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D.C. Hyland

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FOR THE COMMANDER

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ABSTRACT

Difficulties arising from inherent inaccuracies in structural modelling and from the high dimensionality of the dynamical system force a re-examination of the problem of optimal control of large flexible structures within the context of stochastic system theory. In this report, the design of active structural control is formulated as the mean-square optimal control of a linear mechanical system with stochastic parameters. In practice, a complete probabilistic description of modal parameters can never be provided, and a suitable design approach must accept very limited a priori data on parameter statistics. In consequence, we formulate the mean square optimization problem using a complete probability assignment induced by the available data through use of a maximum entropy principle. Furthermore, it is proposed to acknowledge as available the minimum set of a priori statistical data on parameter variations which is needed to preserve any measure of modelling fidelity.

To fix ideas, we specifically address the problem of full state feedback regulation of a linear structural system with statistical variation only in the open loop frequencies. Examining the phenomenology of modal frequency uncertainties we discern the so-called "modal decorrelation times" as the minimum data required to preserve 2nd moment response characteristics at high levels of uncertainty or for high order structural modes.

The decorrelation times are closely related to the damping times of the parameter ensemble averaged modal response and their reciprocals constitute fundamental, albeit unconventional, measures of the variation of the modal frequencies about their nominal or mean values. Choosing to acknowledge the decorrelation times as the available data, the complete probability assignment which is otherwise maximally unconstrained is a white parameter

model in which the noise intensities are inversely proportional to the decorrelation times. The mean-square optimization problem therefore reduces to the solution of a stochastic Riccati equation of a form arising from the state dependent noise problem.

Certain features of the stochastic Riccati equation are next explored. It is shown that under weak restrictions a unique positive semi-definite solution exists for all values of the decorrelation times. Also, 2nd moment stability is guaranteed for the closed-loop system. Thus, the need for design iteration to ensure robustness with respect to stability is largely eliminated within the proposed approach.

Furthermore, in the limit as all decorrelation times approach zero (i.e., uncertainties in all modal frequencies increase without bound), the solution of the stochastic Riccati equation yields a rate feedback control law which is stable for all values of modal frequencies or damping ratios. In the more typical case in which low order structural modes are relatively well known while modelling accuracy degrades for higher order modes, the stochastic Riccati equation produces a control which approaches the asymptotic rate feedback form for high order modes and yet closely resembles "high authority" deterministic plant design for the low order, relatively well-known modes. These two regimes exist as limiting qualitative features of a global control law for which stochastic stability is guaranteed.

The above qualitative features permit a numerical scheme for determination of the optimal gain matrix in which the computational burden is mainly associated with the relatively well-known or "coherent" modes. As long as the "coherent system" is of modest dimension, the stochastic Riccati equation admits of practical numerical treatment for systems of arbitrarily large order. Thus, the proposed design approach eliminates the need for modal truncation with its attendant spillover instability problems.

CONTENTS

OPTIMAL REGULATION OF STRUCTURAL SYSTEMS WITH UNCERTAIN PARAMETERS

1. INTRODUCTION

The advent of the Space Shuttle has prompted considerable attention[1] to the design and control of large, lightweight space platforms. Feasibility studies have identified very stringent tolerances on figure and precision pointing requirements.[2,3,4]

In the face of severe mission requirements, vibration suppression by purely passive structural damping may be inadequate and active electronic control of the structure must be contemplated. Due to its significant influence on mission performance, vehicle elasticity must be carefully accounted for, so that many degrees of freedom are required in the system model. The high dimensionality of the dynamic system renders classical control design techniques excessively cumbersome. Thus, the more systematic approach of "modern" optimal control theory is preferable.

Application of modern control theory to the design of active structural control faces two fundamental problems. In the first place, control effectiveness provided by the optimum controller and estimator gains is sensitive to errors in modelling plant dynamics, i.e., slight variations in the model parameters and errors in environment specification. This is the problem of "robustness." Secondly, there is the difficulty posed by the large dimension of the dynamical model that must be employed. In particular, stability of the closed-loop system cannot be guaranteed if dynamically significant modes are truncated from the plant model. This is the well known problem of control and observation "spillover".[5,6]

If the computational difficulties were not prohibitive, one might attempt to circumvent the above problems by use of a very accurate high-order plant model in a standard Linear-Quadratic formulation. The analysis of high-order structural modes, however, is plagued by two essential difficulties.[7,8] First, truncation of the infinite dimensional system to a finite-dimensional system is always inherent to the finite-element representation. The corresponding errors incurred tend to accumulate with increasing frequency and the calculated mode shapes and frequencies are known to be inaccurate for higher-order modes, even for idealized systems. Secondly, and in addition to mathematical modelling difficulties, the higher modal frequencies and mode shapes are very sensitive to small details of geometry, construction and material properties. Thus, in considering structural response at sufficiently high frequency, the relevant structural details may never be determined and modelled with acceptable accuracy.

Note that if the system order did not pose a problem one could deal with parameter errors within a traditional design philosophy. First design the controller by some method, then characterize robustness with respect to various system properties by determining acceptable bounds on parameter variations. Recent results[9] on the multivariable robustness issue greatly facilitate this process. Unfortunately, since the number of system parameters whose variation must be considered increases rapidly with the system order this approach would seem to entail great complexity.

The difficulties posed by dimensionality and parameter sensitivity stem, in part, from a reliance upon design methods which implicitly assume complete information on system parameters. It is felt that approaches which include measures of parameter uncertainty as part of the control formulation offer significant theoretical advantage.

A number of such controller design methods have been advanced in the last several years, and comparative assessments are to be found in References (10) and (11). A distinct philosophy with regard to the modelling of parameter uncertainties is exemplified by the guaranteed cost control method[12,13] which attempts to bound the effects of uncertainties. Here the system parameters are assumed to lie in a closed bounded region and a modified Riccati equation is devised such that closed-loop system behavior is acceptable for all values of the parameters within specified limits. Recently this has been noted to produce large controller gains and relatively large control effort with overdamped dominant closed-loop poles. Although a special iterative procedure has been devised to remedy this disadvantage,[13] the applicability of this approach to systems having a large number of uncertain parameters has not been demonstrated.

In any case, methods which assume parameter variations within a limited range are basically non-statistical in character. A more thorough-going approach would employ the general concepts of stochastic optimal control,[14-18] allowing parameter probability distributions of unbounded support. We refer specifically to the concepts of dual control and related ideas as clarified by Bar-Shalom and Tse.[16-18] Within this general context, parameter uncertainties are not necessarily accepted at their *a priori* values; instead, a control is considered truely optimal when it has a dual effect - i.e., when it not only affects the state of the system but also reduces the uncertainty of the state.

Unfortunately, the formulation of adaptive control is a nonlinear stochastic control problem[18] and approximations are usually needed to achieve a practical solution. One attractive approximation replaces the actual system parameters by parameters which are uncorrelated in time, thereby making identification impossible and precluding probing action. Not only is the problem rendered solvable but this approach is design conservative in the sense that it bounds the deterioration in performance due to unknown parameters[19] and allows us to obtain an inherent caution in the control.[20]

Within the non-dual approximation and in addition to the well-known state dependent noise formulation for continuous systems,[21,22] much recent work has been directed to the discrete-time case. In particular, the uncertainty threshold principle of Athans and Ku[19,23] and recent results for the general multivariable problem[24] provide valuable qualitative insight in the present problem. Clearly, the possibility of an uncertainty threshold is a critically important issue for active vibration suppression. Moreover, experience with the multivariable case[24] shows that the presence of a large number of uncertain parameters can be the source of prohibitive complexity.

The view espoused here is that a suitable design method must not only be consistent with a non-dual approximation but must incorporate very limited information on parameter statistics thereby reducing the number of measures of parameter uncertainty to a manageable level.

Actually, in the specific area of response estimation, the issue raised by dimensionality and parameter uncertainty and the need for a tractable description of parameter statistics have been addressed, in part, by Statistical Energy Analysis. Motivated by a concern with high frequency vibration and acoustical-structural interaction, the development of SEA has been largely concurrent with that of modern optimal control theory and has given rise to an extensive literature (see References 25 and 26 for a general review of SEA procedures). Broadly speaking, SEA attempts to formulate a "contracted" description of system response consistent with the inherent uncertainties of structural-acoustical modelling.

In its simplest form, SEA divides a complex structure into "subsystems" which are considered as repositories of vibrational energy. Each subsystem consists of a group of "similar" energy storage modes. Modes which play a significant role in transmission, dissipation and storage of energy and which have nearly equal excitation, damping and coupling to other subsystems are assigned to a particular group. Generally, subsystems are associated with a particular frequency band and may be associated with separate structural components. SEA achieves substantial simplification by using the approximate result that power flow between pairs of coupled modes is proportional to the difference in the average modal energies,[27,28] the constant of proportionality being termed the "coupling loss factor." At this juncture, the effect of randomness in the modal frequencies is accounted for by averaging the coupling loss factors over appropriately defined statistics. In taking this most important step, one typically renounces all knowledge of the relative modal frequency locations within each subsystem, and assumes these to be randomly distributed over the subsystem frequency band (the assumption of modal disorder). In this manner, the SEA model incorporates a degree of information regarding system parameters which is commensurate with the limited information actually available. Under these assumptions the over-all subsystem power balance relations are the governing equations and these involve only the average modal energies of the modes within each subsystem. The average modal energies thus constitute the basic dynamical variables and all other measures of second-moment response may be deduced from them. At high frequency, where subsystems contain many modes and the uncertainties in modal frequencies are large, the use of energy and power-flow variables results in a drastic decrease in the number of measures required to characterize the response with an accuracy consistent with the uncertainties in modal parameters. Most importantly, if sufficient modal density exists, the total average energy of all modes above a given frequency may be estimated via simple asymptotic formulae and the problem of dimensionality is circumvented.

Although SEA displays many advantages in estimating high frequency response, its use may be inappropriate at low frequencies. First, the confidence intervals associated with the calculated modal energies are inversely proportional to the number of modes within the subsystems or, equivalently, directly proportional to modal density. At low frequency, the system lacks sufficient modal density to allow predictions with an acceptable degree of certainty. Secondly, the assumption of modal disorder, although it produces a simple model, is unsuitable at low frequency. This assumption would effectively discard all the detailed and relatively accurate information describing the low frequency modes. On the other hand, such information is of great advantage in controller design and must not be ignored.

Although the specific procedures of SEA may not be directly applicable, the underlying philosophy offers an attractive approach to the present problem. In essence, the SEA approach to response estimation is to incorporate incomplete system information within the dynamical model by limiting consideration to randomness in modal frequencies. Since it deals with performance measures defined over the entire system ensemble, SEA incorporates the effect of parameter uncertainties at a fundamental level. But most importantly, this modeling approach has profound consequences for the problem of dimensionality. In optimal regulation as in response estimation, the "curse of dimensionality" is manifested in the great mass of processing of fundamental data (of system models presuming complete information) required for response calculation or formulation of an optimal control policy. The example of SEA intimates the possibility that by use of models which include limited system information, we may so arrange matters that the processing required for control policy formulation may be similarly limited.

Drawing inspiration from the essential ideas of SEA, this report sets forth the basis of a formulation which, it is hoped, will circumvent the difficulties faced by current design methods. Section 2.1 formulates the problem of continuous time, full state-feedback, linear optimal regulation of a structural system with uncertain modal frequencies. Next, it is recognized that the statistical model of parameter uncertainties must be derived from severely limited data. In Section 2.2 we identify a minimal data set (the modal "decorrelation times") which preserves certain asymptotic properties of the open-loop system in the case of large uncertainties or for high order modes. Choosing to acknowledge only this data set, the statistical model which is otherwise maximally unconstrained is determined by use of an entropy principle. In consequence we obtain a white parameter model depending on relatively few measures of frequency uncertainty. Section 2.3 sets forth the solution for optimal linear regulation and a special form of the stochastic Riccati equation arising from the state-dependent noise problem is obtained. The section concludes by summarizing the overall rationale of the proposed design approach.

Section 3 examines the properties of solutions to the stochastic Riccati equation. Specifically addressed are the issues of closed-loop stochastic stability and the influence of uncertainties on the effective dimensionality of the system. In particular, it is found that the computational burden may be reduced to that associated with the relatively few "well-known" modes, permitting the use of high order models and largely eliminating the need for modal truncation.

2. STOCHASTIC OPTIMAL CONTROL UNDER LIMITED PARAMETER INFORMATION

2.1 Problem Statement

To begin formulation of the linear regulator problem it is most suitable to write the equations of motion for the structure normal mode coordinates. The state-space form of these equations may be written:

$$\left.\begin{array}{l} \dot{x} = Ax + Bu + w \ , \ t \ \varepsilon [t_o, t_1] \\ \\ x(t_o) = 0 \\ \\ A \ \varepsilon \ R^{2nx2n}, \ B \ \varepsilon \ R^{2nx\ell}, \ u \ \varepsilon \ R^{\ell} \end{array}\right\} \tag{1}$$

where, for convenience, we restrict consideration to zero initial state. Here x is the vector of modal coordinates and velocities with its odd indexed elements representing modal displacements and the adjacent even indexed elements giving the corresponding modal velocities. With this convention and supposing that linear gyroscopic terms may be neglected and that damping is proportional, the dynamic matrix A assumes the form:

$$A = \underset{k=1,\ldots,n}{\text{block-diag}} \begin{bmatrix} 0 & 1 \\ -\omega_k^2 & -2\eta_k \omega_k \end{bmatrix} \tag{2}$$

where ω_k and η_k are the k^{th} modal frequency and damping ratio. To simplify the developments of this section and without loss of generality we consider only the elastic modes of the system so that all the ω_k are non-zero. Henceforth, we assume small but non-zero structural damping, i.e.,

$$0 < \eta_k << 1$$

for all k.

Furthermore, all the odd numbered rows of the input matrix B are zero:

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ 0 & 0 & \cdots & 0 \\ b_{41} & b_{42} & \cdots & b_{4\ell} \\ \vdots & \vdots & & \vdots \end{bmatrix} \tag{3}$$

where the non-zero elements are proportional to the normal mode shapes at ℓ actuator locations.

w is a vector of white noise disturbance forces. Because of the convention regarding our state space representation, the intensity matrix, v, assumes the form:

$$V_{kj} = 0 \; ; \; k, \; j \text{ odd} \tag{4}$$

where this is assumed non-negative definite.

The control input vector, u, is assumed to arise from linear, full state feedback:

$$u = -K(t) \; x \; (t) \tag{5}$$

where K(t) is a time varying gain matrix.

The standard linear quadratic formulation,[29] assumes complete and accurate knowledge of the system dynamics (A, B, V precisely known) and seeks the K(t) which minimizes the quadratic functional

$$\left.\begin{aligned} &J \triangleq E\left[\int_{t_o}^{t_1} dt\left[x^T R_1 x + u^T R_2 u\right]\right] && \text{a.} \\ &R_1 \geq 0 \ , \ R_2 > 0 && \text{b.} \end{aligned}\right\} \tag{6}$$

subject to the constraints imposed by (1). To fix ideas we have not included terminal state weighting since this involves only trivial modification of the following results.

As a further preliminary step, it is convenient to express the above relations in the eigen-basis of the uncontrolled (K=0) system. In view of the assumption of small damping we may simplify this process by introducing the resonant approximation for A:

$$A = \underset{k=1,\ldots,n}{\text{block-diag}} \begin{bmatrix} -\eta_k\omega_k & 1 \\ -\omega_k^2 & -\eta_k\omega_k \end{bmatrix} \tag{7}$$

so-called because the difference between damped and undamped natural frequencies is neglected. The eigenvalues of (7) differ from those of (2) by terms of second order in the damping, while the eigenvectors differ by terms of first order. Within this replacement, the eigenvector matrix of A is

$$\Phi = \underset{k=1,\ldots,n}{\text{block-diag}} \begin{bmatrix} 1 & 1 \\ i\omega_k & -i\omega_k \end{bmatrix} \tag{8}$$

Defining:

$$\xi \triangleq \Phi^{-1}x \ , \ \tilde{w} \triangleq \Phi^{-1}w \tag{9}$$

and

$$\left.\begin{aligned}
&\mu \triangleq \Phi^{-1}A\Phi = \mathrm{diag}\{\omega_1(i-\eta_1),\omega_1(-i-\eta_1) \\
&\qquad\qquad \ldots\omega_n(i-\eta_n),\omega_n(-i-\eta_n)\} && \text{a.}\\
&\beta \triangleq \Phi^{-1}B && \text{b.}\\
&\kappa \triangleq K\Phi && \text{c.}\\
&v \triangleq \Phi^{-1}V\Phi^{-1H} && \text{d.}\\
&\sigma_1 \triangleq \Phi^H R_1 \Phi && \text{e.}
\end{aligned}\right\} \tag{10}$$

the optimal linear regulator problem, equations (1) through (6), may be re-stated as:

$$\left.\begin{aligned}
&\min_{\substack{\kappa \\ K\ \text{real}}}: \ J = E[\int_{t_o}^{t_1} dt\ \xi^H[\sigma_1 + \kappa^H R_2 \kappa]\xi] && \text{a.}\\
&\sigma_1 \geq 0 \ , \ R_2 > 0 && \text{b.}\\
&\dot{\xi} = (\mu-\beta\kappa)\xi + \tilde{w} \ , \ t \ \varepsilon[t_o,t_1] && \text{c.}\\
&\xi(t_o) = 0 && \text{d.}
\end{aligned}\right\} \tag{11}$$

where $\tilde{w}$ is white noise with intensity matrix v.

Now, the inevitable errors in the finite element model induce statistical variation in all the matrices defined in (10). However, to gain an analytical foothold on the problem, we begin by consideration of the simplest possible case in which only the open-loop eigenvalues (the μ matrix) are subject to random variation. Note that whatever the modelling errors, the real parts of μ will be negative and small. Thus it is likely that the impact of random fluctuation in the open-loop frequencies on closed-loop performance will overshadow randomness in the damping terms.

Consequently, we further limit consideration to random $\mathrm{Im}(\mu)$. Specifically, it is assumed:

$$\left.\begin{aligned} &\mu = \bar{\mu} + \nu(t) && \text{a.}\\ &\bar{\mu} \triangleq \mathrm{diag}\{\bar{\omega}_1(i-\eta_1),\bar{\omega}_1(-i-\eta_1),\ldots,\bar{\omega}_n(i-\eta_n),\bar{\omega}_n(-i-\eta_n)\} && \text{b.}\\ &\nu(t) \triangleq \underset{k=1,\ldots,2n}{\mathrm{diag}}\{i\ \mathrm{Im}(\bar{\mu}_k)\delta_k(t)\} && \text{c.}\end{aligned}\right\}\ (12)$$

where the $\bar{\omega}_k$, k=1...n are the nominal or mean values of the modal frequencies. The $\delta_k(t)$; k=1...2n are assumed real valued, zero-mean and stationary random processes in time and are mutually statistically independent and independent of the disturbance noise $\tilde{w}$.

The above model is consistent with an SEA point of view. The standard SEA formulation tacitly acknowledges that the energetics of modal interactions are most sensitive to relative frequency locations so that randomness in the frequencies is usually emphasized.

Obviously, a general treatment would require that all matrices be random. However, the above restrictions offer an appropriately simple point of departure and permit relatively easy interpretation. Moreover, as will ultimately be seen, the very special problem considered here still exhibits important features.

To include the effect of frequency uncertainties while retaining the form of linear quadratic theory, we again employ (11.a) but extend the averaging operation over the parameter ensemble as well as over the disturbance ensemble. Having done this, it is advantageous to re-phrase (11) in terms of the co-state matrix, ρ, of the associated deterministic plant regulation problem. Then, within the above restrictions, the problem is to determine a κ (such that K is real) to minimize:

$$\bar{J} \triangleq \int_{t_o}^{t_1} dt \; tr[\bar{\rho} v] \tag{13}$$

subject to the constraints

$$\left.\begin{array}{ll} \bar{\rho} = E[\rho] & \text{a.} \\ -\dot{\rho} = (\bar{\mu} + \nu(t) - \beta\kappa)^H \rho + \rho(\bar{\mu} + \nu(t) - \beta\kappa) & \text{b.} \\ \quad + \sigma_1 + \kappa^H R_2 \kappa & \\ \rho(t_1) = 0 & \text{c.} \end{array}\right\} \tag{14}$$

That this statement follows from (10) and (11) is shown by defining $\bar{\rho}$ as in (14b,c), substituting the resulting expression for $(\sigma_1 + \kappa^H R_2 \kappa)$ into (11a) and performing the average over the disturbance ensemble.

