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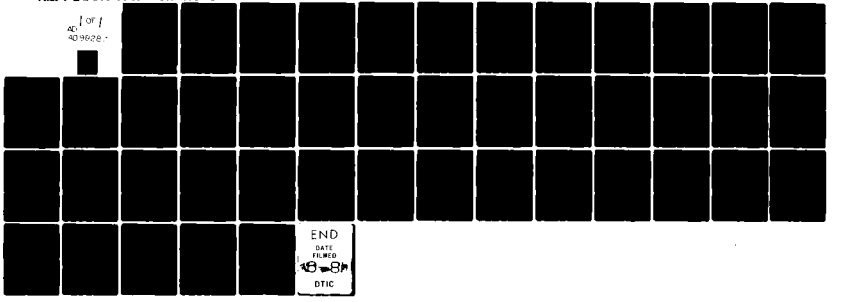

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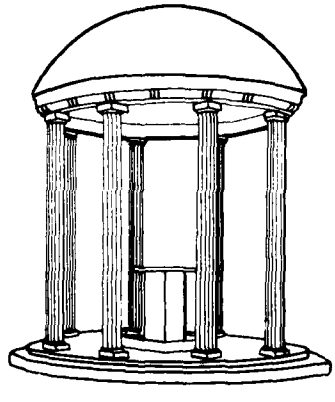

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ACCELERATED ACCURACY IN THE SIMULATION OF MARKOV CHAINS

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Abstract

This paper describes a method of obtaining results from the simulation of a finite state positive recurrent aperiodic Markov chain at a cost considerably below the cost required to achieve the same accuracy with pure random sampling. By reorganizing k independent epochs or tours simulated serially into k replications simulated in parallel, one can induce selected joint distributions across replications that produce the cost-saving benefits. The joint distributions follow from the use of rotation sampling, a special case of the antithetic variate method.

For a finite state nearest neighbor chain the paper shows that even for independent parallel replications the cost of achieving a specified accuracy with serial simulation relative to the cost for parallel simulation has a lower bound $O(k^{\frac{1}{2}})$ as $k \to \infty$. When rotation sampling is used this bound is $O(k^2/(\ln k)^3)$. This lower bound also holds for the more general finite state chain. A simulation of the M/M/1 queueing model with finite capacity n is used to illustrate the effectiveness of the technique for selected values of k,n and activity level ρ.

Introduction

One ostensible attraction that computer based simulation offers to an experimenter is the ability to control the sources of variation in a problem under study. In their monograph on Monte Carlo experiments, Hammersley and Handscomb(1964) enumerate ways in which one can exploit this control to achieve a result with a specified accuracy at less cost than pure random sampling would require. However, that account principally addresses the evaluation of univariate and multivariate integrals, a considerably more well defined problem than one encounters in discrete event simulation. Although studies using one or another of these *variance reducing* techniques are common in the literature on discrete event simulation, only recently has serious attention been devoted to developing methods to exploit this control of variation in conjunction with the general classes of structures encountered there. These recent developments include elimination of the list of scheduled events in a simulation containing an imbedded Markov chain in Hordijk, Iglehart and Schassberger (1976), random number stream manipulation in multifactor experiments in Schruben and Margolin (1978), a perspective on control variates in Lavenberg and Welch (1980) and a reexamination of antithetic variates in Fishman and Huang (1980).

To continue this theoretical development, the present paper shows how one can employ a special case of the antithetic variate technique called rotation sampling to exploit the structure of finite state Markov

chains to produce substantial cost savings in simulation studies designed to achieve a specified level of statistical accuracy. The generality of the approach arises from the observation that many discrete event simulations have an underlying Markov chain structure or a structure close to that of a Markov chain. Regenerative simulation, which includes Markov chains as a special case, offers one example. See Crane and Iglehart (1975) and Fishman (1978).

The proposed technique derives its cost-saving potential from viewing the simulation of k tours in *series* of a finite $(n+1)$ state positive recurrent aperiodic Markov chain as equivalent to the simulation of k replications of the Markov chain in *parallel*. Although the marginal distributions that arise with the two alternative formulations are necessarily the same for corresponding variables, the parallel formulation allows one to induce joint distributions across replications that lead to a significant cost saving. The induced joint distributions follow from the use of rotation sampling, as described in detail in Fishman and Huang (1980). The cost saving arises in two ways. Firstly, for fixed n run time in the correlated case is $O(\ln k)$ in contrast to $O(k)$ for the serial simulation. Secondly, for fixed n the variance of an estimator has an upper bound $O((\ln k/k)^2)$ for the correlated case compared to $O(1/k)$ for the serial case. Moreover, the technique appears to be well suited for use with the aforementioned proposal of Hordijk, Iglehart and Schassberger (1976) for eliminating the list of scheduled events when simulating a system representable as a Markov chain.

Section 1 introduces the Markov chain notation and specializes it to the case of the *nearest neighbor* model. It also formulates the experiment of k independent tours or epochs in series and then shows how one can reformulate the simulation equivalently as k *independent* replications in parallel. This reformulation alone shows that one can achieve a substantive cost saving merely by using an efficient method of generating samples from the binomial distribution.

Section 2 shows how rotation sampling leads to a considerably greater cost saving and Section 3 demonstrates how the results apply in practice. Section 4 considers finite state chains of a more general character than the nearest neighbor model and extends the results of Section 2 appropriately. Section 5 discusses several unresolved issues which include nonconvergence to normality, variance etimation, sequential estimation, infinite state Markov chains and transient simulation.

1. The Nearest Neighbor Model

Consider a positive recurrent aperiodic Markov chain with finite state space $S = (0,1,\ldots,n)$ and $(n+1) \times (n+1)$ transition matrix $\underset{\sim}{p}_n = \| p_{ij} \|$. Since all states 0 through n are *regenerative*, the sample paths of states realized between successive entries into a given state are independent and obey the same probability law. Moreover, a like property holds for functions of these sample paths. For expositional convenience, we take state 0 as the point of regeneration and say that each entry into this state completes an *epoch* or *tour*. Let A_{ij} denote the reward received when a jump occurs from state i to state j, let N_{ijm} denote the number of transitions from state i to state j *between the (m-1)st and mth successive entries into* state 0 (epoch or tour m) for $m \geq 1$ and assume that observation begins with an entry to state 0. Let μ denote the mean reward per epoch. Then

$$R_k = \frac{1}{k} \sum_{m=1}^{k} \sum_{i,j=0}^{n} A_{ij} N_{ijm} \tag{1}$$

is an unbiased estimator of μ and var $R_k \propto 1/k$. Moreover, the number of transitions per epoch T has mean

$$\omega = \omega(\underset{\sim}{p}_n) = \sum_{i,j=0}^{n} E\, N_{ijm} \quad . \tag{2}$$

For convenience of exposition we consider the special but important case of the *nearest neighbor* Markov chain where

$$\underset{\sim}{P}_n = \begin{pmatrix} p_{00} & p_{01} & 0 & \cdots & 0 \\ p_{10} & 0 & p_{12} & & 0 \\ 0 & p_{21} & 0 & p_{23} & 0 \\ \vdots & & \ddots & & \vdots \\ & & & p_{n-1,n-2} & 0 & p_{n-1,n} \\ 0 & \cdots & & & p_{n,n-1} & p_{n,n} \end{pmatrix} \tag{3}$$

with $p_{00} = 0 = 1 - p_{01}$. Section 4 considers the more general finite chain.

