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Relative Risk Aversion.

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Abstract

An individual's preference for risky alternatives is influenced by the strength of preference he feels for the consequences and his attitude toward risk taking. Conventional measures of risk attitude confound these two factors. In this paper we formally separate these factors and explore how this separation might significantly enhance our understanding of decision making under risk.

We introduce a new measure of risk attitude defined relative to strength of preference. This measure is based on comparing an individual's von Neumann-Morgenstern utility function to his strength of preference function. The properties of this measure of relative risk attitude are developed.

The concept of relative risk attitude has several important implications. First, it provides a better description of an individual's attitude toward risk. Second, it provides a better way to combine preferences of various experts in the context of multicriteria decision making. Finally, it provides a better insight into the implications of some commonly employed preference aggregation rules in group decision making.

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## Relative Risk Aversion

### 1. Introduction

Suppose you are offered the following choice: you receive three oranges for sure or receive a lottery in which you get eight oranges with a 0.5 chance and zero oranges with a 0.5 chance. Further suppose you prefer more oranges to less oranges in the range of zero to eight oranges. If you are indifferent between these two options, (three oranges for sure vs. the lottery) then you would be classified as a risk averse individual according to the Pratt-Arrow definition of risk aversion (Pratt [21], Arrow [1]).

Now, we introduce the notion of strength of preference into this example. Strength of preference refers to the intensity of an individual's preference for an alternative or a consequence. This is a concept having an intuitive meaning for most readers that will suffice for this example.

Suppose your strength of preference for acquiring three oranges when you have none is equal to your strength of preference for acquiring five more oranges when you have three. Then your indifference between receiving three oranges for sure and the lottery can be explained by the decreasing marginal value that you place on oranges. The introduction of "risk" in the form of a lottery has no impact on your preferences at all. Rather than calling you a risk averse individual, it seems more descriptive to call you a relatively risk neutral individual. This term is used to indicate that your preferences for risky alternatives, relative to your strength of preference for these certain consequences, are neutral to the introduction of risk.

In making this distinction, we are assuming that at least two factors influence an individual's preferences for risky alternatives; 1) strength of preference for the certain consequences, and 2) an attitude toward risk taking. Our objective is to formally separate these two factors, and to explore how this separation might significantly enhance our understanding of decision making under risk.

In Section 2, we review and formalize the concept of strength of preference. We also discuss some alternative methods for assessing a strength of preference function.

In Section 3, we show how to separate an individual's strength of preference from his risk attitude. This allows us to create a new measure of a risk attitude defined relative to strength of preference. Loosely speaking, Pratt's seminal work on a measure of a risk attitude was based on comparing an individual's von Neumann-Morgenstern utility function [27], to actuarial behavior (Pratt [21]). Our measure of a relative risk attitude is based on comparing an individual's von Neumann-Morgenstern utility function to his strength of preference function. We can develop the properties of this measure of a relative risk attitude in a straightforward manner.

In the fourth section we describe how this new measure of relative risk aversion can enhance our understanding of decision making under risk. Practical applications and theoretical insights are discussed. The conclusions are presented in Section 5.

## 2. Strength of Preference

### 2.1 Overview

Ordinal statements of preference are traditionally accepted as a primitive concept in normative theories of decision-making behavior. If

the objective of a theoretical development is to determine only a rank ordering of alternatives, then this primitive concept is a sufficient assumption.

Most readers will agree, however, that individuals do have strength of preference feelings that are revealed in different ways. The concept of strength of preference has been around for many years (see Stigler [25] for a history). Recent work by Bell and Raiffa [3, 4] and by ourselves (Dyer and Sarin [8, 9], Sarin [23]) suggests that there may be conceptual and operational advantages from formalizing the strength of preference notion. Some of these developments relate to multiattribute utility theory and to the collective choice problem. In subsequent sections of this paper, the need for quantifying strength of preference becomes more obvious.

We will use the term measurable value functions for a preference function that may be used to order the differences in the strength of preference between pairs of alternatives. There are several alternative axiom systems for measurable value functions, including the topological results of Debreu [7] and the algebraic development by Scott and Suppes [24]. Krantz, et al. [19] review these axioms.

