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LANGEVIN EQUATIONS WITH POISSON PERTURBANCES

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ABSTRACT

Let $z(t) \in R^n$ be a generalized Poisson process with parameter λ. In the present paper, the conditions of existence and limiting behavior as $\lambda \to \infty$ or as $\lambda \to 0$ of the stationary distribution of the solution of Langevin equation $dx(t) = Ax(t) + dz(t)$ are investigated. Using these results, the distribution of virtual waiting time in a queueing system with variable service speed is studied.

A

1. Introduction

Let $z(t) = (z_1(t), z_2(t)\ldots, z_n(t))^T \epsilon R^n$ be a generalized Poisson process with parameter λ and jumps $x_1, x_2,\ldots, x_m,\ldots$. Let also $A: R^n \to R^n$ be a linear operator determined by the matrix $A = \| a_{ij} \|_{i,j=1}^{n}$. (We suppose that the basis in R^n is fixed.)

The present paper deals with conditions of existence and limiting properties as $\lambda \to \infty$ or as $\lambda \to 0$, of stationary distribution of the process $x(t) = (x_1(t),\ldots, x_n(t))^T \epsilon R^n$, which satisfies the formal equation

$$dx(t) = Ax(t)dt + dz(t) \tag{1.1}$$

The equation (1.1) is a continuous analog of the autoregression equation

$$x_{m+1} = (\varepsilon A+I)x_m + z_{m+1}$$

where the distribution of independent identically distributed (i.i.d.) random vectors $z_1, z_2,\ldots$ has atom at zero with weight $1 - \varepsilon$. Thus, it can describe some processes connected with radioactive decay, queuing systems with changeable rate of service, etc.

2. Stationary Distribution and the Conditions of its Existence

As usual, we will assume that the differential equation (1.1) with initial condition $x(0) = x_0$ is equivalent to the integral equation.

$$x(t) = x_0 + \int_0^t Ax(u)du + z(t) \tag{2.1}$$

which holds with probability one for all values of t.

In what follows, we assume that the process $z(t)$ has rightcontinuous sample paths with probability one.

Lemma 2.1. The equation (2.1) has a unique solution in the class of measurable processes. This solution is a rightcontinuous strongly Markovian process and can be written in the form

$$x(t) = \exp\{At\}x_0 + \int_0^t \exp\{A(t-u)\}dz(u) \tag{2.2}$$

where the integral on the right-hand side of (2.2) is a Stiltjes integral and exists with probability one.

The proof of this lemma is routine.

Lemma 2.2. The one-dimensional distributions of the process $x(t)$ are infinitely divisible and have the characteristic functions (c.f.)

$$\Psi(s;t) = E\exp\{i(s,x(t))\} =$$

$$\exp\{i(s,\exp\{At\}x_0) - \lambda\int_0^t (1-\varphi(\exp\{A^Tu\}s))du\} \tag{2.3}$$

where $\varphi(s) = E\{\exp i(s,x_1)\}$.

The proof of this lemma follows from the representation (2.2).

Theorem 2.1. The process $x(t)$ possesses limiting distribution as $t \to \infty$ which does not depend on the initial state x_0 if and only if

1) the eigenvalues of A lie in the left halfplane;

2) $E\log(1+|x_1|) < \infty$.

Proof. If $\Psi(s;t) = \Psi(s;t,x_0) \to \Psi(s)$ as $t \to \infty$ and $\Psi(s)$ is continuous, then $\Psi(s)$ does not vanish since it is a c.f. of an infinitely divisible distribution in R^n . Thus

$$\exp\{i(s,\exp\{At\}x_0) = \Psi(s;t,2x_0)\Psi^{-1}(s;t,x_0) \underset{t\to\infty}{\to} 1$$

for all initial values x_0 and $s \in R^n$.

This is possible if and only if the condition 1) holds. In this case, we have the equality

$$\Psi(s) = \exp\{-\lambda\int_0^\infty (1-\varphi(\exp\{A^T u\}s))du\} \tag{2.4}$$

It is easy to check that the infinitely divisible c.f. $\Psi(s;t)$ and $\Psi(s)$ determined by (2.3) and (2.4) have the Lèvy representations

$$\log \Psi(s;t) = i(s,\gamma_t) - Q_t(s) + \int_{|x|>0} [\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1}]N_t(dx) \tag{2.5}$$

$$\log \Psi(s) = i(s,\gamma) - Q(s) + \int_{|x|>0} [\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1}]N(dx) \tag{2.6}$$

where

$$\gamma_t = \lambda\int_0^t \int_{R^n} \exp\{Au\}x(1+(\exp\{Au\}x,\exp\{Au\}x))^{-1}P\{x_1\epsilon dx\}du + \exp\{At\}x_0 ,$$

$$\gamma = \gamma(\lambda) = \lambda\int_0^\infty \int_{R^n} \exp\{Au\}x(1+(\exp\{Au\}x,\exp\{Au\}x))^{-1}P\{x_1\epsilon dx\}du ,$$

