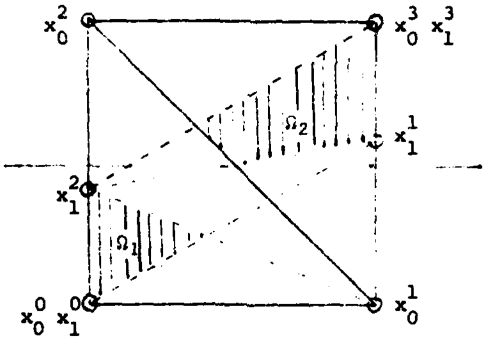
$$\# J_\nu \leq d \,. \tag{42}$$

<u>Then there exists a constant</u> $\kappa_s = c(m,k,p,\kappa)$ <u>such that</u>

$$\kappa_s \Big(\sum_{j \in I\Delta k} |a_j|^p\Big)^{1/p} \leq \Big\|\sum_{j\in I\Delta k} a_j M_j^{(p)}\Big\|_p \leq d^{1/p'} \Big(\sum_{j\in I\Delta k} |a_j|^p\Big)^{1/p} . \tag{43}$$

For univariate spline functions this result is due to C. de Boor [1-3] and has become a basic tool in spline approximation. The following example shows that the assumptions of Theorems 4,5 do not necessarily imply that the B-splines M_j, $j \in I\Delta k$, correspond to a triangulation of $\mathbb{R}^m \times \Delta_k$.

Example 6. Consider the following set of knots which can be extended to a regular perturbation of a triangulation of $\mathbb{R}^2$.



$$x_0^0 = x_1^0 = (-1,-1) \quad x_0^1 = (1,-1),\ x_0^2 = (-1,1),$$
$$x_1^3 = x_1^3 = (1,1),\ x_1^1 = (1,\varepsilon),\ x_1^2 = (-1,-\varepsilon)$$

The balls Ω_i can be chosen in the sets Ω_1, Ω_2 and the coefficient of the B-spline $M(\cdot|x_0^0,x_0^1,x_1^1,x_1^2)$ in the expansion

(16')
$$1 \equiv \sum c_j M_j$$

is given by (c.f. Theorem 3)

$$\frac{1}{6}\ \text{sgn det}\begin{vmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}\ \det\begin{vmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & \varepsilon & -\varepsilon \\ 1 & 1 & 1 & 1 \end{vmatrix} = -\frac{2}{3}\varepsilon .$$

Hence the coefficient may vanish or become negative although the assumptions in Theorems 4,5 are satisfied. Since the B-splines are linearly independent the coefficients in (16') are unique and therefore there cannot exist an associated triangulation of $\mathbb{R}^2 \times \Delta_1$ for $\varepsilon > 0$. This is also the reason why we cannot use the coefficients c_j to normalize the B-splines, i.e. $M_j^{(p)} = (c_j)^{-1/p'} M_j$, *as in the univariate case*, unless we make some more restrictive assumptions on the distribuion of the knots.

Proof of Theorem 5. "$\le$" By (42) we have (c.f. (39), (40))

$$\|\sum a_j M_j^{(p)}\|_{p,U_\nu}^p \le d^{p/p'} \sum_{j\in J_\nu} |a_j|^p \|M_j^{(p)}\|_{p,U_\nu}^p .$$

Summing this inequality with respect to ν and interchanging the order of summation yields

$$\|\sum a_j M_j^{(p)}\|_p^p \leq d^{p/p'} \sum_{j\in I\Delta k} \sum_{\nu\in J_j} |a_j|^p \|M_j^{(p)}\|_{p,U_\nu}^p = d^{p/p'} \sum_{j\in I\Delta k} |a_j|^p .$$

"$\geq$" We use the functionals constructed above (c.f. (36), (38)) and take into account the different normalization of the B-splines

$$\Lambda_j^{(p)} := (\delta_j^{(p)})^{-1} \Lambda_j .$$

We shall show that $\|\Lambda_j^{(p)}\|_{p'}$ is uniformly bounded which implies the local estimates

$$|a_j| \leq c \|\sum a_j M_j^{(p)}\|_{p,\Omega_i} , \quad j\in I\Delta k .$$

Summing this inequality with respect to $j \in I\Delta k$ gives the lower estimate.