Let  $X$  denote the set of all possible consequences, and define  $\lambda^*$  as a quaternary relation on  $X \times X$ . These axioms imply that there exists a real-valued function  $v$  on  $X$  such that, for all  $w, x, y, z \in X$ , the difference in the strength of preference between  $w$  and  $x$  exceeds the difference between  $y$  and  $z$ , written  $wx \lambda^* yz$ , if and only if

$$v(w) - v(x) \geq v(y) - v(z) \quad (1)$$

Further,  $v$  is unique up to a positive linear transformation, so it is a cardinal function. That is if  $v'$  also satisfies (1), then there are real numbers  $a > 0$  and  $b$  such that

$$v'(x) = av(x) + b$$

for all  $x \in X$ . This means that  $v$  provides an interval scale of measurement.

If we relate the binary preference relation  $\succsim$  on  $X$  to the quaternary relation  $\succsim^*$  on  $X \times X$  in the natural way by requiring  $wx \succsim^* yx$  if and only if  $w \succsim y$  for all  $w, x, y \in X$ , then from (1) it is clear that  $w \succsim y$  if and only if  $v(w) \geq v(y)$ . Thus,  $v$  plays the role of an ordinal utility function on  $X$ , but it is distinguished from its order preserving transformations on  $X$ , that are also ordinal utility functions, by virtue of its relation to  $\succsim^*$ . This special property is the reason for calling  $v$  a measurable value function.

While the axiom systems for the existence of a measurable value function are not controversial as descriptors of a rational preference structure, problems arise when attempts are made to assess the measurable value function. A brief review of some of these assessment procedures follows.

## 2.2 Assessment Methods

Three approaches for measuring strength of preference will be reviewed. The first approach assumes that the measurable value function  $v$  is equal to an affine transformation of a von Neumann and Morgenstern [27] utility function  $u$  defined on the same set of consequences, and uses lottery questions to elicit the latter. The second approach is based on the concept of "willingness to pay," while the third simply relies on introspection.

### 2.2.1 Lottery Method

Suppose we accept the following definition of strength of preference:

**Definition 1.** Given  $w, x, y, z \in X$  such that  $w \succ x \succ y \succ z$ , we define  $wy \succ^* xz$  if and only if  $\langle w, z; 0.5 \rangle \succ \langle x, y; 0.5 \rangle$  where  $\langle a, b; 0.5 \rangle$  denotes an even chance lottery between  $a, b \in X$ .

One immediate implication of this definition is that if  $x \sim \langle w, y; 0.5 \rangle$  then  $wx \sim^* xy$ . Loosely speaking, this means that the preferences of an individual for lotteries depend solely on the strength of preference that he feels for the outcomes and not on any extraneous "risk attitude." Harsanyi [14] argues that this should be the case. However, Ellsberg [10], Fishburn [11], and others reject this notion since no element of risk appears in the axiomatic development of strength of preference.

More recently Bell and Raiffa [4] have suggested that a restrictive relationship may exist between the two preference functions  $u$  and  $v$  but they also reject the assumption that they are identical (subject to an affine transformation). Finally, Sarin [23] has shown that Definition 1 is valid if and only if an individual agrees that his rankings of preference differences for outcomes in one uncertain state of the world do not depend on common levels of outcomes in the other states. This condition is not satisfied by the von Neumann-Morgenstern [27] or by the Savage [22] axioms.

### 2.2.2 Willingness to Pay

It may seem reasonable to define  $wx \succ_{\lambda}^* yz$  if an individual is willing to pay more money to exchange  $x$  for  $w$  than to exchange  $z$  for  $y$ . This approach introduces a second criterion into the decision. To avoid confusing difference notation with multiattribute notation we use  $x+\$a$  to indicate an alternative defined on  $X \times A$  where  $x \in X$  is the original outcome of interest and  $a \in A$  is the level of a second criterion introduced as a medium of exchange. This second criterion is commonly expressed in dollars, but in principle it could be some other attribute. To formalize this notion, we state the following:

**Definition 2.** Suppose  $w \sim x+\$a$  and  $y \sim z+\$b$ . Then  $wx \succ_{\lambda}^* yz$  if and only if  $\$a \geq \$b$ .

This method has two limitations. One, it artificially introduces a second attribute in measuring the strength of preference. Second, and more importantly, the two attributes must be difference independent of each other in order to use Definition 2 meaningfully. Difference independence means that the differences in the strength of preference between two levels of one attribute do not depend on the fixed level of the other attribute (see Dyer and Sarin [9]). In short, the measurable value function defined over the two attributes must be additive.