Note that in each epoch independent Bernoulli sampling dictates the branch taken at each step or transition. Let $U_1, U_2, \ldots$ denote a sequence of i.i.d. random variables from $U(0,1)$. Suppose that the chain is in state j after transition $\ell - 1$ and prior to transition ℓ. Then the reward at step ℓ is

$$A_{jj'} I_{[0,p_{jj'})}(U_\ell) + A_{jj''}[1 - I_{[0,p_{jj'})}(U_\ell)] = (A_{jj'} - A_{jj''}) I_{[0,p_{jj'})}(U_\ell) + A_{jj''}$$

where $j' = j'(j) = \max(0,j-1)$, $j'' = j''(j) = \min(n,j+1)$ and

$$I_{[a,b)}(x) = 1 \qquad a \le x < b$$
$$= 0 \qquad \text{otherwise.}$$

Let T_k and S_k denote the time of the kth return to state 0 and the number of states visited on k such epochs respectively. Then

$E\,T_k = k\omega$ and clearly $E\,S_k = E\,T_k = O(k)$. Moreover, the mean computing cost of simulating k epochs is proportional to $E\,S_k$ as $k \to \infty$.

Parallel Simulation

Note that epochs in the aforementioned model occur in *serial* order. An alternative, but entirely equivalent, representation in *parallel* facilitates later development of our proposed variance reduction method. In lieu of running k simulated epochs in sequence, suppose one runs $k \geq n$ independent replications in parallel. Each replication begins with an entry into state 0 and ends with the next entry into state 0. Figure 1 gives an example for $k = 5$ and $n = 3$. Let $K_{i\ell}$ denote the number of replications in state i after transition ℓ on all k replications. One can then regard the k Markov chains operating in parallel as a new Markov chain with $(n+1)$-dimensional state vector $\underset{\sim}{K}_\ell = (K_{0\ell},\ldots,K_{n\ell})$, with $\sum_{j=0}^{n} K_{j\ell} = k$, after transition ℓ and prior to transition $\ell + 1$. For the new chain an epoch begins and ends upon entry into $(k,0,\ldots,0)$. We define this epoch to begin at transition $\ell = 0$. To establish equivalence between the serial and parallel representations, one needs to set $p_{01} = 0 = 1 - p_{00}$ in $\underset{\sim}{p}_n$ immediately after $\ell = 1$, so that states $1,\ldots,n$ are *transient*, state 0 is absorbing and $\underset{\sim}{p}_n$ is an *absorbing* chain. Note that the new state vector has $n + 1$ elements and $\binom{k+n}{k}$ new states exist. See Johnson and Kotz (1977, p. 120).

Let $\{U_{i\ell}: i=1,\ldots,k;\ \ell= 1,2,\ldots\}$ denote a sequence of i.i.d. random variables from $U(0,1)$ where $U_{i\ell}$ determines the branch taken by replication i on step ℓ. Let $i_{j,\ell,1},\ldots,i_{j,\ell,K_{j\ell}}$ denote the

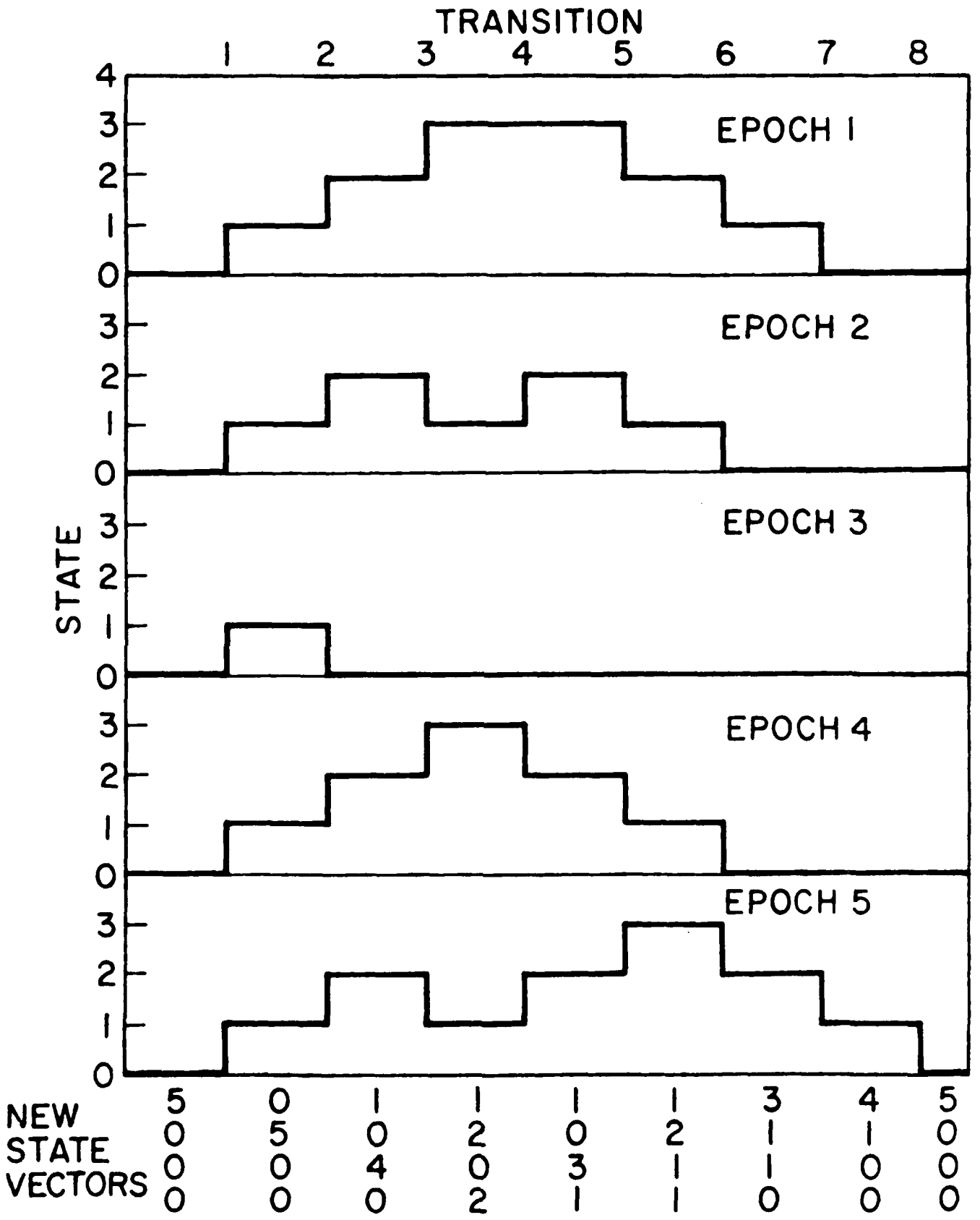

Figure 1. Series and Parallel Representations

$K_{j\ell}$ replications that are in state j after step ℓ. Then the number of replications that move from state j to j' on step $\ell + 1$ is

$$K_{jj',\ell+1} = \sum_{m=1}^{K_{j\ell}} I_{[0,p_{jj'})}(U_{i_{j\ell m},\ell+1}) \qquad K_{j\ell} > 0$$

$$= 0 \qquad K_{j\ell} = 0 . \tag{4}$$

One can now write the sample mean reward corresponding to (1) as

$$\dot{R}_k = \frac{1}{k} \sum_{j=0}^{n} \sum_{\ell=0}^{\infty} (A_{jj'} K_{jj',\ell+1} + A_{jj''} K_{jj'',\ell+1})$$

$$= \frac{1}{k} \sum_{j=0}^{n} [(A_{jj'} - A_{jj''}) \sum_{\ell=0}^{\infty} K_{jj',\ell+1} + A_{jj''} \sum_{\ell=0}^{\infty} K_{j\ell}]$$

where

$$\sum_{m=1}^{k} N_{jj'm} = \sum_{\ell=0}^{\infty} K_{jj',\ell+1} , \qquad \sum_{m=1}^{k} N_{jj''m} = \sum_{\ell=0}^{\infty} K_{jj'',\ell+1}$$

$$K_{001} = 0 , \qquad K_{011} = k ,$$

$$K_{11} = K_{102} + K_{122} = k$$

$$K_{j\ell} = K_{jj',\ell+1} + K_{jj'',\ell+1} \qquad \ell = 1,2,\ldots ; \; j = 1,\ldots,n .$$

