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REGULARIZING EFFECTS FOR

$u_t + A\varphi(u) = 0$ IN $L^1$


10 Michael G./Crandall and Michel/Pierre


6 Regularizing Effects for u subt + A(psi(u))=0 in L1 vector space

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# REGULARIZING EFFECTS FOR $u_t + A\varphi(u) = 0$ IN $L^1$.

Michael G. Crandall and Michel Pierre

Technical Summary Report # 2187
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du/dt + A(psi(u))=0

L1 VECTOR SPACE

psi: R APPROACHES R

## ABSTRACT

Various initial-boundary value problems and Cauchy problems can be written in the form $\frac{du}{dt} + A\varphi(u) = 0$ where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing and $A$ is the linear generator of strongly continuous nonexpansive semigroup $e^{-tA}$ e to the -tA power in an $L^1$ space. For example, if $A = -\Delta$ (subject, perhaps, to suitable next page boundary conditions) we obtain equations arising in flow in a porous medium or plasma physics (depending on the choice of $\varphi$) while if $A = \frac{\partial}{\partial x}$ acting in $L^1(\mathbb{R})$ we have a scalar conservation law. In this paper we show that if $M$, $m > 0$ and $m\varphi'^2 \leq \nu\varphi\varphi'' \leq M\varphi'^2$, where $\nu \in \{1,-1\}$, then (roughly speaking), the norm of $t\,du/dt$ may be estimated in terms of the initial data $u_0$ in $L^1$. Such estimates give information about the regularity of solutions, asymptotic behaviour, etc., in applications.

Side issues, such as the introduction of sufficiently regular approximate problems on which estimates can be made and the assignment of a precise meaning to the operator "$A\varphi$", are also dealt with. These considerations are of independent interest.

AMS(MOS) Subject Classification: 35K15, 35K55

Key Words: regularizing effect, porous flow equations,
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## SIGNIFICANCE AND EXPLANATION

Many models of interesting phenomena yield equations for the evolution of a system of ~~the~~ this abstract form $u' + A\varphi(u) = 0$ where $\varphi$ psi is a nonlinear nondecreasing function and $A$ is an "operator". E.g., $A$ may be the del/del x Laplacian (perhaps under boundary conditions) or $A$ may be $\frac{\partial}{\partial x}$, while $\varphi$ psi may be a power law, $\cancel{\varphi(x) = |x|^{\alpha-1}x}$. Models like this occur in porous flow, plasmas and conservation laws. In this work it is shown that a broad class of such problems are solvable by the nonlinear semigroup theory. The main point, however, is a "regularizing" effect which estimates the speed of the system at time $t > 0$ by the integral of the initial data. This has consequences for the regularity of the solutions of concrete problems and their asymptotic behaviour.

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REGULARIZING EFFECTS FOR $u_t + A\varphi(u) = 0$ IN $L^1$.

Michael G. Crandall and Michel Pierre

Introduction.

When applied to a solution $u$ of the equation

$$u_t - \Delta\varphi(u) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N \tag{1}$$

one of the main results of this paper implies that

$$\int_{\mathbb{R}^N} |u_t(t,x)|dx \le \frac{C}{t} \int_{\mathbb{R}^N} |u(0,x)|dx \tag{2}$$

provided $\varphi$ is nondecreasing, $\varphi(0) = 0$ and has the property

$$0 < m \le \nu \frac{\varphi(r)\ \varphi''(r)}{(\varphi'(r))^2} \le M \quad \text{a.e.} \quad r \in \mathbb{R} \quad \text{for} \quad \nu = 1 \quad \text{or} \quad \nu = -1\ . \tag{3}$$

Indeed, when (3) holds so does (2) and $C$ depends only on the structure constants $m$ and $M$ of (3). Note that the initial data $u(0,x)$ need only belong to $L^1(\mathbb{R}^N)$. The validity of the "$L^1$-regularizing" inequality (2) depends strongly on the properties of the operator "$-\Delta\varphi$" in the space $L^1(\mathbb{R}^N)$. These properties are in fact enjoyed by a large class of operators of the form $A\varphi$ where $\varphi$ is as above and $A$ is a linear operator in an $L^1$ space. Indeed, it is enough that $-A$ be the infinitesimal generator of a strongly continuous nonexpansive semigroup $e^{-tA}$ in $L^1$ such that $0 \le u_0 \le 1$ a.e. implies $0 \le e^{-tA}u_0 \le 1$ a.e. (i.e., $e^{-tA}$ is submarkovian). Thus the results apply to (1) set in a bounded domain with linear homogeneous boundary conditions of Dirichlet or Neumann type imposed on $\varphi(u)$. Similarly, $-\Delta$ can be replaced by more general elliptic operators and we can, for example, also exhibit the conservation law

$$u_t - \varphi(u)_x = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R} \tag{4}$$

as an example of the theory developed here.

The estimate (2) is already known if $\varphi(r) = r$. In this event (1) is the linear heat equation and (2) says that $-\Delta$ generates an analytic semigroup in $L^1(\mathbb{R}^N)$, which is obvious from the solution formula. There has not been much success in developing a general nonlinear analogue of the linear idea of an analytic semigroup and only a few nonlinear

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results with estimates like (2) have been found. We refer to [7] for more comments in this direction as well as to [4] where a large class of homogeneous nonlinearities are exhibited which permit estimates like (2). The main contribution of this paper is the introduction of interesting new classes of such nonlinear examples.

If $\varphi(r) = |r|^{\alpha}\mathrm{sign}\,r$ with $\alpha > 0$, then $\varphi(r)\varphi''(r)/(\varphi'(r))^2 = (\alpha-1)/\alpha$ and (3) holds with $m = [\alpha-1]/\alpha = M$ and $\nu = \mathrm{sign}(\alpha-1)$ if $\alpha \neq 1$. In this case (1) is covered by the results of [7]. Note that we exclude $\alpha = 1$ here. As mentioned in [7], this is not surprising since the proof of our results also applies to (4) and no estimate like (2) holds if $\varphi(r) = r$ in (4).

We also show in this paper that nonnegative solutions of

(5) $$\frac{du}{dt} + A\varphi(u) = 0 ,$$

(which is given a precise sense in the text) satisfy a pointwise estimate

(6) $$u_t \geq -C\frac{u}{t}$$

for the class of operators $A\varphi$ where $A$ is as above and the nondecreasing function $\varphi$ satisfies $\varphi(0) = 0$ and

(7) $$0 < m \leq \frac{\varphi(r)\varphi''(r)}{(\varphi'(r))^2} \quad \text{a.e.} \quad r > 0 .$$

It was previously observed by L. C. Evans and one of the authors that (7) implies (6) for nonnegative solutions of (1). (Pointwise estimates like (6) are enjoyed only by nonnegative solutions.) For $\varphi(r) = r^{\alpha}$, $\alpha > 0$, this was first shown in the case of (1) by Aronson and Benilan [1] while [7] covers a general class of homogeneous nonlinearities. The paper [8] covers (1) for a quite general class of nonlinearities (considerably more general than [7]), but this result requires extensive exploitation of special properties of the Laplace operator. Here our result is more abstract, in the spirit of [7].

The first section is devoted to the abstract results. As usual, the problem of defining "$A\varphi$" (and hence (5)) in a precise sense must be disposed of. Similarly, the appropriate meaning must be given to (2), its abstract analogue, and (6). These matters and the approximations introduced in the proofs of the main results are of substantial independent interest. Several proofs of results used in the sequel are collected in the Appendix.

Section 1.

Throughout this section $\Omega$ denotes a $\sigma$-finite measure space with the measure denoted by "meas". The norm of $L^p(\Omega)$ is denoted by $\| \ \|_p$. The integral of $f \in L^1(\Omega)$ over a measurable $\mathcal{O} \subset \Omega$ is written either as $\int_{\mathcal{O}} f$ or $\int_{\mathcal{O}} f dx$.