At this point, we note that solution of the variational problem of (13) and (14) requires the evaluation of the ensemble average of ρ. In general $\bar{\rho}$ satisfies an infinite dimensional system of ordinary differential equations. Thus, for practical use this system has to be closed at some finite stage. For this purpose, general results are given by Kistner.[30] There appear to be two situations in which the resulting moment equations are tractable. In the first case, the Lie algebra generated by $(\bar{\mu} - \beta\kappa)$ and ν is solvable[31] the greatest simplification being obtained for an Abelian Lie algebra. The second case is that in which $\nu(t)$ is white. Since general conditions on the Lie algebra generated by $(\bar{\mu} - \beta\kappa)$ and ν cannot be guaranteed, the strategy we adopt here consists in replacing the actual statistics of $\nu(t)$ by an equivalent (in a sense to be discussed below) white noise model. Then (13) together with a single, closed equation for $\bar{\rho}$ yield a variational problem whose solution reduces to a special form of the well-known state-dependent noise formulation.

The crux of the matter is the appropriate choice of an approximate probability model for $\nu(t)$. Before pursuing this topic, we first state some preliminary results concerning system (11c,d) and the solution of (14b).

<u>Theorem 1</u>

Suppose that $\nu(t)$ is a stationary zero mean random matrix process. Define an increment in the nonstationary process W by:

$$W(t_1,t_2) \triangleq W(t_2)-W(t_1) \triangleq \int_{t_1}^{t_2} d\tau\ \nu(\tau)\ ;\ t_2 \geq t_1 \tag{15}$$

and assume that $W(t_1,t_2)$ is almost everywhere continuous in t_1,t_2 and bounded for all finite t_2-t_1, and that increments of the form (15) possess joint moments of all orders. Further, suppose that $\kappa(t)$ is bounded and continuous. Then with μ and β as defined by (10) and the foregoing definitions:

A. The transition matrix, $\phi(t,\tau)$, for system (11c,d) is given by:

$$\phi(t,\tau) = \sum_{k=0}^{\infty} \phi_k(t,\tau)\ ;\ t \geq \tau \qquad \text{a.}$$

where

$$\phi_o(t,\tau) = \exp[\bar{\mu}(t-\tau) + W(\tau,t)] \qquad \text{b.} \tag{16}$$

$$\underset{(k>0)}{\phi_k(t,\tau)} = (-1)^k \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \cdots \int_\tau^{\tau_{k-1}} d\tau_k$$

$$\times\ [\phi_o(t,\tau_1)\beta\kappa(\tau_1)\phi_o(\tau_1,\tau_2)\beta\kappa(\tau_2)\cdots\phi_o(\tau_{k-1},\tau_k)\beta\kappa(\tau_k)\phi_o(\tau_k,\tau)] \qquad \text{c.}$$

and where the integrations extend over the left semi-closed intervals.

B. $\phi(t,\tau)$ is almost everywhere continuous and its first and second moments are continuous and differentiable in both arguments.

C. Eqs. (14b,c) possess the unique, positive semi-definite solution:

$$\rho(t) = \int_t^{t_1} d\tau \ \psi(t,\tau) \ ; \ t \ \varepsilon [t_o, t_1] \tag{17}$$

where, for $\tau \geq t$, $\delta \leq \tau - t$:

$$\left.\begin{array}{ll} \psi(t,\tau) = \phi^H(t+\delta,t)\psi(t+\delta,\tau)\phi(t+\delta,t) & \text{a.} \\ \psi(\tau,\tau) = \sigma_1 + \kappa^H(\tau)R_2\kappa(\tau) & \text{b.} \end{array}\right\} \tag{18}$$

D. $\overline{\psi}(t,\tau) \triangleq E[\psi(t,\tau)]$; $t \ \varepsilon \ [t_o, t_1]$ is continuous and differentiable in t and τ.

The proof is contained in Appendix 1. At this point, we remark on the case in which W(t) is a matrix Wiener-Levy process. Expressions (16) through (18) retain validity but the resulting forms of averaged quantities are not implied by the Itô equation:[32]

$$d\xi(t) = (\overline{\mu} - \beta\kappa(t))\xi(t)dt + d\nu(t)\xi(t) + d\tilde{w}(t) \tag{19}$$

but, instead are consistent with the Itô equations with the Stratonovitch correction:[33,34,35]

$$
\left.\begin{aligned}
d\xi(t) &= (\bar{\mu} - \beta\kappa + \tfrac{1}{2}I)\xi(t)dt + \xi(t)\, d\nu(t) + d\tilde{w}(t) \\
I &\triangleq \lim_{\Delta \downarrow 0} \frac{1}{\Delta} E[W^2(t, t+\Delta)]
\end{aligned}\right\} \qquad (20)
$$

In other words, when evaluating response averages we consider white $\nu(t)$ as the limit of a band-limited process as the bandwidth approaches infinity. This result is desirable from a practical point of view, since parameter deviations actually encountered are likely to be piecewise continuous (and may often be random variables constant in time). Qualitative features of second moment response which obtain for $\nu(t)$ continuous should be preserved when $\nu(t)$ is replaced by some equivalent white noise model. With $\nu(t)$ piecewise continuous parameter uncertainties of the form (12), it is easily seen from (16) that the second moment response of the uncontrolled system is always asymptotically stable. While this result is duplicated when (20) is employed, (19) implies that second moment stability is dependent upon the magnitude of the white noise intensity. Thus, in the white noise case, use of (16) through (18) or, equivalently, (20) is indicated.

2.2 Statistical Modelling of Uncertainties

As Eqs.(17) and (18) indicate, a complete specification of the statistical structure of open-loop frequency deviations permits explicit determination of $\bar{\rho}$, so that the optimization scheme is reduced to a problem in the calculus of variations. However, complete data on parameter statistics never exists. In part this arises from the impossibility of devising a scheme of empirical inference sufficiently comprehensive to provide valid statistical estimates of all the characteristic functions of frequency uncertainties. Further, there is the practical difficulty associated with the number of statistical parameters that must enter into the calculation of $\bar{J}$. The number of statistical measures (covariances and perhaps higher order moments) of parameter uncertainty must perforce increase rapidly with the order of the system. For a very large-order system, such as we consider here, the sheer mass of uncertainty measures outdistances our ability to enumerate them all.

This point is abundantly illustrated by the results of Reference (24) wherein the stochastic optimal control problem is solved for a general multi-variable discrete-time system with white parameter uncertainties. The resulting Riccati-like equation requires, as elementary data, the covariances of several fully populated random matrices. For each such matrix this entails specification of $\frac{1}{2}N^2 (N^2+1)$ scalar covariances (N being the system dimension). Clearly, for N large the difficulty of interpreting the meaning and design significance of all these parameters is insuperable.

Thus, in practice, it is necessary to synthesize insensitive controllers given incomplete data on parameter statistics. As in spectral analysis and related fields, a more or less comprehensive probability model must be reconstructed from severely limited data in a manner which is consistent with the data at hand and maximally unpresumptive with regard to unavailable data. The successful principle enunciated by Jaynes[36,37] has immediate application here: the desired probability assignment is the one which, under the constraints imposed by available data, can be realized in the maximum number of ways or, equivalently, maximizes the entropy of the underlying processes.

With the above scheme for reconstructing parameter statistics, it is clear that the resulting form of the optimization problem (i.e., the form of $\overline{\rho}$) depends critically upon the nature of the available data. In practice it may be advantageous to acknowledge as accessible a set of data which is even more restricted than the data actually available but which, under the maximum entropy principle, renders $\overline{\rho}$ tractable. In the following we attempt to discern a data set which (1) is significant to directly observable attributes of system response, (2) constitutes a minimum set for the purpose of maximum entropy probability assignments, and (3) simplifies calculation of the parameter ensemble averaged performance index.

First, since the characteristics of modal frequency uncertainties are inherently associated with the uncontrolled system, let us suppose that the available data is collected from measurements performed on realizations of the uncontrolled system drawn from the parameter ensemble. Such measurements ultimately entail direct observation of the attributes of system response to specified disturbances. For example, modal frequencies are never directly measurable since they are derived quantities which presuppose a dynamical model. Thus, we must imagine that the available data consists of low-order statistics of the uncontrolled system response to the disturbance noise modelled above. Since practical exigencies would preclude estimation of higher order moments we may limit consideration to first and second-order response moments. Clearly the expected value of the co-state matrix must also be considered subject to direct observation, particularly as its value for the uncontrolled system establishes the fiducial level of quadratic cost.

With regard to these quantities, the following results are easily obtained.

Theorem 2

Consider the uncontrolled system:

$$\dot{\xi} = (\bar{\mu} + \nu(t))\xi + \tilde{w}(t)$$
$$\xi(t_o) = \xi_o \qquad (20a)$$

with ξ_o fixed and $\bar{\mu}$, ν and $\tilde{w}$ as defined previously. Define increments of the nonstationary process $\delta_k(0,t)$:

$$\left.\begin{array}{ll} W_{kk}(t_1,t_2) = i\,\mathrm{Im}(\bar{\mu}_k)\,\delta_k(t_1,t_2) & \text{a.} \\ \delta_k(t_1,t_2) \triangleq \delta_k(0,t_2) - \delta_k(0,t_1) \triangleq \int_{t_1}^{t_2} d\tau \;\delta_k(\tau) & \text{b.} \\ \quad k = 1,\ldots,2n \;,\; t_2 \geq t_1 & \end{array}\right\} \qquad (21)$$

with the log-characteristic functions:

$$\Gamma_k(u;t_1,t_2) \triangleq \ln E[\exp[iu\delta_k(t_1,t_2)]] \qquad (22)$$

and define the "modal decorrelation times", T_k:

$$T_k \triangleq (I_k\,\mathrm{Im}(\bar{\mu}_k))^{-1} \triangleq \int_o^{\infty} d\tau \;\left|e^{\Gamma_k(\mathrm{Im}(\bar{\mu}_k);0,\tau)}\right|^2$$
$$k = 1,\ldots,2n \qquad (23)$$

where the I_k are the associated reciprocal time constants.

Further, assuming the same properties for the $W_k(t_1,t_2)$ as in Theorem 1 then for all k; $t \in [t_o,t_1]$:

A. $$|E[\xi_k(t)]|^2 = |\xi_{ok}|^2 \; e^{2Re\bar{\mu}_k(t-t_o)} \left| e^{\Gamma_k(Im\bar{\mu}_k;t_o,t)} \right|^2 \tag{24}$$

$$\left.\begin{aligned} \bar{Q}_{kk} &= v_{kk}\int_{t_o}^{t} d\tau \; e^{2Re\bar{\mu}_k(t-\tau)} + |\xi_{ok}|^2 \; e^{2Re\bar{\mu}_k(t-t_o)} \\ \underset{k\neq j}{\bar{Q}_{kj}} &= v_{kj}\int_{t_o}^{t} d\tau \; e^{(\bar{\mu}_k+\bar{\mu}_j^*)(t-\tau)} e^{\Gamma_k(Im\bar{\mu}_k;\tau,t)+\Gamma_j(Im\bar{\mu}_j;\tau,t)} \\ &\quad + \xi_{ok}\xi_{oj}^* e^{(\bar{\mu}_k+\bar{\mu}_j^*)(t-t_o)} e^{\Gamma_k(Im\bar{\mu}_k;t_o,t)+\Gamma_j(Im\bar{\mu}_j;t_o,t)} \end{aligned}\right\} \tag{25}$$

$$\lim_{|t-t_o|\uparrow\infty} |\bar{Q}_{kj}| \leq |v_{kj}|(T_kT_j)^{1/2} \tag{26}$$

where $\bar{Q}$ denotes the second moment matrix of ξ.

$$\left.\begin{aligned} \bar{\rho}_{kk}(t) &= (\sigma_1)_{kk}\int_{t}^{t_1} d\tau \; e^{2Re\bar{\mu}_k(\tau-t)} \\ \underset{(k\neq j)}{\bar{\rho}_{kj}(t)} &= (\sigma_1)_{kj}\int_{t}^{t_1} d\tau \; e^{(\bar{\mu}_k^*+\bar{\mu}_j)(\tau-t)} e^{\Gamma_j(Im\bar{\mu}_j;t,\tau)+\Gamma_k(Im\bar{\mu}_k;t,\tau)} \end{aligned}\right\} \tag{27}$$

$$\lim_{|t_1-t|\uparrow\infty} |\bar{\rho}_{kj}| \leq |(\sigma_1)_{kj}|(T_kT_j)^{1/2} \tag{28}$$

B.
$$\left.\begin{aligned} &\left|e^{\Gamma_k(Im\bar{\mu}_k;t_o,t)}\right|^2 \leq 1 && \text{a.} \\ &T_k^{-1} \geq 0 && \text{b.} \end{aligned}\right\} \tag{29}$$

where the equalities hold if $\delta_k(t)$ is zero almost everywhere for all t.

Proof

Result (24) follows directly from averaging

$$\xi(t) = \phi_o(t,t_o)\xi_o + \int_{t_o}^{t} \phi_o(t,\tau)d\tilde{w}(\tau)$$

with $\phi_o(t,\tau)$ given by (16b) and using definitions (21) and (22). Similarly, a straight-forward evaluation of $E[\xi\xi^H]$ yields (25), while (26) follows by use of the Schwarz inequality. In view of (17) and (18) with $\kappa=0$:

$$\rho(t) = \int_{t}^{t_1} d\tau\ \phi_o^H(\tau,t)\sigma_1\phi_o(\tau,t)$$

Averaging of this expression leads to (27) and (28).

Equation (29a) expresses an elementary property of the characteristic function[38] while (29b) holds by definition (23). Note that (29a) ensures second moment stability and the existence of constant steady-state values of $\overline{\rho}$ and $\overline{Q}$. Now if $\delta_k(t)$ is zero almost everywhere for all t, then $\delta_m(t_1,t_2)$ also vanishes almost surely for all t_1,t_2. Then Γ_k vanishes and the equality of (29a) holds, whence the integral of (23) is unbounded and the equality of (29b) follows. □

As (24) shows, the mean response provides the magnitudes of all characteristic functions. However, this is still an embarassment of riches, and we must seek a still more restricted data set which is largely independent of the detailed character of Γ_m.

In this connection, (24) and part B of the theorem reveal an important qualitative effect. Aside from small natural dissipation, the energy associated with the mean response of the k^{th} mode is proportional to $|e^{\Gamma m}|^2$ which tends to zero as t tends to infinity. The gross effect of the frequency uncertainty is to introduce a spurious damping (the "decorrelation damping") into the mean response due to progressive decorrelation among individual ensemble members. Of course, as (25a) implies, the modal energy thus lost to the mean response serves to augment the covariance. Thus, the mean response energy gives a measure of the system information retained at any time subsequent to the application of a known disturbance.

Now the ratio of the mean response energy to that predicted by a deterministic model is the magnitude squared of the characteristic function appearing on the right of (24). This ratio is always less than unity and integrable so that any modal frequency uncertainty renders T_k finite. Thus the decorrelation times defined by (23) are the time scales over which a deterministic model retains validity given known initial disturbances, and appear to give natural measures of uncertainty.

To relate the T_k more directly to frequency statistics, suppose, for example, that δ_k is known to be a zero-mean Gaussian random variable with standard deviation σ. This corresponds to a relative uncertainty in Im $\bar{\mu}_k$ with variance σ^2. Using (21) through (23):

$$T_k = \int_0^\infty d\tau \; \exp[-(\mathrm{Im}\bar{\mu}_k)^2\sigma^2\tau^2] = \frac{\sqrt{\pi}}{2}(\sigma \mathrm{Im}\bar{\mu}_k)^{-1}$$

or,

$$I_k = \frac{2}{\sqrt{\pi}}\,\sigma$$

and, in general we may estimate $1/T_k\mathrm{Im}(\bar{\mu}_k)$ as the standard deviation of the frequency deviation relative to its mean value.

Frequency uncertainties may be considered "small" when the I_k are small or when the T_k encompass many periods of natural vibration. For the I_k sufficiently small, the modulation introduced by the characteristic functions, $\exp(\Gamma_k(\mathrm{Im}(\bar{\mu}_k);t_o,t))$, will be overwhelmed by the attenuation due to natural damping. This occurs when the T_k are much larger than the damping time scales, i.e.:

$$T_k \gg (\mathrm{Re}\bar{\mu}_k)^{-1} \;;\; k = 1,\ldots,2n \tag{30}$$

In this regime, as can be seen from (24), (25) and (27); the influence of frequency uncertainties is negligible and a deterministic model of the plant may be used.

In the more interesting case in which (30) is not satisfied, various qualitative features of $\bar{Q}$ and $\bar{\rho}$ may be deduced from (25) and (27).

For this discussion and henceforth, we suppose that the state vector ξ is so arranged that

$$\bar{\omega}_1 \le \bar{\omega}_2 \le \ldots \le \bar{\omega}_n$$

and define, for convenience, the integer-valued function $N(k) \varepsilon [1,2,\ldots,n]$ such that

$$\overline{\omega}_{N(k)} = |\mathrm{Im}\overline{\mu}_k| ; \; k = 1,\ldots,2n$$

Further, we impose various restrictions likely to be satisfied in practice:

a. $|\overline{\mu}_{2k}-\overline{\mu}_{2k-2}| / \sqrt{\overline{\omega}_k\overline{\omega}_{k-1}}$ non-increasing with increasing k

b. $0 \le \eta_k \le \eta \; ; \; \forall k$

c. $T_k|\mathrm{Im}\overline{\mu}_k|$ monotone decreasing with increasing k,

i.e., $I_k > I_{k-1} \; ; \; \forall k$

Condition (a) is satisfied in the usual case wherein modal frequency separation (considered as a function of frequency) is bounded by some finite power of the nominal modal frequency. Condition (b) postulates an upper bound on the modal damping coefficients. (c) states, in essence, that the uncertainty of open-loop frequencies relative to nominal values increase for the higher order modes. Actually, the degradation of accuracy with increasing mode number need not be monotone, but condition (c) nevertheless reflects the overall trend.

Consider now the steady state behavior ($t \uparrow 0$ and $t_1 \uparrow \infty$ in (25) and (27), respectively) of the covariance and expected cost matrices.

Corollary 1

Consider the model order, n, arbitrarily large. Assume conditions (a) through (c) above and denote by $\bar{Q}^{(D)}$ and $\bar{\rho}^{(D)}$ the results obtained under a deterministic model ($\Gamma=0$). Then for the uncontrolled system in the steady state:

A. With K_c the smallest integer such that:

$$\bar{\omega}_{N(k_c)} \geq \frac{|\bar{\mu}_{k_c} + \bar{\mu}^*_{k_c} - 2|}{I_{k_c}} \triangleq \omega_c \tag{31}$$

then for $k \geq k_c$, $j = k + 2$:

$$\left.\begin{aligned} \frac{|\bar{Q}_{kj}|}{\sqrt{\bar{Q}_{kk}\bar{Q}_{jj}}} &< \frac{|\bar{Q}^{(D)}_{kj}|}{\sqrt{\bar{Q}^{(D)}_{kk}\bar{Q}^{(D)}_{jj}}} & \text{a.} \\ \frac{|\bar{\rho}_{kj}|}{\sqrt{\bar{\rho}_{kk}\bar{\rho}_{jj}}} &< \frac{|\bar{\rho}^{(D)}_{kj}|}{\sqrt{\bar{\rho}^{(D)}_{kk}\bar{\rho}^{(D)}_{jj}}} & \text{b.} \end{aligned}\right\} \tag{32}$$

B. Similarly, there exists a smallest integer, k_{cL} such that (32) holds for $k \geq k_{cL}$ and $j \in (k+2, \ldots, k+2L)$.