Simulation of k independent replications of the original Markov chain with transition matrix (3) in terms of the new Markov chain for

$\ell = 1,2,\ldots$ takes the form

$$
\begin{aligned}
K_{0,\ell+1} &= K_{0\ell} + K_{10,\ell+1} \\
K_{1,\ell+1} &= K_{21,\ell+1} \\
K_{j,\ell+1} &= K_{j+1,j,\ell+1} + K_{j-1,\ell} - K_{j-1,j-2,\ell+1} \qquad j = 2,\ldots,n-1 \qquad (5) \\
K_{n,\ell+1} &= K_{n-1,\ell} - K_{n-1,n-2,\ell+1} + K_{n\ell} - K_{n,n-1,\ell+1}
\end{aligned}
$$

with $K_{01} = 0$, $K_{011} = K_{11} = k$, $K_{j1} = 0$ for $j = 2,\ldots,n$ and where $K_{jj',\ell+1}$ has the binomial distribution $B(K_{j\ell},p_{jj'})$ for $j = 1,\ldots,n$. Therefore, the distribution of $\underset{\sim}{K}_{\ell+1}$ given $\underset{\sim}{K}_{\ell}$ depends on the n aforementioned binomial distributions. Also, it is clear that $\text{var } R_k = \text{var } \dot{R}_k \propto 1/k$.

The computational complexity of simulating k parallel replications depends on two factors, the cost of generating $\{K_{jj'\ell}, K_{jj''\ell} \mid K_{j,\ell-1} ; j=1,\ldots,n\}$ and on the absorption time T_k . If one uses Bernoulli sampling to generate this vector, then at each transition the cost is $O(kn)$. However, algorithms now exist for directly sampling $K_{jj'\ell}$ and $K_{jj''\ell}$ from the binomial distribution with cost $O(1)$ so that $\underset{\sim}{K}_{\ell}$ can be generated at a cost $O(n)$. See Ahrens and Dieter (1980) and Fishman (1978).

Let $\dot{S}_k$ denote the total number of transient states $1,\ldots,n$ visited on all transitions up to absorption. Then the mean cost of simulating k parallel replications is proportional to $E\,\dot{S}_k$ if one adopts efficient binomial sampling. Also, $E\,\dot{S}_k \le n\,E\,\dot{T}_k$. However, note that, in practice, the cost of visiting a state is greater in the parallel case than it is in the serial case.

More importantly, we now demonstrate that merely reorganizing a serial simulation into a parallel one leads to a cost saving that grows with $k^{\frac{1}{2}}$.

Theorem 1. Let $V(A,B)$ denote the cost of using an estimator A to estimate mean reward relative to the cost of using estimator B, given var A = var B. Then for fixed n

$$(i) \quad E\,\dot{T}_k \le k^{\frac{1}{2}}\ (E\,T^2)^{\frac{1}{2}} .$$

$$(ii) \quad V(R_k, \dot{R}_k) \ge O(k^{\frac{1}{2}}) \quad .$$

See the Appendix for proof.

The results of this theorem imply that mean absorption time is substantially shorter for parallel simulation and that the relative cost of serial simulation grows with $k^{\frac{1}{2}}$ for fixed n . The next section shows that yet a more favorable situation can be created for parallel simulation by inducing an appropriate joint distribution across the Bernoulli trials within each state.

2. Rotation Sampling in Parallel Simulation

Consider the array

$$\begin{array}{llll} U_{11} & U_{12} & U_{13} & \cdots\cdots\ U_{1\ell_1} \\ U_{21} & U_{22} & U_{23} & \cdots\cdots\cdots\ U_{2\ell_2} \\ \vdots & \vdots & \vdots & \vdots \\ U_{k1} & U_{k2} & U_{k3} & \cdots\ U_{k\ell_k} \end{array} \qquad (6)$$

where ℓ_j denotes the number of transitions on replication j that occurs in the cycle beginning with an exit from state 1 and ending upon entry into state 0. Now observe that while elements within a row need to be independent, elements within a column need not be. All that is required within a column is that elements have the marginal distribution $U(0,1)$.

Consider the transition $\ell = 2$. In the previously described case, $U_{11},\ldots,U_{k1}$ are independent, so that $\text{var } K_{102} = \text{var } K_{122} = kp_{10}(1-p_{10})$. Now suppose one imposes the column restriction for $i = 1,\ldots,k$

$$
\begin{aligned}
U_{i1} &\equiv U_{11} + \frac{i-1}{k} \pmod 1 \\
&= U_{11} + \frac{i-1}{k} \qquad \text{if} \qquad 0 \le U_{11} < 1 - \frac{i-1}{k} \\
&= U_{11} + \frac{i-1}{k} - 1 \qquad \text{if} \qquad 1 - \frac{i-1}{k} \le U_{11} < 1
\end{aligned} \tag{7}
$$

so that $U_{11},\ldots,U_{k1}$ each have $U(0,1)$ but are dependent. This sampling plan is a special case of *rotation sampling* and is discussed in detail in Fishman and Huang (1980).

For the proposed sampling plan, let replication i jump to state 0 if $U_{i1} < p_{10}$ and to state 2 if $U_{i1} \ge p_{10}$. Let $K'_{jj',\,\ell+1}$ and $K'_{jj'',\,\ell+1}$ denote numbers of replications that move from j to j' and j'', respectively, on transition $\ell+1$ with this new sampling plan. Then for $\ell+1 = 2$, $E\,K'_{102} = E\,K_{102} = kp_{10}$ and $E\,K'_{122} = E\,K_{122} = k(1-p_{10})$.

But

$$\text{var}(K'_{102} \mid K'_{11} = k) = \text{var}(K'_{122} \mid K'_{11} = k) = (\overline{kp}_{10})(1-\overline{kp}_{10})$$

$$\leq \text{var}(K_{102} \mid K_{11} = k) = \text{var}(K_{122} \mid K_{11} = k) \qquad (8)$$

where $K'_{j\ell}$ denotes the number of replications in state j after transition ℓ and prior to transition $\ell+1$ and $\overline{kp}_{10} = kp_{10} \pmod 1$. See Theorem 3 of Fishman and Huang (1980). Note that under this new sampling plan K'_{102} and K'_{122} have variance $\leq 1/4$, regardless of k, an attractive feature of the induced joint distribution[†] on $U_{11},\ldots,U_{k1}$.

One can now extend this correlated sampling plan to all subsequent transitions by grouping replications by state at each transition. Recall that $i_{j\ell m}$ is the mth replication on transition ℓ that is in state j for $K'_{j\ell} > 0$ and $m = 1,\ldots,K'_{j\ell}$. Then replication $i_{j\ell m}$ jumps to state j' if $U_{i_{j\ell m}\ell} < p_{jj'}$ where for $K'_{j\ell} > 0$

$$U_{i_{j\ell m}\ell} \equiv U_{i_{j,\ell,1}\ell} + \frac{m-1}{K'_{j\ell}} \pmod 1 \qquad (9).$$

Then one has:

Theorem 2. Assume that one uses (9) in a *parallel* simulation of the

[†]Fishman and Huang (1981) show that (8) is the minimum achievable variance for the sum of k Bernoulli random variables, each with the same marginal distribution.

Markov chain with $\underset{\sim}{p}_n$ as in (3). Then

$$\text{(i)} \qquad K'_{jj',\ell+1} = \lfloor K'_{j\ell}\, p_{jj'} \rfloor \qquad \text{w.p.} \quad 1 - \overline{K'_{j\ell}\, p_{jj'}}$$

$$= \lfloor K'_{j\ell}\, p_{jj'} \rfloor + 1 \qquad \text{w.p.} \quad \overline{K'_{j\ell}\, p_{jj'}} \ .$$

$$\text{(ii)} \qquad \operatorname{var}(K'_{jj',\ell+1} \mid K'_{j\ell}) = (\overline{K'_{j\ell}\, p_{jj'}})\ (1 - \overline{K'_{j\ell}\, p_{jj'}}) \le 1/4 \ .$$

$$\text{(iii)} \qquad \operatorname{var} K'_{jj',\ell+1} = O(1) \ .$$

See the Appendix for the proof.