### 2.2.3 Direct Ordering of Intervals

If we reject Definitions 1 and 2 as providing operational assessment procedures for measurable value functions, we are left with introspection based on the assumption that strength of preference is a primitive concept. Several assessment procedures accept this viewpoint, including direct rating, the use of the direct ordered metric, and exchange questions (see Fishburn [12] for a review), but none of these approaches can be verified by actually observing choices by the decision maker. These limitations are discussed by Fishburn [13]. The challenge in applying these approaches is to improve introspection so that a subject can comfortably provide coherent responses with confidence that they do reflect the strength of his preferences.

## 3. Risk Attitude and Strength of Preference

A von Neumann-Morgenstern utility function  $u$  confounds an individual's risk attitude with the strength of preference he feels for the outcomes. If our purpose is simply to rank lotteries, there is no need to dissect

these two factors. However, a better insight into decision making under risk is provided if these two factors are separated. Further, it may be desirable or even necessary to identify a risk attitude and strength of preference separately in some contexts. Moreover, there is some emerging literature in psychology (e.g., see Coombs [6], Pollatsek and Tversky [20]) that attempts to measure "risk" as a basic attribute of a lottery. Our work may provide a basis for synthesizing this line of research with traditional developments in utility theory.

### 3.1 Relative Risk Attitude

We will introduce the concept of a relative risk attitude to analyze the impact of the introduction of risk on an individual's preferences for lotteries. We will assume that both the von Neumann-Morgenstern utility function  $u(x)$  and the measurable value function  $v(x)$  are monotonically increasing in  $x$ , and that they are continuously twice differentiable.

Pratt's measure of a risk attitude, defined as  $r(x) = -u''(x)/u'(x)$  is well known. We define a similar measure for the measurable value function  $v(x)$ . We call  $m(x) = -v''(x)/v'(x)$  the coefficient of value satiation. Loosely speaking,  $m(x)$  is a local measure of the strength of preference of an individual at a level of an asset  $x$ . It seems natural to say that  $m(x) = 0$  indicates constant marginal value at  $x$ ,  $m(x) > 0$  indicates decreasing marginal value at  $x$ , and  $m(x) < 0$  indicates increasing marginal value at  $x$ . Alternatively, we interpret these three conditions as an indication that the preference differences associated with additional units of the criterion at  $x$  are constant, decreasing, and increasing, respectively.

Some insights for the interpretation of  $m(x)$  can be provided by considering the value of a small increase  $h$  in the level of the asset  $x$ . Given our regularity conditions, there exists an increment  $\Delta$  such that

$$v(x+h) = v(x) + \Delta v'(x) \quad (2)$$

If the marginal value of additional units of  $x$  remains constant, then  $h = \Delta$ .

We define  $s(x,h) = h - \Delta$  as the satiation sacrifice for the increase. The interpretation of  $s(x,h)$  requires some caution. Like strength of preference itself, we feel that it must be accepted as a result of personal introspection. If we do invoke the notion of willingness to pay, then we could determine  $x+h \sim x+\$a$ , and  $x+\Delta \sim x+\$b$ , and interpret  $s(x,h)$  in terms of  $\$a - \$b$ .

We are interested in understanding the relationship between this satiation sacrifice  $s(x,h)$  and  $m(x)$ . Expanding  $v$  around  $x$ , we can obtain\*

$$v(x+h) = v(x) + hv'(x) + \frac{1}{2}h^2v''(x) + O(h^3) \quad (3)$$

and

$$v(x+\Delta) = v(x) + \Delta v'(x) + \frac{1}{2}\Delta^2v''(x) + O(\Delta^3) \quad (4)$$

Substituting from (2) in (4) and simplifying, we have

$$v(x+h) - v(x+\Delta) = -\frac{1}{2}\Delta^2v''(x) - O(\Delta^3) \quad (5)$$

Subtracting (4) from (3) gives

$$v(x+h) - v(x+\Delta) = (h-\Delta)v'(x) + \frac{1}{2}(h^2-\Delta^2)v''(x) + O(h^3) - O(\Delta^3) \quad (6)$$

Thus, from (5) and (6)

$$s(x,h) \cong \frac{1}{2}h^2m(x)$$

so that the decision maker's satiation sacrifice for a small increment  $h$  is approximately  $m(x)$  times half the square of  $h$ .