C. Given ε_ρ, $\varepsilon_Q > 0$, there exists a $k_u > k_c$ sufficiently large that:

$$\frac{|\overline{Q}_{kj}|}{\sqrt{\overline{Q}_{kk}\overline{Q}_{jj}}} < \varepsilon_Q$$

$$\frac{|\overline{\rho}_{kj}|}{\sqrt{\overline{\rho}_{kk}\overline{\rho}_{jj}}} < \varepsilon_\rho$$

for $k > k_u$ and all $j \neq k$.

<u>Proof</u>

We need consider only $\overline{Q}$ since the results for $\overline{\rho}$ follow analogously. In the steady-state:

$$\left.\begin{aligned} \overline{Q}_{\substack{kj \\ k\neq j}} &= v_{kj}\int_0^\infty d\tau\, e^{(\overline{\mu}_k+\overline{\mu}_j^*)\tau}\, e^{\Gamma_k(\mathrm{Im}\overline{\mu}_k;0,\tau)+\Gamma_j(\mathrm{Im}\overline{\mu}_j;0,\tau)} \\ \overline{Q}_{kk} &= v_{kk}\int_0^\infty d\tau\, e^{2\mathrm{Re}\overline{\mu}_k\tau} \end{aligned}\right\} \qquad (33)$$

where the above integrals exist by condition (b). Clearly the diagonal elements of $\overline{Q}$ are unaffected by frequency uncertainties and $\overline{Q}_{kk}$ and $\overline{Q}_{kk}^{(D)}$ are identical. From (26):

$$|\overline{Q}_{kj}| \leq |v_{kj}|(T_kT_j)^{\frac{1}{2}}$$

while, by direct calculation:

$$|\overline{Q}_{kj}^{(D)}| = \frac{|v_{kj}|}{|\overline{\mu}_k+\overline{\mu}_j^*|}$$

so that:

$$|\overline{Q}_{kj}|/|\overline{Q}_{kj}^{(D)}| \le (T_k T_j)^{\frac{1}{2}} |\overline{\mu}_k + \overline{\mu}_j^*|$$

$$= \frac{|\overline{\mu}_k + \overline{\mu}_j^*|}{\sqrt{I_k I_{k\pm 2} \overline{\omega}_m \overline{\omega}_{m\pm 1}}}$$

where k=2m or 2m-1. In view of (31) and conditions (a) and (c), (32) follows directly. Similarly, by virtue of (a) and (c), we may determine a smallest integer, k_{c2} such that

$$\overline{\omega}_{N(k_{c2})} \ge \frac{|\overline{\mu}_{k_{c2}} + \overline{\mu}_{k_{c2}-4}|}{I_{k_{c2}}}$$

and use (26) and direct calculation of $\overline{Q}_{kj}^{(D)}$ to show (32) for $k \ge k_{c2}$, $j \in (k+2, k+4)$. Repetition of this argument proves the assertion of part B.

Finally part C follows by use of (26), (28) and condition (c). □

Note that parts A and B did not give results for k even and j odd or vice-versa. However in this case:

$$|\overline{Q}_{kj}| = O\left(\frac{|v_{kj}|}{|\overline{\omega}_{N(k)} + \overline{\omega}_{N(j)}|} \int_0^\infty d\tau \left| e^{\Gamma_k(\mathrm{Im}\overline{\mu}_k;0,\tau) + \Gamma_j(\mathrm{Im}\overline{\mu}_j;0,\tau)} \right|\right)$$

$$\le O\left(\frac{1}{|\overline{\omega}_{N(k)} + \overline{\omega}_{N(j)}|} |v_{kj}| (T_k T_j)^{\frac{1}{2}}\right)$$

where the first line follows from Reimann's lemma[39], and the second by the Schwarz inequality. Thus, from (26) we may estimate:

$$|\bar{Q}_{kj}| \sim O\left(\frac{|\bar{Q}_{k,j\pm1}|}{|\bar{\omega}_{N(k)}+\bar{\omega}_{N(j)}|}\right)$$

so that for k and j not both odd or even, $|\bar{Q}_{kj}|$ is negligible in any case.

The Corollary shows that the main effect of frequency uncertainties is to suppress cross-correlation among the open-loop modes. Indeed, as part C reveals, the portions of $\bar{Q}$ and $\bar{\Omega}$ corresponding to sufficiently high order modes tend to become diagonalized under the decorrelating effect of uncertainties.

These results allow a division of the open-loop modes into various qualitative regimes. An important line of demarcation is provided by ω_c as defined by (31). We shall term the quantity ω_c the <u>coherence limit</u> in frequency since it locates the onset of reduced inter-modal correlation.

Suppose k and j exist such that $\bar{\omega}_{N(k)}, \bar{\omega}_{N(j)} << \omega_c$. Then as one sees from (33):

$$|\bar{Q}_{kj}| \simeq |\bar{Q}_{kj}^{(D)}|$$

so that in this <u>quasi-deterministic range</u> frequency uncertainties have little influence.

On the other hand, for modes much above the coherence limit, frequency uncertainties tend to obliterate modal cross-correlation. In this regime, as Corollary 1 shows, there exist k, j sufficiently large that the correlation coefficient $\bar{Q}_{kj}/\sqrt{\bar{Q}_{kk}\bar{Q}_{jj}}$ is as small as desired. We may say that:

$$\bar{\omega}_{N(k)} >> \omega_c$$

defines an incoherent range in which the open-loop modes are uncorrelated, $\bar{Q}$ approximately diagonal and (from (25a)) independent of parameter statistics.

Of course, the same general behavior can be deduced for $\bar{\rho}$. In particular, the sub-block of $\bar{\rho}$ corresponding to modes in the incoherent range is approximately diagonal and independent of the Γ_k $(Im(\bar{\mu}_k); o,\tau)$; k=1...2n. Thus the specific form of the Γ_k has no influence on that portion of the cost contributed by the incoherent range, while the total cost is primarily dependent on the location of the incoherence limit. We conclude that most qualitative features of open-loop response are dictated by the magnitudes of the T_k relative to the other time scales of the problem.

At the very least, we must require that any approximating probability model of the $\delta_k(t)$; k=1...2n should be capable of duplicating the general behavior described above - in particular, it should preserve the time scales of decorrelation damping, provide a correct estimate of the coherence limit and satisfy the bound given by (26) for the cross-correlations of high order modes. This is possible only if the decorrelation times are admitted as fundamental data.

Thus, henceforth, we propose to acknowledge only the T_k (k=1...2n) as the "available" data. At this point note that in practice:

$$\mu_{2m} = \mu^{*}_{2m-1} \; ; \; m = 1,\ldots,n$$

so that:

$$\delta_{2m}(t) = \delta_{2m-1}(t) \; ; \; m = 1,\ldots,n$$

Since the simplest model requires use of the T_k (k=1...2n) only, we shall acknowledge this relation only to the extent that:

$$T_{2m} = T_{2m-1} \tag{34}$$

With this choice, it remains to construct a full probability model which presumes as little as possible regarding the unavailable data. In other words, it is desired to determine the probability assignment which maximizes the entropy of the processes $\delta_k(t)$, k=1...2n subject to the constraints implied by (23) and (34).

In general, the entropy functional to be maximized should be defined on the joint probability distribution functional of the processes $\delta_k(t)$, k=1,...n for all t on the real line. To avoid the use of such an unwieldy entity, we shall proceed heuristically and define the <u>maximum entropy probability assignment induced by the decorrelation times</u> as the set of all joint statistics of all finite sets of increments (21) obtained from the solution of the following problem in the limit as T and N tend to infinity:

"Under the Restrictions:

$$\left.\begin{array}{lr} \text{(a)} \quad \dfrac{T}{N} \displaystyle\sum_{m=1}^{N} \left| e^{\Gamma_k(\mathrm{Im}\,\bar{\omega}_k;0,mT/N)} \right|^2 = T_k & \text{a.} \\ \text{with:} \quad T \in [T_k,\infty) \; , \; T_{2m} = T_{2m-1} \; ; \; m = 1,\ldots,n & \text{b.} \end{array}\right\} \quad (35)$$

(b) The increments $\delta_k(t_1,t_2)$; $t_1,t_2 \geq 0$ are stationary

(c) $\delta_k(t_1,t_2)$; $t_1,t_2 \geq 0$ are zero-mean and possess finite, non-zero second moments for $|t_2-t_1| > 0$

choose the joint distribution, F, of the increments

$$\delta_k(0,m\,\tfrac{T}{N}) \ ; \ k = 1,\ldots,2n \ ; \ m = 1,\ldots,N \tag{36}$$

so that the entropy:

$$H \triangleq -\int dF \ln P$$

is maximized."

-where P denotes the probability density and it is seen that the limit of the sum in (35) is the right side of (23).

Note that restrictions (a), (b) and (c) differ from the form usually assumed for the available information in the problem of determining the maximum entropy probability distribution.[36,37] However, in the limit as N and then T approach infinity, the solution is quite simple and intuitively plausible. Its derivation is contained in Appendix 2. Here we summarize the conclusion as follows:

Theorem 3

Assuming that the processes

$$\delta_k(0,t) \triangleq \int_0^t \delta_k(\tau)d\tau \qquad (37)$$
$$k=1,\ldots,2n$$

possess finite, non-zero variances for all $t\varepsilon(0,\infty)$ and stationary increments, the maximum entropy probability assignment induced by the data:

$$\left.\begin{aligned} &T_k \triangleq (\mathrm{Im}\bar{\mu}_k I_k)^{-1} = \int_0^\infty d\tau \left| e^{\Gamma_k(\mathrm{Im}\bar{\mu}_k;0,\tau)} \right|^2 \\ &k = 1,\ldots,n \\ &T_{2m} = T_{2m-1} \ ; \ m = 1,\ldots,n \end{aligned}\right\} \qquad (38)$$

is the one under which the $\delta_k(o,t)$; $k=1,\ldots,2n$ are independent Wiener-Levy processes with intensities $I_k/\bar{\omega}_{N(k)}$:

$$\left.\begin{aligned} &E[\delta_k^2(0,t)] = \frac{I_k}{\bar{\omega}_{N(k)}}|t| \\ &I_{2m} = I_{2m-1} \ ; \ m = 1,\ldots,n \end{aligned}\right\} \qquad (39)$$

With the T_k as available data, the maximum entropy probability model gives independent modal frequency uncertainties - a property that was assumed in Section 2.1. Furthermore, this statistical model satisfies the assumptions of Theorem 1 and those results may be employed directly.

Note that:

$$\Gamma_k(\mathrm{Im}\bar{\mu}_k;0,\tau) = -\tfrac{1}{2}\bar{\omega}_{N(k)} I_k|\tau|$$

Substitution of this into (24), (25) and (27) shows that all the qualitative features noted above for $E[\xi_k]$, $\bar{Q}$ and $\bar{\rho}$ are preserved.

In this connection, it is convenient to compare the result obtained for $\bar{\rho}$ under the above white parameter model with $\bar{\rho}$ as computed under a complete statistical model having the same decorrelation times. As a practical matter we must limit consideration to the uncontrolled system. With this restriction, suppose that, in reality, the open loop frequency deviations, δ_k, are normal random variables with variances $\sigma_k{}^2$. Then by application of (27), the steady state value of $\bar{\rho}_{kj}(k\neq j)$ is found to be

$$\bar{\rho}_{kj} = \bar{\rho}_{kj}^{(G)} \stackrel{\Delta}{=} \sigma_{1kj}\int_0^\infty d\tau\ e^{-\frac{1}{2}(\bar{\omega}_{N(k)}^2\sigma_k^2+\bar{\omega}_{N(j)}^2\sigma_j^2)\tau^2 + (\bar{\mu}_j+\bar{\mu}_k^*)\tau} \qquad (39)^a$$

and the diagonal elements of $\bar{\rho}$ are unaffected by frequency uncertainties and need not be considered further. Of particular concern in this comparison is the behavior of $\bar{\rho}$ for large uncertainties and/or high order modes (in view of conditions (a) and (c)). Thus, consider the specific case:

$$\sqrt{\bar{\omega}_k^2\sigma_k^2 + \bar{\omega}_j^2\sigma_j^2} >> |\bar{\mu}_j+\bar{\mu}_k^*|$$

Then (39)a yields approximately:

$$\bar{\rho}_{kj}^{(G)} \simeq \frac{\sqrt{\frac{\pi}{2}}\,\sigma_{1kj}}{\sqrt{\sigma_k^2\bar{\omega}_{N(k)}^2+\sigma_j^2\bar{\omega}_{N(j)}^2}} \tag{40)a}$$

Now the decorrelation times may be computed as:

$$T_k = \frac{\sqrt{\pi}}{2}\,(\sigma_k\bar{\omega}_{N(k)})^{-1}$$

so that for large σ_k's, the maximum entropy statistical model yields:

$$\underset{k\neq j}{\bar{\rho}_{kj}} \simeq \bar{\rho}_{kj}^{(w)} \triangleq \frac{\sqrt{\pi}\,\sigma_{1kj}}{\bar{\omega}_{N(k)}\sigma_k+\bar{\omega}_{N(j)}\sigma_j} \tag{41)a}$$

Comparison of (40)a and (41)a readily shows:

$$|\bar{\rho}_{kj}^{(G)}| \leq |\bar{\rho}_{kj}^{(w)}| \tag{42)a}$$

where equality holds only if $\sigma_k\,\bar{\omega}_{N(k)} = \sigma_j\,\bar{\omega}_{N(j)}$.

Thus, at least in the present comparison, the maximum entropy statistical model correctly models the suppression of off-diagonal elements of $\bar{\rho}$ due to frequency uncertainties. In fact, as (42)a shows the model of Theorem 3 somewhat underestimates this diagonalizing effect.

Most importantly, the $\delta_k(t)$ are modelled as white noise so that posterior learning is impossible and the stochastic control problem is nondual. Although physically unrealistic, the white parameter uncertainty model entails an extra degree of caution in the control and provides a worst case situation from the point of view of parameter identification. Indeed the model may be used to determine performance degradation due to parameter uncertainty and to assess the need for identification and adaptive algorithms.

2.3 Determination of the Optimal Gain

Let us now resume consideration of the optimization problem. Adopting the statistical model set forth in Theorem 3, we obtain:

Theorem 4

With the maximum entropy probability assignment induced by the decorrelation times as given in Theorem 3 and $\kappa(t)$ bounded and continuous in $t \, \varepsilon [t_o, t_1]$, $\bar{\rho}$ as defined in (14) is the unique, positive semi-definite solution of

$$\begin{aligned} -\dot{\bar{\rho}}(t) &= (\bar{\mu} - \beta\kappa(t) - \tfrac{1}{2}\mathcal{I})^H \bar{\rho}(t) \\ &\quad + \bar{\rho}(t)(\bar{\mu} - \beta\kappa(t) - \tfrac{1}{2}\mathcal{I}) \\ &\quad + \mathcal{I}\{\bar{\rho}\} + \sigma_1 + \kappa^H(t) R_2 \kappa(t) \\ \bar{\rho}(t_1) &= 0 \end{aligned} \tag{40}$$

where

$$\mathcal{I} \triangleq \operatorname{diag}[\bar{\omega}_1 I_2, \bar{\omega}_1 I_2, \ldots, \bar{\omega}_n I_{2n}, \bar{\omega}_n I_{2n}] \tag{41}$$

and where, for any square matrix, M:

$$\{M\} \triangleq \operatorname{diag}[M_{11}, M_{22}, \ldots, M_{2n,2n}] \tag{42}$$

Proof

With the statistical model of Theorem 3, $W(t_1,t_2)$ as defined in (21) satisfies the restrictions of Theorem 1 and the results of this Theorem may be used. In particular, (18) yields:

$$\left.\begin{aligned} \psi(t,\tau) &= \phi^H(t+\delta,t)\psi(t+\delta,\tau)\phi(t+\delta,t) && \text{a.} \\ \psi(t+\delta,\tau) &= \phi^H(\tau,t+\delta)[\sigma_1+\kappa^H(\tau)R_2\kappa(\tau)]\phi(\tau,t+\delta) && \text{b.} \end{aligned}\right\} \quad (43)$$

where $\delta>0$. (16) shows that $\phi(t+\delta,t)$ depends upon $\delta_k(t_1,t_2)$ only for $t_1,t_2 \in (t, t+\delta]$, while $\phi(\tau, t+\delta)$ depends upon $\delta_k(t_1,t_2)$ only for $t_1,t_2 \in (\tau, t+\delta]$. Since these intervals are disjoint and the increments of $\delta_k(o,t)$ are independent, the ensemble average of (43a) becomes:

$$\overline{\psi}(t,\tau) = E[\phi^H(t+\delta,t)\overline{\psi}(t+\delta,\tau)\phi(t+\delta,t)] \qquad (44)$$

where

$$\overline{\psi}(t,\tau) \triangleq E[\psi(t,\tau)]$$

Now examine (16). Keeping in mind that (1) the ϕ_o's appearing in the integrals of (16c) are each dependent on increments of the $\delta_k(o,t)$ over mutually disjoint intervals, (2) that partial sums of (16a) are almost everywhere convergent, and (3) the $\delta_k(o,t)$ are Gaussian with variances (39), it is seen that the contribution of $\sum_{k=2}^{\infty} \phi_k(t+\delta, t)$ to $\phi(t+\delta, t)$ produces terms of of order δ^2 on the right side of (44).

(44) may thus be written:

$$\overline{\psi}(t,\tau) = E[(\phi_o(t+\delta,t) - \int_t^{t+\delta} d\tau_1 \phi_o(t+\delta,\tau_1)\beta\kappa(\tau_1)\phi_o(\tau_1,t))^H$$

$$\times\ \overline{\psi}(t+\delta,\tau)(\phi_o(t+\delta,t) - \int_t^{t+\delta} d\tau_1 \phi_o(t+\delta,\tau_1)\beta\kappa(\tau_1)\phi_o(\tau_1,t))] + O(\delta^2)$$

Similarly, using t'.e expression (16b) for the ϕ_o's appearing above, we have:

$$\overline{\psi}(t,\tau) = E[\Lambda^H \overline{\psi}(t+\delta,\tau)\Lambda] + O(\delta^2)$$

$$\Lambda \triangleq I + \overline{\mu}\delta + W(t,t+\delta) + \tfrac{1}{2}W^2(t,t+\delta) - \int_t^{t+\delta} d\tau_1 \beta\kappa(\tau_1)$$

After expanding out, rearranging and dividing by δ:

$$- \frac{1}{\delta}[\overline{\psi}(t+\delta,\tau) - \overline{\psi}(t,\tau)]$$

$$= (\overline{\mu} - \beta\kappa(t) + \frac{1}{2\delta}E[W^2(t,t+\delta)])^H \overline{\psi}(t+\delta,\tau)$$

$$+ \overline{\psi}(t+\delta,\tau)(\overline{\mu} - \beta\kappa(t) + \frac{1}{2\delta}E[W^2(t,t+\delta)])$$

$$+ \frac{1}{\delta}E[W^H(t,t+\delta)\overline{\psi}(t+\delta,\tau)W(t,t+\delta)] + O(\delta)$$

Next use (21a) and Theorem 3 to evaluate the above averages, then pass to the limit $\delta \downarrow 0$. Recalling that $\frac{\partial}{\partial t}\overline{\psi}(t,\tau)$ exists by Theorem 1.D, we obtain:

$$- \frac{\partial}{\partial t}\overline{\psi}(t,\tau) = (\overline{\mu} - \beta\kappa(t) - \tfrac{1}{2}I)^H\ \overline{\psi}(t,\tau)$$

$$+ \overline{\psi}(t,\tau)(\overline{\mu} - \beta\kappa(t) - \tfrac{1}{2}I) + I\{\overline{\psi}(t,\tau)\} \qquad (45a)$$

with I given as in (41). Also, (18b) yields directly:

$$\overline{\psi}(\tau,t) = \sigma_1 + \kappa^H(\tau)R_2\kappa(\tau) \qquad (45b)$$

Finally, integration of all terms in (45) over $\tau \in [t,t_1]$ and use of (17) gives (40).