The results in Theorem 2 are of considerable importance. In particular, (i) implies that at most only one Bernoulli trial from $\mathrm{Ber}(\overline{K'_{j\ell}\, p_{jj'}})$ is needed for state j. Therefore, one can effect the proposed parallel sampling plan with a maximum of n Bernoulli trials. Also note that for large k the sample path at the beginning of the simulation is virtually deterministic. Property (ii) establishes the independence of the conditional variance from $K'_{j\ell}$ in contrast to the independent case where this conditional variance is proportional to $K'_{j\ell}$. In particular, observe that $\operatorname{var}(K'_{jj',\ell+1} \mid K'_{j\ell}) = 0$ when $\overline{K'_{j\ell} p_{jj'}} = 0$. Property (iii) removes the conditionality on $K'_{j\ell}$ and shows that for every ℓ the orders of magnitude of $\operatorname{var} K'_{jj',\ell+1}$ and $\operatorname{var} K'_{jj'',\ell+1}$ are unaffected by k.

Simulation of the Markov chain for $\ell = 1,2\ldots$ now takes the form

$$
\begin{aligned}
K'_{0,\ell+1} &= K'_{0\ell} + K'_{10,\ell+1} \\
K'_{1,\ell+1} &= K'_{21,\ell+1} \\
K'_{j,\ell+1} &= K'_{j+1,j,\ell+1} + K'_{j-1,\ell} - K'_{j-1,j-2,\ell+1} \qquad j = 2,\ldots,n-1 \\
K'_{n,\ell+1} &= K'_{n-1,\ell} - K'_{n-1,n-2,\ell+1} + K'_{n\ell} - K'_{n,n-1,\ell+1}
\end{aligned} \tag{10}
$$

with $K'_{01} = 0$, $K'_{011} = K'_{11} = k$, $K'_{j1} = 0$ for $j = 2,\ldots n$ and $K'_{jj',\ell+1} = \lfloor K'_{j\ell} p_{jj'} \rfloor + C_{j,\ell+1}$ where $C_{j,\ell+1}$ has the Bernoulli distribution $Ber(\overline{K'_{j\ell} p_{jj'}})$. Note that the principal distinction between (5) and (10) is that (5) relies on binomial sampling, whereas (10) uses Bernoulli sampling. However, the sample paths generated on a single replication of the chain (3) follow the same probability laws regardless of the sampling plan.

The sample mean reward is now

$$
R'_k = \frac{1}{k} \sum_{j=0}^{n} \sum_{\ell=1}^{\infty} (A_{jj'} K'_{jj'\ell} + A_{jj''} K'_{jj''\ell}) \tag{11}
$$

and the ostensible objective of analysis is to show that the cost of independent serial simulation relative to the cost of correlated parallel simulation for $\text{var } R_{k_1} = \text{var } R'_{k_2}$ increases with k_1 ($k_2 < k_1$). To demonstrate this result we need to study T'_k, the time at which all k correlated replications are absorbed (absorption time).

Theorem 3. Let

$$T'_{kr} = \min\ (t:\ \sum_{j=1}^{n} K'_{jt} \le r) \qquad r = \lfloor 1/p_{12} \rfloor < k\ ,$$

$$Y_{kr} = T'_{k} - T'_{kr}\ ,$$

$$M_{j\ell} = K'_{jj',\ell} - K'_{j,\ell-1}\, p_{jj'} \qquad \ell = j,\ j+1,\ldots$$

$$= 0 \qquad \ell < j \qquad j = 1,\ldots n,$$

and

$$p_{j0}^{(\ell)} = \text{probability of moving from } j \text{ to } 0 \text{ in } \ell \text{ steps.}$$

Then

(i) Absorption can occur at t if and only if

$$\underset{\sim}{K}'_{t-1} \in \{(k-m,m,0,\ldots,0); \qquad m = 1,\ldots,r\}\ .$$

(ii) $$K'_{0t} = M_{1t} + \sum_{\ell=1}^{t-1} M_{1\ell}\,(1 - \sum_{i=1}^{t-\ell} p_{20}^{(i)})$$

$$+ \sum_{j=2}^{n} \sum_{\ell=1}^{t-j'} M_{j\ell} \sum_{i=1}^{t-\ell} (p_{j'0}^{(i)} - p_{j''0}^{(i)}) + k \sum_{i=1}^{t} p_{10}^{(i)}\ .$$

(iii) There exist $b \ge 1$ and $\rho \in (0,1)$ such that

$$-b + k\,O(\rho^{t}) < k - K'_{0t} \le b + k\,O(\rho^{t})\ .$$

(iv) $T'_{kr} = O(\ln k)$. w.p. 1 .

(v) $E\,T'_{kr} = O(\ln k)$.

(vi) $Y_{kr} = O(1)$ w.p. 1 .

(vii) $E\, T_k' = O(\ln k)$.

See the Appendix for proof.

Result (i) shows that total absorption can occur if and only if K_{t-1} is in one of $\lfloor 1/p_{12} \rfloor$ states. Also, result (ii) reveals that the rate of entry into the absorbing state has a deterministic component which is linear in k and a stochastic component whose magnitude is independent of k. Together, results (i) and (ii) enable one to establish parts (iii) through (vii). In particular, note that the ratio of mean absorption times is $E(T_k)/E(T_k') = O(k/\ln k)$, clearly favoring correlated parallel simulation (CPS) for equal k's . However, it remains to show how $\text{var } R_k'$ compares to $\text{var } R_k$ and how S_k', the number of states visited in CPS, compares to S_k.

Theorem 4. Let c denote the mean computing cost of visiting a state in the simulation using rotation sampling relative to that cost in a serial simulation. For CPS and fixed n

(i) $\quad \text{var } R_k' \cdot O((\ln k/k)^2)$

(ii) $\quad E\, S_k' = O(\ln k)$

(iii) $\quad V(R_{k_1}, R_{k_2}') = \dfrac{E\, S_{k_1}}{E\, S_{k_2}'} \cdot \dfrac{\text{var } R_{k_1}}{\text{var } R_{k_2}'} \cdot \dfrac{1}{c}$

$$\geq O(k_2^2 \,/\, (\ln k_2)^3) \text{ and } k_2 < k_1 \quad .$$

See the Appendix for the proof of part (i) . Part (ii) follows from part (v)

of Theorem 3 and part (iii) follows from the observation that

$E\, S_{k_1} \cdot \text{var}\, R_{k_1} = O(1)$.

Part (iii) is the result of ultimate interest in this study. It shows that a lower bound $O(k_2^2 / (\ln k_2)^3)$ exists on the cost of achieving a specified accuracy with independent serial tours relative to the cost of achieving the same accuracy ($\text{var}\, R_{k_1} = \text{var}\, R'_{k_2}$) with correlated parallel replications. Moreover, one can easily show that this bound looks like $O(k_2^2)$ as the accuracy requirement increases (smaller variance). The next section demonstrates how these bounds fare in practice.

3. An Illustration

This section describes a simulation designed to show how the theoretical results of Section 2 fare in practice. Consider a single server queueing system with independent and identically distributed exponential interarrival times with rate λ, independent and identically distributed exponential service times with rate $\omega > \lambda$ and finite capacity n. Here jobs that arrive when n jobs are already in the system merely go away. Corresponding to this continuous time representation one can view this sytem as a nearest neighbor Markov chain with $p_{jj'} = \omega/(\lambda+\omega)$ $j=1,\ldots,n$.