Recall that a (possibly nonlinear) mapping $A: D(A) \subset X \to X$ in a Banach space $X$ is accretive if for each $\lambda > 0$ $(I+\lambda A)^{-1}$ is a nonexpansive mapping of $R(I+\lambda A)$ (the range of $I+\lambda A$) into $X$. If $A$ is accretive and $R(I+\lambda A) = X$ for $\lambda > 0$ (equivalently, $R(I+A) = X$), then $A$ is m-accretive. If $A$ is linear and densely defined, then $A$ is m-accretive if and only if $-A$ is the infinitesimal generator of a (linear) strongly continuous nonexpansive semigroup $e^{-tA}$ on $X$. More generally, if $A$ is accretive and $R(I+\lambda A) \supseteq \overline{D(A)}$ for $\lambda > 0$ it determines a (in general, nonlinear) strongly continuous nonexpansive semigroup $e^{-tA}$ on $\overline{D(A)}$. (We use the notation $e^{-tA}$ in the linear and nonlinear cases.) See, e.g. [2], [6], [9].

We assume a densely defined linear operator $A: D(A) \subset L^1(\Omega) \to L^1(\Omega)$ is given which satisfies

(A1) $$A \text{ is m-accretive in } L^1(\Omega)$$

and

(A2) $$\begin{cases} \text{If } \lambda > 0,\ f \in L^1(\Omega),\ a,b \in R \text{ and } a \le f \le b \text{ a.e.,} \\ \text{then } a \le (I+\lambda A)^{-1} f \le b \text{ a.e.} \end{cases}$$

Since $A$ is linear, densely defined and m-accretive, (A2) is equivalent to $0 \le f \le 1 \Longrightarrow 0 \le e^{-tA} f \le 1$. (Actually, (A1) and (A2) imply $D(A)$ is dense ([10]).) It was proved in [5] that for linear m-accretive $A$'s as above, (A2) is equivalent to

(A3) $$\begin{cases} \text{If } \beta \text{ is a maximal monotone graph in } \mathbb{R} \times \mathbb{R} \text{ with } 0 \in \beta(0),\ u \in D(A), \\ Au \in L^p(\Omega),\ 1 \le p \le \infty,\ v \in L^{p/(p-1)}(\Omega),\ v(x) \in \beta(u(x)) \text{ a.e. then} \\ \int_\Omega v(x) Au(x) dx \ge 0 . \end{cases}$$

The proper interpretation of "$A\varphi$" is discussed next. Set

$$P_0 = \{\varphi: \mathbb{R} \to \mathbb{R};\ \varphi \text{ is continuous, nondecreasing and } \varphi(0) = 0\}.$$

For any $\varphi \in P_0$ and $B: D(B) \subseteq L^1(\Omega) \to L^1(\Omega)$ the operator $B\varphi$ in $L^1(\Omega)$ is first defined in the obvious way:

$$
(1.1)\qquad \begin{cases} D(B\varphi) = \{u \in L^1(\Omega);\ \ \varphi(u) \in D(B)\} \\ \\ \forall u \in D(B\varphi)\ ,\ \ B\varphi(u) = B(\varphi(u)). \end{cases}
$$

The proposition below summarizes some results which follow easily from the results and arguments of, e.g., [5] .

<u>Proposition 1.</u> Let $A$ be linear, densely defined and satisfy (A1), (A2). Let $\varphi \in P_0$. Then:

(i) $A\varphi$ is accretive in $L^1(\Omega)$ .

(ii) For each $\varepsilon > 0$ and $\lambda > 0$, $\varepsilon I + A(I+\lambda A)^{-1}$ satisfies (A1), (A2) .

(iii) For each $\varepsilon > 0$ , $(\varepsilon I+A)\varphi$ is m-accretive in $L^1(\Omega)$ .

(iv) For $\lambda > 0$ , $(I+\lambda A\varphi)^{-1}$ is an order-preserving nonexpansive mapping of $R(I+\lambda A\varphi)$ into $L^1(\Omega)$. Moreover, $f \in R(I+\lambda A\varphi)$ $a,b \in R$ and $a \leq f \leq b$ a.e. implies $a \leq (I+\lambda A\varphi)^{-1}f \leq b$ a.e.

(v) $\|u\|_p \leq \|(I+\lambda A\varphi)(u)\|_p$ for $u \in D(A\varphi)$, $1 \leq p \leq \infty$.

The main omission of Proposition 1 is the assertion that $A\varphi$ is m-accretive. In general this fails even if $A$ satisfies (A1), (A2). However, the pair $(A,\varphi)$ typically determines an m-accretive operator $A_\varphi$ which extends $A\varphi$ and $(A,\varphi)$ always determines an accretive operator $A_\varphi$ for which $R(I+\lambda A_\varphi) \supset L^1(\Omega)^+$ as is stated in the next proposition.

<u>Proposition 2.</u> Let $A$ be linear, densely defined and satisfy (A1), (A2). Let $\varphi \in P_0$ and assume at least one of the conditions:

(i) $\varphi$ is strictly increasing,

(ii) $\exists r_0 > 0$ , $K$ such that $|\varphi(r)| \leq K|r|$ for $|r| \leq r_0$ ,

or

(iii) $\text{meas}\ (\Omega) < \infty$ .

Then there is an m-accretive operator $A_\varphi$ in $L^1(\Omega)$ which extends $A\varphi$ such that for every $\lambda > 0$ and $f \in L^1(\Omega)$

$$
(1.2)\qquad \lim_{\varepsilon \downarrow 0} (I+\lambda(\varepsilon I+A)\varphi)^{-1}f = (I+\lambda A_\varphi)^{-1}f.
$$

Moreover, for <u>every</u> $\varphi \in P_0$ there exists an accretive operator $A_\varphi$ in $L^1(\Omega)$ which extends $A\varphi$ such that (1.2) holds for every $\lambda > 0$ and $f \in L^1(\Omega)^+ = \{f \in L^1(\Omega): f \geq 0\}$.

Proposition 2 is tangential to our main concerns and is discussed and proved in the Appendix.

Each $\varphi$ we deal with will allow the application of Proposition 2, and we take $A_\varphi$ to be the correct interpretation of $A\varphi$ in (5). Solutions of (5) are then understood in the sense of nonlinear semigroup theory - i.e. $u(t) = e^{-tA_\varphi}u(0)$. An important fact for our presentation is the:

<u>Convergence Theorem:</u> Let $G_n$, $n = 1,2,\cdots,\infty$ be a sequence of accretive operators in $L^1(\Omega)$ such that $\overline{D(G_n)} \supset D(G_\infty)$ and $R(I+\lambda G_n) \supset \overline{D(G_n)}$ for $n = 1,2,\cdots,\infty$ and $\lambda > 0$. Assume

$$\lim_{n\to\infty} (I+\lambda G_n)^{-1}f = (I+\lambda G_\infty)^{-1}f$$

for $f \in D(G_\infty)$ and $\lambda > 0$. Then whenever $f_n \in \overline{D(G_n)}$ and $f_n \to f_\infty \in \overline{D(G_\infty)}$ we have

$$\lim_{n\to\infty} e^{-tG_n}f_n = e^{-tG_\infty}f_\infty$$

uniformly for bounded $t \geq 0$. (All convergences are in $L^1(\Omega)$.)

This theorem is a special case of known results (see, e.g., [6] for references). It follows from Proposition 1 and the convergence theorem that $\lim_{\varepsilon\downarrow 0} e^{-t(\varepsilon I+A)\varphi}u_\varepsilon = e^{-tA_\varphi}u$ uniformly for bounded $t \geq 0$ whenever $u_\varepsilon \in D((\varepsilon I+A)\varphi)$ converges to $u \in \overline{D(A_\varphi)}$.