$$Q_t(s) = Q(s) = 0 ,$$

$$N_t(B) = \lambda\int_0^t P\{\exp\{Au\}x_1\epsilon B\}du ,$$

$$N(B) = N(B;\lambda) = \lambda\int_0^\infty P\{\exp\{Au\}x_1\epsilon B\}du .$$

It follows from the theorem proved in [4, p. 188] that $\Psi(s,t) \to \Psi(s)$ if and only if

a) $N_t(B) \to N(B)$ as $t \to \infty$ for continuity sets B of N

lying in $R^n/S_\varepsilon, S_\varepsilon = \{x: |x| \leq \varepsilon\}$;

b) $\gamma_t \to \gamma$ as $t \to \infty$;

c) $$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \int_{0<|x|<\varepsilon} (x,x) N_t(dx) = 0 .$$

Using the scheme of the proof of the theorem 1 in [5] we can show that the condition a) is equivalent to the condition 2) of the theorem.

Since $N_t(B) \leq N(B)$, we have

$$0 \leq \int_{0<|x|<\varepsilon} |x| N_t(dx) \leq \int_{0<|x|\leq\varepsilon} |x| N(dx) \tag{2.7}$$

Thus, using the estimations

$$\int_{0<|x|<\varepsilon} xN(dx) = \int_0^\varepsilon r d_r N(S_r) = \int_0^\varepsilon N(S_\varepsilon/S_r) dr \leq$$

$$\int_0^\varepsilon N(R^n/S_r) dr = \lambda \int_0^\varepsilon \int_0^\infty P\{|\exp\{Au\} x_1| > r\} du dr \leq$$

$$\lambda \int_0^\varepsilon \int_0^\infty P\{|x_1|+1 > cr \exp\{au\}\} du dr =$$

$$\lambda \int_0^\varepsilon \int_0^\infty P\{a^{-1} \log [(|x_1|+1)(rc)^{-1}] > u\} du dr =$$

$$\lambda \int_0^\varepsilon E\, a^{-1} \log [(|x_1|+1)(rc)^{-1}] dr =$$

$$\lambda a^{-1} (\varepsilon E \log (|x_1|+1) - c^{-1} \varepsilon \log \varepsilon) \tag{2.8}$$

which are valid for small values of ε and some $a > 0$ $c > 0$, we can easily deduce c). The condition b) can be checked in the same manner.

3. Limiting Behavior of the Stationary Distribution as $\lambda \to \infty$

Although the formula (2.4) gives the explicit form of c.f. of stationary distribution of $x(t)$, it is of interest to investigate its limiting behavior as $\lambda \to \infty$. In this part of the paper, we will study the limiting distributions of random vectors $b^{-1}(\lambda)(x(t)-a(\lambda))$ under the assumption that $x(t)$ has a stationary distribution. (Here $a(\lambda) \in R^n$ and $b(\lambda) > 0$ are nonrandom functions.) The class of nondegenerate limiting distributions for such vectors coincides with the class of stable distributions in R^n, which were investigated by P. Lèvy [3], F. Feldhaim [1] and E. L. Rvačeva [4]. It has been shown there that stable distributions in R^n are infinitely divisible and their c.f. have the form

$$\varphi(s) = \begin{cases} \exp\{-|s|^{\alpha}[c_1(s/|s|)+ic_2(s/|s|)] + i(\beta,s)\}, & 0 < \alpha \leq 2, \alpha \neq 1 \\ \exp\{-|s|[c_1(s/|s|)+ic_2^1(s)] + i(\beta,s)\}, & \alpha = 1 \end{cases} \qquad (3.1)$$

where $c_1(s/|s|) = c\int |\cos(\widehat{s,w})|^{\alpha} dH(w)$,

$$c_2(s/|s|) = -c\tan(2^{-1}\pi\alpha)\int \cos(\widehat{s,w})|\cos(\widehat{s,w})|^{\alpha-1} dH(w),$$

$$c_2^1(s) = 2\pi^{-1}\int \cos(\widehat{s,w}) \log(|s||\cos(\widehat{s,w})|) dH(w),$$

$\beta \in R^n$ is a constant vector, w denotes a point on the unit sphere (and the vector joining the origin to it), H is a finite measure on the unit sphere, and the domain of the integration is the entire surface of the unit sphere. The number α is called the characteristic exponent of the distribution. If $\alpha = 2$, we have the multidimensional normal distribution.

Rvačeva [4, p. 192] showed that for the nondegenerate stable laws in R^n the Lèvy representations of their c.f.

$$\varphi(s) = \exp\{i(\beta,s) - 2^{-1}Q(s) + \int_{|x|>0} (\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1}) dN_0(x)$$

have such characteristics:

a) for $\alpha = 2$, $N_0(B)$ is constant, $Q(s) = 2(s,s)C_1(s/|s|)$ (3.2)

b) for $0 < \alpha < 2$, $N_0(B) = R^{-\alpha}H(W)$, $Q(s) = 0$ (3.3)

for every set B of the form $\{x: |x| > R , w \in W\}$, W being a subset of the surface of the unit sphere.