The linearity of this equation guarantees the uniqueness of the solution, and the positive-semidefiniteness of $\rho(t)$ noted in Theorem 1.c implies the same property for $\bar{\rho}(t)$. □

Under the maximum entropy statistical model we thus obtain a modified Lyapunov equation for $\bar{\rho}$ which must be appended to the variational problem (13) as a constraint. Clearly as the decorrelation times approach infinity, the matrix I approaches zero and (40) reduces to the familiar Lyapunov equation for a deterministic plant.

With finite decorrelation times, the qualitative structure of (40) should be noted. First, considering only the diagonal elements of (40), we have

$$- \frac{d}{dt}\{\bar{\rho}\} = \bar{\mu}^H\{\bar{\rho}\} - \{(\beta\kappa)^H\bar{\rho}\}$$

$$+ \{\bar{\rho}\}\bar{\mu} - \{\bar{\rho}\beta\kappa\} + \{\sigma_1 + \kappa^H R_2 \kappa\}$$

Thus the terms arising from frequency uncertainties do not appear explicitly and the diagonal elements of $\bar{\rho}$ depend on the decorrelation times only through their dependence upon the off-diagonal elements.

On the other hand, considering only off-diagonal elements, (40) takes the form:

$$-\dot{\bar{\rho}} = (\bar{\mu} - \beta\kappa - \tfrac{1}{2}I)^H \bar{\rho} + \bar{\rho}(\bar{\mu} - \beta\kappa - \tfrac{1}{2}I)$$

$$+ \sigma_1 + \kappa^H R_2 \kappa$$

The off-diagonal elements of $\bar{\rho}$ are directly influenced by the uncertainty terms, but only through the matrix $(\bar{\mu} - \beta\kappa - \frac{1}{2}I)$. It is easily shown that under the statistical model of Theorem 3, the mean response (averaged over the parameter ensemble), $\bar{\xi}$, is given by

$$\dot{\bar{\xi}} = (\bar{\mu} - \beta\kappa - \tfrac{1}{2}I)\bar{\xi}$$

Consequently $(\bar{\mu} - \beta\kappa - \frac{1}{2}I)$ is the dynamic matrix of the "mean system" and the term $-\frac{1}{2}I$ represents the effect of decorrelation damping.

Thus, while the diagonal elements of $\bar{\rho}$ are not directly affected by parameter uncertainties, the off-diagonal elements are subject to the decorrelation damping of the mean system. These features lead to the same suppression of off-diagonal elements of $\bar{\rho}$ (and $\bar{Q}$) as noted previously for the uncontrolled system.

The problem outlined in (13) and (14) is now reduced to one in the calculus of variations. With (13) and (40) we may proceed directly to obtain:

<u>Theorem 5</u>

Under conditions (40), the performance index:

$$\bar{J} = \int_{t_o}^{t_1} dt\ \mathrm{tr}[\bar{\rho}v] \tag{46}$$

is minimized for all $v \geq 0$ if and only if:

$$\kappa = R_2^{-1}\beta^H\bar{\rho} \tag{47}$$

where $\bar{\rho}$ is the positive semi-definite, hermitian solution of

$$
\left.\begin{aligned}
-\dot{\bar{\rho}} &= (\bar{\mu}-\tfrac{1}{2}I)^H\bar{\rho} + \bar{\rho}(\bar{\mu}-\tfrac{1}{2}I) + I\{\bar{\rho}\} \\
&\quad + \sigma_1 - \bar{\rho}\sigma_2\bar{\rho} \\
\bar{\rho}(t_1) &= 0
\end{aligned}\right\} \tag{48}
$$

where

$$
\sigma_2 \triangleq \beta R_2^{-1}\beta^H \tag{49}
$$

<u>Proof</u>

The necessary stationary conditions may be derived by introducing a multiplier matrix, Q, and requiring that the first variation of

$$
\begin{aligned}
H = \int_{t_o}^{t_1} dt\ \mathrm{tr}[\bar{\rho}v + Q^H[\dot{\bar{\rho}} + (\bar{\mu}-\beta\kappa-\tfrac{1}{2}I)^H\bar{\rho} \\
+ \bar{\rho}(\bar{\mu}-\beta\kappa-\tfrac{1}{2}I) + I\{\bar{\rho}\} + \sigma_1 + \kappa^H R_2\kappa]]
\end{aligned} \tag{50}
$$

with respect to independent variations in $\bar{\rho}$ and κ vanish, imposing also the terminal condition (40b). Vanishing of the first variation with respect to $\bar{\rho}$ gives:

$$
\left.\begin{aligned}
\dot{Q} &= (\bar{\mu}-\beta\kappa-\tfrac{1}{2}I)Q + Q(\bar{\mu}-\beta\kappa-\tfrac{1}{2}I)^H + I\{Q\} + v \\
Q(t_o) &= 0
\end{aligned}\right\} \tag{51}
$$

which may be recognized as the equation determining the second moment matrix. Equating to zero the first variation of (50) with respect to κ yields:

$$
(R_2\kappa - \beta^H\bar{\rho})Q = 0
$$

For this to hold for all v, (47) must follow. Finally, substitution of (47) into (40) produces (48).

To prove sufficiency of (47) and (48), first define:

$$\kappa_o \triangleq R_2^{-1}\beta^H\bar{\rho}_o$$

where $\bar{\rho}_o$ satisfies (48). In general, we may set

$$\kappa = \kappa_o + \tilde{\kappa}$$
$$\bar{\rho} = \bar{\rho}_o + z$$

With this substitution, (40) becomes:

$$\begin{aligned} -\dot{z} &= (\bar{\mu}-\tfrac{1}{2}I-\beta\kappa)^H z + z(\bar{\mu}-\tfrac{1}{2}I-\beta\kappa) + I\{z\} + \tilde{\kappa}^H R_2\tilde{\kappa} \\ z(t_1) &= 0 \end{aligned} \tag{52}$$

Next, define q(t) by:

$$\begin{aligned} \dot{q} &= (\bar{\mu}-\tfrac{1}{2}I-\beta\kappa)q + q(\bar{\mu}-\tfrac{1}{2}I-\beta\kappa) + I\{q\} \\ q(t_o) &= q_o > 0 \end{aligned} \tag{53}$$

where, by construction, $q > 0$ on $t \in [t_o,t_1]$. Then, with (52) and (53):

$$\frac{d}{dt}\,\mathrm{tr}[zq] = -\mathrm{tr}[\tilde{\kappa}^H R_2\tilde{\kappa}q] \leq 0 \;;\; \forall\; t \in [t_o,t_1],\; \tilde{\kappa}\neq 0 \tag{54}$$

where the inequality is implied by $q > 0$, $R_2 > 0$. From the terminal condition on z:

$$(\mathrm{tr}[zq])_{t=t_1} = 0$$

Combined with (54), this implies:

$$\mathrm{tr}[zq] \geq 0 \ ; \ t \ \varepsilon [t_o, t_1]$$

and since $q > 0$:

$$\rho - \rho_o \geq 0 \ ; \ \forall \ t \ \varepsilon [t_o, t_1], \ \tilde{\kappa} \neq 0$$

Therefore, since $v \geq 0$:

$$\bar{J}(\kappa) \geq \bar{J}(\kappa_o)$$

which shows that $\bar{J}$ is minimized with (47) and (48). □

So far the requirement that K be real has not been mentioned. But this condition is inconsequential in view of the following result:

Corollary 2

Let $\bar{\rho}$ be any hermitian solution of the terminal value problem (48). Then the matrix, $\bar{P}$, of $\bar{\rho}$ expressed in the modal coordinate basis:

$$\bar{P} \triangleq \Phi^{-1H} \bar{\rho} \Phi^{-1} \tag{55}$$

is real and, consequently,

$$K = R_2^{-1} B^T \bar{P} \tag{56}$$

is real.

The elementary but laborious proof is given in Appendix 3.

As a consequence, subsequent theoretical developments may be carried out within the eigen-basis of A, employing the complex equation (48). In numerical work, however, it is of obvious advantage to present (48) in the original modal coordinate basis. Use of (55), (48) and previous definitions gives:

$$\left.\begin{aligned} -\dot{\overline{P}} &= (\overline{A}-\tfrac{1}{2}I)^T\overline{P} + \overline{P}(\overline{A}-\tfrac{1}{2}I) + D[I,\overline{P}] \\ &\quad + R_1 - \overline{P}\; B\; R_2^{-1}B^T\overline{P} \\ \overline{P}(t_1) &= 0 \end{aligned}\right\} \tag{57}$$

where, for real $\overline{P}$:

$$D[I,\overline{P}] = \underset{k=1,\ldots,n}{\text{block-diag}}\left[\frac{I_{2k}}{2\overline{\omega}_k}(\overline{P}_{2k-1,2k-1}+\overline{\omega}_k^{-2}\overline{P}_{2k,2k})\begin{bmatrix}\overline{\omega}_k^{-2} & 0\\ 0 & 1\end{bmatrix}\right] \tag{58}$$

This expression for $D[I,\overline{P}]$ serves to illustrate the relative simplicity of the complex form given by (48).

2.4 Recapitulation of the Proposed Approach

In addressing the problem of optimal control of uncertain structural systems in the preceding sections, we have developed an approach which is an explicit expression of a general design philosophy which differs significantly from the traditional view. At this point it is well to contrast the proposed formulation with conventional application of modern control theory and outline the ramifications of the new design strategy.

Such a comparison is depicted in general terms by Fig. 1. Clearly, the traditional approach has a rich and diverse background, yet we must paint it with broad strokes in order to reveal its essential premises.

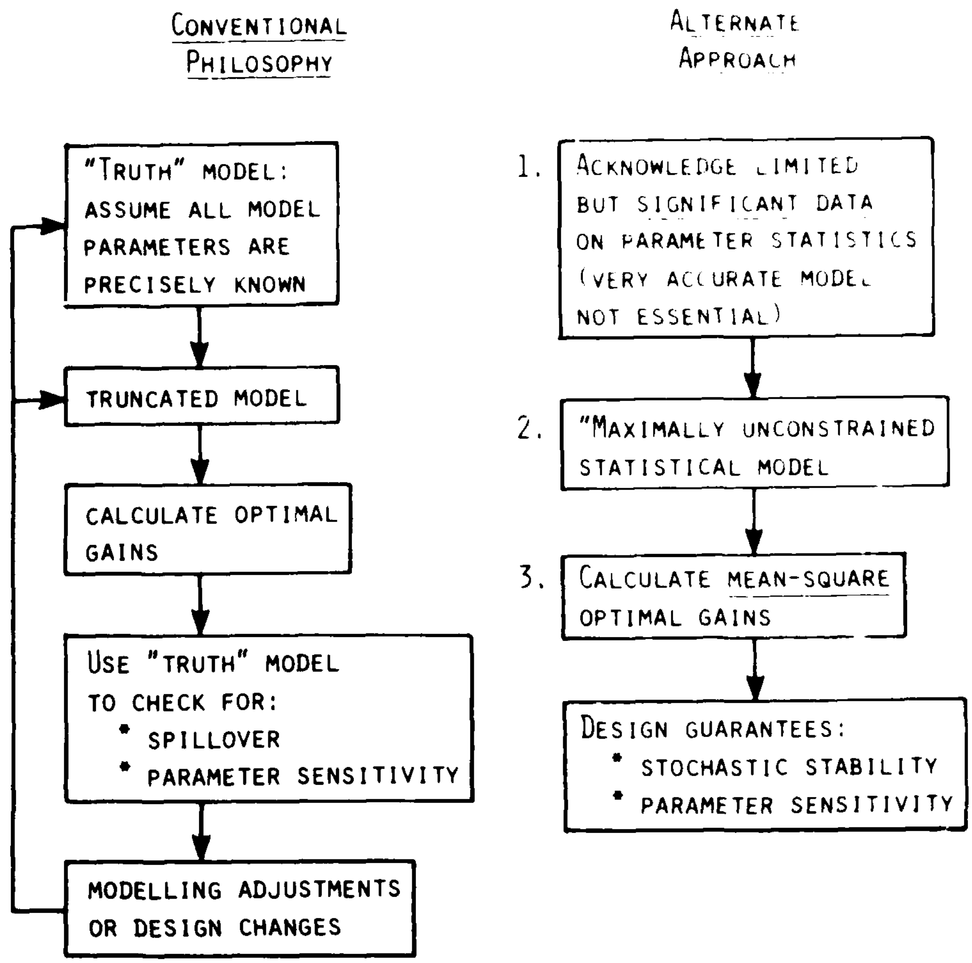

Fig. 1. Stochastic optimal control under limited system information.

Perhaps the defining feature of the conventional design philosophy is its assumption, at the outset, of a very high order and supposedly accurate structural model. This "truth model" gives a highly detailed picture of system dynamics, but, in view of the inevitable parameter uncertainties and modeling errors, most of these details are spurious. Actually what is needed is a description of system dynamics at a level of detail commensurate with the information available on system parameters.

Nevertheless, the conventional approach proceeds by assuming the "truth model" to be an accurate description of reality. Next, the obstacle interposed by the large dimension of the truth model must be circumvented. This necessarily entails model order reduction by truncation of modal coordinates. Although modal truncation may be carried out by a variety of techniques, all methods require, as an initial step, truncation of the full-order truth model based on open-loop properties.

Once an approximating model has been obtained of sufficiently low dimension, the control law may be computed - either by standard LQ theory or by a suitable variant. At this point, the conventional approach obtains a controller design but no general assurance of elementary system properties - such as stability. Inevitably, there must ensue an often cumbersome checking procedure wherein the full-order truth model is employed in a detailed design evaluation. This results in adjustments in the model or the truncation procedure - and often requires iteration of design parameters merely to ensure stability of the nominal system.

The alternate approach proposed here is markedly different. At the outset it is recognized that a precisely accurate description of the structure never exists and that system parameters are always subject to some level of uncertainty. Moreover we can never possess a complete probablistic description of system parameters but must accept highly restricted statistical data.

Thus, as indicated on the right of Fig. 1, the alternate approach begins with a severely limited set of parameter data. In practice, we are limited to the nominal or mean values of system parameters and to statistical measures of the variation about mean values. There is some latitude of choice with regard to measures of parameter variation and it is expeditious to acknowledge as available a data set which is also significant to the overall fidelity of the model.

Having identified a significant and limited data set which we choose to acknowledge as available, a complete probability model is still required for computation of the control law. Consistency requires that we induce this complete probability model uniquely from the acknowledged data. To do this we follow the doctrine that the complete statistical assignment be consistent with acknowledged data and maximally unconstrained otherwise, i.e., we appeal to a maximum entropy principle. This introduces an element of design conservatism since the statistical "degrees of freedom" of the maximum entropy model are far greater in number than under the actual parameter statistics. Thus stability properties found under the maximum entropy model are very likely to hold over the actual parameter ensemble.

At this stage (with specification of items 1 and 2 in Fig. 1) we possess the "system model" employed by our alternate approach. This model is necessarily of high dimension, as is the truth model, but implicitly accounts for the substantial uncertainty in high order modal parameters while requiring relatively little elementary data as input.

With the complete probability model, we must compute the control law based upon a performance measure defined on the entire parameter ensemble. The simplest choice appears to be the average of a quadratic criterion over the parameter ensemble since it provides us with a straightforward mechanism whereby almost sure stability may be guaranteed.

Under the restrictions of full-state feedback and uncertainties only in the open-loop frequencies, the preceding sections have given specific form to items 1 through 3 indicated by Fig. 1. To recapitulate, we first attempted to identify a data set which was significant to modelling fidelity by examining the phenomenology of frequency uncertainties as reflected in the mean response, the covariance and the expected cost. As a result, we identified the modal decorrelation times as the parameter data set that must be acknowledged as available if the open-loop system second moment response is to be adequately modelled. Indeed, various qualitative features such as (1) the location of the "coherence limit," (2) the behavior of the portions of the expected cost matrix and second moment matrix pertaining to modes in the "incoherent range," and (3) the gross effect of "decorrelation damping" depend explicitly upon the decorrelation times. Choosing to acknowledge only these quantities as available data, we constructed a full probability model for frequency uncertainties which is otherwise maximally unconstrained. The resulting statistical model is an equivalent white noise parameter model which permits formulation of closed equations for the expected cost matrix and explicit solution of the linear regulator problem.

One clear advantage to the present formulation is the relatively small number of parameter statistical measures that must be provided and their direct relationship to system characteristics. Secondly, the white parameter model yields a non-dual problem. Thus, while accepting frequency uncertainties at their *a priori* levels, the approach is consistent with the more general context of stochastic optimal control.

In addition, certain features arising from the specific form of (48) are essential to the overall rationale and will be elucidated in the sequel. To anticipate these results, it will be shown that under weak restrictions, (48) possesses a unique positive semi-definite solution in the steady-state case for all values of the decorrelation times. This precludes the existence of an uncertainty threshold. Also, with the gain (47), second moment stability is guaranteed for the closed-loop system. Thus the need for design iteration to ensure robustness with respect to stability is largely eliminated within the present approach.

Finally the form of the modified Riccati equation has important implications for the dimensionality issue. We shall find that by confronting parameter uncertainties directly, we shall also greatly reduce the need for modal truncation with its attendant spillover problems.

3. LINEAR REGULATION UNDER MODAL FREQUENCY UNCERTAINTIES: RAMIFICATIONS OF THE STOCHASTIC DESIGN APPROACH

3.1 Introduction

In the remainder of this report various ramifications of the design approach outlined previously are investigated. Specifically, we resume consideration of linear full state feedback regulation in the presence of open-loop frequency uncertainties and develop the properties of solutions to the terminal value problem (48).

First, we recapitulate the formulation and specify the restrictions under which the work will proceed. It must be noted that previous results retain validity if $\omega_k = 0$ in (2) for some k, i.e., if rigid body modes are explicitly recognized. In the following the specification:

$$\bar{\mu} = \begin{bmatrix} \bar{\mu}_r & 0 \\ 0 & \bar{\mu}_e \end{bmatrix} \varepsilon\ C^{2nx2n} \tag{59}$$

$$n \triangleq n_r + n_e$$

will be assumed, where $\bar{\mu}_r$ is the dynamic matrix of n_r rigid body degrees of freedom:

$$\bar{\mu}_r \triangleq \text{block-diag} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \varepsilon\ R^{2n_r x 2n_r} \tag{60}$$

and $\bar{\mu}_e$ encompasses the elastic odes:

$$\left.\begin{aligned}
&\bar{\mu}_e \triangleq \mathrm{diag}\{\bar{\omega}_1(i-\eta_1),\ \bar{\omega}_1(-i-\eta_1),\\
&\qquad \ldots \bar{\omega}_{n_e}(i-\eta_{n_e}),\ \bar{\omega}_{n_e}(-i-\eta_{n_e})\}\\
&0 < \bar{\omega}_1 < \bar{\omega}_2 < \ldots < \bar{\omega}_{n_e}
\end{aligned}\right\} \tag{61}$$

although the results are readily generalized, we assume for simplicity that all the $\bar{\omega}_k$ are distinct. Further, non-zero structural damping will be assumed for all elastic modes:

$$\eta_k > 0 \ ;\ k=1,\ldots,n_e \tag{62}$$

Consistently with (59), the matrix of inverse decorrelation times now assumes the form:

$$\left.\begin{aligned}
&I = \begin{bmatrix} 0 & 0 \\ 0 & I_e \end{bmatrix}\\
&I_e \triangleq \mathrm{diag}[\bar{\omega}_1 I_2, \bar{\omega}_1 I_2, \ldots, \bar{\omega}_{n_e} I_{2n_e}, \bar{\omega}_{n_e} I_{2n_e}]
\end{aligned}\right\} \tag{63}$$

with the I_k as defined in (38).