Our main goal is to estimate the speed of the semigroup $e^{-tA_\varphi}$ generated by $-A_\varphi$ under suitable assumptions on $\varphi$. We will prove:

<u>Theorem 3.</u> Let $A$ be linear, densely defined, satisfy (A1), (A2) and $\varphi \in P_0$. Assume

(1.3) $\qquad \varphi \in C^1(\mathbb{R}\backslash\{0\})$, $\varphi'$ is locally Lipschitz on $\mathbb{R}\backslash\{0\}$,

and

(1.4) $\qquad$ There exists $m, M > 0$ and $\nu \in \{-1,1\}$ such that

$$m(\varphi'(r))^2 \leq \nu\varphi(r)\varphi''(r) \leq M(\varphi'(r))^2 \quad \text{a.e. } r \in \mathbb{R}.$$

Then $\varphi$ satisfies either (i) or (ii) of Proposition 2 and for $S(t) = e^{-tA_\varphi}$, $u_0 \in \overline{D(A_\varphi)}$, (1.5) $$\lim_{h \downarrow 0} \frac{\|S(t+h)u_0 - S(t)u_0\|_1}{h} \le \frac{C}{t} \|u_0\|_1$$

where $C = 2(M+1)(m+2M)/m^2$ .

**Remarks:**

($\alpha$) The assumption (1.4) is a natural generalization of the condition $\varphi\varphi''/(\varphi')^2 = C \neq 0$ which is the homogeneous case treated in [7]. Note that $\nu = 1$ and $\nu = -1$ correspond to quite different behaviours of $\varphi$. For instance, if $\nu = 1$ then $\varphi$ is convex on $[0,\infty)$ while if $\nu = -1$ it is concave.

One can easily see that for $\varphi \in P_0$ , (1.3) and (1.4) are equivalent to

(1.6) $$\begin{cases} \varphi \in C^1(\mathbb{R} \setminus \{0\}), \ \varphi/\varphi' \text{ is Lipschitz continuous on } \mathbb{R} \text{ and} \\ \qquad \nu - m \ge \nu\left(\frac{\varphi}{\varphi'}\right)' \ge \nu - M \\ (\text{where } \varphi/\varphi' \text{ is understood to vanish if } \varphi(r) = \varphi'(r) = 0 \text{ or } r = 0), \end{cases}$$

or

(1.7) $$\begin{cases} r \to \frac{\nu}{1-\nu m} |\varphi(r)|^{1-\nu m} \ (\log|\varphi(r)| \text{ if } \nu m = 1) \text{ is convex and} \\ r \to \frac{\nu}{1-\nu M} |\varphi(r)|^{1-\nu M} \ (\log|\varphi(r)| \text{ if } \nu M = 1) \\ \text{is concave on each of } (-\infty,0) \text{ and } (0,\infty) . \end{cases}$$

Note that $\nu = 1$ implies $m \le 1$ . Also note that if $\nu = 1$ , the convexity implies $|\varphi(r)| \le K|r|$, $K = \max(\varphi'(r_0+), \ \varphi'(r_0-))$ on $|r| \le r_0$ so Proposition 2(ii) holds, while if $\nu = -1$ either $\varphi \equiv 0$ on $[0,\infty)$ or $\varphi$ is strictly increasing by (1.6) and Proposition 2(i) holds.

($\beta$) It would be interesting to know if the existence of the upper bound $M$ in (1.4) is necessary to have an estimate like (1.5). Our next result shows one needs only $m$ if the initial data is nonnegative and $\nu = 1$ .

**Theorem 4.** Let $A$ be linear, densely defined, satisfy (A1), (A2) and $\varphi \in P_0$ . Assume $m > 0$, $m \neq 1$, $\nu \in \{-1,1\}$ and

(1.8) $$r \to \frac{\nu}{1-\nu m} \ \varphi(r)^{1-\nu m} \text{ is convex on } (0,\infty).$$

Let $A_\varphi$ be as in Proposition 2 and $S(t) = e^{-tA_\varphi}$. Then for $u_0 \geq 0$, $u_0 \in \overline{D(A_\varphi)}$

$$(1.9) \qquad t \to \nu t^{\nu/m} \varphi(S(t)u_0(x)) \text{ is nondecreasing a.e. } x \in \Omega .$$

If also $\nu = 1$ (so $m < 1$), then

$$(1.10) \qquad \lim_{h \downarrow 0} \frac{S(t+h)u_0 - S(t)u_0}{h} \leq \frac{2(1-m)}{mt} \| u_0 \|_1 .$$

Remarks: Notice that (1.9) is a weak formulation of

$$\nu t \frac{du}{dt} \geq - \frac{1}{m} \frac{\varphi(u)}{\varphi'(u)}$$

where $u = S(t)u_0$. If $\nu = 1$, then $\varphi(u)/\varphi'(u) \leq (1 - m)u$, so we obtain

$t\frac{du}{dt} \geq - \frac{(1-m)}{m} u$. This means $t \to t^{\frac{1-m}{m}} S(t)u_0$ is nondecreasing, which may be deduced from (1.9) directly when $\nu = 1$.

We begin the proofs of Theorems 3 and 4. While the formal manipulations which are the basis of the main estimates are quite straightforward, there are considerable difficulties concerning regularity to be overcome. We use a four-layered approximation process to dispose of these difficulties. One has been introduced already, namely the approximation of $A_\varphi$ by $(\varepsilon I+A)\varphi$. To this we add the regularization of $A$ itself by its Yosida approximation $A_\lambda = \lambda^{-1}(I - (I+\lambda A)^{-1}) = A(I+\lambda A)^{-1}$ and, in turn, the replacement of $\varphi$ by its Yosida approximation $\varphi_\alpha = \alpha^{-1}(I - (I+\alpha\varphi)^{-1})$. A fourth approximation process is introduced later. We recall that $A_\lambda$ is m-accretive, defined on all of $L^1(\Omega)$ and bounded. Moreover, by Proposition 1, $\| (I+\lambda A)^{-1} f \|_p \leq \|f\|_p$ for $f \in L^p(\Omega) \cap L^1(\Omega)$, $1 \leq p \leq \infty$ and $\lambda > 0$. Thus $A_\lambda : L^1(\Omega) \cap L^p(\Omega) \to L^1(\Omega) \cap L^p(\Omega)$ and $A_\lambda$ on this domain is accretive and Lipschitz continuous in the $L^p(\Omega)$ norm. The next lemma handles the problem of passing to the limit in the approximation of $\varphi$ by $\varphi_\alpha$ as $\alpha \downarrow 0$.

Lemma 5. Let $\varphi \in P_0$, $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$, $\varphi(u_0) \in L^1(\Omega)$. Let $\varepsilon, \lambda, \alpha > 0$, and $B = \varepsilon I + A_\lambda$. Then the problems

$$(1.11) \qquad \frac{du}{dt} + B\varphi_\alpha(u_\alpha) = 0 , \quad u_\alpha(0) = u_0 ,$$

and

$$(1.12) \qquad \frac{du}{dt} + B\varphi(u) = 0 , \quad u(0) = u_0 ,$$

have unique solutions $u_\alpha$, $u \in W^{1,\infty}([0,\infty):L^1(\Omega))$.

Moreover

(1.13)
$$\begin{cases} \lim\limits_{\alpha \downarrow 0} u_\alpha = u & \text{in} \quad C([0,T]:L^1(\Omega)) \\ \lim\limits_{\alpha \downarrow 0} \dfrac{du_\alpha}{dt} = \dfrac{du}{dt} & \text{in} \quad L^1(0,T:L^1(\Omega)) \end{cases}$$

for every $T > 0$.