Since $\|\phi_\alpha\|_{p'} \leq c$ we have by (38)

(44) $$\|\Lambda_j^{(p)}\|_{p'} \leq c\, (\delta_j^{(p)})^{-1} \sum_{|\alpha|\leq k} h_i^{-m/p-|\alpha|} \; D^\alpha C_j(0|\mathbf{x}-x_i) .$$

We first observe that

(45) $$(\delta_j^{(p)})^{-1} \leq c\, h_i^{-m/p'} .$$

To see this, consider the set of knots $\{x_{r_0}^{j_0}, \ldots, x_{r_n}^{j_n}\}$ of the B-spline M_j. This set must contain points $\tilde{x}^0,\ldots,\tilde{x}^m$ such that $\Omega_i \cap \sigma(\tilde{x}^0,\ldots,\tilde{x}^m) \neq \emptyset$, since otherwise $\Omega_i \cap \operatorname{supp} M_j = \emptyset$ contradicting (25). (26) implies $S_{k|\Omega_i} = P_{k|\Omega_i}$ in particular $M_{j|\Omega_i} \in P_{k|\Omega_i}$. Hence we must have $\Omega_i \subset \sigma(\tilde{x}^0,\ldots,\tilde{x}^m)$ since M_j is not smooth across the (m-1) dimensional faces of this simplex. We associate with M_j the simplex

$$\tilde{\sigma}_j := \sigma((\tilde{x}^0,e_0),\ldots,(\tilde{x}^m,e_0),(\overset{*}{x}{}^1,e_1),\ldots,(\overset{*}{x}{}^k,e_k))$$

where $\{\overset{*}{x}{}^1,\ldots,\overset{*}{x}{}^k\} = \{x_{r_0}^{j_0},\ldots,x_{r_n}^{j_n}\} \setminus \{\tilde{x}^0,\ldots,\tilde{x}^m\}$. Clearly

$$\operatorname{vol} \tilde{\sigma}_j = \operatorname{vol} \sigma((\tilde{x}^0,e_0),\ldots, (\tilde{x}^m,e_0),(0,e_1),\ldots,(0,e_k)) \sim h_i^m .$$

and by Definition 1 we obtain

$$\|M_j\|_p \leq c\, h_i^{-m} \quad h_i^{m/p} = c\, h_i^{-m/p'}$$

which implies the estimate (45).

Now consider the term $D^\alpha C_j(0|\mathbf{x}-x_i)$. By its definition (c.f. Theorem 3) and (22) it is the sum of terms of the form

$$D^\alpha [\prod_{\nu=1}^{m} b_\nu \prod_{\nu=1}^{k} (1+ z^\nu y)]_{|y=0}$$

where $|b_\nu|,\ |z^\nu| \le \kappa h_i$. This yields the estimate

$$|D^\alpha C_j(0|\mathbf{X}-x_i)| \le c(m,k)(\kappa h_i)^{m+|\alpha|}. \tag{46}$$

Combining the estimates (45), (46) we obtain

$$\|\Lambda_j^{(p)}\|_{p'} \le c(m,k,p)\kappa^{m+k}. \tag{47}$$

<u>Corollary 3.</u> Let $S_{k,\mathbf{X},I}$ be a space of splines not necessarily satisfying conditions (25)-(27). Assume that for a family of bounded functionals $L_j : L_p \to \mathbb{R}$, $\operatorname{supp} L_j \subset \operatorname{supp} M_j$, $\|L_j\|_p \le c$, the associated "quasiinterpolant"

$$Qf := \sum_{j\in I\Delta k} (L_j f) M_j^{(p)} \tag{48}$$

reproduces polynomials of total degree $\le r$, i.e.

$$Qp = p, \quad p \in P_r. \tag{49}$$

Then we have

$$\|Qf - f\|_{p,\Omega} \le (dc+1)\, \operatorname{dist}_{L_p(\tilde{\Omega})}(f, P_r) \tag{50}$$

where $\tilde{\Omega} := \cup\{\operatorname{supp} M_j \mid \operatorname{supp} M_j \cap \Omega \ne \phi\}$ and d is an upper bound for $\#J_j, \#J_\nu$ (c.f. (39), (40)).