Under conditions (59) through (63), we consider the terminal value problem arising from Theorem 5:

$$\left.\begin{aligned} -\dot{\bar{\rho}} &= \bar{\mu}_m^H\bar{\rho} + \bar{\rho}\,\bar{\mu}_m + I\{\bar{\rho}\} + \sigma_1 - \bar{\rho}\sigma_2\bar{\rho} \\ \bar{\rho}(t_1) &= 0 \end{aligned}\right\} \tag{64}$$

where $\left.\begin{aligned} \bar{\mu}_m &\stackrel{\Delta}{=} \bar{\mu} - \tfrac{1}{2}I \\ \sigma_2 &\stackrel{\Delta}{=} \beta R_2^{-1}\beta^H \end{aligned}\right\}$ (65)

$$\left.\begin{aligned} \sigma_1 &\geq 0 \\ R_2 &> 0 \end{aligned}\right\} \tag{66}$$

$$\left.\begin{aligned} \beta &\stackrel{\Delta}{=} \begin{bmatrix} b_r \\ \beta_e \end{bmatrix} \\ b_r &= \begin{bmatrix} 0 & \cdots & 0 \\ b_{r11} & & b_{r1\ell} \\ \vdots & & \\ b_{rn_r1} & \cdots & b_{rn_r\ell} \end{bmatrix} \end{aligned}\right\} \tag{67}$$

and where $\sigma_1 \in C^{2nx2n}$ is as defined by (10e) and (6b) and β_e is as defined by (10b), (8) and (3).

With the control gain:

$$\kappa = R_2^{-1}\beta^H\bar{\rho} \tag{68}$$

the second moment matrix of the closed loop system is determined by:

$$\left.\begin{aligned} \dot{Q} &= (\bar{\alpha} - \tfrac{1}{2}I)Q + Q(\bar{\alpha} - \tfrac{1}{2}I)^H + I\{Q\} + v \\ Q(t_o) &= Q_o \geq 0\ ,\ v \geq 0 \\ &\text{where} \\ \bar{\alpha} &\stackrel{\Delta}{=} \bar{\mu} - \beta\kappa \end{aligned}\right\} \tag{69}$$

as (51) in the proof of Theorem 5 shows. Here, we assume that the initial state is non-zero and statistically independent of disturbance and parameter noise.

The next section reviews various concepts and results concerned with stochastic stability. Sufficient conditions for the existence and uniqueness of steady state solutions to (64) are established in Sections 3.3 through 3.5. Much of this material parallels corresponding developments in Refs. (40) and (41), except that several restrictions previously stated can be removed for the problem considered here. In addition, Section 3.5 achieves assurance of various stochastic measures of stability. For the steady state case, Section 3.6 gives certain well known numerical procedures. Finally, the section concludes with consideration of asymptotic properties for high uncertainty levels and high order modes in Sections 3.7 through 3.9.

3.2 Stochastic Stability and Extended Lyapunov Equations

Preliminary results concerning the stability properties of the closed-loop stochastic system (1) must be given. Any boundedness or convergence property used in deterministic system theory can be translated for stochastic systems in different ways, depending on the type of stochastic convergence one wishes to consider.

The convergence of sample solutions of a stochastic dynamic system of the form (1) to the null solution are most frequently characterized by the following stability definitions:[42]

1. Almost sure exponential stability: The sample solutions converge exponentially to zero as t tends to infinity with probability one.

2. p^{th} moment exponential stability: With p a positive integer and $p_1 \ldots p_{2n}$ any set of non-negative integers such that

$$\sum_{k=1}^{2n} p_k = p$$

and for all initial states, the p^{th} moments:

$$E[x_1^{p_1}(t)\, x_2^{p_2}(t) \ldots x_{2n}(t)^{p_{2n}}]$$

converge exponentially to zero.

3. p^{th} mean exponential stability: For $p > 0$ and all initial states,

$$E[||x(t)||^p]$$

converges exponentially, where $||\ldots||$ denotes some vector norm.

Clearly, p^{th} moment exponential stability is useful as a convergence property only if p is an even integer and, with p even, definitions 2 and 3 are actually equivalent. Moreover, for $p > \tilde{p} > 0$, p^{th} mean exponential stability implies $\tilde{p}^{th}$ mean exponential stability and almost sure stability.

Since the system performance is characterized by a quadratic functional, it suffices for our purpose to consider only second moment (or equivalently second mean) exponential stability. Obviously, for second moment exponential stability it is necessary and sufficient that

$$\lim_{t-t_o \uparrow \infty} Q(t) = 0 \tag{70}$$

for all $Q_o \geq 0$ and $v = 0$ in (69). If the system is second moment stable then $Q(t)$ with $v \geq 0$ has a unique equilibrium value

$$X = \lim_{t-t_o \uparrow \infty} Q(t) \tag{71}$$

which from (69) is the positive semi-definite solution of

$$\left.\begin{aligned} &0 = X(\bar{\alpha} - \tfrac{1}{2}I)^H + (\bar{\alpha} - \tfrac{1}{2}I)X + I\{X\} + v \\ &v \geq 0 \end{aligned}\right\} \tag{72}$$

Particularly germane to the existence of steady-state solutions of the stochastic Riccati equation (64) is the character of solutions to the adjoint of (72):

$$\left.\begin{aligned} &0 = \Lambda(\bar{\alpha} - \tfrac{1}{2}I) + (\bar{\alpha} - \tfrac{1}{2}I)^H\Lambda + I\{\Lambda\} + S \\ &S \geq 0 \end{aligned}\right\} \tag{73}$$

Conditions previously given[43] for the existence and uniqueness of positive semi-definite solutions to stochastic Lyapunov equations of the form (72) or (73) may be stated in terms of <u>equivalent coefficient matrices</u> defined as follows: Since Q is hermitian we may set up the vector

$$\hat{Q}^H \triangleq [Q_{11}, \sqrt{2}\, Q_{12}, Q_{22}, \sqrt{2}\, Q_{13}, \ldots] \tag{74}$$

so that only the upper triangular portion of Q is used and $\hat{Q} \in C^{n(2n+1)}$. Vector $\hat{v}$ is similarly defined on v in (69). Then (69) may be written:

$$\dot{\hat{Q}} = \Delta\hat{Q} + \hat{V} \tag{75}$$

where Δ results from representation (74) and is termed the "equivalent coefficient matrix of Q in $[Q(\bar{\alpha}-\tfrac{1}{2}I)^H+(\bar{\alpha}-\tfrac{1}{2}I)Q+I\{Q\}]$." To display its definition explicitly, we denote Δ in (75) by

$$\Delta_Q[Q(\bar{\alpha} - \tfrac{1}{2}I)^H + (\bar{\alpha} - \tfrac{1}{2}I)Q + I\{Q\}]$$

Proceeding similarly for Λ, Equation (73) may be written:

$$0 = \Delta_\Lambda[\Lambda(\bar{\alpha} - \tfrac{1}{2}I) + (\bar{\alpha} - \tfrac{1}{2}I)^H\Lambda + I\{\Lambda\}]\hat{\Lambda} + \hat{S}$$

Clearly, from the above definitions:

$$\Delta_\Lambda = \Delta_Q^H \tag{76}$$

and a necessary and sufficient condition for second moment exponential stability is that Δ_Q (or equivalently Δ_Λ) be asymptotically stable.

The relations between the stability of Δ_Λ and the character of solutions of (72) and (73) may be illustrated by the following two results. First we have:

Theorem 6

Consider (73) with $S \geq 0$ ($S \leq 0$). If

$$\Delta_\Lambda[\Lambda(\bar{\alpha} - \tfrac{1}{2}I) + (\bar{\alpha} - \tfrac{1}{2}I)^H\Lambda + I\{\Lambda\}]$$

is asymptotically stable then there exists a unique positive semi-definite (negative semi-definite, resp.) solution to (73).

Proof

Consider the case $S \geq 0$ first. For any $\xi_o \in C^{2n}$, define $Q(t)$ by (69) with $v = 0$, $t_o = 0$, and

$$Q_o \triangleq \xi_o \xi_o^H$$

so that $\hat{Q}$ is determined by

$$\left.\begin{aligned} \dot{\hat{Q}} &= \Delta_Q \hat{Q} \\ \hat{Q}(0) &= \hat{Q}_o \end{aligned}\right\} \quad (77)$$

If Δ_Λ is asymptotically stable, then so is Δ_Q by virtue of (76). Consequently (77) possesses the unique solution:

$$\hat{Q}(t) = e^{\Delta_Q t}\hat{Q}_o$$

with $Q(t)$ positive semi-definite on $[0, \infty)$.

Similarly, with Δ_Λ asymptotically stable,

$$0 = \Delta_{\hat{\Lambda}} \hat{\Lambda} + \hat{S}$$

has the unique solution:

$$\hat{\Lambda} = -\Delta_\Lambda^{-1} \hat{S} = \int_0^\infty dt\ e^{\Delta_Q^H t} \hat{S}$$

With these results we have:

$$\xi_o^H \Lambda \xi_o = tr\{Q_o \Lambda\} = \hat{Q}_o^H \hat{\Lambda} = \int_0^\infty dt\ \hat{Q}^H(t) \hat{S}$$

Furthermore, the inequalities:

$$\hat{Q}^H(t)\hat{S} = tr\{Q(t)S\} \geq \lambda_{MIN}\{S\} ||Q(t)||_2 \geq 0 \tag{78}$$

hold for $t \in [0, \infty)$, where $\lambda\{M\}$ denotes an eigenvalue of matrix M and $||\ldots||_2$ denotes the spectral norm. Hence:

$$\xi_o^H \Lambda \xi_o \geq 0 , \forall \xi_o \neq 0$$

and the assertion for $S \geq 0$ is shown.

For $S \le 0$, we may replace (78) by

$$\hat{Q}^H(t)\hat{S} = tr\{Q(t)S\} \le 2n\, \lambda_{MAX}\{S\}||Q(t)||_2 \le 0$$

so that:

$$\xi_o^H \Lambda \xi_o \le 0 \; , \; \forall \xi_o \ne 0$$

which completes the proof. □

To a considerable extent, the converse of the above may be shown:

<u>Theorem 7</u>

Suppose that $(\bar{\alpha}-\frac{1}{2}I)$ is asymptotically stable, $S \ge 0$ and $(\bar{\alpha}-\frac{1}{2}I, S^{\frac{1}{2}})$ is completely reconstructible. If (73) possesses a positive definite solution, then

$$\Delta_\Lambda[\Lambda(\bar{\alpha}-\tfrac{1}{2}I) + (\bar{\alpha}-\tfrac{1}{2}I)^H\Lambda + I\{\Lambda\}]$$

is asymptotically stable.

<u>Proof</u>

By Lemma 3.1 of Reference (44), complete reconstructibility of $(\bar{\alpha}-\frac{1}{2}I, S^{\frac{1}{2}})$ implies that

$$\int_o^\infty dt\, e^{(\bar{\alpha}-\frac{1}{2}I)^H t}\, S\, e^{(\bar{\alpha}-\frac{1}{2}I)t}$$

is positive definite. With this condition and the remaining assumed conditions, the stated conclusion follows by application of Theorem 5-3 of Ref. (41).

3.3 The Time-Dependent Equation - Convergence to the Steady-State

Here we investigate certain properties of (64) with a view to establishing conditions under which the solution approaches a steady state value as t_1 tends to infinity.

The first preliminary result allows the problem to be reduced to a consideration of the completely reconstructible portion of the system.

Lemma 1

A. Under the conditions specified in Section 3.1, suppose $(\bar{\mu}_m, \sigma_1^{\frac{1}{2}})$ is detectable. Then the state vector may be rearranged to assume the form:

$$\xi = \begin{bmatrix} \hat{\xi} \\ \xi_u \end{bmatrix} \tag{79}$$

so that $\hat{\xi}$ contains the rigid body modes, ξ_u comprises the unreconstructible subspace of $\bar{\mu}_m$, and (partitioning all matrices accordingly):

$$\left.\begin{aligned} \bar{\mu}_m &= \begin{bmatrix} \hat{\bar{\mu}}_m & 0 \\ 0 & \bar{\mu}_{mu} \end{bmatrix}, \quad \hat{\bar{\mu}}_m \equiv \hat{\bar{\mu}} - \tfrac{1}{2}\hat{I} \\ \sigma_1 &= \begin{bmatrix} \hat{\sigma}_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{80}$$

where $(\hat{\bar{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ is completely reconstructible and $\hat{I}$ is a diagonal matrix comprising those entries of I corresponding to the reconstructible portion of the system.

B. With all matrices partitioned in a manner consistent with (79), and with conditions (65) and (66), the solution of (64) exists, is positive semi-definite on $(-\infty, t_1]$ and assumes the form:

$$\bar{\rho} = \begin{bmatrix} \hat{\bar{\rho}} & 0 \\ 0 & 0 \end{bmatrix} \tag{81}$$

where $\hat{\bar{\rho}}$ is the solution to

$$\left.\begin{aligned} -\frac{d}{dt}\hat{\bar{\rho}} &= \hat{\bar{\mu}}_m^H \hat{\bar{\rho}} + \hat{\bar{\rho}}\hat{\bar{\mu}}_m + \hat{I}\{\hat{\bar{\rho}}\} + \hat{\sigma}_1 - \hat{\bar{\rho}}\hat{\sigma}_2\hat{\bar{\rho}} \\ \hat{\bar{\rho}}(t_1) &= 0 \end{aligned}\right\} \tag{82}$$

where:

$$\left.\begin{aligned} \beta &\triangleq \begin{pmatrix} \hat{\beta} \\ \beta_u \end{pmatrix} \\ \hat{\sigma}_2 &\triangleq \hat{\beta} R_2^{-1} \hat{\beta}^H \end{aligned}\right\} \tag{83}$$

The proof is contained in Appendix 4.

Now, upon consideration of (82), Theorem 6-5 of Reference (41) yields:

<u>Lemma 2</u>

Consider (82) and (83) with $(\hat{\bar{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ completely reconstructible. In addition to the conditions of Lemma 1, suppose that:

$$\left.\begin{aligned}
0 &= \hat{\bar{\mu}}_m^H \Lambda + \Lambda \hat{\bar{\mu}}_m + \hat{I}\{\Lambda\} + \hat{\sigma}_1 - \Lambda \hat{\sigma}_2 \Lambda \\
\hat{\sigma}_2 &\triangleq \hat{C} R_2^{-1} \hat{\beta}^H \\
R_2 &> 0 \ , \ \hat{\sigma}_1 \geq 0
\end{aligned}\right\} \qquad (84)$$

has a unique positive semi-definite solution, Λ_∞, and that

$$\Delta_Q[(\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty)Q + Q(\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty)^H + \hat{I}\{Q\}]$$

is asymptotically stable. Then $\hat{\bar{\rho}}(t) \geq 0$ for all $t \leq t_1$ and:

$$\lim_{t_1 \uparrow \infty} \hat{\bar{\rho}}(t) = \Lambda_\infty \qquad (85)$$

Further, with the reduction afforded by Lemma 1, conditions for second moment stability can be simplified as follows:

Theorem 8

Under the assumptions of Lemma 2, and denoting the steady-state solution of (64) by $\bar{\rho}$; then both $\bar{\mu}_m - \sigma_2 \bar{\rho}$ and:

$$\Delta_Q[(\bar{\mu}_m - \sigma_2 \bar{\rho})Q + Q(\bar{\mu}_m - \sigma_2 \bar{\rho})^H + I\{Q\}]$$

are asymptotically stable.

The lengthy but elementary proof is relegated to Appendix 5.

From the above result and Lemma 2, the situation with respect to convergence of $\overline{\sigma}$ to a steady state solution and overall second moment stability under the resulting steady state control law is seen to hinge on the existence and uniqueness of a positive semi-definite solution of (84) for which $\Delta_Q[(\hat{\overline{\mu}}_m-\hat{\sigma}_2\Lambda)Q+Q(\hat{\overline{\mu}}_m-\hat{\sigma}_2\Lambda)^H+\hat{I}\{Q\}]$ is asymptotically stable. This matter is addressed in the next section.

3.4 The Time-Independent Equation: Existence and Uniqueness of Solutions

Here we determine sufficient conditions for the validity of the assumptions of Lemma 2. Specifically, we investigate the conditions under which

$$\left.\begin{aligned} 0 &= \hat{\overline{\mu}}_m^H\Lambda + \Lambda\hat{\overline{\mu}}_m + \hat{I}\{\Lambda\} + \hat{\sigma}_1 - \Lambda\hat{\sigma}_2\Lambda \\ \hat{\sigma}_2 &\triangleq \hat{\beta}R_2^{-1}\hat{\beta}^H \ , \quad \hat{\overline{\mu}}_m \triangleq \hat{\overline{\mu}} - \tfrac{1}{2}\hat{I} \\ R_2 &> 0 \ , \quad \hat{\sigma}_1 \geq 0 \end{aligned}\right\} \qquad (86)$$

possesses a unique positive definite solution for which $\Delta_Q[(\hat{\overline{\mu}}_m-\hat{\sigma}_2\Lambda)Q+Q(\hat{\overline{\mu}}_m-\hat{\sigma}_2\Lambda)^H+\hat{I}\{Q\}]$ is asymptotically stable.

Before proceeding, however, it is necessary to state various preliminary results. The first of these concerns an elementary property of positive semi-definite matrices[(45)].

<u>Lemma 3</u>

Every sequence, $\{X_i\}$, of hermitian positive semi-definite matrices bounded below (above, resp.) with $(X_{i+1}-X_i)$ negative semi-definite (positive semi-definite, resp.) for each i converges to a positive semi-definite limit.

The second result concerns the preservation of detectability (Theorem 3.6(ii) of Reference (44)):

Lemma 4

If $M \geq 0$ and $(A, M^{\frac{1}{2}})$ is detectable (reconstructible) then for all $Q \geq 0$, $N > 0$ and any B and F the pair $(A + BF, (M + Q + F^H NF)^{\frac{1}{2}})$ is detectable (reconstructible).

Further, it is well to recall a basic result for quadratic optimization of deterministically parametered systems (Theorem 12.2 of Reference (44)):

Lemma 5

Under previous definitions of $\hat{\bar{\mu}}_m$ and $\hat{\beta}$, consider

$$\left.\begin{aligned} 0 &= \hat{\bar{\mu}}_m^H \Lambda + \Lambda \hat{\bar{\mu}}_m + S - \Lambda \hat{\sigma}_2 \Lambda \\ \hat{\sigma}_2 &\triangleq \hat{\beta} R_2^{-1} \hat{\beta}^H \end{aligned}\right\} \qquad (87)$$

with

$S \geq 0$, $R_2 > 0$

$(\hat{\bar{\mu}}_m, \hat{\beta})$ stabilizable

$(\hat{\bar{\mu}}_m, S^{\frac{1}{2}})$ completely reconstructible

Then (87) has a unique positive definite solution, Λ_∞, and $(\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty)$ is asymptotically stable.

The final preliminary result illustrates the importance of stabilizability and is indispensable to the main Theorem of this section:

Lemma 6

With $\hat{\bar{\mu}}_m$, $\hat{\beta}$, $\hat{\mathcal{I}}$ and $\hat{\sigma}_2$ as previously defined and $(\hat{\bar{\mu}}, \hat{\beta})$ stabilizable, there exists a $Y > 0$ such that

$$\Delta_\Lambda[\Lambda(\hat{\bar{\mu}} - \tfrac{1}{2}\hat{\mathcal{I}} - \hat{\sigma}_2 Y) + (\hat{\bar{\mu}} - \tfrac{1}{2}\hat{\mathcal{I}} - \hat{\sigma}_2 Y)^H \Lambda + \hat{\mathcal{I}}\{\Lambda\}]$$

is asymptotically stable. If, in addition, the system possesses no rigid body modes, Y may be chosen as any positive diagonal matrix.

The proof is given in Appendix 6. With this lemma we are in a position to demonstrate the main result.