Alternatively,  $m(x)$  may be interpreted by relating it to  $\lambda(x,h)$  defined as the ratio of the value increase in going from  $x-h$  to  $x$  to the value in-

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\*In an expansion,  $O(\ )$  means "terms of order at most." For these expansions, we assume that  $v$  has a third derivative that is continuous and bounded over the interval  $(x, x + \max(x+h, x+\Delta))$ .

crease in going from  $x-h$  to  $x+h$ . Notice that if the measurable value function is linear, then  $\lambda(x,h) = 0.5$ . It follows from this definition that

$$v(x) = \lambda(x,h) v(x+h) + (1-\lambda(x,h)) v(x-h). \quad (7)$$

Expanding  $v(x+h)$  and  $v(x-h)$  and substituting in (7), for small values of  $h$  we obtain

$$\lambda(x,h) \cong 0.5 + \frac{1}{2}h m(x)$$

Recall that Pratt [21] showed that the risk premium for a small actuarially neutral risk is approximately  $r(x)$  times half the variance of the risk, and the probability premium is approximately  $r(x)$  times  $\frac{1}{2}h$ , emphasizing the close parallel between these concepts.

Suppose  $v_1$  and  $v_2$  are two measurable value functions with local value satiation coefficients  $m_1$  and  $m_2$ , respectively. If at a point  $x$ ,  $m_1(x) > m_2(x)$  then  $v_1$  exhibits a faster decrease in marginal value than  $v_2$ ; that is, the corresponding satiation sacrifices satisfy  $s_1(x,h) > s_2(x,h)$  for sufficiently small  $h$ . This statement paraphrases Pratt's conclusions regarding the local risk aversion of two individuals. We state one further analogous result without proof:

Theorem 1: Let  $m_i(x)$  and  $s_i(x,h)$  be the local coefficient to the measurable value function  $v_i$ ,  $i = 1, 2$ . Then the following conditions are equivalent, in either the strong form (indicated in brackets), or the weak form (with the bracketed material omitted).

- (a)  $m_1(x) \geq m_2(x)$  for all  $x$  [and  $>$  for at least one  $x$  in every interval].
- (b)  $s_1(x,h) \geq [>] s_2(x,h)$  for all  $x$  and  $h$ .
- (c)  $\lambda_1(x,h) \geq [>] \lambda_2(x,h)$  for all  $x$  and  $h > 0$ .
- (d)  $v_1(v_2^{-1}(t))$  is a [strictly] concave function of  $t$ .
- (e)  $\frac{v_1(y) - v_1(x)}{v_1(w) - v_1(z)} \leq [<] \frac{v_2(y) - v_2(x)}{v_2(w) - v_2(z)}$  for all  $z, w, y, x$  with  $z < w \leq x < y$ .

The same equivalences hold if attention is restricted to an interval that contains  $x$ ,  $x+h$ ,  $x-h$ ,  $v_2^{-1}(t)$ ,  $z$ ,  $w$ , and  $y$ . Theorem 1 shows that global properties corresponding to local strength of preference attitudes also hold in the natural sense.

It should be obvious by now that we could continue to develop a theory of marginal strength of preference in a manner analogous to Pratt's development for the von Neumann-Morgenstern utility function. Instead, our interest here is in linking strength of preference with risky behavior in a logical manner. Therefore, we focus on the relationship between  $u$  and  $v$  for a single decision maker. To avoid extraneous notation, we assume that  $u$  and  $v$  are restricted to an interval  $[x^0, x^*] = X' \subset X$  and that preferences are strictly increasing over this interval. Further, we scale both  $u$  and  $v$  so that  $u(x^0) = v(x^0)$  and  $u(x^*) = v(x^*)$ . This choice of scaling is not necessary for our development.

We begin our synthesis by observing that if  $v(x) = u(x)$  for all  $x \in X'$  then the introduction of risk has no apparent effect on the individual's preferences, so we say that he is relatively risk neutral. When  $v(x) = u(x)$  for all  $x \in X'$ , then  $m(x) = r(x)$  over this same interval. When  $v(x) \neq u(x)$ , we can use the relationship between  $m$  and  $r$  to define a local measure of relative risk aversion.

Definition 3. At  $x \in X'$ , an individual is relatively risk averse if  $m(x) < r(x)$ , relatively risk prone if  $m(x) > r(x)$ , and relatively risk neutral if  $m(x) = r(x)$ .