<u>Proof of Lemma 5.</u> By Proposition 1, $B\varphi_\alpha$ and $B\varphi$ are m-accretive in $L^1(\Omega)$. Moreover $B\varphi_\alpha \to B\varphi$ as $\alpha \downarrow 0$ in the sense $(I+\lambda B\varphi_\alpha)^{-1}f \to (I+\lambda B\varphi)^{-1}f$ for $f \in L^1(\Omega)$, $\lambda > 0$. Indeed, if $f \in L^1(\Omega)$ and

$$v_\alpha + \lambda B\varphi_\alpha(v_\alpha) = f \ , \quad v + \lambda B\varphi(v) = f$$

we also have $v_\alpha - v + \lambda B\varphi_\alpha(v_\alpha) - \lambda B\varphi_\alpha(v) = \lambda B(\varphi(v) - \varphi_\alpha(v))$. Since $B\varphi_\alpha$ is accretive this *yields*

$$\|v_\alpha - v\|_1 \leq \lambda \|B\| \|\varphi(v) - \varphi_\alpha(v)\|_1$$

and the right-hand side tends to $0$ as $\alpha \downarrow 0$ because $\varphi(v) \in L^1(\Omega)$, $|\varphi_\alpha(v)| \leq |\varphi(v)|$ and $\varphi_\alpha(v) \to \varphi(v)$ a.e. by standard properties of the Yosida approximation $\varphi_\alpha$. Let $T_\alpha(t) = e^{-tB\varphi_\alpha}$ and $T(t) = e^{-TB\varphi}$. Since $B\varphi_\alpha \to B\varphi$, $u_\alpha(t) = T_\alpha(t)u_0 \to T(t)u_0 = u(t)$ in $L^1(\Omega))$ as $\alpha \downarrow 0$ uniformly for bounded $t \geq 0$. Now $B\varphi_\alpha$ is Lipschitz continuous so $u_\alpha \in C^1([0,\infty):L^1(\Omega))$ and

$$\left\| \frac{du_\alpha}{dt}(t) \right\|_1 \leq \|B\varphi_\alpha(u_0)\|_1 \quad \text{for} \quad t \geq 0$$

by the accretivity of $B\varphi_\alpha$. As $|\varphi_\alpha(u_0)| \leq |\varphi(u_0)|$, $B\varphi_\alpha(u_0)$ is bounded in $L^1(\Omega)$ independently of $\alpha > 0$ and hence so are $du_\alpha/dt$ and $\varphi_\alpha(u_\alpha) = -B^{-1}\left(\frac{du_\alpha}{dt}\right)$ bounded in $L^\infty(0,\infty:L^1(\Omega))$. Moreover

(1.14)
$$\int_0^T \int_\Omega \varphi_\alpha(u_\alpha)dxdt = \int_\Omega B^{-1}(u_0 - u_\alpha(T))dx.$$

Since $B$ enjoys the property (A2) together with $A$, $\|u_\alpha(t)\|_\infty \leq \|u_0\|_\infty$. It follows that $\varphi_\alpha(u_\alpha)$ is bounded in $L^\infty(\Omega)$ uniformly in $\alpha$, $t \geq 0$ and then, by interpolation, in every $L^p(\Omega)$. We conclude that $\frac{du_\alpha}{dt}$ is bounded in $L^\infty(0,\infty:L^2(\Omega))$. This together with $u_\alpha \to u$ in $C([0,\infty):L^1(\Omega))$, shows $\frac{du}{dt} \in L^\infty(0,\infty:L^2(\Omega))$ and $du_\alpha/dt \to du/dt$ weakly in $L^2(0,T:L^2(\Omega))$ for each $T > 0$. From $\varphi_\alpha(u_\alpha) = B^{-1}\left(-\frac{du_\alpha}{dt}\right)$ and the boundedness of $B^{-1}$ in $L^2(\Omega)$ it then follows that $\varphi_\alpha(u_\alpha) \to -B^{-1}\left(\frac{du}{dt}\right)$ weakly in $L^2(0,T:L^2(\Omega))$. On the other hand, $\varphi_\alpha(u_\alpha) \to \varphi(u)$ in measure and so $\varphi(u) = -B^{-1}(\frac{du}{dt})$, which establishes (1.12) and its consequence

$$(1.15) \qquad \int_0^T \int_\Omega \varphi(u)dxdt = \int_\Omega B^{-1}(u_0 - u(T)).$$

Assume now $u_0 \geq 0$ so $u_\alpha, u \geq 0$. By (1.14), (1.15) and $u_\alpha \to u$ in $C([0,\infty):L^1(\Omega))$ we conclude $\varphi_\alpha(u_\alpha) \to \varphi(u)$ in measure and

$$\int_0^T \int_\Omega \varphi_\alpha(u_\alpha)dxdt \to \int_0^T \int_\Omega \varphi(u)dxdt .$$

Since $\varphi_\alpha(u_\alpha), \varphi(u) \geq 0$ this implies $\varphi_\alpha(u_\alpha) \to \varphi(u)$ in $L^1(0,T:L^1(\Omega))$. If $u_0$ is not of fixed sign we may estimate $\varphi_\alpha(u_\alpha)$ by $\varphi_\alpha(v_\alpha) \leq \varphi_\alpha(u_\alpha) \leq \varphi_\alpha(w_\alpha)$ where $w_\alpha = T_\alpha(t)u_0^+$, $v_\alpha = T_\alpha(t)(-u_0^-)$. Since $\varphi_\alpha(v_\alpha)$ and $\varphi_\alpha(w_\alpha)$ converge in $L^1(0,T:L^1(\Omega))$ and $\varphi_\alpha(u_\alpha)$ converges in measure, $\varphi_\alpha(u_\alpha)$ converges in $L^1$. By the continuity of $B$

$$\frac{du_\alpha}{dt} = -B\varphi_\alpha(u_\alpha) \to \frac{du}{dt} \text{ in } L^1(0,T:L^1(\Omega)).$$

This completes the proof of the Lemma.

The next lemma, which establishes the desired estimates on solutions of (1.12) with a little extra regularity on $\varphi$, contains the heart of the proof.

Lemma 6. Let $\varphi \in P_0$, $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\varphi(u_0) \in L^1(\Omega)$. Let $u$ be the solution of (1.12).

(i) Let $\varphi \in C^2(\mathbb{R} \setminus \{0\})$ and satisfy (1.4). Then

$$(1.16) \qquad \|\frac{du}{dt}\|_1 \leq \frac{C(m,M)}{t} \|u_0\|_1$$

with $C(m,M) = 2(M+1)(m+2M)/m^2$.

(ii) Let $\varphi \in C^2(0,\infty)$, $\varphi/\varphi' \in C^1([0,\infty))$ and satisfy (1.8). Let $u_0 \geq 0$. Then

$$(1.17) \qquad \nu \frac{du}{dt} \geq -\frac{1}{mt} \frac{\varphi(u)}{\varphi'(u)} .$$

Remarks: If (1.8) is satisfied with $\nu = 1$ (and hence $m < 1$) and if $\varphi^{-1}(0) = [0,r_0]$, then for $r > r_0$

$$\frac{\varphi(r)^{1-m}}{1-m} \leq \frac{\varphi'(r)}{\varphi(r)^m}(r-r_0) \quad \text{or} \quad \frac{\varphi(r)}{\varphi'(r)} \leq (1-m)(r-r_0) .$$

If $\nu = -1$, then $r_0 = 0$ and $(\varphi/\varphi')(0+) = 0$, but $(\varphi/\varphi')'$ is not necessarily bounded in a neighborhood of $0$. Because of this we impose the extra condition

$(\varphi/\varphi') \in C^1([0,\infty))$ in (ii). Note that the stronger condition (1.4) implies (1.6) and so

$$(\nu-m) \geq \nu(\frac{\varphi}{\varphi'})' \geq \nu-M, \quad |\frac{\varphi(r)}{\varphi'(r)}| \leq (M+1)|r|$$

on $R\setminus\{0\}$.

*Proof of Lemma 5.* Throughout the computations to follow we will use the fact that if $p: R \to R$ is Lebesgue measurable and bounded, $j(r) = \int_0^r p(s)ds$ and $w \in W^{1,1}(0,T:L^1(\Omega))$, then $j(w) \in W^{1,1}(0,T: L^1(\Omega))$ and

$$\frac{d}{dt} j(w) = p(w) \frac{dw}{dt} \text{ a.e.}$$

In particular, the above relation with $p$ equal to the characteristic function of a null set $N \subset R$ (so $j \equiv 0$) implies that $\frac{dw}{dt}(t,x) = 0$ a.e. on $\{(t,x): w(t,x) \in N\}$.

The above is well-known when $L^1(\Omega) = R$. For the reader's convenience a proof for this case is given in Lemma a.1 of the Appendix. The general case follows by use of Fubini's theorem.