As a figure of merit we take mean number of customers in system (μ) the analytical expression for which appears in Gross and Harris (1974, p.67). In the serial model we estimate this quantity by

$$\hat{\mu}_k = \frac{\frac{1}{k}\left[\frac{1}{(\lambda+\omega)}\sum_{m=1}^{k}\sum_{j=1}^{n} j(N_{jj'm} + N_{jj''m})\right]}{\frac{1}{k}\left[\frac{k}{\lambda} + \frac{1}{\lambda+\omega}\sum_{m=1}^{k}\sum_{j=1}^{n}(N_{jj'm} + N_{jj''m})\right]} . \qquad (12)$$

In the parallel model using rotation sampling we estimate μ by

$$\mu_k' = \frac{\frac{1}{k}\left[\frac{1}{\lambda+\omega}\sum_{\ell=1}^{T_k'}\sum_{j=1}^{n} j(K'_{jj'\ell} + K'_{jj''\ell})\right]}{\frac{1}{k}\left[\frac{k}{\lambda} + \frac{1}{\lambda+\omega}\sum_{\ell=1}^{T_k'}\sum_{j=1}^{n}(K'_{jj'\ell} + K'_{jj''\ell})\right]} \quad . \quad (13)$$

Note that (12) and (13) are *ratio* estimators in contrast to (1) and (11) which are *linear* estimators. As a result, (12) and (13) are biased estimators of μ and one additional inquiry that we make here concerns which estimator has the smaller bias. Also, it is of interest to study the extent to which the bounding results in Theorem 4 apply to ratio estimation.

We are interested in observing how $E S_k / E S_k'$, $\text{var}\,\hat{\mu}_k / \text{var}\,\hat{\mu}_k'$ and $V(\hat{\mu}_k, \hat{\mu}_k')$ vary with k, n and $\rho = \lambda/\omega$. For convenience and without loss of generality we set $\omega = 1$. For each value of k given in Table 1, 1000 independent replications were performed for each experimental layout.

Table 1

Experimental Layout

$k = 2^m$

ρ \ n	1	3	7
0.5	m=1,...,8	m=2,...,9	m=3,...,10
0.9	m=1,...,8	m=2,...,9	m=3,...,10

This represents 1000 replications on each of the 48 experiments corresponding to a unique triplet (k,n,ρ) .

Table 2 shows selected ratios. Scrutinizing these results one sees that $E S_k / E S_k$, $\text{var } \hat{\mu}_k / \text{var } \hat{\mu}_k'$ and $V(\hat{\mu}_k, \hat{\mu}_k')$ all favor correlated sampling. In particular, for fixed ρ and n , increasing k increases these ratios. For fixed ρ and k , increasing n shows a degradation in these ratios. This is consistent with our theoretical results. However, the most interesting observation occurs for fixed k and n . Here the ratio $\text{var } \hat{\mu}_k / \text{var } \hat{\mu}_k'$ shows little effect due to changing ρ from 0.05 to 0.9. Whether or not this observation denotes an invariance in the performance of our proposed method with changes in ρ remains a topic for future investigation.

As prevously mentioned, use of the ratio estimators (12) and (13) enable one to study relative bias. The ratios $(E\hat{\mu}_k - \mu)/(E\hat{\mu}_k' - \mu)$ in Table 2 show that correlated sampling always produces a smaller absolute bias. However, the empirical results do not permit one to predict how the relative bias changes with k, n and ρ .

Several other results not available in Table 2 deserve mention. Firstly, for the 48 triplets considered the bias in $\hat{\mu}_k'$ did not exceed 0.1μ and usually was considerably smaller. Secondly, for fixed ρ and n a plot of $E T_k / E T_k'$ versus $k/\ln k$ showed the behavior dictated by (iii) of Theorem 3. Thirdly, a plot of $\text{var } \hat{\mu}_k'$ versus $(\ln k/k)^2$ revealed a linear relationship as $k \to \infty$ confirming

Table 2

Simulation Results for Empirical Example Based on 1000 Independent Replications

$$\frac{E\,S_k}{E\,S_k'}$$

k \ n	ρ=0.5 1	ρ=0.5 3	ρ=0.5 7	ρ=0.9 1	ρ=0.9 3	ρ=0.9 7
2	1.49			1.39		
4	2.23	1.52		2.05	1.27	
8	3.61	1.95	1.74	3.25	1.51	1.26
16	6.13	2.51	2.28	5.44	2.04	1.38
32	10.32	3.45	2.91	9.29	2.87	1.68
64	18.63	5.25	4.02	16.52	4.30	2.08
128	33.02	8.18	5.62	29.14	7.04	2.78
256	58.93	13.39	8.37	53.12	11.90	4.06
512		23.13	12.39		20.68	6.37
1024			18.64			10.60

$$\frac{\text{var}\ \hat{\mu}_k}{\text{var}\ \hat{\mu}_k'}$$

k \ n	ρ=0.5 1	ρ=0.5 3	ρ=0.5 7	ρ=0.9 1	ρ=0.9 3	ρ=0.9 7
2	1.38			1.95		
4	2.20	1.63		3.23	1.98	
8	4.09	2.22	1.32	4.18	2.36	1.62
16	6.80	2.96	1.65	7.04	3.44	1.82
32	10.43	4.05	1.95	10.73	4.49	2.28
64	26.21	6.44	2.68	19.71	6.54	2.47
128	34.40	11.10	3.68	37.05	10.22	4.03
256	73.19	16.69	6.08	67.26	16.91	6.16
512		33.84	9.53		28.34	8.78
1024			13.55			14.68

$$V(\hat{\mu}_k, \hat{\mu}_k') = \frac{E\,S_k}{E\,S_k'} \cdot \frac{\text{var}\ \hat{\mu}_k}{\text{var}\ \hat{\mu}_k'}$$

k \ n	ρ=0.5 1	ρ=0.5 3	ρ=0.5 7	ρ=0.9 1	ρ=0.9 3	ρ=0.9 7
2	2.06			2.70		
4	4.90	2.48		6.63	2.50	
8	14.77	4.38	2.31	13.63	3.56	2.03
16	37.52	7.43	3.77	38.28	7.02	2.51
32	107.68	13.96	5.67	99.73	12.88	3.82
64	488.33	33.81	10.77	325.61	28.12	5.15
128	1136.03	90.75	20.72	1079.64	71.93	11.18
256	4313.15	223.57	50.90	3573.16	201.21	25.00
512		782.75	117.86		586.12	55.92
1024			252.56			155.69

$$\frac{E\,\hat{\mu}_k - \mu}{E\,\hat{\mu}_k' - \mu}$$

k \ n	ρ=0.5 1	ρ=0.5 3	ρ=0.5 7	ρ=0.9 1	ρ=0.9 3	ρ=0.9 7
2	1.61			1.48		
4	3.60	1.70		3.12	2.02	
8	2.45	1.05	1.71	-19.95	2.60	1.51
16	10.04	4.51	1.55	-4.61	2.08	2.23
32	-8.36	-6.70	4.42	-5.87	2.38	1.98
64	8.87	1.64	3.50	51.34	-20.38	3.13
128	11.29	8.06	-4.52	-6.45	-52.94	3.25
256	58.03	14.61	2.48	4.92	424.09	-14.64
512		25.96	-1.35		99.39	20.48
1024			-12.67			23.60

the applicability of the lower bound $O((\ln k/k)^2)$ for this ratio estimator as well. Fourthly, empirical observation showed that for fixed ρ and n var $T_k' = O(1)$ as $k \to \infty$. This too remains a topic for future study. Lastly, we comment about c, the mean computing cost associated with visiting a state in the parallel simulation using rotation sampling relative to that mean cost in the serial simulation. For $n=1$, $c \approx 3.5$ as k increases; for $n=3$ and 7, $c \approx 2.5$ as k increases. This is not a serious issue for three reasons. Firstly increasing k firmly establishes the superiority of parallel simulation. Secondly, these estimates of c come from simulation experience in which many interim statistics, which are not normally collected in a simulation, were recorded for evaluation purposes. This additional collection naturally added to computing cost. Thirdly, since no attempt was made to optimize the programming code used, one expects that more careful attention to code would reduce c.