Notice that if  $r(x) > m(x) > 0$  for all  $x \in X'$ , then both  $u$  and  $v$  are concave, but  $u$  is relatively "more concave" than  $v$ . Some additional insight into this concept is obtained by defining a utility function  $u_v: V \rightarrow \mathbb{R}^1$  where the attribute  $V$  takes on the "value" of an attribute outcome  $x$ ; that

is,  $v: X \rightarrow V$ . As noted by Keeney and Raiffa, [17, pp. 220-221], we can assess  $u_v[v(x)]$  in principle, but in practice we would expect the individual to directly consider the outcome  $x$  associated with  $v(x)$  rather than  $v(x)$  itself. For consistency, therefore, we define  $u_v[v(x)] = u(x)$  for all  $x \in X$ . Finally, we let  $r_v(v) = -u_v'[v(x)]/u_v''[v(x)]$  where the differentiation is taken with respect to  $v$ .

**Theorem 2.** At  $x \in X$ , an individual is relatively risk averse if and only if  $r_v(v) > 0$ , relatively risk prone if and only if  $r_v(v) < 0$ , and relatively risk neutral if and only if  $r_v(v) = 0$ .

**Proof:** Since  $u(x) = u_v[v(x)]$ ,

$$u'(x) = u_v'[v(x)]v'(x), \text{ and}$$

$$u''(x) = u_v''[v(x)]v''(x) + u_v'[v(x)](v'(x))^2$$

Dividing the third equation by the second and rearranging terms, we obtain

$$r(x) - m(x) = v'(x)r_v(v) \quad (8)$$

Since  $v'(x) > 0$ , the theorem is proved.

The conceptual value of this result should be obvious. If a value function  $v(x)$  exists and risk is introduced, then a utility function defined on  $v(x)$  should be concave if an individual is relatively risk averse for all  $x \in X$ .

To explore one further implication of a relative risk attitude, we introduce an equal difference point for each interval.

**Definition 4.** For any interval  $(y, z) \in X$ , the point  $x_e$  such that  $yx_e \sim^* x_ez$  is the equal difference point for  $(y, z)$ . It should be clear that  $v(x_e) = 0.5(v(y) + v(z))$ .

**Corollary 1.** Suppose  $y, z \in X$  such that not  $y \sim z$ ,  $x_c$  is the certainty equivalent for an even chance lottery between  $y$  and  $z$ , and  $x_e$  is the equal difference point for  $(y, z)$ . Then

- (a) If  $x_e \succ x_c$  the individual is relatively risk averse.
- (b) If  $x_c \succ x_e$  the individual is relatively risk prone.
- (c) If  $x_c \sim x_e$  the individual is relatively risk neutral.

Proof: If  $x_e \succ x_c$ , then

$$\begin{aligned} u_v(v(x_e)) &= u_v((0.5)v(y) + (0.5)v(z)) \\ &> (0.5)u_v(v(y)) + (0.5)u_v(v(z)) \end{aligned}$$

which implies that  $u_v$  is a concave function of  $v$ , and hence  $r_v > 0$ .

Similar arguments apply for the other cases.

We can interpret  $\pi_v = x_e - x_c$  as a "relative risk premium". It follows immediately from the corollary that if  $\pi_v > 0$  the individual is relatively risk averse, if  $\pi_v < 0$ , the individual is relatively risk prone, and if  $\pi_v = 0$ , the individual is relatively risk neutral.

If  $\pi$  is the risk premium corresponding to  $u$ , it is possible to have  $\pi > 0$  while  $\pi_v < 0$ . Again an individual could be risk averse in the classic sense of Pratt's definition, but relatively risk prone.

### 3.2 Examples

The preceding comment emphasizes the intimate interaction between strength of preference and a relative risk attitude. This relationship can be illuminated by considering several special cases.

One case of particular interest is constant relative risk aversion. Bell and Raiffa [4] have argued that a rational individual should exhibit constant relative risk aversion in risky situations. The implication of this argument is that either

$$\begin{aligned} u(x) &\sim -e^{-kv(x)} && \text{iff } r_v(v) = k > 0, \\ u(x) &\sim v(x) && \text{iff } r_v(v) = 0, \text{ or} \\ u(x) &\sim e^{kv(x)} && \text{iff } r_v(v) = k < 0 \end{aligned}$$

where the notation  $\sim$  between utility functions means that one must be a linear transformation of the other. We now investigate the implications of this restriction on the general relationship between  $u$  and  $v$ , and some other possible relationships as well.

Case 1 ( $r(x) = 0$ ). If  $r(x) = 0$ , from (8) we obtain

$$r_v(v) = -m(x)/v'(x)$$

If  $m(x) = 0$ , then  $r_v(v) = 0$ . That is, if the decision maker is willing to make risky decisions based on the actuarial value of the alternatives and he feels that the additional units have constant marginal value, then he is relatively risk neutral as we would expect.