The main part of the proof of the lemma is the introduction of the function

$$v = tu_t + \rho \frac{\varphi(u)}{\varphi'(u)}, \tag{1.18}$$

where $\rho \in R$ is a parameter to be chosen, and the study of the equation satisfied by $v$. Here and below, the subscript $t$ denotes differentiation in $t$.

It is first assumed that $\varphi$ is locally Lipschitz on $R$ for (i) and on $[0,\infty)$ for (ii). (This is implied by the assumptions if $\nu = 1$; if $\nu = -1$ we later approximate $\varphi$ by $\varphi_\alpha$.) Since $u \in W^{1,\infty}(0,T:L^1(\Omega))$, we have $\varphi(u) \in W^{1,1}(0,T:L^1(\Omega))$ and $\varphi(u)_t = \varphi'(u)u_t$. As $B$ is linear and continuous (1.12) proves that $u \in W^{2,1}(0,T:L^1(\Omega))$ and

$$u_{tt} + B(\varphi'(u)u_t) = 0. \tag{1.19}$$

Differentiating (1.18) we find

$$v_t = u_t + tu_{tt} + \rho(\frac{\varphi}{\varphi'})'(u)u_t. \tag{1.20}$$

Taken together, (1.19) and (1.20) imply

$$\left\{ \begin{aligned} & tv_t + B(t\varphi'(u)v) + G(u)v = \rho G(u) \frac{\varphi(u)}{\varphi'(u)} \\ & \text{where } G(r) = -\rho(\frac{\varphi}{\varphi'})'(r) + \rho - 1 \end{aligned} \right. \tag{1.21}$$

Set

$$\operatorname{sign} r = \begin{cases} \{1\} & r > 0 \\ [-1,1] & r = 0 \\ \{-1\} & r < 0 \end{cases}, \quad \operatorname{sign}^- r = \begin{cases} \{0\} & r > 0 \\ [0,-1] & r = 0 \\ \{-1\} & r < 0 . \end{cases}$$

A selection out of $\operatorname{sign} v$ means a measurable function $\alpha$ such that $\alpha(x) \in \operatorname{sign} v(t,x)$ a.e. $x$, etc. To prove (i), multiply (1.21) by a selection out of $\operatorname{sign} v$ (which is a subset of $\operatorname{sign}\varphi'(u)v$) and use the accretivity of $B$ in $L^1(\Omega)$ to conclude

$$t \frac{d}{dt} \int_\Omega |v| + \int_\Omega |G(u)||v| \le |\rho| \int_\Omega |G(u)| \frac{|\varphi(u)|}{\varphi'(u)} . \tag{1.22}$$

If we choose $\rho = 2\nu/m$ the assumptions on $\varphi$ imply

$$1 \le G(r) \le \frac{2M}{m} - 1 , \quad \frac{|\varphi(r)|}{\varphi'(r)} \le (M+1)|r| . \tag{1.23}$$

The estimates (1.23) used in (1.22) and integration in time of the result yields

$$t \int_\Omega |v| + \int_0^t \int_\Omega (G(u)-1)|v| \le \frac{2}{m}\frac{2M}{m}(M+1) \int_0^t \int_\Omega |u| \le \frac{4M(M+1)}{m^2}\, t \,\|u_0\|_1 , \tag{1.24}$$

where the last inequality comes from the accretivity of $B\varphi$ in $L^1(\Omega)$ which implies $\|u\|_1$ is nonincreasing. From (1.24) we have

$$\|v\|_1 \le \frac{4M(M+1)}{m^2} \|u_0\|_1$$

and this with the definition (1.18) of $v$ implies

$$t\|u_t\|_1 \le \frac{4M(M+1)}{m^2} \|u_0\|_1 + \frac{2}{m} \int_\Omega \frac{|\varphi(u)|}{\varphi'(u)} \le \frac{2(M+1)}{m} \left(1 + \frac{2M}{m}\right)\|u_0\|_1 ,$$

whence the result.

For (ii), we chose $\rho = \nu/m$ which implies $G(r) \geq 0$ on $(0,\infty)$. Then multiply (1.21) by a selection out of $\nu \operatorname{sign}^-(\nu v)$ (which is a subset of $\nu \operatorname{sign}^-(\nu\varphi'(u)v)$) and use (A3) for $B$ to conclude

$$-t \frac{d}{dt} \int_\Omega (\nu v)^-(t) \geq 0 .$$

Since $\nu v(0) = \frac{1}{m} \frac{\varphi(u_0)}{\varphi'(u_0)} \geq 0$, the above implies $\nu v(t) \geq 0$. Recalling the definition (1.18) of $v$ this implies

$$\nu t u_t \geq -\frac{1}{m}\frac{\varphi(u)}{\varphi'(u)} . \tag{1.25}$$

This implies $\nu t \varphi(u)_t \geq -\varphi(u)/m$ which is equivalent to $(\nu t^{\nu/m}\varphi(u))_t \geq 0$.

When $\varphi$ is not locally Lipschitz on $R$ (or $[0,\infty)$ for (ii)) we approximate $\varphi$ by its Yosida approximation $\varphi_\alpha$ and $u$ by the $u_\alpha$ of (1.11). Unfortunately, $\varphi_\alpha$ need not satisfy (1.4). Indeed, $\varphi_\alpha(r) = \varphi(\gamma_\alpha(r))$ where $\gamma_\alpha(r) = (I+\alpha\varphi)^{-1}(r)$ and so

$$\begin{cases} \varphi'_\alpha(r) = \varphi'(\gamma_\alpha(r))/(1+\alpha\varphi'(\gamma_\alpha(r))) \quad \text{and} \\[2ex] \dfrac{d}{dr}\dfrac{\varphi_\alpha(r)}{\varphi'_\alpha(r)} = 1 - \dfrac{\varphi(\gamma_\alpha(r))\varphi''(\gamma_\alpha(r))}{[\varphi'(\gamma_\alpha(r))^2][1+\alpha\varphi'(\gamma_\alpha(r))]} . \end{cases}$$

It follows that if (1.4) holds, $G_\alpha$ is defined as in (1.21) with $\varphi_\alpha$ in place of $\varphi$ and $\rho = 2\nu/m$ then

$$\left|\frac{d}{dr}\frac{\varphi_\alpha}{\varphi_\alpha'}\right| \leq 1 + M \quad \text{and} \quad |G_\alpha(r)| \leq 2M/m . \tag{1.26}$$

Since $\varphi_\alpha$ is Lipschitz, computations leading to (1.22) are valid with $u_\alpha, v_\alpha = t\,u_{\alpha t} + \rho\varphi_\alpha(u_\alpha)/\varphi'_\alpha(u_\alpha)$, $G_\alpha$ in place of $u,v,G$ and integration together with (1.26) gives

$$t\int_\Omega |v_\alpha| + \int_0^t \int_\Omega (G_\alpha(u_\alpha)-1)|v_\alpha| \leq \frac{4M}{m^2}(1+M)t\|u_0\|_1 . \tag{1.27}$$

By Lemma 4, $u_{\alpha t} \to u_t$ in $L^1(0,T:L^1(\Omega))$. It also follows from (1.26) and $u_\alpha \to u$ in $C([0,T]:L^1(\Omega))$ that $\varphi_\alpha(u_\alpha)/\varphi'_\alpha(u_\alpha) \to \varphi(u)/\varphi'(u)$ in $L^1(0,T:L^1(\Omega))$. Hence $v_\alpha \to v$ in $L^1(0,T:L^1(\Omega))$. Since $\varphi \in C^2(R/\{0\})$, $G_\alpha(r)$ converges to $G(r)$ for $r \neq 0$. Hence $G_\alpha(u_\alpha)|v_\alpha|$ (interpreted as $0$ at points where $u_\alpha$ vanishes, since $v_\alpha = 0$ a.e. on $\{(t,x): u_\alpha(t,x) = 0\}$) converges to $G(u)|v|$ in $L^1(0,T:L^1(\Omega))$. Thus one may pass to the limit in (1.27) to obtain the desired conclusion.