Theorem 3.1. If for some suitably chosen nonrandom functions, $a(\lambda) \in R^n$ and $b(\lambda) > 0$ the distribution of the vector $b^{-1}(\lambda)(x(t)-a(\lambda))$ weakly converges as $\lambda \to \infty$ to a nondegenerate distribution Π, then, Π is a stable distribution in R^n with characteristic exponent α , $0 < \alpha \leq 2$, and $b(\lambda)$ is a regularly varying function with exponent α^{-1} .

Proof. It follows from the formula (2.6) that the c.f. of the vector $x(t)$ has the form $\Psi(s) = \exp\{\lambda K(s)\}$, where $K(s)$ does not depend on λ. Thus, we can consider $x(t) = x(t;\lambda)$ as the value of a homogeneous process with independent increments at the moment λ. This implies the statement of the theorem. Later, we will need the following result.

Lemma 3.1. If the i.i.d. vectors x_k belong to the domain of attraction of a stable law in R^n with characteristic exponent α then $|x_k|$ belong to the domain of attraction of a stable law in R^n with the same exponent α , $0 < \alpha \leq 2$.

The proof of this lemma follows immediately from the theorems 4.1 and 4.2 of the work [4].

Remark 3.1. We will also use the fact that the norming functions $b_1(n)$ and $b_2(n)$ for which the sequences $b_1^{-1}(n)(x_1+\ldots+x_n-a_n)$ and $b_2^{-1}(n)(|x_1|+\ldots+|x_n|-a_n^1)$ weakly converge can be chosen equal, $b_1(n) = b_2(n) = b(n)$. This follows from theorem 2.3 of the work [4].

We will consider the cases $\alpha < 2$ and $\alpha = 2$ separately.

Theorem 3.2. The distribution of the vector $b^{-1}(\lambda)(x(t;\lambda)-a(\lambda))$ weakly converges to the stable law in R^n with characteristic exponent

α , $0 < \alpha < 2$ and spectral Lèvy measure N_0 if and only if the distribution of the vector x_1 belongs to the domain of attraction of the stable law with the same exponent α and spectral Lèvy measure M. The measures N_0 and M determine each other uniquely by the equality

$$N_0(B) = \int_0^{\infty} M(\exp\{-Au\}B)\,du \ .$$

Proof. Necessity. From (2.6) we have

$$\log E \exp\{i(s,b^{-1}(\lambda)(x(t;\lambda)-a(\lambda)))\} = \\ i(s,a_1(\lambda)) + \int_{|x|>0} (\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1})dN(b(\lambda)x) \quad (3.4)$$

where $a_1(\lambda) = b^{-1}(\lambda)(\gamma(\lambda)-a(\lambda)- \int_{|x|>0} x(1+(x,x))^{-1}dN(x))$

$$+ \int_{|x|>0} x(1+(x,x))^{-1}dN(b(\lambda)x) \ .$$

(Note that the integrals converge because of inequality (2.8).) It follows from theorem 1.2 [4, p. 188] that the required convergence is possible if and only if

$$\text{a)} \quad N(b(\lambda)B) = N(\lambda;b(\lambda)B) \underset{\lambda\to\infty}{\to} N_0(B) \ , \qquad (3.5)$$

where N_0 is determined by (3.3) and $B \subset R^n/S_\varepsilon$ is its continuity set;

$$\text{b)} \quad \lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \int_{|x|<\varepsilon} (x,x)dN(xb(\lambda)) = 0; \qquad (3.6)$$

$$\text{c)} \quad \lim_{\lambda\to\infty} \gamma(\lambda) = \gamma_0 \ .$$

The condition (3.5) implies the weak compactness of the family of measures

$$\lambda P\{\exp\{Ar\}x_1 \in b(\lambda)B\} \qquad (3.7)$$

in R^n/S_ε , $\varepsilon > 0$. To prove this, we consider the Borel sets of the form $B = \bigcup_{v \geq 0} \exp\{-Av\}S$, where S is a hypersurface in R^n and the sets $\exp\{-Av\}S$ do not intersect for different values of v. For such sets we have

$$N(b(\lambda)B) = \lambda \int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda)B\}du =$$

$$\lambda \int_0^\infty P\{x_1 \in b(\lambda) \bigcup_{v \geq u} \exp\{-Av\}S\}du = \lambda \int_0^\infty \int_u^\infty P\{x_1 \in b(\lambda)d \exp\{-Av\}S\}du =$$

$$\lambda \int_0^\infty \int_0^v duP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} = \lambda \int_0^\infty vP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} \to N_0(B) .$$