If  $m(x) > 0$ , since  $v'(x) > 0$  is decreasing, then  $r_v(v) < 0$  must be increasing relative to  $m(x)$ . That is, if the decision maker makes risky decisions based on actuarial values but feels that the marginal values of additional units are decreasing, then he must be relatively risk prone in such a way that his relative risk attitude compensates for the decreasing marginal value. This would be the case, for example if  $v(x) = \sqrt{x}$ , and  $u_v(v(x)) = v^2$ .

Case 2 ( $r(x) = c$ ). Again using (8), we find

$$r_v(v) = (c - m(x))/v'(x)$$

If  $c > 0$  and  $c > m(x)$ , then  $r_v(v) > 0$  as we expect. Now suppose that the decision maker has a constant marginal value strength of preference, so that  $m(x) = \ell \neq c$ , which implies that  $v(x) \sim -e^{-\ell x}$  if  $\ell > 0$ .

Then

$$r_v(v) = (c - \ell)/\ell e^{-\ell x}$$

so  $r_v(v)$  cannot be constant also. That is, Bell and Raiffa's argument that  $r_v(v)$  should be constant is inconsistent with the possibility that  $r(x)$  and  $m(x)$  are constant and not equal to one another.

Case 3 ( $r(x) = c$ ,  $r_v(v) = k$ ;  $c, k > 0$ ). An important extension of Case 2 is the implication of a constant Pratt-Arrow risk attitude and a constant relative risk attitude. From (8)

$$m(x) + kv'(x) - c = 0$$

Solving this second order differential equation for  $v(x)$ , we obtain

$$v(x) = \frac{1}{k}(cx - \log(\frac{k}{c} + de^{cx}))$$

where  $d$  is an arbitrary constant of integration, and

$$m(x) = \left(\frac{k}{dc^2} e^{-cx} + \frac{1}{c}\right)^{-1}$$

which is increasing if  $d > 0$ . Notice that if  $d = 0$ ,  $m(x) = 0$  so it is possible to have a constant Pratt-Arrow risk attitude and a constant relative risk attitude with  $c = k$  and a linear value function  $v(x) \sim x$ . Otherwise,  $m(x)$  cannot be constant at a nonzero value.

Case 4 ( $r(x)$  strictly decreasing). Again using (8), it is easy to see that if  $m(x)$  and  $r_v(v)$  are strictly decreasing, then so is  $r(x)$ . If  $r_v = k > 0$ , then

$$r(x) = kv'(x) + m(x)$$

is decreasing if  $m(x)$  is decreasing. A more interesting case is  $r_v = k > 0$  and  $v(x) \sim -e^{-\ell x}$ ,  $\ell > 0$ . Notice that

$$r(x) = k\ell e^{-\ell x} + \ell$$

is decreasing even when constant marginal value and a constant relative risk attitude exist simultaneously. This combination may help to explain the appeal of decreasing Pratt-Arrow risk aversion as an appropriate description of a risk attitude.

Case 5 ( $r(x) = c/x$ ,  $r_v = k$ ;  $c > 1$ ,  $k > 0$ ). Consider the case of proportional risk aversion in the Pratt-Arrow sense and constant relative risk aversion. Making the appropriate substitutions and solving (8) as a second order differential equation, we find

$$v(x) = \frac{1}{k} \log \frac{x^{c-1}}{\frac{k}{c-1} + d x^{c-1}}$$

where  $d$  again is an arbitrary constant of integration, and

$$m(x) = \frac{\left(\frac{k}{c-1}x + cdx^c\right)}{x\left(\frac{k}{c-1}x + dx^c\right)}$$

which is decreasing in  $x$  if  $d > 0$ . Notice that  $m(x) = x^{-1}$  if  $d = 0$ .

It should be clear from these examples that (8) can be used to explore the relationships among various attitudes towards risk and marginal value.

A summary of these results is provided in Table 1.

#### 4. Implications

There are several important implications of the concept of relative risk aversion. We organize our discussion of these implications under the headings of a theory of decision making, multicriteria decision making, and group decision making.

##### 4.1 Theory of Decision Making

Intuitively we think of a risk averse individual as one who prefers to avoid taking chances, or who behaves conservatively in the face of risk. The Pratt-Arrow definition of risk aversion confounds an individual's attitude toward risk and his strength of preference for outcomes. The concept of relative risk aversion is a better description of an individual's attitude toward risk. This concept could be used to empirically examine several conjectures regarding decision making under conditions of risk.