Theorem 9

Assuming $(\bar{\mu}_m, \sigma_1^{\frac{1}{2}})$ detectable, adopt decomposition (79) and define $\hat{\sigma}_1 \geq 0$, $\hat{\sigma}_2$, $\hat{\bar{\mu}}_m$ and $\hat{\mathcal{I}}$ as in Lemma 1 so that, in particular, $(\hat{\bar{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ is completely reconstructible. Then, if $(\hat{\bar{\mu}}, \hat{\beta})$ is stabilizable:

$$0 = \hat{\bar{\mu}}_m^H \Lambda + \Lambda \hat{\bar{\mu}}_m + \hat{\mathcal{I}}\{\Lambda\} + \hat{\sigma}_1 - \Lambda \hat{\sigma}_2 \Lambda \tag{88}$$

has one and only one positive definite solution, Λ_∞, and:

$$\Delta_Q[(\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty)Q + Q(\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty)^H + \hat{\mathcal{I}}\{Q\}]$$

and $\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_\infty$ are asymptotically stable.

Proof

To show existence of a positive definite solution we define an infinite sequence $\{\Lambda_i\}$ by:

$$\left.\begin{array}{ll} 0 = T_i + \hat{\sigma}_1 + \Lambda_i \hat{\bar{\mu}}_m + \hat{\bar{\mu}}_m^H \Lambda_i + \hat{I}\{\Lambda_i\} - \Lambda_i \hat{\sigma}_2 \Lambda_i & \text{a.} \\ \alpha_i = \hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_i & \text{b.} \\ \Sigma_i \alpha_i + \alpha_i^H \Sigma_i + \hat{I}\{\Sigma_i\} = T_i & \text{c.} \\ \Lambda_{i+1} = \Lambda_i + \Sigma_i & \text{d.} \end{array}\right\} \quad (89)$$

for each i. First, assume there exists a positive definite Λ_i for which T_i is positive semi-definite. Then, rearranging (89a), we have that Λ_i exists as a positive definite solution of

$$0 = \hat{\bar{\mu}}_m^H \Lambda_i + \Lambda_i \hat{\bar{\mu}}_m + s - \Lambda_i \hat{\sigma}_2 \Lambda_i \qquad \text{a.}$$

where

$$\left.\begin{array}{ll} s \triangleq T_i + \hat{\sigma}_1 + \hat{I}\{\Lambda_i\} \geq 0 & \text{b.} \\ (\hat{\bar{\mu}}_m,\ s^{1/2}) \text{ completely reconstructible} & \text{c.} \\ (\hat{\bar{\mu}}_m,\ \hat{\beta}) \text{ stabilizable} & \text{d.} \end{array}\right\} \quad (90)$$

Note that $\lambda_R\{\hat{\bar{\mu}}_m\} \leq \lambda_R\{\hat{\bar{\mu}}\}$ so that the unstable subspace of $\hat{\bar{\mu}}$ contains that of $\hat{\bar{\mu}}_m$ and (90d) is implied by stabilizability of $(\hat{\bar{\mu}},\ \hat{\beta})$. Moreover (90c) is implied by Lemma 4 (setting $Q = T_i + \hat{I}\{\Lambda_i\}$). Conditions (90b,c,d) conform to the assumptions of Lemma 5; whence, $(\bar{\mu}_m - \hat{\sigma}_2 \Lambda_i)$ is asymptotically stable.

Rearranging (89a) once again, Λ_i is, by assumption, a positive definite solution of

$$
\left.
\begin{aligned}
&0 = \Lambda_i(\overline{\alpha} - \tfrac{1}{2}\hat{I}) + (\overline{\alpha} - \tfrac{1}{2}\hat{I})^H\Lambda_i + I\{\Lambda_i\} + s \\
&\text{where:} \\
&\overline{\alpha} \triangleq \hat{\overline{\mu}} - \hat{\sigma}_2\Lambda_i \\
&s \triangleq T_i + \hat{\sigma}_1 + \Lambda_i\hat{\sigma}_2\Lambda_i \geq 0 \\
&(\overline{\alpha} - \tfrac{1}{2}\hat{I},\ s^{\frac{1}{2}}) \text{ completely reconstructible}
\end{aligned}
\right\} \quad (91)
$$

The last condition follows from reconstructibility of $(\hat{\overline{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ and application of Lemma 4 (setting $N = R_2^{-1} > 0$, $F = \hat{\beta}^H\Lambda_i$, $Q = T_i \geq 0$ and $B = -\hat{\beta}R_2^{-1}$). Since, in addition, $(\overline{\alpha} - \frac{1}{2}I) = \hat{\overline{\mu}}_m - \hat{\sigma}_2\Lambda_i$ is stable, Theorem 7 implies that

$$\Delta_\Lambda[\Lambda\alpha_i + \alpha_i^H\Lambda + \hat{I}\{\Lambda\}]$$

is asymptotically stable.

Elimination of T_i from (89a) and (89c) gives:

$$0 = \Lambda_{i+1}\alpha_i + \alpha_i^H\Lambda_{i+1} + \hat{I}\{\Lambda_{i+1}\} + \hat{\sigma}_1 + \Lambda_i\hat{\sigma}_2\Lambda_i \tag{92}$$

Since $\Delta_\Lambda[\Lambda\alpha_i + \alpha_i^H\Lambda + \hat{I}\{\Lambda\}]$ is asymptotically stable, Σ_i and Λ_{i+1} are seen to be negative and positive semi-definite, respectively, by Theorem 6. Moreover reconstructibility of $(\hat{\bar{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ implies $(\alpha_i, (\hat{\sigma}_1 + \Lambda_i\hat{\sigma}_2\Lambda_i + \hat{I}\{\Lambda_{i+1}\})^{\frac{1}{2}})$ reconstructible so that:

$$\Lambda_{i+1} = \int_0^\infty dt\ e^{\alpha_i^H t}[\hat{\sigma}_1 + \Lambda_i\hat{\sigma}_2\Lambda_i + \hat{I}\{\Lambda_{i+1}\}]e^{\alpha_i t}$$

is nonsingular. Manipulation of (89) yields:

$$T_{i+1} = \Sigma_i\hat{\sigma}_2\Sigma_i \tag{93}$$

and T_{i+1} is therefore positive semi-definite.

Thus, in summary, under the assumption that $\Lambda_i > 0$ exists for which T_i in (89a) is positive semi-definite, it follows that $(\Lambda_{i+1} - \Lambda_i) \leq 0$, $\Lambda_{i+1} > 0$ and $T_{i+1} \geq 0$. In other words:

$$\Lambda_\infty = \lim_{i\to\infty} \Lambda_i$$

exists by Lemma 3. Since $T_i \downarrow 0$ this limit is a positive definite solution of (88) for which $\Delta_Q[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_\infty)Q + Q(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_\infty)^H + \hat{I}\{Q\}]$ is asymptotically stable.

It remains to show that $\Lambda_o > 0$ exists for which $T_o \geq 0$. By Lemma 6, a $Y > 0$ exists such that

$$\Delta_\Lambda[\Lambda(\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y)^H\Lambda + \hat{I}\{\Lambda\}]$$

is asymptotically stable, since $(\hat{\bar{\mu}}, \hat{\beta})$ is stabilizable. Now a $W \geq 0$ can always be found such that:

$$[W - (\hat{\sigma}_1 + Y\hat{\bar{\mu}}_m + \hat{\bar{\mu}}_m^H Y + \hat{I}\{Y\} - Y\hat{\sigma}_2 Y)] \geq 0 \tag{94}$$

Also, let Z be the unique positive semi-definite solution of

$$0 = Z(\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y)^H Z + \hat{I}\{Z\} + W \tag{95}$$

in accordance with Theorem 6. Letting:

$$\Lambda_o = Y + Z > 0$$

(89a) becomes

$$T_o = [W - (\hat{\sigma}_1 + Y\hat{\bar{\mu}}_m + \hat{\bar{\mu}}_m^H Y + \hat{I}\{Y\} - Y\hat{\sigma}_2 Y)] + Z\hat{\sigma}_2 Z \geq 0 \tag{96}$$

Thus $\Lambda_o > 0$ and $T_o \geq 0$ and this completes the proof of existence.

To prove uniqueness, let Λ_1 and Λ_2 be any two distinct positive definite solutions of (88). With reasoning analogous to that following (90) we may conclude that both $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_1$ and $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_2$ are asymptotically stable by Lemma 5. Also, rearranging (88) into a form analogous to (91), it is seen that

$\Delta_\Lambda[\Lambda(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_1) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_1)^H\Lambda + \hat{I}\{\Lambda\}]$ and $\Delta_\Lambda[\Lambda(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_2) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_2)^H\Lambda + \hat{I}\{\Lambda\}]$ are asymptotically stable by Theorem 7. Define:

$$Z \stackrel{\Delta}{=} \Lambda_2 - \Lambda_1 \tag{97}$$

Manipulation of (88) yields:

$$\left.\begin{aligned} 0 &= Z(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_1) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_1)^H Z + \hat{I}\{Z\} - Z\hat{\sigma}_2 Z \\ 0 &= Z(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_2) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_2)^H Z + \hat{I}\{Z\} + Z\hat{\sigma}_2 Z \end{aligned}\right\} \tag{98}$$

Since $Z\hat{\sigma}_2 Z \geq 0$, Z is both positive and negative semi-definite by Theorem 6 and hence $Z = 0$. □

3.5 Existence and Uniqueness of Steady-State Solutions - Closed-Loop Stability

With the foregoing developments we are now ready to determine sufficient conditions for existence and uniqueness of steady state solutions of (64). The preceding results may be combined and summarized as follows:

Theorem 10

Consider:

$$\left.\begin{aligned} -\dot{\bar{\rho}} &= \bar{\mu}_m^H\bar{\rho} + \bar{\rho}\,\bar{\mu}_m + I\{\bar{\rho}\} + \sigma_1 - \bar{\rho}\sigma_2\bar{\rho} \\ \bar{\rho}(t_1) &= 0 \end{aligned}\right\} \tag{99}$$

where:

$$\left.\begin{aligned} \bar{\mu}_m &\triangleq \bar{\mu} - \tfrac{1}{2}I \\ \sigma_2 &\triangleq \beta R_2^{-1} \beta^H \end{aligned}\right\} \qquad (100)$$

and where $\bar{\mu}$, I, β, σ_1 and R_2 are as defined in Section 3.1.

If:

$$\left.\begin{array}{ll} \sigma_1 \geq 0 \ , \ R_2 > 0 & \text{a.} \\ (\bar{\mu}, \beta) \text{ stabilizable} & \text{b.} \\ (\bar{\mu}, \sigma_1^{\frac{1}{2}}) \text{ detectable} & \text{c.} \end{array}\right\} \qquad (101)$$

then:

A. (99) has a unique positive semi-definite solution for all $t \leq t_1$

B. $\lim_{t_1 \uparrow \infty} \bar{\rho}(t) = \Lambda$ where Λ is the unique positive semi-definite solution of

$$0 = \bar{\mu}_m^H \Lambda + \Lambda \bar{\mu}_m + I\{\Lambda\} + \sigma_1 - \Lambda \sigma_2 \Lambda \qquad (102)$$

C. With $\kappa = R_2^{-1} \beta^H \Lambda$, the closed loop system is second moment exponentially stable and almost surely exponentially stable for all $I \geq 0$.

Proof

Part A has been shown in Lemma 1, and so we pass immediately to Part B. First note that $\lambda_R\{\bar{\mu}_m\} \leq \lambda_R\{\bar{\mu}\}$ so that the unstable subspace of $\bar{\mu}_m$ is contained in that of $\bar{\mu}$. Then, condition (101c) implies:

$$(\bar{\mu}_m, \sigma_1^{\frac{1}{2}}) \text{ detectable}$$

Thus we can employ the reconstructibility canonical form of the system arising from decomposition (79). By Lemma 1, the positive semi-definite solution of (99) can be reduced to the form

$$\bar{\rho} = \begin{bmatrix} \hat{\bar{\rho}} & 0 \\ 0 & 0 \end{bmatrix}$$

with $\hat{\bar{\rho}}$ satisfying (82), where $(\bar{\mu}_m, \hat{\sigma}_1^{\frac{1}{2}})$ is completely reconstructible. Since the reconstructible subspace contains the unstable subspace and $(\bar{\mu}, \beta)$ is stabilizable, $(\hat{\bar{\mu}}, \hat{\beta})$ is also stabilizable. Thus the conditions of Theorem 9 are met and (84) has a unique positive definite solution, Λ_∞, and

$$\Delta_Q[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_\infty)Q + Q(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_\infty)^H + \hat{I}\{Q\}]$$

is asympototically stable. Then, by Lemma 2:

$$\lim_{t_1 \uparrow \infty} \bar{\rho}(t) = \Lambda_\infty$$

Therefore, under conditions (101), we conclude that:

$$\lim_{t_1 \uparrow \infty} \bar{\rho}(t) = \begin{bmatrix} \Lambda_\infty & 0 \\ 0 & 0 \end{bmatrix} \tag{103}$$

where Λ_∞ is the unique positive definite solution of (84).

Obviously, (103) is also a positive semi-definite solution to (102), but it remains to show that (103) is the only such solution. Partition all matrices in (102) in accordance with (79); in particular set:

$$\Lambda \triangleq \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{12}^H & \Lambda_2 \end{bmatrix} \tag{104}$$

and use (80) and (83). Then the (2,2) sub-block of (102) may be written:

$$\left.\begin{aligned} 0 = \bar{\mu}_{mu}^H \Lambda_2 + \Lambda_2 \bar{\mu}_{mu}^H + \mathcal{I}_u\{\Lambda_2\} \\ -(\hat{\beta}^H \Lambda_{12} + \beta_u^H \Lambda_2)^H R_2^{-1} (\hat{\beta}^H \Lambda_{12} + \beta_u^H \Lambda_2) \end{aligned}\right\} \tag{105}$$

Note that if Λ is positive semi-definite, so is Λ_2. On the other hand, it was shown in the proof of Theorem 8 that $\Lambda_{Q2}[\bar{\mu}_{mu} Q_2 + Q_2 \bar{\mu}_{mu}^H + \mathcal{I}_u\{Q_2\}]$ or, equivalently, $\Delta_{\Lambda 2}[\bar{\mu}_{mu}^H \Lambda_2 + \Lambda_2 \bar{\mu}_{mu}^H + \mathcal{I}_u\{\Lambda_2\}]$ is asymptotically stable. Then from (105), Theorem 6 and $R_2 > 0$, Λ_2 is negative semi-definite unless $(\hat{\beta}^H \Lambda_{12} + \beta_u^H \Lambda_2)$ vanishes, in which case Λ_2 also vanishes. Thus, for Λ_2 to be non-negative, we must have:

$$
\left.\begin{array}{ll}
\Lambda_2 = 0 & \text{a.} \\
\hat{\beta}^H \Lambda_{12} = 0 & \text{b.}
\end{array}\right\} \quad (106)
$$

Under these conditions, the (1,2) sub-block of (102) assumes the form:

$$0 = (\hat{\bar{\mu}}_m - \hat{\sigma}_2 \Lambda_1)^H \Lambda_{12} + \Lambda_{12} \bar{\mu}_{mu}$$

whence Λ_{12} vanishes. Therefore, any positive semi-definite solution of (102) must assume the form:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$$

But with this result, (102) reduces to (84) and (84) has been shown to possess the unique positive definite solution Λ_∞. Consequently (103) is the unique positive semi-definite solution of (102), which completes the proof of Part B.

Because of the result of Theorem 9, the assumptions of Lemma 2 are fulfilled. Thus, with the steady state control gain $\kappa = R_2^{-1} \beta^H \Lambda$,

$$\Lambda_Q [(\bar{\mu}_m - \beta\kappa) Q + Q(\bar{\mu}_m - \beta\kappa)^H + \mathcal{I}\{Q\}]$$

is asymptotically stable by Therorem 8 and the conclusion of Part C follows. $\square$

The situation with respect to the existence and uniqueness of steady state solutions to (99) is thus seen to be entirely analogous to that of the ordinary Riccati equation for a deterministic plant. Note that if the system lacks rigid body modes, stabilizability and detectability are assured so that (101a) form the only ancillary conditions. In the general case, we require only that the rigid body modes be controllable and reconstructible - conditions readily imposed in practice.

In place of guaranteed stability for the nominal system which results from the solution of the deterministic plant regulation problem we here obtain the stronger assurance of second moment stability. This is indirectly a consequence of the use of a mean-quadratic performance criterion, i.e., existence of a mean-square optimal control necessarily demands second moment stability.

Furthermore, due to the particular form of the uncertainties considered, even stronger guarantees of stochastic stability may be obtained. The following result:

<u>Theorem 11</u>

Assume the conditions of Theorem 10 and that $\bar{\rho}$ is the positive semi-definite solution to (102). Define:

$$S \triangleq \sigma_1 - \tfrac{1}{2}I\bar{\rho} + I\{\bar{\rho}\} - \tfrac{1}{2}\bar{\rho}I \tag{107}$$

Then:

$$\begin{gathered} 0 \le S \le \sigma_1 \\ (\bar{\mu},\ S^{\frac{1}{2}}) \text{ detectable} \end{gathered} \tag{108}$$

the proof of which is given in Appendix 7, leads to the conclusion:

Theorem 12

With $\bar{\rho}$ the positive definite solution to (102) under the conditions of Theorem 10, then $(\bar{\mu} - \sigma_2\bar{\rho})$ is asymptotically stable and the closed loop stochastic system is almost surely exponentially stable and p^{th} mean exponentially stable for all $p > 0$ and all values of the decorrelation times.

Proof

By rearrangement of (102), $\bar{\rho}$ is seen to exist as the positive semi-definite solution of

$$0 = \bar{\mu}^H\bar{\rho} + \bar{\rho}\,\bar{\mu} + S - \bar{\rho}\sigma_2\bar{\rho} \;;\; S \geq 0 \tag{109}$$

where $(\bar{\mu}, \beta)$ is stabilizable by assumption and $(\bar{\mu}, S^{\frac{1}{2}})$ is detectable by Theorem 11. Consequently (see Theorem 12.2 of Reference (44)) $(\bar{\mu} - \sigma_2\bar{\rho})$ is asymptotically stable. Since, in addition, the random portion of μ is skew-hermitian (see Appendix 6) the remaining conclusions follow by application of Criterion 6 of Reference (46). □

Thus, the stochastic regulator design guarantees a very considerable measure of stochastic stability.

3.6 Constant Gain Stochastic Design: Computational Procedures

Because of the relative ease with which constant gain controllers may be implemented, attention is henceforth restricted to the steady state case. In the following, $\bar{\rho}$ denotes the positive semi-definite solution to the steady state version of (99):

$$0 = \bar{\mu}_m^H \bar{\rho} + \bar{\rho}\,\bar{\mu}_m + I\{\bar{\rho}\} + \sigma_1 - \bar{\rho}\sigma_2\bar{\rho}$$

$$\bar{\mu}_m \triangleq \bar{\mu} - \tfrac{1}{2}I \tag{110}$$

$$\sigma_2 \triangleq \beta R_2^{-1}\beta^H$$

under the conditions of Theorem 10. In this section various convergent iterative procedures for the numerical solution of (110) are set forth.