Similarly, $N(b(\lambda) \exp\{-Ar\}B) = \lambda \int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda) \exp\{-Ar\}B\}du =$

$$\lambda \int_0^\infty P\{x_1 \in b(\lambda) \bigcup_{v \geq u+r} \exp\{-Av\}S\}du = \lambda \int_0^\infty \int_{u+r}^\infty P\{x_1 \in b(\lambda)d \exp\{-Av\}S\}du =$$

$$\lambda \int_r^\infty \int_0^{v-r} duP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} = \lambda \int_r^\infty (v-r)P\{x_1 \in b(\lambda)d \exp\{-Av\}S\} =$$

$$\lambda \int_r^\infty vP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} - \lambda rP\{x_1 \in b(\lambda) \exp\{-Ar\}B\} \to N_0(\exp\{-Ar\}B) .$$

Since

$$\int_r^\infty vP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} \leq \int_0^\infty vP\{x_1 \in b(\lambda)d \exp\{-Av\}S\} ,$$

the last two relations imply the weak compactness of the family (3.7) in R^n/S_ε , $\varepsilon > 0$.

In this case the family of measures

$$\lambda P\{x_1 \in b(\lambda)B\} \qquad (3.8)$$

is also weakly compact in R^n/S_ε , $\varepsilon > 0$.

Since the estimate

$$\lambda \int_t^\infty P\{\exp\{Au\}x_1 \in b(\lambda)B\}du = \lambda \int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda)\exp\{-At\}B\}du \to$$

$$N_0(\exp\{-At\}B) < \delta$$

holds for large values of t if $B \subset R^n/S_\varepsilon$, we can choose from (3.8) a weakly convergent subsequence

$$\lambda_k P\{x_1 \in b(\lambda_k)B\} \to M(B) \qquad (3.9)$$

and interchange the signs of integral and limit in the relation

$$N_0(B) = \lim_{k\to\infty} N(b(\lambda_k)B) = \lim_{k\to\infty} \lambda_k \int_0^\infty P\{x_1 \in b(\lambda_k)\exp\{-Au\}B\}du$$

to obtain the equality

$$N_0(B) = \int_0^\infty M(\exp\{-Au\}B)du \,. \qquad (3.10)$$

Since the equality (3.10) implies

$$M(B) = \lim_{r\to 0} r^{-1}(N_0(B)-N_0(\exp\{-Ar\}B)) \,,$$

the measure M is determined uniquely and the sequence (3.8) weakly converges in R^n/S_ε, $\varepsilon > 0$,

$$\lambda P\{x_1 \in b(\lambda)B\} \underset{\lambda\to\infty}{\to} M(B) \,, \qquad (3.11)$$

$$B \subset R^n/S_\varepsilon \text{ is a continuity set of } M.$$

Using the condition (3.6) we can easily deduce that for almost all values of u

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (x,x)P\{\exp\{Au\}x_1 \in b(\lambda)dx\} = 0$$

This is equivalent to

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (x,x)P\{x_1 \epsilon\, b(\lambda)dx\} = 0 \qquad (3.12)$$

Now we can use the obvious continuous analog of theorem 2.3 in the work [4] which states that the conditions (3.11) and (3.12) are sufficient for x_1 to belong to the domain of attraction of a stable law. Since norming function $b(\lambda)$ did not change, this stable law has the same characteristic exponent α.

Sufficiency. Let the sequence $b^{-1}(n)(x_1+\ldots+x_n-a_n)$ be weakly convergent to a stable law in R^n with characteristic exponent α, $0 < \alpha < 2$. It follows from lemma 3.1, remark 3.1 and theorem 1 [2, p. 313] that in this case

$$P\{|x_1|>x\} = x^{-\alpha}L(x) ,$$

where $L(x)$ is a slowly varying function. Without loss of generality $b(n)$ can be chosen monotone and satisfying the relation

$$nL(b(n))b^{-\alpha}(n) \to 1 \qquad \text{as} \quad n \to \infty .$$

Therefore, as it follows from the properties of the regularly varying functions (see [2, ch. VIII])

$$\lambda \int_t^\infty P\{|\exp\{Au\}x_1|>\varepsilon b(\lambda)\}du \le \lambda \int_t^\infty P\{|x_1|>c\varepsilon b(\lambda) \exp\{au\}\}du =$$

$$\lambda a^{-1} \int_{c\varepsilon b(\lambda)\exp\{at\}}^\infty P\{|x_1|>z\}z^{-1}dz \le c_1\lambda a^{-1}P\{|x_1|>c\varepsilon b(\lambda) \exp\{at\}\} =$$

$$c_1\lambda a^{-1}(c\varepsilon b(\lambda) \exp\{at\})^{-\alpha}L(c\varepsilon b(\lambda) \exp\{at\}) \le c_2\varepsilon^{-\alpha}\exp\{-a\alpha t\}$$

for sufficiently large values of λ.