To obtain (ii) we also approximate $u$ by $u_\alpha$ as above. The estimates (1.26) need not hold now, but since $\varphi/\varphi' \in C^1([0,\infty])$, $(\varphi_\alpha/\varphi'_\alpha)'$ and $G_\alpha$ remain locally bounded on $[0,\infty)$ uniformly in $\alpha$. Hence the convergence assertions above remain valid and we can pass to the limit in the inequality

$$-t\frac{d}{dt}\int_\Omega (\nu v_\alpha)^- - \int_\Omega G_\alpha(u_\alpha)(\nu v_\alpha)^- \geq -\frac{1}{m}\int_\Omega (G_\alpha(u_\alpha))^- \frac{\varphi_\alpha(u_\alpha)}{\varphi'_\alpha(u_\alpha)}$$

which is deduced from the $\alpha$-version of (1.21) with $\rho = \nu/m$ as before. Since $G(u) \geq 0$, we then obtain that $\int_\Omega (\nu v)^-$ is nondecreasing and finish as before.

<u>Proof of Theorem 3:</u>

There are three steps of the proof remaining. We first show that if $\varphi$ satisfies (1.3) and (1.4), then it can be locally approximated by functions $\varphi_n$ satisfying the hypothesis of Lemma 5 in such a way that $e^{-tB\varphi_n}$ converges suitably to $e^{-tB\varphi}$. Then we show that $B\varphi = (\varepsilon I+A_\lambda)\varphi$ converges to $(\varepsilon I+A)\varphi$ as $\lambda \downarrow 0$. Finally we deduce (1.5) as a consequence of (1.16) in the various limits.

We know that (1.4) may be restated as the Lipschitz continuity of $\varphi/\varphi'$ (entended as zero on $\{\varphi=0\}$) together with

(1.28) $$\nu-m \geq \nu\left(\frac{\varphi}{\varphi'}\right)' \geq \nu-M .$$

Let $g \in C([0,\infty))$ be of locally bounded variation, $g(0) = 0$, and consider the approximations $g_n$, $n = 1,2,\cdots$ given by

(1.29) $$g_n(r) = T_n g(r) = n\int_0^r e^{n(s-r)}g(s)ds = g(r) - \int_0^r e^{n(s-r)}dg(s)$$

so that

(1.30) $$g'_n(r) = n\int_0^r e^{n(s-r)}dg(s) .$$

From (1.29) we see that $T_n g$ is $C^1$ and converges as $n \to \infty$ uniformly to $g$ on compact sets. Moreover, if $g$ is nondecreasing, then $g_n(r)$ is nondecreasing in $r$ as well as $n$ and $g_n$ increases to $g$. Moreover, from (1.30) we see that $0 \leq g' \leq K$ implies $0 \leq g'_n \leq K$. Set $g(r) = (\nu-m)r - \nu\varphi(r)/\varphi'(r)$ and define $\varphi_n$ on $[0,\infty)$ by $\nu\varphi_n/\varphi'_n = (\nu-m)r - T_n g$, $\varphi_n(r_1) = \varphi(r_1)$, where $r_1$ is chosen so that $\varphi(r_1) > 0$ and large enough for what comes later. Since

(1.31) $$\varphi_n(r) = \varphi(r_1)\exp\int_{r_1}^{r}\frac{\varphi_n'(s)}{\varphi_n(s)}\,ds\ ,$$

the above consideration imply that $\varphi_n$ satisfies the assumptions of Lemma 5 (with the same $M$ and $m$ as $\varphi$ ) and $\varphi_n$ decreases (respectively, increases) to $\varphi$ on $[0,r_1]$ if $\nu = 1$ (respectively, $\nu = -1$). We could likewise arrange that $\varphi_n$ converge monotonically to $\varphi$ in the opposite sense by choosing $g(r) = (\nu-M)r - \nu\varphi(r)/\varphi'(r)$ (which is nonincreasing), $\varphi_n/\varphi_n' = (\nu-M)r - T_n g$. The analogous process is done on $(-\infty,0]$ to define $\varphi_n$ on $\mathbb{R}$ so that $\varphi_n$ converges monotonically to $\varphi$ on $[-r_1,r_1]$.

Now, for $f \in L^1(\Omega)$ with $\|f\|_\infty < r_1$ , let $\lambda > 0$ and $u_n$ be the solution of $u_n + \lambda B\varphi_n(u_n) = f$ By Proposition 1(v) applied to $B$ , $\|u_n\|_p \le \|f\|_p$ for $p = 1,\infty$. Hence $u_n$ is bounded in $L^1(\Omega)$ and has its values in the interval for which $n \to \varphi_n(r)$ is monotone. Since $B^{-1}$ is bounded, $\varphi_n(u_n) = \lambda^{-1}B^{-1}(f - u_n)$ is also bounded in $L^1(\Omega)$. Since $n \to \varphi_n(r)$ is monotone so is $n \to \varphi_n(u_n)$ (Lemma a.2 of the Appendix). Hence $\varphi_n(u_n)$ converges in $L^1(\Omega)$ and so does $u_n$ by continuity of $B$ . The limit $u$ clearly satisfies $u + \lambda B\varphi(u) = f$ . It follows from the convergence theorem that $e^{-tB\varphi_m}u_0 \to e^{-tB\varphi}u_0$ whenever $u_0 \in L^1(\Omega)$, $\|u_0\|_\infty < r_1$ .

For the second step, let $u_\lambda, u$ solve

$$u_\lambda + (\varepsilon I+A_\lambda)\varphi(u_\lambda) = f\ ,\quad u + (\varepsilon I+A)\varphi(u) = f\ ,$$

respectively, where $f \in L^1(\Omega)$ and $\varphi \in P_0$ . Rewriting the second equation as $u + (\varepsilon I+A_\lambda)\varphi(u) = f + A_\lambda\varphi(u) - A\varphi(u)$ and using this, the first equation and accretivity we find

$$\|u-u_\lambda\|_1 \le \|A_\lambda\varphi(u) - A\varphi(u)\|_1\ .$$

Since $A$ is linear, densely defined and m-accretive, $A_\lambda v \to Av$ as $\lambda \downarrow 0$ for all $v \in D(A)$ and we conclude that $u_\lambda \to u$ in $L^1(\Omega)$ . Thus $(\varepsilon I+A_\lambda)\varphi \to (\varepsilon I+A)\varphi$.

Now let $\varphi$ satisfy (1.3), (1.4) and $u_0 \in \overline{D((\varepsilon I+A)\varphi)}$. Choose $u_0^j \in L^1(\Omega) \cap L^\infty(\Omega)$ whose support is of finite measure so that $u_0^j \to u_0$ in $L^1(\Omega)$ . With $\varphi_n$ as above, we fix $j$ and let $r_1 > \|u_0^j\|_\infty$ . Set

$$u_n = e^{-tB\varphi_n} u_0^j .$$

By Lemma 6

$$\| \frac{du_n}{dt} \|_1 \le \frac{C(m,M)}{t} \|u_0^j\|_1 .$$

Since $e^{-tB\varphi_n} u_0^j \to e^{-tB\varphi} u_0^j$ uniformly for bounded $t$ and $u^j(t) = e^{-tB\varphi} u_0^j \in W^{1,1}([0,\infty):L^1(\Omega))$ by Lemma 5, the above inequality is correct with $u^j$ in place of $u_n$. Moreover, since $t \to \|\frac{du^j}{dt}(t)\|_1$ is nonincreasing, we have

$$\|u^j(t+h) - u^j(t)\|_1 \le \frac{h}{t} C(m,M)\|u_0^j\|_1 . \tag{1.31}$$

Now $e^{-tB\varphi} u_0^j \to e^{-tB\varphi} u_0$ as $j \to \infty$, so (1.31) holds with $u_\varepsilon = e^{-t(\varepsilon I+A)\varphi} u_0$ in place of $u^j$. We may then send $\varepsilon$ to $0$ to find (1.5).

<u>Proof of Theorem 4.</u>

The property (1.9) can be obtained from Lemma 6(ii) by successive approximations as above. The assumption (1.8) implies that $(\nu-m)r - \nu\varphi(r)/\varphi'(r)$ is nondecreasing. As above, it is the increasing limit of $C^1$ nondecreasing functions on $[0,\infty]$ and we can construct $\varphi_n$ satisfying the assumptions of Lemma 6(ii) converging monotonically to $\varphi$ on any $[0,r_1]$. The rest is as above.