The first method is suggested by Theorems 9 and 10:

<u>Theorem 13</u>

Using the decomposition of Lemma 1, choose $Y > 0$ such that:

$$\Delta_\Lambda[\Lambda(\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y)^H \Lambda + \hat{I}\{\Lambda\}]$$

is asymptotically stable and define $W \geq 0$ and $Z \geq 0$ so that:

$$[W - (\hat{\sigma}_1 + Y\hat{\bar{\mu}}_m + \hat{\bar{\mu}}_m^H Y + \hat{I}\{Y\} - Y\hat{\sigma}_2 Y)] \geq 0 \tag{111}$$

$$0 = Z(\hat{\bar{\mu}}_m - \sigma_2 Y) + (\hat{\bar{\mu}}_m - \hat{\sigma}_2 Y)^H Z + \hat{I}\{Z\} + W \tag{112}$$

Then:

$$\lim_{k\uparrow\infty} \bar{\rho}_i = \bar{\rho} \tag{113}$$

where:

$$\left.\begin{aligned} 0 &= (\bar{\mu}_m - \sigma_2\bar{\rho}_k)^H \bar{\rho}_{k+1} + \bar{\rho}_{k+1}(\bar{\mu}_m - \sigma_2\bar{\rho}_k) \\ &\quad + \mathcal{I}\{\bar{\rho}_{k+1}\} + \sigma_1 + \bar{\rho}_k\sigma_2\bar{\rho}_k \\ \bar{\rho}_o &= \begin{bmatrix} Y+Z & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \qquad (114)$$

<u>Proof</u>

With (80) through (83) from Lemma 1, induction on k from (114) shows that

$$\bar{\rho}_k = \begin{bmatrix} \hat{\bar{\rho}}_k & 0 \\ 0 & 0 \end{bmatrix}$$

for all $k \geq 0$. The sequence $\{\hat{\bar{\rho}}_k\}$ is seen to be defined by relations (89). Furthermore, (111), (112) and the choice for Y ensure that T_o defined by (96) is positive semi-definite. Referring to the proof of Theorem 9, this guarantees the convergence of the sequence (89). Thus, sequence (114) converges to a positive semi-definite limit. By (103) this limit is the solution of (110). □

The above result corresponds to the Newton-Raphson method and therefore exhibits quadratic convergence in a sufficiently small neighborhood of the solution of (110). The primary difficulty arises from solution of a stochastic Lyapunov equation on each iteration. Also, of course, proper choice of the starting value may occasion some inconvenience.

Ambiguity in the starting value is removed in the following method.

Theorem 14

$$\lim_{k\uparrow\infty} \bar{\rho}_k = \bar{\rho} \tag{115}$$

where the positive semi-definite sequence $\{\bar{\rho}_k\}$ is defined by

$$\left.\begin{aligned} 0 &= \bar{\mu}_m^H \bar{\rho}_{k+1} + \bar{\rho}_{k+1}\bar{\mu}_m + I\{\bar{\rho}_k\} + \sigma_1 - \bar{\rho}_{k+1}\sigma_2\bar{\rho}_{k+1} \\ \bar{\rho}_o &= 0 \end{aligned}\right\} \tag{116}$$

Proof

Using the representation of Lemma 1, (116) is seen to yield:

$$\bar{\rho}_k = \begin{bmatrix} \Lambda_k & 0 \\ 0 & 0 \end{bmatrix} \tag{117}$$

for all $k \geq 0$. By substitution, the Λ_k are found to satisfy:

$$\left.\begin{aligned} 0 &= \hat{\bar{\mu}}_m^H \Lambda_{k+1} + \Lambda_{k+1}\hat{\bar{\mu}}_m + \hat{I}\{\Lambda_k\} + \hat{\sigma}_1 - \Lambda_{k+1}\hat{\sigma}_2\Lambda_{k+1} \\ \Lambda_o &= 0 \;;\; \Lambda_k > 0 \;,\; \forall\, k > 0 \end{aligned}\right\} \tag{118}$$

Now let:

$$\bar{\rho} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$$

be the positive semi-definite solution to (110) where Λ is the positive definite solution of (88) and $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda$ is asymptotically stable by Theorem 9. (88) and (116) may be manipulated to yield:

$$0 = (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H(\Lambda - \Lambda_{k+1}) + (\Lambda - \Lambda_{k+1})(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda) + \hat{I}\{\Lambda - \Lambda_k\} + (\Lambda - \Lambda_{k+1})\hat{\sigma}_2(\Lambda - \Lambda_{k+1}) \tag{119}$$

Thus, by Lemma 12.1 of Reference (44), if $\Lambda - \Lambda_k$ is positive semi-definite then so is $\Lambda - \Lambda_{k+1}$. Furthermore if Λ_k is positive definite, then because $(\hat{\bar{\mu}}_m, \hat{\sigma}_1^{\frac{1}{2}})$ is completely reconstructible, so is $(\hat{\bar{\mu}}_m, (\hat{\sigma}_1 + \hat{I}\{\Lambda_k\})^{\frac{1}{2}})$ by Lemma 4. Thus Lemma 5 is applicable to (118), whence $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_{k+1}$ is asymptotically stable. Manipulation of (118) yields:

$$0 = (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_{k+1})^H(\Lambda_{k+1} - \Lambda_k) + (\Lambda_{k+1} - \Lambda_k)(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda_{k+1}) + \hat{I}\{\Lambda_k - \Lambda_{k-1}\} + (\Lambda_{k+1} - \Lambda_k)\hat{\sigma}_2(\Lambda_{k+1} - \Lambda_k) \tag{120}$$

so that if $(\Lambda_k - \Lambda_{k-1})$ is positive semi-definite then $(\Lambda_{k+1} - \Lambda_k)$ is also positive semi-definite. Thus, by Lemma 3, the sequence $\{\Lambda_k\}$ converges to Λ provided that $\Lambda_1 - \Lambda_o \geq 0$. But this is assured by the choice $\Lambda_o = 0$ and by virtue of Lemma 5. Consequently, with (117), the sequence $\{\overline{\rho}_k\}$ converges to the positive semi-definite solution of (110). $\square$

This method may be termed the "perturbed weighting" method since, in effect, it involves sequential modification of the state weighting matrix, σ_1. Obviously, the problem of determining a suitable starting value is eliminated. Moreover, each iteration involves solution of the standard Riccati equation. On the other hand, this method exhibits only linear convergence in contrast to the quadratic convergence of the Newton method.

3.7 Asymptotic Properties for Large Uncertainties

Having determined that a unique positive semi-definite solution of (110) exists for all positive I, it is natural to enquire what behavior $\overline{\rho}$ attains for large uncertainties, i.e., for very small decorrelation times. In examining this question we are mainly concerned with the resulting form of the control law for the elastic modes. Thus, in the remainder of this report, it is assumed that the system possesses no rigid body modes, i.e., that $\overline{\mu} = \overline{\mu}_e$, $n = n_e$. This restriction permits a relatively straightforward development and, moreover, the qualitative form of the results to be derived below is expected to hold even in the presence of rigid body modes.

The principal effects of frequency uncertainty for small values of the decorrelation times are connected with the distinct character (noted after the proof of Theorem 4) of the diagonal and off-diagonal portions of equation (110). Since we mean to treat the diagonal and off-diagonal elements of $\overline{\rho}$ separately, it is expeditious to introduce the notation:

$$\langle M\rangle \triangleq M - \{M\} \tag{121}$$

for any square matrix M. In addition, define:

$$\{\rho^*\}_k \triangleq \frac{1}{\{\sigma_2\}_k} [\mathrm{Re}\bar{\mu}_k + ((\mathrm{Re}\bar{\mu}_k)^2 + \{\sigma_1\}_k\{\sigma_2\}_k)^{\frac{1}{2}}] \tag{122}$$

$\{\rho^*\}$ may be recognized as the solution for the diagonal portion of $\bar{\rho}$ obtained from the diagonal portion of (110) with $\langle\bar{\rho}\rangle$ ignored.

With these definitions, we first establish certain bounds on $\{\bar{\rho}\}$ and $\langle\bar{\rho}\rangle$ as follows:

<u>Theorem 15</u>

Under the conditions of Theorem 10, and with $\bar{\rho}$ the positive semi-definite solution of (110) with $\bar{\mu} = \bar{\mu}_e$:

A.
$$|\langle\bar{\rho}\rangle_{kj}| \leq \frac{2(\{\sigma_1\}_k\{\sigma_1\}_j)^{\frac{1}{2}}}{|\bar{\mu}_k^* + \bar{\mu}_j - \frac{1}{2}(I_k + I_j)|} \tag{123}$$

B. Defining
$$\begin{aligned} B_k &\triangleq \{\sigma_2\langle\bar{\rho}\rangle\}_k + \{\langle\bar{\rho}\rangle\sigma_2\}_k \\ C_k &\triangleq \{\langle\bar{\rho}\rangle\sigma_2\langle\bar{\rho}\rangle\}_k \end{aligned} \tag{124}$$

, if:

$$\{\sigma_1\}_k - C_k > 0 \tag{125}$$

then:

$$|\{\bar{\rho}\}_k - \{\rho^*\}_k| \le \frac{|C_k + B_k\{\rho^*\}_k|}{|B_k + 2\{\rho^*\}_k\{\sigma_2\}_k - 2\mathrm{Re}\bar{\mu}_k|} \tag{126}$$

C. If $(\sigma_2)_{kk} \neq 0$, then:

$$|\{\bar{\rho}\}_k - \{\rho^*\}_k| \le 2\sqrt{\frac{\sigma_{1kk}}{\sigma_{2kk}}}\,[\Gamma_k + \Gamma_k^2] \tag{127}$$

$$\Gamma_k \triangleq \sum_{\ell \neq k} \frac{(\sigma_{1\ell\ell}\sigma_{2\ell\ell})^{\frac{1}{2}}}{|\bar{\mu}_\ell^* + \bar{\mu}_k - \frac{1}{2}(I_\ell + I_k)|} \tag{128}$$

Proof

As a preliminary step, decompose each term of (110) into its diagonal and off-diagonal elements to obtain:

$$\left.\begin{aligned} 0 &= 2\mathrm{Re}\bar{\mu}_k\{\bar{\rho}\}_k + \{\sigma_1\}_k - \{\bar{\rho}\sigma_2\bar{\rho}\}_k && \text{a.} \\ 0 &= \langle\bar{\rho}\rangle_{kj}[\bar{\mu}_k^* + \bar{\mu}_j - \tfrac{1}{2}(I_k + I_j)] + \langle\sigma_1\rangle_{kj} - \langle\bar{\rho}\sigma_2\bar{\rho}\rangle_{kj} && \text{b.} \end{aligned}\right\} \tag{129}$$

Now consider part A. Since $\sigma_2 \ge 0$, $\bar{\rho}\,\sigma_2\bar{\rho}$ is positive semi-definite. Consequently $\{\bar{\rho}\,\sigma_2\bar{\rho}\}$ is non-negative and:

$$|\langle\bar{\rho}\sigma_2\bar{\rho}\rangle_{kj}| \le (\{\bar{\rho}\sigma_2\bar{\rho}\}_k\{\bar{\rho}\sigma_2\bar{\rho}\}_j)^{\frac{1}{2}} \; ; \; k \neq j \tag{130a}$$

Also, from (129a):

$$|\{\bar{\rho}\sigma_2\bar{\rho}\}_k| \le \{\sigma_1\}_k \tag{130b}$$

since $\mathrm{Re}\bar{\mu} < 0$. Then (130a) yields:

$$|\langle\bar{\rho}\sigma_2\bar{\rho}\rangle_{kj}| \le (\{\sigma_1\}_k\{\sigma_1\}_j)^{\frac{1}{2}} \tag{131}$$

Using this inequality in (129b) and noting that because $\sigma_1 \ge 0$,

$$|\sigma_{1kj}| \le (\{\sigma_1\}_k\{\sigma_1\}_j)^{\frac{1}{2}}$$

we obtain (123).

With the above results, inequalities on $\{\bar{\rho}\}$ may be shown, starting with part B. Expand the quadratic term in (129a) to get:

$$\{\bar{\rho}\}_k^2\{\sigma_2\}_k + \{\bar{\rho}\}_k[B_k - 2\mathrm{Re}\bar{\mu}_k] + C_k - \{\sigma_1\}_k = 0 \tag{132}$$

with B_k and C_k as defined in (124). Defining:

$$\Delta = \{\bar{\rho}\} - \{\rho^*\} \tag{133}$$

and using (132) with (122), we obtain for Δ:

$$\Delta_k^2\{\sigma_2\}_k + [B_k+2\{\rho^*\}_k\{\sigma_2\}_k-2\mathrm{Re}\bar{\mu}_k]\Delta_k + C_k+\{\rho^*\}_kB_k = 0 \qquad (134)$$

Solving (132) for $\{\bar{\rho}\}_k$ and (134) for Δ_k:

$$\left.\begin{aligned} \{\bar{\rho}\}_k &= \tfrac{1}{2}[-\tilde{\alpha}\pm\sqrt{\alpha^2-4\beta}] \ge 0 && \text{a.} \\ \Delta_k &= \tfrac{1}{2}[-\alpha\pm\sqrt{\alpha^2-4\beta}] && \text{b.} \end{aligned}\right\} \qquad (135)$$

$$\left.\begin{aligned} \tilde{\alpha} &\triangleq \frac{1}{\{\sigma_2\}_k}[B_k-2\mathrm{Re}\bar{\mu}_k] && \text{a.} \\ \alpha &= \tilde{\alpha} + 2\{\rho^*\}_k && \text{b.} \\ \beta &= \frac{1}{\{\sigma_2\}_k}[C_k+\{\rho^*\}_kB_k] && \text{c.} \end{aligned}\right\} \qquad (136)$$

where the same sign appears before the radical in both (135a) and (135b). Suppose that the negative sign is to be taken in (135). Then, (135a) implies:

$$2\{\rho^*\}_k \ge \alpha + \sqrt{\alpha^2-4\beta}$$

or; rearranging:

$$0 \ge \tilde{\alpha} + \sqrt{\tilde{\alpha}^2-4[C_k-\{\sigma_1\}_k]/\{\sigma_2\}_k}$$

But in view of condition (125), the right hand side above is inherently positive, implying a contradiction. Therefore, the positive signs must be chosen in (135).

$$\left.\begin{aligned} \Delta_k &= \tfrac{1}{2}[-\alpha + \sqrt{\alpha^2-4\beta}] && \text{a.} \\ \alpha &= \frac{1}{\{\sigma_2\}_k}[B_k-2\mathrm{Re}\bar{\mu}_k] + 2\{\rho^*\}_k && \text{b.} \\ \beta &= \frac{1}{\{\sigma_2\}_k}[C_k+\{\rho^*\}_k B_k] \geq 0 && \text{c.} \end{aligned}\right\} \qquad (137)$$

Note that (132) may be rewritten:

$$0 = (\bar{\mu}-\{\sigma_2\}\{\bar{\rho}\}-\{\sigma_2\langle\bar{\rho}\rangle\})^H\{\bar{\rho}\} + \{\bar{\rho}\}(\bar{\mu}-\{\sigma_2\}\{\bar{\rho}\}-\{\sigma_2\langle\bar{\rho}\rangle\})$$

$$+ \{\bar{\rho}\}^2\{\sigma_2\} + \{\sigma_1\}-\{\langle\bar{\rho}\rangle\sigma_2\langle\bar{\rho}\rangle\}$$

Now, $\{\bar{\rho}\}$ exists as a positive-definite matrix, while $\{\sigma_1\} - \{\langle\bar{\rho}\rangle\sigma_2\langle\bar{\rho}\rangle\}$ is positive by assumption (125). Consequently $(\bar{\mu} - \{\sigma_2\}\{\bar{\rho}\} - \{\sigma_2\langle\bar{\rho}\rangle\})$ is asymptotically stable, i.e., α is positive.

With this property, the inequalities:

$$1 - \frac{|x|}{2} \leq \sqrt{1+x} \leq 1 + \frac{|x|}{2}, \ \forall \ x \text{ real}$$

may be used in conjunction with (137) to obtain:

$$\Delta_k \; \varepsilon \; - \frac{|\beta|}{\alpha} \; , \; \frac{|\beta|}{\alpha} \tag{138}$$

With substitution of (137b,c), the result (126) follows.

Part C. can be shown more easily. From (132):

$$\left.\begin{aligned} \{\sigma_2\}_k\{\bar{\rho}\}_k^2 &\geq \{\sigma_1\}_k - |C_k|_M + \{\bar{\rho}\}_k[-|B_k| + 2\mathrm{Re}\bar{\mu}_k] \\ \{\sigma_2\}_k\{\rho\}_k^2 &\leq \{\sigma_1\}_k + \{\rho\}_k[|B_k|_M + 2\mathrm{Re}\bar{\mu}_k] \end{aligned}\right\} \tag{139}$$

where:

$$\left.\begin{aligned} |B_k|_M &\triangleq 2 \sum_{\ell\neq k} \sqrt{\sigma_{2kk}\sigma_{2\ell\ell}}\,|\rho_{\ell k}|_M \\ |C_k|_M &\triangleq \sum_{\ell\neq k}\sum_{m\neq k} \sqrt{\sigma_{2\ell\ell}\sigma_{2mm}}\,|\rho_{\ell k}|_M|\rho_{mk}|_M \end{aligned}\right\} \tag{140}$$

and $|\rho_{\ell k}|_M$ denotes the upper bound given by (123). Note also that:

$$4\{\sigma_2\}|C_k|_M = |B_k|_M^2$$

Consequently, the second inequality of (139) actually gives an upper bound for $|\{\bar{\rho}\}_k - \{\rho^*\}_k|$. From (139b):

$$\{\bar{\rho}\}_k \le \frac{1}{2\{\sigma_2\}_k}[|B_k|_M + 2\mathrm{Re}\bar{\mu}_k + \sqrt{(|B_k|_M + 2\mathrm{Re}\bar{\mu}_k)^2 + 4\{\sigma_1\}_k\{\sigma_2\}_k}]$$

so that:

$$|\{\bar{\rho}\}_k - \{\bar{\rho}^*\}_k| \le \frac{1}{2\{\sigma_2\}_k}[|B_k|_M + \sqrt{|B_k|_M^2 + 4\{\sigma_1\}_k\{\sigma_2\}_k} - \sqrt{\{\sigma_1\}_k\{\sigma_2\}_k}] \quad (141)$$

Using:

$$\sqrt{1+x} \;\; -1 \le \frac{x}{2}, \; \forall\, x \ge 0$$

in (141) and substituting expressions (140), the result (127) is obtained. □

Of the bounds given in Parts B. and C., C. is the simpler but does not apply for $\sigma_{2kk} = 0$. On the other hand (126) is applicable for sufficiently high uncertainty levels and $\sigma_{2kk} \ge 0$ and gives a closer bound for small values of σ_{2kk}.

The presence of $(I_k + I_j)$ in the denominator of (123) clearly shows that one effect of frequency uncertainties is to suppress the off-diagonal elements of the expected cost. In this connection, use of the above bounds easily leads to the following conclusions:

Theorem 16

Assuming the conditions of Theorem 10 and introducing a positive scaling parameter, J, into I:

$$I = J\,\tilde{I} \; ; \; J \geq 0 \; , \; \tilde{I} > 0 \tag{142}$$

then:

A. $$\lim_{J \to \infty} \bar{\rho} = \{\rho^*\} \tag{143}$$

with $\{\rho^*\}$ given by (122).

B. The control $u = -Kx$, where

$$\left.\begin{aligned} K &= \kappa \Phi^{-1} && \text{a.} \\ \kappa &= R_2^{-1} \beta^H \{\rho^*\} && \text{b.} \end{aligned}\right\} \tag{144}$$

is a rate feedback law, i.e., the odd indexed columns of K vanish.

C. With κ given by (144b), and $\tilde{\mu}$ any diagonal matrix with negative real part, the system:

$$\dot{\xi} = (\tilde{\mu} - \beta\kappa)\xi \; ; \; \xi(t_o) = \xi_o \tag{145}$$

is asymptotically stable.