Thus, the condition

$$nP\{x_1 \in b(n)B\} \underset{n\to\infty}{\to} M(B) , \tag{3.13}$$

which must be satisfied (see [4, th. 2.2]) implies

$$N(b\ (\lambda)B) = \lambda \int_0^\infty P\{\exp\ \{Au\}x_1 \in b(\lambda)B\}du \to \int_0^\infty M(\exp\ \{-Au\}B)du \tag{3.14}$$

To complete the proof, we have to show that the condition

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ nb^{-2}(n) \int_{|x|<\varepsilon b(n)} (x,x)P\{x_1 \in dx\} = 0 \tag{3.15}$$

implies

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ \lambda b^{-2}(\lambda) \int_{|x|<\varepsilon b(\lambda)} \int_0^\infty (x,x)P\{\exp\ \{Au\}x_1 \in dx\} = 0 \tag{3.16}$$

In accordance with theorem 2.3 of the work [4], this conclusion and the relation 3.14) will be sufficient for our aim. Using again the properties of regularly varying functions, we have

$$\lambda b^{-2}(\lambda) \int_{|x|<\varepsilon b(\lambda)} \int_0^\infty (x,x)P\{\exp\ \{Au\}x_1 \in dx\}du =$$

$$2\lambda b^{-2}(\lambda) \int_0^{\varepsilon b(\lambda)} \int_0^\infty [P\{|\exp\ \{Au\}x_1| \geq y\} - P\{|\exp\ \{Au\}x_1| > \varepsilon b(\lambda)\}]ydudy \leq$$

$$2\lambda b^{-2}(\lambda) \int_0^{\varepsilon b(\lambda)} \int_0^\infty P\{|\exp\ \{Au\}x_1| \geq y\}ydudy \leq$$

$$2\lambda b^{-2}(\lambda) \int_0^\infty \int_0^{\varepsilon b(\lambda)} P\{|x_1| \geq cy\exp\ \{au\}\}ydydu =$$

$$2c^{-2}\lambda b^{-2}(\lambda) \int_0^\infty \int_0^{c\varepsilon b(\lambda)\exp\{au\}} P\{|x_1| > v\}v\ \exp\ \{-2au\}dvdu \leq$$

$$c_1\lambda b^{-2}(\lambda)\int_0^\infty (c\varepsilon b(\lambda)\exp\{au\})^{2-\alpha}L(c\varepsilon b(\lambda)\exp\{au\})\exp\{-2au\}du =$$

$$c_2\lambda\varepsilon^2\int_{\varepsilon c b(\lambda)}^\infty z^{-1-\alpha}L(z)dz \le c_3\lambda\varepsilon^2(\varepsilon c b(\lambda))^{-\alpha}L(\varepsilon c b(\lambda)) \le c_4\varepsilon^{2-\alpha}$$

for sufficiently large λ and some $c_4 > 0$, $a > 0$.

The last inequality enables us to prove (3.16). The theorem is proved.

Theorem 3.3. The distribution of the vector $b^{-1}(\lambda)(x(t,\lambda)-a(\lambda))$ weakly converges to the normal law in R^n with c.f. $\exp\{-2^{-1}Q_0(s)\}$ if and only if the distribution of the vector x_1 belongs to the domain of attraction of the normal law with c.f. $\exp\{-2^{-1}Q(s)\}$. The quadratic forms $Q_0(s)$ and $Q(s)$ determine each other uniquely by the equality

$$Q_0(s) = \int_0^\infty Q(\exp\{A^T u\}s)du$$

Proof. In accordance with theorems 1.2 and 2.3 of the work [4] and the formulae obtained above, we have to prove that the relations

$$N(\lambda;b(\lambda)B) \underset{\lambda\to\infty}{\to} 0 \quad , \quad B \subset R^n/S_\varepsilon \, , \tag{3.17}$$

$$\lim_{\varepsilon\to 0}\varlimsup_{\lambda\to\infty}\int_{0<|x|<\varepsilon}(s,x)^2 dN(xb(\lambda)) = \lim_{\varepsilon\to 0}\varliminf_{\lambda\to\infty}\int_{0<|x|<\varepsilon}(s,x)^2 dN(xb(\lambda)) = Q_0(s) \tag{3.18}$$

are equivalent to the relations $\lambda P\{|x_1| > \varepsilon b(\lambda)\} \underset{\lambda\to\infty}{\to} 0$ (3.19)

$$\lim_{\varepsilon\to 0}\varlimsup_{\lambda\to\infty}\lambda\int_{0<|x|<\varepsilon}(s,x)^2 P\{x_1 \in b(\lambda)dx\} = \lim_{\varepsilon\to 0}\varliminf_{\lambda\to\infty}\lambda\int_{0<|x|<\varepsilon}(s,x)^2 P\{x_1\in b(\lambda)dx\} =$$

$$= Q(s) \tag{3.20}$$

Necessity. The condition (3.19) follows from (3.17) in the same manner as the condition (3.11) followed from (3.5). The condition (3.18) implies the equicontinuity as $\lambda \to \infty$ and as $\varepsilon \to 0$ with respect to s of the expressions

$$\lambda \int_{0<|x|<\varepsilon} (s,x)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dx\}$$

for almost all $u \in R^1$.