For (1.10), if $u$ is a solution of (1.12), we use that

$$\int_\Omega |u_t| = \int_\Omega (u_t + 2(u_t)^-)$$

by (A.3) applied to $B$, $\int_\Omega u_t \le 0$. By (1.17)

$$\int_\Omega u_t^- \le (\frac{1}{m} - 1)\frac{1}{t}\int_\Omega u \le (\frac{1-m}{m})\frac{1}{t} \int_\Omega u_0 .$$

Hence $\int_\Omega |u_t| \le 2 \frac{1-m}{mt} \|u_0\|_1$ and (1.10) follows.

## Appendix

**Proof of Proposition 2.**

Let $\varphi \in P_0$ and $A$ satisfy the hypotheses of Proposition 2. We may simply define $A_\varphi$ by

(a.1) $$g \in A_\varphi(u) \text{ if } \exists\lambda > 0 \text{ and } f \in L^1(\Omega) \text{ such that if } u_\varepsilon = (I + \lambda((\varepsilon I+A)\varphi))^{-1}f \text{ then } \lim_{\varepsilon\downarrow 0} u_\varepsilon = u \text{ and } g = \lambda^{-1}(f - u).$$

To see that $A_\varphi$ extends $A\varphi$, observe that if $u \in D(A\varphi)$, $f = u + \lambda A\varphi(u)$, and $u_\varepsilon = (I + \lambda(\varepsilon I+A)\varphi)^{-1}f$ then

$$\|u_\varepsilon - u\|_{L^1} = \|(I + \lambda(\varepsilon I+A)\varphi)^{-1} f - (I + \lambda(\varepsilon I+A)\varphi)^{-1}(f + \varepsilon\varphi(u))\|_{L^1}$$

$$\le \varepsilon\|\varphi(u)\|_{L^1} \to 0$$

so $\lambda^{-1}(f - u_\varepsilon) \to \lambda^{-1}(f - u) = A\varphi(u)$. To see that $A\varphi$ is accretive, set $C_\varepsilon = (\varepsilon I+A)\varphi$ for $\varepsilon > 0$. Now if $g \in A_\varphi u$, (a.1) implies the existence of $u_\varepsilon$ such that $u_\varepsilon \to u$ and $C_\varepsilon u_\varepsilon \to g$, i.e. $A_\varphi \subset \liminf_{\varepsilon\downarrow 0} C_\varepsilon$. But the limit inferior of a family of accretive operators is clearly accretive.

We next show $R(I+\lambda A\varphi) \supset L^1(\Omega)^+$, $(I+\lambda A_\varphi)^{-1}L^1(\Omega)^+ \subset L^1(\Omega)^+$ and $(I+\lambda(\varepsilon I+A)\varphi)^{-1}f \to (I+\lambda A\varphi)^{-1}f$ for $\lambda > 0$ and $f \in L^1(\Omega)^+$. This merely requires showing that if $\lambda > 0$, $f \in L^1(\Omega)^+$ and $u_\varepsilon$ solves

(a.2) $$u_\varepsilon + \lambda(\varepsilon\varphi(u_\varepsilon) + A\varphi(u_\varepsilon)) = f$$

then $u_\varepsilon \ge 0$ and $\lim_{\varepsilon\downarrow 0} u_\varepsilon$ exists. Now $u_\varepsilon \ge 0$ follows from Proposition 1, as does the estimate

(a.3) $$\|u_\varepsilon + \lambda\varepsilon\varphi(u_\varepsilon)\|_1 = \|u_\varepsilon\|_1 + \lambda\varepsilon\|\varphi(u_\varepsilon)\|_1 \le \|f\|_1 .$$

Moreover, we show that $u_\varepsilon$ is nonincreasing in $\varepsilon$. This monotonicity and the estimate (a.3) imply $\lim_{\varepsilon\downarrow 0} u_\varepsilon$ exists. Indeed, if $\varepsilon > \eta > 0$ we have (because $\varphi(u_\varepsilon) \geq 0$)

$$u_\varepsilon + \lambda(\eta\varphi(u_\varepsilon) + A\varphi(u_\varepsilon)) = f - \lambda(\varepsilon - \eta)\varphi(u_\varepsilon) \leq f$$

$$u_\eta + \lambda(\eta\varphi(u_\eta) + A\varphi(u_\eta)) = f .$$

Now by Proposition 1, $(I+\lambda(\eta I+A)\varphi)^{-1}$ is order preserving and thus $u_\varepsilon \leq u_\eta$.

Remark: It would be nice (especially below) if $\varepsilon\varphi(u_\varepsilon) \to 0$ in $L^1(\Omega)$, in which case the current task would be quite simple. However, examples show this to be false in general.

The final assertion of Proposition 2 has already been verified. We consider next the case in which $|\varphi(r)| \leq K|r|$ on $|r| \leq r_0$. We now seek to show that if $u_\varepsilon$ solves (a.2) and $f \in L^1(\Omega)$ is arbitrary, then $\lim_{\varepsilon\downarrow 0} u_\varepsilon$ exists. Since $(I + \lambda(\varepsilon I+A)\varphi)^{-1})$ is nonexpansive, it suffices to choose $f$ from the dense set $L^1(\Omega) \cap L^\infty(\Omega)$. Then $\|u_\varepsilon\|_\infty \leq \|f\|_\infty$ and there is another constant $K_1$ such that $\|\varphi(r)\| \leq K_1|r|$ on $|r| \leq \|f\|_\infty$. Hence $\varphi(u_\varepsilon)$ is bounded in $L^1(\Omega)$ by $K_1\|u_\varepsilon\|_1$. Since (a.3) still holds, $\varepsilon\varphi(u_\varepsilon) \to 0$ in $L^1(\Omega)$ and

$$\|u_\varepsilon - u_\eta\|_1 = \|(I+\lambda A\varphi)^{-1}(f-\lambda\varepsilon\varphi(u_\varepsilon)) - (I+\lambda A\varphi)^{-1}(f-\lambda\eta\varphi(u_\eta))\|_1$$

$$\leq \lambda(\|\varepsilon\varphi(u_\varepsilon)\|_1 + \|\eta\varphi(u_\eta)\|_1)$$

so $u_\varepsilon$ is Cauchy in $L^1(\Omega)$ as $\varepsilon \downarrow 0$. The case $\operatorname{meas}\Omega < \infty$ is similar, since then $\varepsilon\varphi(u_\varepsilon) \to 0$ in $L^\infty(\Omega)$ implies the convergence in $L^1(\Omega)$.

Remark: The above proof shows that $A_\varphi$ is the closure of $A\varphi$ in these cases. With $\Omega = R$, $\varphi(r) = r^3$, $A = 0$ we have an example where $A_\varphi \neq A\varphi$.