<u>Proof.</u> Denote by p the best approximation to f on $\tilde{\Omega}$. By (49) we have

$$\|Qf-f\|_p \le \|Q(f-p)\|_p + \|f-p\|_p.$$

The first term can be estimated by (c.f. (39), (40))

$$\|Q(f-p)\|_p^p = \sum_\nu \Big\|\sum_{j\in J_\nu} L_j(f-p) M_j^{(p)}\Big\|_{p,U_\nu}^p \le \sum_\nu d^{p/p'} \sum_{j\in J_\nu} \Big(\sum_{\mu\in J_j} c^p \|f-p\|_{p,U_\mu}^p\Big) \|M_j^{(p)}\|_{p,U_\nu}^p.$$

Interchanging the order of summation we obtain by (41)

$$\le d^{p/p'} \sum_{\substack{j \\ \operatorname{supp} M_j\ \Omega\ne\phi}} \sum_{\mu\in J_j} \sum_{\nu\in J_j} c^p \|f-p\|_{p,U_\mu}^p \|M_j^{(p)}\|_{p,U_\nu}^p \le d^{p/p'} \sum_j \sum_{\mu\in J_j} c^p \|f-p\|_{p,U_\mu}^p$$

$$= d^{p/p'} \sum_\mu \sum_{j\in J_\mu} c^p \|f-p\|_{p,U_\mu}^p \le d^{p/p'} dc^p \sum_\mu \|f-p\|_{p,U_\mu}^p \le d^{p/p'+1} c^p \|f-p\|_{p,\tilde{\Omega}}^p.$$

Corollary 3 is the essential error estimate for local approximation schemes (c.f. [1,4] for the univariate and [18] for the tensor product case). The dual functionals $\Lambda_j^{(p)}$ are one particular choice for L_j . But, they are merely of theoretical importance. Efficient schemes should be based on point evaluation, etc. Of course, in principal, such methods could be constructed along the same lines. In order to get a smooth approximation the major problem, however, is to choose the knots so that the derivatives of the B-splines are not too big. Moreover there should be efficient ways for computing the values of $L_j f$, M_j and precise estimates for the constants involved. These problems however exceed the scope of this paper and we give only one example illustrating the above concept. The scheme constructed below is in the univariate case due to M. J. Marsden and I. J. Schoenberg [20] and for two variables to T. N. T. Goodman and S. L. Lee [15].

Example 7. Consider the following special cases of identity (19') (c.f. also Theorem 3 and (21))

$$1 \equiv \sum_{j\in I\Delta k} c_j M_j$$

(51)
$$(x-\tau)_\nu = \sum_{j\in I\Delta k} k^{-1} \frac{\partial}{\partial y_\nu} c_j(y|\mathbf{X}-\tau)_{|y=0} \quad M_j(x)$$

and let

$$\xi_j := c_j^{-1} \quad k^{-1} \quad \frac{\partial}{\partial y_\nu} c_j(y|\mathbf{X})_{|y=0} \quad .$$

We define the approximation scheme

(52)
$$Qf = \sum_j f(\xi_j)\, c_j\, M_j$$

which by (51) reproduces linear functions. If we furthermore assume

(53)
$$c_j > 0 \ , \quad \xi_j \in \operatorname{supp} M_j$$

then Q is a positive linear operator and we obtain by a slight modification of Corollary 3

(54)
$$\|Qf-f\|_\infty \le c(m,k)\, h^2\, |f|_{2,\infty}$$

where $h := \sup_j \operatorname{diam} \operatorname{supp} M_j$.

In the univariate case, conditions (53) are automatically satisfied. In general we obtain from (51), if the B-splines are linearly independent,

$$\xi_j - \tau = C_j^{-1} k^{-1} \frac{\partial}{\partial y_\nu} C_j(y|X-\tau)|_{y=0} .$$

Let τ be the center of supp M_j. By (46) it is therefore necessary for $\xi_j \in$ supp M_j that

$$C_j^{-1} \leq c\ (\text{diam supp } M_j)^{-m}$$

which leads to restrictions for the distribution of the knots (c.f. Example 6).

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It is shown that the span of a collection of multivariate B-splines, $S_k := \operatorname{span}_{j\in J} M(\cdot|x^{j_0},\cdots,x^{j_{k+m}})$, contains polynomials of total degree $\le k$ if the index set $\{j=(j_0,\cdots,j_{k+m})\}_{j\in J}$ has a certain combinatorial structure, in particular the knots $\{x^\nu\}\subset \mathbb{R}^n$ can be chosen essentially arbitrarily. Under mild assumptions on the distribution of the knots, a dual basis $\{\lambda_j\}_{j\in J}$ for S_k is constructed which has local support, i.e., (continued)

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20. Continued

supp $\lambda_j \subset$ supp M_j, and the condition number of the B-spline basis is estimated. This leads to good local approximation schemes.