4. More General Markov Chain

We now consider the more general case of an aperiodic positive recurrent $n+1$ state Markov chain where transitions to more than two states are possible, $p_{00} = 0$ and where we select state 0 as the regeneration point. Then over k independent epochs the sample mean reward is

$$R_k = \frac{1}{k} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{m=1}^{k} A_{ij} N_{ijm} \qquad (14)$$

which again has var $R_k \propto 1/k$.

To obtain an equivalent formulation for k parallel, but independent, replications we begin our simulation with k departures from state 0. Let $\{m_{jr};\ r = 1,\ldots s_j\}$ denote the ordered sequence $(m_{jr} < m_{j,r+1})$ of the s_j states that a replication can enter from state j for $j = 1,\ldots,n$ and define $q_{ij} = \sum_{m=0}^{j} p_{im}$. Then the simulation is for $q_{jm_{j,0}} \equiv 0$:

1. Move replication i to state $m_{0,r}$ if $q_{0,m_{0,r-1}} \le U_{i1} < q_{0,m_{0,r}}$ for $r = 1,\ldots,n$ and $i = 1,\ldots,k$ where $U_{i1} \sim u(0,1)$.
2. Modify $\underset{\sim}{p}_n$ so that $p_{00} = 1$ and $p_{0j} = 0$ for $j = 1,\ldots,n$.
3. On step $\ell+1$ where $\ell \ge 1$ and for $j = 1,\ldots,n$, if $K_{j\ell} > 0$ move replication $i_{j\ell m}$ to state m_{jr} if $q_{jm_{j,r-1}} \le U_{i_{j\ell m},\ell+1} < q_{jm_{j,r}}$ where $U_{i_{j\ell m},\ell+1} \sim u(0,1)$ for $m = 1,\ldots,K_{j\ell}$ and $r = 1,\ldots,s_j$.
4. Stop at step $T^* = \min\{\ell:\ K_{0\ell} = k;\ \ell = 1,2,\ldots\}$.

The sample mean reward is

$$R_k = \frac{1}{k} \sum_{j=0}^{n} \sum_{r=1}^{s_j} \sum_{\ell=0}^{T^*-1} A_{jm_{jr}}\ K_{j,m_{jr},\ell+1}\ . \tag{15}$$

If one imposes the restriction (9), the simulation remains as above in principle with K'_{ij} replacing K_{ij} in steps 1 through 4. The sample mean reward is then

$$R'_k = \frac{1}{k} \sum_{j=0}^{n} \sum_{r=1}^{s_j} \sum_{\ell=0}^{T'_k-1} A_{jm_{jr}}\ K'_{j,m_{jr},\ell+1}\ .$$

Moreover, one has:

Theorem 5 . If the restriction in (9) is imposed, then

(i) For $r = 1,\ldots,s_j$

$$K'_{j,m_{jr},\ell+1} = \lfloor Q \rfloor - \lfloor P \rfloor - 1 \quad \text{w.p. } \max(\bar{P},\bar{Q}) - \bar{Q}$$

$$= \lfloor Q \rfloor - \lfloor P \rfloor \quad \text{w.p. } 1 - 2\max(\bar{P},\bar{Q})$$

$$= \lfloor Q \rfloor - \lfloor P \rfloor + 1 \quad \text{w.p. } \max(\bar{P},\bar{Q}) - \bar{P}$$

where $P = K'_{j\ell}q_{jm_{j,r-1}}$, $Q = K'_{j\ell}q_{jm_{jr}}$, $\bar{P} \equiv P \pmod 1$ and $\bar{Q} \equiv Q \pmod 1$.

(ii) $\operatorname{var}(K'_{ij,\ell+1} \mid K'_{i\ell}) = (\overline{K'_{i\ell}p_{ij}})(1 - \overline{K'_{i\ell}p_{ij}}) \le 1/4$.

(iii) $\operatorname{var} K'_{ij,\ell+1} = O(1)$.

See the Appendix for the proof.

As observed for the nearest neighbor model, a simulation using (9) for large k has transitions near the beginning that are essentially deterministic. Also, the random component of a transition from one state to another has the Bernoulli distribution, since examination of part (i) of Theorem 5 shows that only two of the three outcomes are possible.

With regard to the total absorption time T'_k and $\operatorname{var} R'_k$, it is not difficult to show that they behave as in the nearest neighbor case as a function of k and n . However, S'_k , the number of states visited, calls for special attention. Whereas one has $E\,S'_k \le n\,E\,T'_k$ for the

nearest neighbor model, it is here $E S_k' \le E(T_k') \sum_{j=1}^{n} s_j \le n^2 \, E T_k'$.

This results from the observation that s_j computations are necessary to allocate the $K_{j\ell}'$ replications in state j on transition ℓ.

We summarize these observations in Theorem 6.

Theorem 6. For the more general finite state Markov chain

$$\text{(i)} \qquad E T_k' = O(\ln k)$$

$$\text{(ii)} \qquad \text{var } R_k' \le O((\ln k/k)^2)$$

$$\text{(iii)} \qquad E S_k' \le O(\ln k)$$

$$\text{(iv)} \qquad V(R_k, R_k') \ge O(k^2/(\ln k)^3) \quad .$$

Proofs follow in a manner analogous to those for earlier theorems and, to conserve space, are omitted here.

5. Unresolved Issues

The theoretical results of Sections 1 and 2 together with empirical observations of Section 3 unequivocally demonstrate the value of rotation sampling for reducing variance in the simulation of Markov chains. We now address several ancillary problems reasonable solutions to which are necessary before one can move totally from theory to practical implementation.

Normality

When simulating k independent epochs in series, an experimenter ultimately relies on the central limit theorem to derive an acceptable approximating distribution theory for confidence intervals. For independent parallel replications, there is some plausibility in assuming normality for at least some states when k is large. This follows from the observation that the binomial distribution converges to the normal as $k \to \infty$. For correlated parallel replications using (9), the Bernoulli distributions in each state for the sum of all members indicates clearly that no convergence to normality exists, regardless of the magnitude of k. In summary, *superconvergence* of variance is achieved at the loss of convergence to normality.

Variance Estimation

Simulating independent epochs serially also enables one to estimate the variance needed to assess accuracy. Regrettably, at the current writing no comparable ease of estimation exists for k correlated parallel replications. For R_k' as in (11) a conceivable upper bound estimate on $\text{var } R_k'$ is $(n A T_k'/2k)^2$ where $A=\sup_{j=1,\ldots,n} \max(|A_{jj'}|,|A_{jj''}|)$. The value of this estimate remains to be studied in detail. For the ratio estimator (13), no comparable proposal is currently available.

Sequential Estimation

Although one should not minimize the importance of the loss of convergence to normality and the ability to estimate $\text{var } R_k'$, at least one issue of importance in simulation mitigates the seriousness of the loss. If one sets out to estimate, say, $E R_k'$ to within an accuracy of $\pm\delta$, then a sequential estimation procedure inevitably plays a role. A principal result in this area is in Chow and Robbins (1965) and is described in Fishman (1978, Ch. 2) for the special circumstances of discrete event simulation. Let $R_k'^{(m)}$ denote the mth independent *macroreplication* of the new Markov chain were each macroreplication consists of k correlated parallel replications of the original chain. Then one runs m^* macroreplications with outcomes $R_k'^{(1)}, R_k'^{(2)}, \ldots, R_k'^{(m^*)}$, termination occurring after the macroreplication (m^*) dictated by the stopping rule associated with the procedure. By making k large one achieves the benefit of correlated sampling and allows an experimenter to opt for a very small δ. Since as $\delta \to 0$ the relevance of the Chow-Robbins result for practice grows, one sees that in this context the accelerated convergence that correlated replications induces in $\text{var } R_k'$ compensates for the loss of normal convergence and the ability to estimate $\text{var } R_k'$ directly.