The remaining case is the one in which $\varphi$ is strictly monotone. Again let $f \in L^1(\Omega) \cap L^\infty(\Omega)$ and $u_\varepsilon$ solve (a.2). Let $u_{\varepsilon+}, u_{\varepsilon-}$ solve

$$u_{\varepsilon\nu} + \lambda(\varepsilon\varphi(u_{\varepsilon\nu}) + A\varphi(u_{\varepsilon\nu})) = \nu f^\nu,\ \nu = \pm .$$

By the order preserving properties, $u_{\varepsilon-} \le u_\varepsilon \le u_{\varepsilon+}$. Moreover, by the first case treated above, $u_{\varepsilon\nu}$ converges monotonically as $\varepsilon \downarrow 0$ to $u_\nu \in L^1(\Omega)$ and so $u_- \le u_\varepsilon \le u_+$. Therefore, by the dominated convergence theorem, it is enough to show that for $\delta > 0$

$$\lim_{\varepsilon,\eta\to 0} \operatorname{meas}\{|u_\varepsilon - u_\eta| > \delta\} = 0.$$

Since $u_\varepsilon, u_\eta$ are bounded and $\varphi$ is strictly monotone, there is a $\mu > 0$ such that $\{|u_\varepsilon - u_\eta| > \delta\} \subset \{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$. Now

$$(u_\varepsilon - u_\eta) + \lambda A(\varphi(u_\varepsilon) - \varphi(u_\eta)) = (\eta\varphi(u_\eta) - \varepsilon\varphi(u_\varepsilon)) .$$

Let $p(r) = 1$ if $r > \mu$, $p(r) = -1$ if $r < -\mu$ and $p(r) = 0$ if $|r| \le \mu$. Multiply the above by $p(\varphi(u_\varepsilon) - \varphi(u_\eta))$, integrate, and use (A.3) to conclude

$$\int_{\{|\varphi(u_\varepsilon)-\varphi(u_\eta)|>\mu\}} |u_\varepsilon - u_\eta| \le \int_{\{|\varphi(u_\varepsilon)-\varphi(u_\eta)|>\mu\}} (|\eta\varphi(u_\eta) - \varepsilon\varphi(u_\varepsilon)|$$

Now let $\kappa > 0$ be such that $\{|u_\varepsilon - u_\eta| > \kappa\} \supset \{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$. There is such a $\kappa$ because $\varphi$ is continuous. We have, by the above,

$$\kappa \operatorname{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\} \le \|\eta\varphi(u_\eta) - \varepsilon\varphi(u_\varepsilon)\|_2 \operatorname{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta) > \mu\}^{1/2} .$$

But $\varepsilon\varphi(u_\varepsilon)$ is bounded in $L^1(\Omega)$ and tends to zero in $L^\infty(\Omega)$. Thus $\varepsilon\varphi(u_\varepsilon) \to 0$ in $L^p(\Omega)$, $1 < p < \infty$ and we conclude that $\operatorname{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$ and so $\operatorname{meas}\{|u_\varepsilon - u_\eta| > \delta\})$ tends to zero as $\varepsilon, \eta \to 0$, thus completing the proof.

Remarks:

(a) We do not know if $(I + \lambda(\varepsilon I + A)\varphi)^{-1}$ converges as $\varepsilon \downarrow 0$ for every $\varphi \in P_0$.

($\beta$) The definition of $A_\varphi$ is consistent with known examples. One important case is $\Omega = R^N$ and $A = -\Delta$. The construction of [3] coincides with ours when Proposition 2 applies, however in [3] precise information on the domain of $A_\varphi$ is obtained and more general $\varphi$'s are permitted.

<u>Lemma a.1.</u> Let $p: R \to R$ be Lebesgue measurable, bounded and $j(r) = \int_0^r p(s)ds$. Let $w \in W^{1,1}(0,T: L^1(\Omega))$. Then $j(w) \in W^{1,1}(0,T:L^1(\Omega))$ and

(a.4) $$\frac{d}{dt} j(w) = p(w)\frac{dw}{dt} \text{ a.e.}$$

<u>Proof.</u> Let us treat the case $L^1(\Omega) = R$. Then the general case follows by using Fabini's theorem and looking directly at $\lim_{h\downarrow 0} (j(w(t+h)) - j(w(t)))/h$.

One has to prove

(a.5) $$-\int_0^T \psi'(t)j(w(t))dt = \int_0^T \psi(t)p(w(t))w'(t)dt \quad \forall\ \psi \in C_0^\infty(0,T)$$

with the proof demonstrating the measurability of $p(w)w'$ so that the equation has a meaning. Notice that if (a.4) holds for a sequence $(p_n, j_n)$ in place of $(p,j)$ and $p_n$ converges boundedly everywhere to $p$, then (a.5) holds for $(p,j)$.

The relation (a.5) is obvious if $p$ is continuous. If $O$ is open in $R$ and $p$ is the characteristic function $\chi_O$ of $O$, then $p$ is the increasing limit of continuous functions. Hence (a.5) holds with $p = \chi_O$.

If $N \subset R$ is a null set, then there is a decreasing sequence $O_n$ of open sets such that $O_n \supset N$ and $\text{meas}O_n \to 0$. Let $N' = \bigcap_n O_n$ so that $\chi_{N'} = \lim_{n\to\infty} \chi_{O_n}$ is the decreasing limit of characteristic functions of open sets. By the above remarks, (a.5) holds with $j = 0$ and $p = \chi_{N'}$, so $0 = \chi_{N'}(w)w'$ a.e. and $w' = 0$ a.e. on $\{t \in (0,T): w(t) \in N \subset N'\}$

If $E \subset (0,T)$ is measurable, then there exists a decreasing sequence $O_n$ of open sets such that $O_n \supset E$ and $\text{meas}\{\bigcap_n O_n \setminus E\} = 0$. Set $E' = \bigcap_n O_n$. We have, by the above remarks, $\chi_{E'}(w)w' = \chi_E(w)w' + \chi_{E'\setminus E}(w)w' = \chi_E(w)w'$ a.e. and the validity of (a.5) for $p = \chi_{E'}$ implies the validity for $\chi_E$. Since any bounded measurable function is the uniform limit of a sequence of simple functions the proof is complete.

Lemma a.2. Let $B$ be linear, densely defined and satisfy (A.1), (A2). Let $\varphi, \psi \in P_0$ and $\varphi(r) \geq \psi(r)$ for all $r$. Let $u \in D(B\varphi)$, $v \in D(B\psi)$ and

$$u + B\varphi(u) = v + B\psi(v).$$

Then $\varphi(u) \geq \psi(v)$.

Proof. We have

$$u - v + B(\varphi(u) - \psi(v)) = 0 . \tag{a.6}$$

Set

$$p(x) = \begin{cases} -1 & \text{if} \quad \varphi(u(x)) < \psi(v(x)) \\ 0 & \quad\quad \text{otherwise} . \end{cases}$$

Then $p(x) \in \beta(\varphi(u(x)) - \psi(v(x))$ where

$$\beta(r) = \begin{cases} \{0\} & \text{if} \quad r > 0 \\ [0,1] & \text{if} \quad r = 0 \\ \{0\} & \text{if} \quad r < 0 . \end{cases}$$

Moreover, $(u - v)p(\varphi(u) - \psi(v)) = |u - v|$ on $\{\varphi(u) < \psi(v)\}$ by the monotonicity of $\varphi$ and $\varphi \geq \psi$, while $(u - v)p(\varphi(u) - \psi(v)) \geq 0$. Multiplying (a.6) by $p(\varphi(u) - \psi(u))$ and integration with the use of (A.3) yields

$$\int_{\{\varphi(u) < \psi(v)\}} |u - v| \leq 0$$

so $\varphi(u) \geq \psi(v)$.

Remark: If it is known that $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq r_1$ and $\varphi(r) \geq \psi(r)$ holds for $|r| \leq r_1$ we clearly have the same conclusion.

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Various initial-boundary value problems and Cauchy problems can be written in the form $\frac{du}{dt} + A\varphi(u) = 0$ where $\varphi:\mathbb{R}\to\mathbb{R}$ is nondecreasing and $A$ is the linear generator of strongly continuous nonexpansive semigroug $e^{-tA}$ in an $L^1$ space. For example, if $A = -\Delta$ (subject, perhaps, to suitable boundary conditions) we obtain equations arising in flow in a porous medium or plasma physics (depending on the choice of $\varphi$) while if $A = \frac{\partial}{\partial x}$ acting in $L^1(\mathbb{R})$ we have a scalar conservation law. In this paper we show that if $M, m > 0$ and $m\varphi'^2 \le \nu\varphi\varphi'' \le M\varphi'^2$, where $\nu \in \{1,-1\}$, then (roughly speaking), the norm of $t\,du/dt$ may be estimated in terms of the

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20. Abstract (continued)

initial data $u_0$ in $L^1$. Such estimates give information about the regularity of solutions, asymptotic behaviour, etc., in applications.

Side issues, such as the introduction of sufficiently regular approximate problems on which estimates can be made and the assignment of a precise meaning to the operator "$A\varphi$", are also dealt with. These considerations are of independent interest.

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