Table 1. Summary of Example Cases

	$\underline{r(x)}$	$\underline{r_v(v)}$	$\underline{m(x)}$
Case 1	0 0	0 <0	0 >0
Case 2	$c > 0$ and $c > m(x)$ $c > 0$	0 $(c-l)/le^{-lx}$	- $l \neq c$
Case 3	$c > 0$	$k > 0$	$(\frac{k}{dc^2} e^{-cx} + \frac{1}{c})^{-1}$
Case 4	strictly decreasing strictly decreasing strictly decreasing	strictly decreasing $k > 0$ $k > 0$	strictly decreasing decreasing $l > 0$
Case 5	$c/x, c > 1$	$k > 0$	$(\frac{k}{c-1}x + cdx^c)$ $x(\frac{k}{c-1}x + dx^c)$

Several empirical studies (e.g., see Swalm [26]) have assessed utility functions of real-world managers that are often described as s-shaped, so that the Pratt-Arrow measure of risk aversion indicates a change from risk averse to risk seeking behavior over the range of outcomes. This switch in risk attitude commonly occurs at the status quo level of wealth, and decision makers are said to be risk seeking for losses and risk averse for gains (Kahneman and Tversky, [16]). A hypothesis we consider worthy of empirical testing is the following: The measurable value function of individuals tends to be s-shaped around the status quo point, but the relative risk attitude is constant. If this hypothesis is true, it may significantly alter our understanding of how risk affects decision making.

It seems intuitively appealing to expect that an individual's risk attitude will differ in different decision-making roles. For example, the decision maker may adopt one attitude when making decisions as a manager of a corporation, and a very different attitude when making personal investment decisions (another hypothesis that could be tested). In the same role, however, we conjecture the following: An individual's relative risk attitude is independent of the attribute on which his preferences are assessed. In applications of multiattribute utility theory, utility functions have been assessed that indicate a decision maker is risk prone on one attribute and risk averse on another using the Pratt-Arrow measure. When we adjust these responses for the differences in the measurable value functions, we may find a consistent relative risk attitude.

#### 4.2 Multicriteria Decision Making

The previous hypothesis leads naturally into the implications of the concept of relative risk aversion for multicriteria decision making. The

assessment of a multicriteria utility function is often practical only if it can be decomposed into easily assessed single-criterion conditional utility functions (Keeney and Raiffa [17]). In real-world applications, these conditional utility functions are sometimes assessed independently from different experts, and the trade-offs among the criteria are resolved by the group of experts or by another individual representing the decision-making organization. The resulting multiattribute utility function may combine distinctly different relative risk attitudes of the experts.

We feel that it is more appropriate to obtain only measurable value functions from the experts, and then transform these value functions into utility functions by using a relative risk attitude obtained from the consensus of the group of experts or from the individual representing the decision maker.

Even in the case of a single decision maker, the concept of relative risk aversion may be a useful assessment aid. In some problem situations, a decision maker may find it easy to respond to lottery questions on some attributes but not on others. For example, a decision maker considering alternate automobiles may be able to answer lottery questions on the attribute "cost" but not on the attribute "appearance." One approach would be to assess both  $u(x)$  and  $v(x)$  on the cost attribute  $x$ , and derive the implied relative risk attitude function  $r_v$ . The utility function  $u(y)$  on the appearance attribute  $y$  can then be constructed from  $v(y)$  and  $r_v$ . Of course, care must be taken to ensure a proper normalization of the functions by a conjoint scaling of the two attributes. For a further discussion see Dyer and Sarin [9].

### 4.3 Group Decision Making

In group decision making preferences of individuals are aggregated by means of a preference aggregation rule (PAR), and this PAR is used for evaluating alternatives. Following Dyer and Sarin [8] we denote a group PAR under risky situations by  $W_U$  and under riskless situations by  $W_V$ . Two important forms for  $W_U$  and  $W_V$  are additive and multiplicative. That is, for the risky PAR

$$W_U(x) = \sum_{i=1}^n k_i u_i(x), \text{ or } 1 + kW_U(x) = \prod_{i=1}^n (1 + k k_i u_i(x))$$

and for the riskless PAR

$$W_V(x) = \sum_{i=1}^n \lambda_i v_i(x), \text{ or } 1 + \lambda W_V(x) = \prod_{i=1}^n (1 + \lambda \lambda_i v_i(x))$$

where,  $\lambda_i$ ,  $k_i$ ,  $\lambda$ , and  $k$  are scaling constants, and  $u_i$  and  $v_i$  are respectively each individual's utility and measurable value function.