Finite State Space

The results of this paper clearly show that the lower bound on the relative benefit of correlated parallel sampling diminishes for fixed

k and increasing n . Although the seriousness of this weakening of the boundary remains a topic for further research, one can draw solace from Table 3 which gives the mean number of states occupied per transition for the simulations of Section 4. Observe the relatively slow rate of increase with regard to k .

Table 3

Sample Mean Number of Occupied States per Transition

n \ k	8	16	32	64	128	256	512	1024
ρ=0.5								
3	1.20	1.40	1.67	1.92	2.10	2.20	2.30	
7	1.17	1.31	1.53	1.77	2.04	2.38	2.76	3.22
ρ=0.9								
3	1.46	1.76	2.01	2.18	2.28	2.38	2.44	
7	1.36	1.68	2.18	2.79	3.49	4.05	4.47	4.80

Transient Simulation

As described here, the simulation we have in mind aims at estimating a steady-state characteristic of a Markov chain. Alternatively one may be interested in the paths between states i and j and function of these paths. Since the choice of regenerative simulation via state 0 was merely a convenience of exposition, it is not difficult to see that in the transient case one can construct correlated parallel replications which have all the desirable features given in Sections 3 and 5.

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APPENDIX

Proof of Theorem 1. Let

$$p_{j0}^{(\ell)} = \text{pr(moving from state } j \text{ to state } 0 \text{ in } \ell \text{ steps)} \quad j > 0.$$

Since the parallel replications remain independent, one has

$$E\,\dot{T}_k^m = k \sum_{t=1}^{\infty} t^m \left(\sum_{\ell=1}^{t} p_{10}^{(\ell)}\right)^{k-1} p_{10}^{(t)} \qquad m=1,2$$

$$< k\,E\,T^m \ .$$

Then

$$E\,\dot{T}_k^2 = \text{var}\,\dot{T}_k + E^2\,\dot{T}_k \leq k\,E\,T^2$$

from which it follows that

$$E\,\dot{T}_k < k^{1/2}\,(E\,T^2)^{1/2} \ .$$

For part (ii) note the specification that $\text{var}\,R_k = \text{var}\,\dot{R}_k$. Since $E\,T_k = k\,\omega(p_n)$ one has for fixed n

$$V(R_k,\dot{R}_k) = \frac{E\,S_k}{E\,\dot{S}_k} \geq O(k^{1/2}) \ .$$

Proof of Theorem 2

One can write $K'_{jj',s+1}$ as

$$K'_{jj',s+1} = \sum_{\ell=1}^{K'_{js}} I_{[0,p_{jj'})}(U_{j,\ell,l}^{s} + \sum_{m=1}^{m-1} K'_{js} \pmod 1) \quad . \tag{A.1}$$

From Lemma 5.1 of Fishman and Huang (1980), we have equivalently,

$$K_{jj',s+1} = \sum_{m=1}^{K'} I_{[0,p)} \binom{U+m-1}{K'} \tag{A.2}$$

where $K' = K'_{js}$, $U = K'U_{j,\ell,l}^{s} \pmod 1$ and $p = p_{jj'}$. Let $P_t = pr(K'_{jj',s+1} = t)$. Then

$$P_0 = 1 - \min(1, K'p)$$

$$P_1 = \min(1,K'p) - \max(0, \min(1, K'p - 1))$$

$$= 1 - P_0 - \max(0, \min(1, K'p - 1))$$

$$P_2 = 1 - P_0 - P_1 - \max(0, \min(1, K'p - 2))$$

$$\vdots$$

$$P_t = 1 - \sum_{t'=0}^{t-1} P_{t'} - \max(0, \min(1, K'p - t))$$

so that

$$pr(K'_{jj',s+1} = t) = 1 - \max(0, \min(1, K'p - t)) \qquad t = 0,1,\ldots,K' \; .$$

Therefore,

$$\text{pr}(K'_{jj',\ell+1} \le t) = 0 \qquad 0 \le t \le \lfloor K'p \rfloor - 1$$

$$\text{pr}(K'_{jj',\ell+1} = \lfloor K'p \rfloor) = 1 - K'p + \lfloor K'p \rfloor = 1 - \overline{K'p}$$

$$\text{pr}(K_{jj',\ell+1} = \lfloor K'p \rfloor + 1) = \overline{K'p}$$

which establishes (i). Note that $K'_{jj',\ell+1} - \lfloor K'p \rfloor$ has the Bernoulli distribution $\text{Ber}(\overline{K'p})$.

Proof of (ii) follows immediately. Since $E(K'_{jj',\ell+1}|K') = \lfloor K'p \rfloor + \overline{K'p} - K'p$ and $E(K'_{jj',\ell+1}|K')^2 = \lfloor K'p \rfloor^2(1 - \overline{K'p}) + (\lfloor K'p \rfloor + 1)^2 \overline{K'p}$ we have $\text{var}(K'_{jj',\ell+1}|K'_{j\cdot}) = (\overline{K'_{j\ell}p_{jj'}})(1 - \overline{K'_{j\ell}p_{jj'}}) \le 1/4$. It also follows from Theorem 3 of Fishman and Huang (1980).

We prove (iii) by induction. Observe that

$$\text{var } K'_{jm,\nu+1} = E\ \text{var}(K'_{jm,\nu+1}|K'_{j\ell}) + p_{jm}\ \text{var } K'_{j\ell} \qquad m = j', j''$$

Since

$$\text{var}(K'_{jm,\nu+1}|K'_{j\ell}) = (K'_{j\nu}p_{jm})(1 - K'_{j\ell}p_{jm}) \le 1/4\ , \tag{A.3}$$

$$E\ \text{var}(K'_{jm,\nu+1}|K'_{j\ell}) \le 1/4\ . \tag{A.4}$$

Also

$$\operatorname{var}(K'_{jm,1} \mid K'_{j0}) = 0 \qquad j = 0,1,\ldots,n$$

$$\operatorname{var} K'_{j0} = 0$$

and

$$\operatorname{var} K'_{j,\ell+1} = \operatorname{var} K'_{j'j\ell} + \operatorname{var} K'_{j''j\ell} + 2 \operatorname{cov}(K'_{j'j\ell}, K'_{j''j\ell})$$

$$\leq [(\operatorname{var} K'_{j'j\ell})^{\frac{1}{2}} + (\operatorname{var} K'_{j''j\ell})^{\frac{1}{2}}]^2 \qquad (A.5)$$

so that

$$\operatorname{var} K'_{jm,\ell+1} \quad \frac{1}{4} + p'_{jm}[(\operatorname{var} K'_{j'j,\ell-1})^{\frac{1}{2}} + (\operatorname{var} K'_{j''j,\ell-1})^{\frac{1}{2}}]^2$$

$$\frac{1}{4} + r^2[(\operatorname{var} K'_{j'j,\ell-1})^{\frac{1}{2}} + (\operatorname{var} K'_{j''j,\ell-1})^{\frac{1}{2}}]^2 \qquad (A.6)$$

where $r = \sup p_{ij} \quad i,j = 1,\ldots,n$.