Consequently, we obtain the equicontinuity with respect to s of the expressions

$$\lambda \int_{0<|x|<\varepsilon} (s,x)^2 P\{x_1 \in b(\lambda)dx\} \tag{3.21}$$

and can suppose that a subsequence of (3.21) converges to the quadratic form $Q(s)$. Since

$$\lambda \int_t^\infty \int_{0<|x|<\varepsilon} (s,x)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du =$$

$$\lambda \int_0^\infty \int_{0<|\exp\{At\}z|<\varepsilon} (\exp\{A^T t\}s,z)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dz\}du \to$$

$$Q_0(\exp\{A^T t\}s) < c\exp\{-at\}(s,s) ,$$

we obtain from (3.18) that

$$Q_0(s) = \int_0^\infty Q(\exp\{A^T u\}s)du \tag{3.22}$$

Since the equality (3.22) implies

$$Q(s) = \lim_{r\to 0} r^{-1}(Q_0(s) - Q_0(\exp\{A^T r\}s)$$

the sequence (3.21) converges to $Q(s)$ and the condition (3.20) is fulfilled.

Sufficiency. If the conditions (3.19) and (3.20) are satisfied, then in accordance with lemma 3.1 and remark 3.1, x_1 belongs to the domain of attraction of a normal law in R^1 and $b(\lambda) = \lambda^{1/2}L(\lambda)$, where L is a slowly varying function. Then the inverse function $b^{-1}(\lambda)$ has the representation $b^{-1}(\lambda) = \lambda^2 L_1(\lambda)$, where $L_1(\lambda)$ is a slowly varying function too. In this case, denoting $b(\lambda)$ by μ we obtain

$$N(\lambda;b(\lambda)(R^n/S_\varepsilon)) = \lambda\int_0^\infty P\{|\exp\{Au\}x_1| > b(\lambda)\varepsilon\} \le$$

$$\lambda\int_0^\infty P\{|x_1| > c\varepsilon b(\lambda)\exp\{au\}\}du = a^{-1}b^{-1}(\mu)\int_{c\varepsilon\mu}^\infty P\{|x_1| > v\}v^{-1}dv \le$$

$$a^{-1}b^{-1}(\mu)\delta\int_{c\varepsilon\mu}^\infty (vb^{-1}(v))^{-1}dv \le c_1 b^{-1}(\mu)\delta(\mu^2\varepsilon^2 L_1(\mu\varepsilon))^{-1} \to$$

$$c_1\delta\varepsilon^{-2} \quad \text{as} \quad \lambda \to \infty .$$

Since δ can be chosen small, $N(\lambda;b(\lambda)(R^n/S_\varepsilon)) \to 0$ as $\lambda \to \infty$ and the condition (3.17) is satisfied. We divide the rest of the proof into two parts.

1. $E(x_1,x_1) < \infty$. In this case, $E(x(t,\lambda),x(t,\lambda)) < \infty$ and consequently $x(t,\lambda)$ belongs to the domain of attraction of a normal law. Indeed, we have to check that

$$\int_0^\infty \int_{R^n} (x,x)P\{\exp\{Au\}x_1 \in dx\}du < \infty .$$

We have

$$\int_0^\infty \int_{R^n} (x,x)P\{\exp\{Au\}x_1 \in dx\}du = 2\int_0^\infty \int_0^\infty P\{|\exp\{Au\}x_1| > y\}ydydu \le$$

$$2\int_0^\infty \int_0^\infty P\{|x_1| > cy\exp\{au\}\}ydydu =$$

$$2\int_0^\infty \int_0^\infty P\{|x_1| > v\}vc^{-2}\exp\{-2au\}dvdu = c^{-2}(2a)^{-1}E(x_1,x_1) < \infty .$$

2. $E(x_1,x_1) = \infty$. In this case

$$x^2P\{|x_1| > x\} = o(\int_{|y|<x} y^2dP\{|x_1| < y\})$$

and

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (s,x)^2P\{x_1 \in b(\lambda)dx\} = \lim_{\varepsilon\to 0} \underline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (s,x)^2P\{x_1 \in b(\lambda)dx\} = Q(s)$$