The concept of relative risk aversion can be used to provide an important insight into the implications of these PAR's.

**Theorem 3:** If  $W_U$  and  $W_V$  are either additive or multiplicative, then each individual  $i$  in the group is either relatively risk averse if  $\lambda > k$ , or relatively risk prone if  $\lambda < k$ , or relatively risk neutral if  $\lambda = k$ .

**Proof:** The relationship between  $u_i$  and  $v_i$  can be derived as shown in Dyer and Sarin [8, Theorem 4]. We consider the case  $k \neq 0$ ,  $\lambda \neq 0$  since the proof is similar for the other cases. If  $k \neq 0$ ,  $\lambda \neq 0$ , then

$$1 + k k_i u_i(x) = (1 + \lambda \lambda_i v_i(x))^c,$$

where

$$c = \log(1+k)/\log(1+\lambda).$$

From this relationship we obtain

$$r_{u_i}(v_i(x)) = - \frac{u_i'(v_i(x))}{u_i(v_i(x))} = \frac{(1-c)\lambda\lambda_i}{1+\lambda\lambda_i v_i}$$

If  $\lambda > k$ , then  $c < 1$  if  $\lambda > 0$ , and  $c > 1$  if  $\lambda < 0$ ; similarly, if  $\lambda < k$ , then  $c > 1$  if  $\lambda > 0$ , and  $c < 1$  if  $\lambda < 0$ . This gives the desired result.

The following corollary provides a similar result for the relative risk attitude of the group.

Corollary 2: If  $W_u$  and  $W_v$  are either additive or multiplicative, then the group is either relatively risk averse if  $\lambda > k$ , or relatively risk prone if  $\lambda < k$ , or relatively risk neutral if  $\lambda = k$ .

One implication of Theorem 3 is that for an additive PAR to be appropriate, each individual in the group must be relatively risk neutral. This seems to be an extremely strong requirement. However, Harsanyi [15] has argued that each  $u_i$  should not be influenced by an individual's attitude toward risk.

If the group PAR's are multiplicative, then each individual does not have to be indifferent to risk in the sense of the case of the additive PAR. It does imply, however, that each individual must exhibit a homogeneous attitude toward the introduction of risk into the group decision. This follows directly from Theorem 3 since common values of  $\lambda$  and  $k$  determine whether each individual is relatively risk averse or relatively risk prone.

It should be noted that the above conditions are necessary for the additive or multiplicative group PAR but not sufficient. It would be interesting to explore both necessary and sufficient conditions in terms of relative risk attitude that imply a particular form for the group PAR.

Corollary 2 provides some implications for two interesting aspects of the group behavior: "equity consciousness" and "risk attitude." When  $\lambda = 0$ , the group has no egalitarian preferences, and, for example, would equally prefer an outcome  $(v_1 = 0, v_2 = 1)$  to another outcome  $(v_1 = 0.5, v_2 = 0.5)$

if  $\lambda_1 = \lambda_2$ . In this situation the value of  $k$  merely reflects the group's relative attitude toward risk. From Corollary 2, it easily follows that if  $k = 0$  the group is relatively risk neutral, if  $k > 0$ , the group is relatively risk prone, and if  $k < 0$  the group is relatively risk averse. This contrasts with the arguments of Harsanyi [15] and Keeney and Kirkwood [18] who attribute a two-person group's preference for an even chance lottery that results in utility functions values  $(1, 1)$  or  $(0, 0)$  over the even chance lottery that results in either  $(1, 0)$  or  $(0, 1)$  entirely to its posterior equity consciousness. In our framework such a preference may be due exclusively to the relative risk proneness of the group unless  $\lambda$  is also greater than 0.

## 5. Summary and Conclusions

In this paper we have shown how the concept of a relative risk attitude may be used to separate the effects of risk from strength of preference in a choice situation. Some of the implications and topics for further research that follow from the concept of relative risk aversion were identified in the previous section.

There seems to be a renewed interest in the relationship between choices under conditions of risk and certainty. Evidence is provided by the recent work of Bell and Raiffa [3, 4], von Winterfeldt, Barron, and Fischer [28], Barrager [2], and Chew and MacCrimmon [5]. We are hopeful that the concept of a relative risk attitude will contribute to the understanding of this topic.

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