Beginning with $\ell = 1$, one can easily show that the magnitudes of

$\operatorname{var} K'_{jm,1}$, $\operatorname{var} K'_{jm,2}$, $\operatorname{var} K'_{jm,3}$, ... for $j = 0,\ldots,n$

are independent of k . Therefore, it follows by induction that

$$\operatorname{var} K'_{jm,\ell+1} = O(1) .$$

Proof of Theorem 3

Let Q_t denote the probability of total absorption on transition t. Then

$$Q_t = pr(K'_{0,t-1} < k,\ K'_{0t} = k)$$

$$= \sum_{m=0}^{k-1} pr(K'_{0t} = k \mid K'_{0,t-1} = m)\ pr(K'_{0,t-1} = m) \quad .$$

Recall that for given $K'_{j\ell}$

$$K'_{jj',\ell+1} = \lfloor K'_{j\ell}\ p_{jj'} \rfloor \qquad \text{w.p. } 1 - \overline{K'_{j\ell}\ p_{jj'}}.$$

$$= \lfloor K'_{j\ell}\ p_{jj'} \rfloor + 1 \qquad \text{w.p. } \overline{K'_{j\ell}\ p_{jj'}} \quad .$$

so that

$$pr\ (K'_{0t} = k \mid K'_{0,t-1} = m) = \overline{(k-m)\ p_{10}}\ pr(K'_{1,t-1} = k-m \mid K'_{0,t-1} = m)$$

$$m = k - r, \ldots, k - 1$$

$$= 0 \qquad \text{otherwise.}$$

Therefore, $Q_t = 0$ unless $K'_{t-1} \in \{(k-m, m, 0, \ldots, 0), m=1, \ldots, r\}$, which proves (i).

Part (ii) follows from merely writing down the transition pattern into each state and regrouping terms. To prove (iii) one notes that

(a) $-1 < M_{j\ell} < 1$

(b) $M_{jj''} = 0 \qquad j=1,\ldots,n$

(c) $1 - \sum_{i=1}^{t} p_{j0}^{(i)} = O(\beta_j^t) \qquad 0 < \beta_j < 1 \qquad j=1,\ldots,n$

(d) $p_{j'0}^{(i)} \cdot p_{j''0}^{(i)} \qquad i=j',\ j'+1,\ldots$.

Note (c) is a standard result for transitions from a transient state to an absorbing state (e.g. Cox and Miller, 1965). Observe that

$$|M_{1t} + \sum_{\ell=1}^{t-1} M_{1\ell} (1 - \sum_{i=1}^{t-\ell} p_{20}^{(i)}) + \sum_{j=2}^{n} \sum_{\ell=1}^{t-j'} M_{j\ell} \sum_{i=1}^{t-\ell} (p_{j'0}^{(i)} - p_{j''0}^{(i)})|$$

$$\leq 1 + \sum_{j=1}^{n} \sum_{\ell=1}^{t-j'} (1 - \sum_{i=1}^{t-\ell} p_{j''0}^{(i)}) \leq b$$

where $b \geq 1$ and is finite for fixed n and all $t > 1$. Therefore,

$$-b + k(1 - \sum_{i=1}^{t} p_{10}^{(i)}) < k - K'_{0t} < b + k(1 - \sum_{i=1}^{t} p_{10}^{(i)})$$

so that for $\rho = \beta_1$

$$-b + k\, O(\rho^t) < k - K'_{0t} < b + k\, O(\rho^t) \quad ,$$

which proves part (iii) .

Observe that $r \le b + k\, O(\rho^{T'_{kr}})$ and $-b + k\, O(\rho^{T'_{kr}}) \le 1$. Therefore, $O(\ln k) \le T'_{kr} \le O(\ln k)$ which establishes (iv) and (v). Now observe that for $t \ge T'_{kr}$ the number of replications remaining in the transient set is independent of k. Therefore, $Y_{kr} = O(1)$ w.p. 1 so that $E\, T'_k = O(\ln k)$, proving (vi) and (vii).

Proof of Theorem 4(i)

Let

$$D_\ell = \sum_{j=1}^{n} (A_{jj'} K'_{jj'\ell} + A_{jj''} K'_{jj''\ell}) \quad .$$

From (iii) of Theorem 2 one has $\text{var}\, D_\ell = O(1)$. From (iii) of Theorem 3 there exists a constant b_1 such that $T'_{kr} \le b_1 \ln k$. Therefore, one can represent R'_k as

$$R'_k = \frac{1}{k} \sum_{1 \le \ell \le b_1 \ln k} D_\ell + \varepsilon$$

where ε is a random variable whose distribution is independent of k. Finally

$$\text{var}\, R'_k \le O((b_1 \ln k/k)^2) = O((\ln k/k)^2) ,$$

which establishes (i).

Proof of Theorem 5

Let

$$L(q,p) = \sum_{\ell=1}^{K} I_{[q,p)}\left(\frac{U+\ell-1}{K}\right)$$

where $0 \le q < p \le 1$ and $U \sim U(0,1)$. Observe that

$$pr[L(q,p) = m] = \sum_{j=0}^{K-m} pr[L(0,q) = j,\ L(0,p) = j+m] \qquad m = 0,\ldots,K \ . \quad (A.7)$$

From part (1) in Theorem 2 we have

$$pr[L(0,q) = \lfloor Kq \rfloor] = 1 - \overline{Kq}$$

$$pr[L(0,p) = \lfloor Kp \rfloor] = 1 - \overline{Kp}$$

$$pr[L(0,q) = \lfloor Kq \rfloor + 1] = \overline{Kq}$$

$$pr[L(0,p) = \lfloor Kp \rfloor + 1] = \overline{Kp}$$

so that

$$pr[L(0,q) = \lfloor Kq \rfloor , L(0,p) = \lfloor Kp \rfloor] = 1 - \max(\overline{Kq},\overline{Kp})$$

$$pr[L(0,q) = \lfloor Kq \rfloor , L(0,p) = \lfloor Kp \rfloor + 1] = \max(0, \overline{Kp} - \overline{Kq})$$

$$pr[L(0,q) = \lfloor Kq \rfloor + 1, L(0,p) = \lfloor Kp \rfloor + 1] = \min(\overline{Kp},\overline{Kq})$$

$$pr[L(0,q) = \lfloor Kq \rfloor + 1, L(0,p) = \lfloor Kp \rfloor] = \max(0, \overline{Kq} - \overline{Kp}) .$$

Substituting into (A.7) gives

$$\begin{aligned} pr[L(q,p) &= \lfloor Kp \rfloor - \lfloor Kq \rfloor - 1] = \max(0, \overline{Kq} - \overline{Kp}) \\ pr[L(q,p) &= \lfloor Kp \rfloor - \lfloor Kq \rfloor] = 1 - \max(\overline{Kp}, \overline{Kq}) + \min(\overline{Kp}, \overline{Kq}) \\ pr[L(q,p) &= \lfloor Kp \rfloor - \lfloor Kq \rfloor + 1] = \max(0, \overline{Kp} - \overline{Kq}) . \end{aligned} \qquad (A.8)$$

Then part (i) holds for $q = q_{j,m_{j,r-1}}$, $p = q_{j,m_{jr}}$, $L(q,p) = K'_{j,m_{jr},\ell+1}$ and $K = K'_{j\ell}$.

Part (ii) follows immediately from (A.8) and part (iii) follows in a manner similar to part (iii) of Theorem 2.

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This paper describes a method of obtaining results from the simulation of a finite state positive recurrent aperiodic Markov chain at a cost considerably below the cost required to achieve the same accuracy with pure random sampling. By reorganizing k independent epochs or tours simulated serially into k replications simulated in parallel, one can induce selected joint distributions across replications that produce the cost-saving benefits. The joint distributions follow from the use of rotation sampling, a special case of the

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20. antithetic variate method.

For a finite state nearest neighbor chain the paper shows that even for independent parallel replications the cost of achieving a specified accuracy with serial simulation relative to the cost for parallel simulation has a lower bound $\Omega(k^{\frac{1}{2}})$ as $k \to \infty$. When rotation sampling is used this bound is $O(k^2/(\ln k)^3)$. This lower bound also holds for the more general finite state chains. A simulation of the M/M/1 queueing model with finite capacity n is used to illustrate the effectiveness of the technique for selected values of k, n and activity level ρ.


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