Thus, noting $\mu_2(z) = \int_0^z x^2dP\{|x_1| < x\}$, we have

$$\lambda \int_t^\infty \int_{|x|<\varepsilon} (s,x)^2P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du \le$$

$$c\lambda \int_t^\infty \int_{|x|<\varepsilon} (x,x)P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du \le$$

$$2c\lambda b^{-2}(\lambda) \int_t^\infty \int_0^{\varepsilon b(\lambda)} P\{|x_1| \ge c_1y \exp\{au\}\}ydydu =$$

$$2c\lambda b^{-2}(\lambda) \int_t^\infty \int_0^{c_1\varepsilon b(\lambda)\exp\{au\}} P\{|x_1| \ge v\}c_1^{-2}v \exp\{-2au\}dvdu =$$

$$c_2\lambda b^{-2}(\lambda) \int_t^\infty \mu_2(c_1\varepsilon b(\lambda) \exp\{au\}) \exp\{-2au\}du =$$

$$\lambda c_3b^{-2}(\lambda) \int_{\varepsilon b(\lambda)\exp\{at\}}^\infty \mu_2(z)z^{-3}\varepsilon^2b^2(\lambda)dz \le$$

$$c_4\lambda\varepsilon^2\mu_2(\varepsilon b(\lambda) \exp\{at\})(\varepsilon b(\lambda) \exp\{at\})^{-2} \to c_5 \exp\{-2at\} \qquad (3.23)$$

(We have used theorems 1 and 1a from [2. p. 312-314] which give us the following properties of $\mu_2(z)$:

a) $\mu_2(z)$ is a slowly varying function and

b) $\lambda b^{-2}(\lambda)\mu_2(b(\lambda)) \to C_6$ as $\lambda \to \infty$ $0 < C_6 < \infty$.)

Now condition (3.18) follows from (3.20) and (3.23).

Remark 3.2. It follows from the proof of theorems 3.2 and 3.3 that the norming functions $b(\lambda)$ and $b_1(n)$ for which the distributions of $(x(t,\lambda)-a(\lambda))/b(\lambda)$ and $(x_1+\ldots+x_n-a_n)/b_1(n)$ weakly converge can be chosen equal: $b(n) = b_1(n)$.

4. Limiting Behavior of the Stationary Distribution as $\lambda \to 0$

In this part, we will suppose that the matrix A is similar to the diagonal matrix $\Lambda = \| \delta_{ij}\lambda_i \|$, where λ_i , $1 \le i \le n$, are the eigenvalues of A, i.e., $A = T\Lambda T^{-1}$, and T is a nonsingular matrix with real-valued elements. In this case, $\lambda_i < 0$, $i \le 1 \le n$. We will show that the module of $x(t,\lambda)$ tends to zero with an exponential speed.

Lemma 4.1. If $\lambda \to 0$, then $x(\lambda) \overset{P}{\to} 0$. The proof follows from formula (2.4).

Let us denote $\eta = (\eta_1,\ldots,\eta_n) = T^{-1}x_1$,

$\zeta = (\zeta_1,\ldots,\zeta_n) = T^{-1}x(t,\lambda)$,

$p_i = P\{\eta_i = 0\}$, $\text{sign } x = (\text{sign } x_1,\ldots \text{sign } x_n)$ if $x = (x_1,\ldots x_n)$,

$\nu_i = \lambda\lambda_i^{-1}$.

For simplicity we consider only the particular case $p_i = 0$, $1 \le i \le n$.

Theorem 4.1. If $p_i = 0$ for all i, $1 \le i \le n$, then the distribution of

$$(\text{sign } \zeta \; , \; |\zeta_1|^{-\nu_1},\ldots,|\zeta_n|^{-\nu_n})$$

weakly converges as $\lambda \to 0$ to the distribution of

$$(\text{sign } \eta \; ,\alpha,\ldots\alpha)$$

where α has the uniform distribution on the interval $(0,1)$ and does not depend on η.

Proof. The distribution of $x(t,\lambda)$ coincides with the distribution of the vector

$$\xi = \int_0^\infty \exp\{Au\} dz(u) \tag{4.1}$$

We can suppose that the process $z(t)$ is determined by the values of its jumps $x_1, x_2, \ldots$ and by the lengths of the intervals between jumps $\lambda^{-1}\tau_1, \lambda^{-1}\tau_2, \ldots$ where all x_i and τ_i are independent, $P\{\tau_i > x\} = \exp\{-x\}, \quad x > 0$.

Thus, the formula (4.1) implies

$$\xi = \exp\{\lambda^{-1}A\tau_1\}x_1 + \exp\{\lambda^{-1}A(\tau_1+\tau_2)\}x_2 + \ldots = \exp\{\lambda^{-1}A\tau_1\}(x_1+\xi^1) \ , \tag{4.2}$$

where τ_1 , x_1 , and ξ^1 are independent and the distributions of ξ and ξ^1 coincide. It follows from (4.2) that

$$T^{-1}\xi = \exp\{\lambda^{-1}\Lambda\tau_1\}T^{-1}(x_1+\xi^1)$$

or

$$\kappa = \exp\{\lambda^{-1}\Lambda\tau_1\}(\eta+\kappa^1) \quad \text{where} \quad \kappa = T^{-1}\xi \ ,$$

$$\kappa^1 = T^{-1}\xi^1 \quad \text{and the distributions of } \kappa \text{ and } \zeta \text{ coincide} \tag{4.3}$$

According to lemma 4.1 $\kappa^1 \xrightarrow[\lambda\to 0]{P} 0$ and we have from (4.3)

$$\operatorname{sign} \kappa = \operatorname{sign} (\eta+\kappa^1) \xrightarrow[\lambda\to 0]{P} \operatorname{sign} \eta \ ,$$

$$|\kappa_i|^{-\nu_1} = |\exp\{\lambda^{-1}\lambda_i\tau_i\}(\eta_i+\kappa_i^1)|^{-\nu_1} \xrightarrow[\lambda\to 0]{P} \exp\{-\tau_1\}$$

Since $\exp\{-\tau_1\}$ has uniform distribution on $(0,1)$ and does not depend on η, the statement of the theorem easily follows from the well-known properties of convergence in probability and weak convergence.

5. Applications to a Queueing System with Changeable Service Rate

In this part, we consider the case $n = 1$, $x_0 \geq 0$, $x_i \geq 0$. We write equation (1.1) in the form

$$dx(t) = -\mu x(t)dt + dz(t) \tag{5.1}$$

where $\mu > 0$.

Equation (5.1) is connected with a following queueing system. Input flow is a Poisson flow with parameter λ. To serve the n'th customer, the server has to spend x_n units of work. If $x(t)$ is the amount of work necessary to serve the customers present in the system at the moment t, then service rate equals to $\mu x(t)$. We will investigate the distribution of the virtual waiting time Θ subject to the condition that system has stationary distribution. (It is assumed that system has FIFO service discipline.)

Let $y(t)$ be the amount of work performed by the moment t. Then

$$y(0) = 0\,,$$

$$dy(t) = \mu x(t)dt \tag{5.2}$$

Thus

$$dy(t) = dz(t) - dx(t)$$

and

$$y(t) = z(t) - x(t) + x_0 = \int_0^t (1-e^{-\mu(t-u)})dz(u) + x_0(1-e^{-\mu t})$$

where x_0 has the distribution defined by (2.4).

If we denote by x the amount of work necessary to serve the considered customer then

$$P\{\Theta > t\} = P\{y(t) < x_0 / x(0+) = x_0 + x\} =$$

$$P\{\int_0^t (1-e^{-\mu(t-u)})dz(u) + (x_0+x)(1-e^{-\mu t}) < x_0\} =$$

$$P\{\int_0^t (1-e^{-\mu(t-u)})dz(u) + x(1-e^{-\mu t}) < e^{-\mu t}x_0\} \qquad (5.3)$$

Theorem 5.1. If $Ex = m_1 < \infty$, then $\mu\Theta \Rightarrow 1$ as $\lambda \to \infty$.

Proof. The proof follows from the relations

$$\lambda^{-1}\int_0^t (1-e^{-\mu(t-u)})dz(u) \Rightarrow \int_0^t (1-e^{-\mu(t-u)})m_1 du =$$

$$m_1 t - m_1\mu^{-1}(1-e^{-\mu t}) ,$$

$$\lambda^{-1}x_0 = \lambda^{-1}e^{-\mu t}\int_{-\infty}^0 e^{\mu u}dz(u) \Rightarrow e^{-\mu t}\int_{-\infty}^0 e^{\mu u}m_1 du = \mu^{-1}e^{-\mu t}m_1 ,$$

$$\lambda^{-1}x(1-e^{-\mu t}) \Rightarrow 0 \quad \text{as} \quad \lambda \to \infty .$$

Theorem 5.2. If the distribution of x belongs to the domain of attraction of a stable law with exponent α, $1 < \alpha \leq 2$, then there exists a regularly function $f(\lambda)$ with exponent $1-\alpha^{-1}$ such that the distribution of $(\Theta-1/\mu)f(\lambda)$ weakly converges as $\lambda \to \infty$ to the stable law with exponent α and c.f.

$$h(s) = \exp\{(e^{i\pi\alpha/2}(\alpha\mu)^{-1}(m_1 e)^{-\alpha} + \int_0^{\mu^{-1}} e^{-i\pi\alpha/2}(1-e^{-\mu u})^{\alpha}du)s^{\alpha}\} \qquad , s \geq 0 .$$

Proof is analogous and follows from the theorems 3.2, 3.3 and representation (5.3).

Theorem 5.3. *If the distribution of x belongs to the domain of attraction of a stable law with exponent* α, $0 < \alpha < 1$, *then*

$$\lim P\{\Theta > t\} = P\{(Z_1/Z_2)^{\alpha} > \alpha \int_0^t (e^{\mu t} - e^{\mu u}) du\} ,$$

where Z_1 *and* Z_2 *are positive independent identically distributed random variables and have stable distribution with exponent* α.

Proof is analogous and follows from the theorem 3.2 and representation (5.3).

Theorem 5.4. $\lim_{\lambda\to\infty} P\{\Theta^{\lambda/\mu} < t\} = t$, *if* $t \in (0,1)$.

Proof follows from the representation (5.3) and theorem 4.1.

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