Proof

A. It is clear that the bound given by (123) decreases monotonically with increasing J. Hence:

$$\lim_{J\to\infty} |\langle\bar{\rho}_{kj}\rangle| = 0 \tag{146}$$

Moreover, there exists a J sufficiently large that condition (125) is satisfied and the bound (126) may be employed. In view of (146), the limit of the right side of (126) is zero. $\square$

B. From the definition (10e) of σ_1 and the form of Φ given by (8), it is seen that:

$$(\sigma_1)_{2m-1,2m-1} = (\sigma_1)_{2m,2m} \; ; \; m = 1,\ldots,n \tag{147a}$$

Similarly, from (10b), (8) and (49):

$$(\sigma_2)_{2m-1,2m-1} = (\sigma_2)_{2m,2m} \; ; \; m = 1,\ldots,n \tag{147b}$$

Consequently, from (12b) and (122):

$$\{\rho^*\}_{2m-1} = \{\rho^*\}_{2m} \; ; \; m = 1,\ldots,n \tag{147c}$$

and K may be written:

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$$K = R_2^{-1}B^T\Phi^{-1H}\{\rho^*\}\Phi^{-1}$$

$$= R_2^{-1}B^T \times \underset{m=1,\dots,n}{\text{block-diag}} \frac{\rho^*_{2m,2m}}{2\bar{\omega}_m^2}\begin{bmatrix}\bar{\omega}_m^2 & 0 \\ 0 & 1\end{bmatrix} \tag{148}$$

Thus, since the odd rows of B vanish, the odd columns of K also vanish. □

C. Define:

$$\mathcal{L} \triangleq \tfrac{1}{2}\xi^H\{\rho^*\}\xi > 0 \;, \; \forall\; \xi \neq 0$$

Then, for the system (145):

$$\frac{d}{dt}\mathcal{L} = (\{\rho^*\}^{\frac{1}{2}}\xi)^H[\mathrm{Re}\tilde{\mu} - \{\rho^*\}^{\frac{1}{2}}\beta R_2^{-1}\beta^H\{\rho^*\}^{\frac{1}{2}}]\{\rho^*\}^{\frac{1}{2}}\xi$$

$$< 0$$

since $\tilde{\mu} < 0$ and R_2, $\{\rho^*\} > 0$. Therefore, $\mathcal{L}$ is a Lyapunov function for (145) and the stated conclusion follows. □

The stochastic Riccati equation does indeed have an asymptotic solution, of a particularly simple form, in the limit as all decorrelation times tend to zero. According to Parts B and C of the above Theorem, the resulting rate feedback control law renders the closed-loop system stable for all values of the open loop frequencies. This feature appears to be a natural consequence of the inclusion of parameter uncertainties as an intrinsic part of the model and illustrates the qualitative principle: "If nothing is known regarding the open loop modal frequencies then the mean-square optimal control is the control law which is inherently dissipative under frequency uncertainties." Furthermore, Part C. shows that the asymptotic form of the control also guarantees stability in the face of uncertainties in the modal damping ratios. Thus a greater degree of robustness has been obtained than was originally sought.

3.8 Asymptotic Properties for High-Order Modes: Incoherence and Decoupling

Theorem 16 pertains only to the case in which the decorrelation times are all uniformly small. Analogous results are to be expected when uncertainties in the low frequency modes are small while modelling accuracy degenerates for modes of increasing order. Supposing the modes to be arranged in order of increasing nominal frequency, we anticipate that correlation between distinct modes will be suppressed to a greater and greater degree the higher are the orders of the modes involved. Similarly, the expected cost matrix will become increasingly diagonalized toward the lower right hand corner and that portion corresponding to very high order and poorly known modes will approach the asymptotic form (143).

To assess the degree to which off-diagonal elements of $\bar{\rho}$ are reduced in magnitude, these must be scaled properly relative to the corresponding diagonal elements. Thus, for the purpose of the following development, we introduce the co-state correlation coefficient matrix, $\check{\rho}$:

$$\check{\rho} \triangleq \{\bar{\rho}\}^{-\frac{1}{2}}\,\bar{\rho}\{\bar{\rho}\}^{-\frac{1}{2}} \qquad \text{a.}$$

so that:

$$\check{\rho}_{kj} = \bar{\rho}_{kj}/\sqrt{\bar{\rho}_{kk}\bar{\rho}_{jj}} \qquad \text{b.} \qquad (149)$$

is clearly analogous to the state correlation coefficient formed from the covariance and has similar properties.

With (149), and as a consequence of the bounds given in Theorem 15, one obtains:

Theorem 17

Arrange modes in order of increasing nominal frequency and suppose that n, the number of modes retained in the model, is arbitrarily large. Considering only the reconstructible modes partition the state vector thus:

$$\xi = \begin{pmatrix} \xi_C \\ \xi_I \end{pmatrix} ; \; \xi_C \in C^{2N_C} \tag{150}$$

and define the corresponding partitions of $\bar{\rho}$ and $\check{\rho}$ by:

$$\left.\begin{aligned} \bar{\rho} &= \begin{bmatrix} \bar{\rho}_C & \bar{\rho}_{CI} \\ \bar{\rho}_{CI}^H & \bar{\rho}_I \end{bmatrix} \\ \check{\rho} &= \begin{bmatrix} \check{\rho}_C & \check{\rho}_{CI} \\ \check{\rho}_{CI}^H & \check{\rho}_I \end{bmatrix} \end{aligned}\right\} \tag{151}$$

Assume that the damping coefficients, η_k are non-zero and bounded for all k; that the input matrix, B, and the control weighting matrix, R_2, are bounded from above and from below, respectively, and the following restrictions:

1. I_k monotone increasing with increasing k

2. $\underline{\sigma}_1 \, \bar{\omega}^2_{N(k)} \le \sigma_{1kk} \le \bar{\sigma}_1 \, \bar{\omega}^2_{N(k)}$, $\forall k$

   where $\underline{\sigma}_1$, $\bar{\sigma}_1 > 0$

3. $\frac{1}{m} (\bar{\omega}_{m+1} - \bar{\omega}_m)$ nondecreasing with m

Then, given $\varepsilon > 0$, there exists an N_c sufficiently large that:

$$\left.\begin{array}{lr} \frac{1}{\{\rho_I^*\}_k} \left| \{\bar{\rho}_I\}_k - \{\rho_I^*\}_k \right| < \varepsilon & \text{a.} \\ \left| \langle \check{\rho}_I \rangle_{jk} \right| < \varepsilon & \text{b.} \\ \left| \check{\rho}_{CI_{jk}} \right| < \varepsilon & \text{c.} \end{array}\right\} \quad (152)$$

where $\{\rho_I^*\}$ is (143) evaluated for the "I" modes. Moreover, each of the quantities on the left of (152) are bounded by monotonically decreasing functions of k.

The proof is given in Appendix 8.

Conditions (1) through (3) of the above Theorem demand particular note. Clearly, condition (1) demands that the decorrelation times expressed as multiples of the corresponding natural periods of vibration decrease with increasing modal order - reflecting a general decline of modelling accuracy for the higher order modes. Condition (2) assumes an "energy weighting" on the state and represents a rather extreme assignment of cost for the high order modes. Often in practice, displacements alone would be included in the performance index, resulting in a more modest rate of increase of σ_{1kk} with k. Finally, condition (3) requires that the modal frequency separation considered as a function of frequency be bounded from below by a constant, i.e., that modal density decrease with increasing frequency. This condition is also likely to hold in practice, at least above a certain frequency.

Inspection of Appendix 8 reveals that the rates of increase of modal density and modal state weighting for which Theorem 17 retains validity are related. In particular, for σ_{1kk} increasing as a power of $\bar{\omega}_k$ less than two, the above results will hold and will retain validity even when modal density increases with increasing frequency. In any case, however, the Theorem as stated is applicable to most situations of interest.

We may say, in view of (152), that the partitioning (150) apportions the modes into "coherent" and "incoherent" systems. This assignment of modes is uniquely defined by the maximum tolerable correlation level, ε, which may typically be set $<<1$. Further, N_c may be termed the "coherence limit" associated with the correlation level ε. Thus the qualitative features of the mean-square optimally controlled system are analogous to those identified in Section 2.2 for the uncontrolled system.

The coherence limit, $N_c(\varepsilon)$, (where we now display the functional dependence upon ε explicitly) may be estimated through use of the bounds given by Theorem 15. However, useful and convenient results may be obtained with approximate forms of these bounds. Very simply, if ε is small, then for $k \gtrsim 2N_c$, $\{\bar{\rho}\}_k$ may be approximated by $\{\rho^*\}_k$ in evaluating $|\langle \check{\rho} \rangle_{kj}|$ from (123). This gives the following approximate determination of $N_c(\varepsilon)$:

$$N_c(\varepsilon): \ |\langle\check{\rho}\rangle_{2N_c,j}| \lesssim \frac{2(\{\sigma_1\}_{2N_c}\{\sigma_1\}_j/\{\rho^*\}_{2N_c}\{\rho^*\}_j)^{\frac{1}{2}}}{|\bar{\mu}^*_{2N_c} + \bar{\mu}_j - \frac{1}{2}(I_{2N_c} + I_j)|} < \varepsilon \ , \ \forall j \quad (153)$$

Furthermore if for $k \sim 2N_c$ the damping is known to be small in the sense:

$$(\mathrm{Re}\bar{\mu}_k)^2 << \{\sigma_1\}_k\{\sigma_2\}_k \quad (154)$$

we have $\{\rho^*\}_k \simeq (\sigma_{1kk}/\sigma_{2kk})^{\frac{1}{2}}$ so that (153) becomes

$$N_c(\varepsilon): \ |\langle\check{\rho}\rangle_{2N_c,j}| \lesssim \frac{2(\{\sigma_1\}_{2N_c}\{\sigma_2\}_{2N_c}\{\sigma_1\}_j\{\sigma_2\}_j)^{\frac{1}{4}}}{|\bar{\mu}^*_{2N_c} + \bar{\mu}_j - \frac{1}{2}(I_{2N_c} + I_j)|} < \varepsilon \ , \ \forall j \quad (155)$$

The above expressions (particularly (155)) reveal the dependence of $N_c(\varepsilon)$ upon the various design parameters. Clearly if higher control authority is demanded (i.e., $\{\sigma_2\}_k$ or $\{\sigma_1\}_k$ increased) $N_c(\varepsilon)$ is increased. Most important, $N_c(\varepsilon)$ strongly depends on the rate of increase of the reciprocal decorrelation times, I_k, with k, and the more rapidly increasing are the I_k, the smaller is $N_c(\varepsilon)$. In fact, the dependence of I_k upon k might be chosen in an ad hoc fashion so as to achieve a desired coherence limit.

Obviously if ε is sufficiently small, $\bar{\rho}_{CI}$ is negligible so that modal frequency uncertainties effectively decouple the coherent and incoherent systems. Moreover, $\bar{\rho}_I$ is approximately $\{\rho *\}$ as given by (143). Thus, for the incoherent system, (110) naturally gives rise to a rate feedback control law which is stable regardless of the values of modal frequencies or damping ratios. For sufficiently large values of the reciprocal decorrelation times and fixed $\varepsilon << 1$, $N_c(\varepsilon)$ dirinishes to less than unity; the whole system becomes incoherent and the entire control law reduces to the asymptotic form given in Theorem 16.

The above properties can greatly simplify computation of the control gain for system models of very high dimension. We first note, without dwelling on the matter in any detail here, that the expressions given in Theorem 15 or their approximate counterparts may be used to bound the contributions of $\bar{\rho}_{CI}$ and $\bar{\rho}_I - \{\rho_I^*\}$ to κ. These contributions decline with increasing N_c and are convergent even as n tends to infinity.

Thus, under the assumptions of Theorem 17, the calculation of κ may proceed as follows. First, using the bounds given in Theorem 15, determine N_c sufficiently large that the magnitude of all elements in the error:

$$R_2^{-1}[\beta_C^H \beta_I^H] \begin{bmatrix} 0 & \bar{\rho}_{CI} \\ \bar{\rho}_{CI}^H & \bar{\rho}_I - \{\rho_I^*\} \end{bmatrix} \tag{156}$$

incurred in the approximation:

$$\kappa \simeq R_2^{-1}[\beta_C^H \beta_I^H] \begin{bmatrix} \overline{\rho}_C & 0 \\ 0 & \{\overline{\rho}_I^*\} \end{bmatrix} \tag{157}$$

may be considered negligible. Then compute κ from (157), determining $\overline{\rho}_c$ from

$$0 = \overline{\mu}_{cm}^H \overline{\rho}_c + \overline{\rho}_c \overline{\mu}_{cm} + I_c\{\overline{\rho}_c\} + \sigma_{1c} - \overline{\rho}_c \sigma_{2c} \overline{\rho}_c \tag{158}$$

by either of the methods of Section 3.6 and calculating $\{\overline{\rho}_I^*\}$ from (122).

(158) is the upper left sub-block of (110) when partitioned in accordance with (150), and is a stochastic Riccati equation of order $2N_c \times 2N_c$. If the number of coherent modes, N_c, is modest, either the Newton method or the perturbed weighting method may be employed at reasonable computational cost. The calculation of $\{\rho_I^*\}$, since it proceeds from simple analytical expressions, entails no computational burden whatsoever. Note that the error incurred in $\overline{\rho}_I \simeq \{\rho_I^*\}$ is greatest on the upper left corner and diminishes toward the lower right. Consequently, somewhat above the coherence limit, as many modes as desired may be accommodated without incurring significant computational effort.

We conclude that if the modeled uncertainties increase with modal order and the number of coherent, relatively well known modes is modest, the solution of the stochastic Riccati equation and the control gain may be computed with satisfactory accuracy for systems of arbitrary order.

3.9 Efficient Computational Procedures for the Coherent System

Let us retain the conditions of Theorem 17 and consider the stochastic Riccati equation for the coherent system:

$$0 = \bar{\mu}_{cm}^{H}\bar{\rho}_c + \bar{\rho}_c\bar{\mu}_{cm} + I_c\{\bar{\rho}_c\} + \sigma_{1c} - \bar{\rho}_c\sigma_{2c}\bar{\rho}_c$$

$$\bar{\rho}_c \in C^{2N_c \times 2N_c}$$

Here we explore the numerical treatment of (158) when the number of coherent modes is too large to permit efficient application of the methods of Section 3.6.

In this connection, the diagonalization effect of frequency uncertainties noted previously suggests a novel iterative numerical technique:

$$\left.\begin{aligned} 0 &= (\bar{\mu}_m - \{\sigma_2\bar{\rho}_k\})^H \bar{\rho}_{k+1} + \bar{\rho}_{k+1}(\bar{\mu}_m - \{\sigma_2\bar{\rho}_k\}) + I\{\bar{\rho}_{k+1}\} \\ &\quad + \sigma_1 + \{\sigma_2\bar{\rho}_k\}^H \bar{\rho}_k + \bar{\rho}_k\{\sigma_2\bar{\rho}_k\} - \bar{\rho}_k\sigma_2\bar{\rho}_k \\ \bar{\rho}_o &= \{\rho^*\} \end{aligned}\right\} \quad (159)$$

where, for convenience, the "c" subscript has been dropped. The rationale for (159) can be understood through examination of the diagonal and off-diagonal portions separately:

$$0 = (\bar{\mu} - \tfrac{1}{2}I - \{\sigma_2\bar{\rho}_k\})^H\langle\bar{\rho}_{k+1}\rangle + \langle\bar{\rho}_{k+1}\rangle(\bar{\mu} - \tfrac{1}{2}I - \{\sigma_2\bar{\rho}_k\})$$

$$+ \langle\sigma_1\rangle + \{\sigma_2\bar{\rho}_k\}^H\langle\bar{\rho}_k\rangle + \langle\bar{\rho}_k\rangle\{\sigma_2\bar{\rho}_k\} - \langle\bar{\rho}_k\sigma_2\bar{\rho}_k\rangle \qquad \text{a.} \qquad (160)$$

$$0 = (\bar{\mu} - \{\sigma_2\bar{\rho}_k\})^H\{\bar{\rho}_{k+1}\} + \{\bar{\rho}_{k+1}\}(\bar{\mu} - \{\sigma_2\bar{\rho}_k\})$$

$$+ \{\sigma_1\} + 2\mathrm{Re}\{\bar{\rho}_k\}\{\sigma_2\bar{\rho}_k\} - \{\bar{\rho}_k\sigma_2\bar{\rho}_k\} \qquad \text{b.}$$

where $\{\bar{\rho}_o\} = \{\rho^*\}$

$$\langle\bar{\rho}_o\rangle = 0 \qquad (161)$$

At least for modes corresponding to the lower right entries of $\bar{\rho}$, the reciprocal decorrelation times will be relatively large. Thus the presence of $-\tfrac{1}{2}I$ in the coefficient matrix of $\langle\bar{\rho}_{k+1}\rangle$ in (160a) suggests that $\langle\bar{\rho}_{k+1}\rangle$ will be "small" in the sense $||\langle\bar{\rho}_{k+1}\rangle|| << ||\{\bar{\rho}_{k+1}\}||$. Note that $(\bar{\mu} - \tfrac{1}{2}I - \{\sigma_2\bar{\rho}_k\})$ is the eigenvalue matrix of the mean system evaluated to within a first order perturbation in the control. Equation (160a) can thus be seen as a perturbation approximation to $\langle\bar{\rho}\rangle$ for the mean system with its decorrelation damping $(-\tfrac{1}{2}I)$.

If indeed $||\langle\check{\rho}\rangle||$ is small, the contribution of $\langle\bar{\rho}_k\rangle$ to $\bar{\rho}_k$ in (160b) will also be "small" so that $\langle\bar{\rho}_k\rangle$ provides a perturbation on what would be Newton's method for determination of $\{\bar{\rho}\}$ in the case $\langle\bar{\rho}\rangle = 0$. Obviously, to within the approximation that $\langle\bar{\rho}_k\rangle$ is negligible, (160) yields $\{\bar{\rho}_1\} = \{\bar{\rho}_o\} = \{\bar{\rho}\}$ since $\{\rho^*\}$ satisfies (158) identically in the case $\langle\bar{\rho}\rangle = 0$.

The above reasoning motivates our use of the term "asymptotic refinement" for the method of (159) since it represents a sequential correction, by simple substitutions, of the asymptotic solution given by Theorem 16.

One clear advantage of (159) is that the coefficient matrices of $\langle\bar{\rho}_{k+1}\rangle$ and $\{\bar{\rho}_{k+1}\}$ in the Lyapunov equations which must be solved on each iteration are already diagonal. Thus, each element of $\bar{\rho}_{k+1}$ is determined separately:

$$\left.\begin{aligned}
\underset{k\neq j}{\langle\bar{\rho}_{k+1}\rangle_{\ell m}} &= \frac{\langle\sigma_1\rangle_{\ell m}+\{\sigma_2\bar{\rho}_k\}^H_\ell\langle\bar{\rho}_k\rangle_{\ell m}+\langle\bar{\rho}_k\rangle_{\ell m}\{\sigma_2\bar{\rho}_k\}_m-\langle\bar{\rho}_k\sigma_2\bar{\rho}_k\rangle_{\ell m}}{-\bar{\mu}^*_\ell+\{\sigma_2\bar{\rho}_k\}^*_\ell-\bar{\mu}_m+\{\sigma_2\bar{\rho}_k\}_m+\tfrac{1}{2}(I_\ell+I_m)} && \text{a.}\\[2ex]
\{\bar{\rho}_{k+1}\}_\ell &= \frac{\{\sigma_1\}_\ell+2\mathrm{Re}\{\bar{\rho}_k\}_\ell\{\sigma_2\bar{\rho}_k\}_\ell-\{\bar{\rho}_k\sigma_2\bar{\rho}_k\}}{-2\mathrm{Re}(\bar{\mu}_\ell+\{\sigma_2\bar{\rho}_k\})} && \text{b.}
\end{aligned}\right\} \quad (162)$$

Thus, when it does converge, the asymptotic refinement method is considerably faster than the methods of Section 3.6 since, apart from evaluations of the form (162) it requires only a few matrix multiplications on each iteration. For consistency in this approach, we must at least require:

$$\left.\begin{aligned}
\frac{\langle\bar{\rho}_1\rangle_{\ell m}}{\sqrt{\{\rho^*\}_\ell\{\rho^*\}_m}} &= \frac{(\langle\sigma_1\rangle_{\ell m}-\{\rho^*\}_\ell\langle\sigma_2\rangle_{\ell m}\{\rho^*\}_m)/(\{\rho^*\}_\ell\{\rho^*\}_m)^{\frac{1}{2}}}{-\bar{\mu}^*_\ell-\bar{\mu}_m+\{\sigma_2\}_\ell\{\rho^*\}_\ell+\{\sigma_2\}_m\{\rho^*\}_m+\tfrac{1}{2}(I_\ell+I_m)}\\
&<< 1
\end{aligned}\right\} \quad (163)$$

Apart from this rule of thumb, conditions for convergence of (159) remain the object of further research. At present we can only appeal to computational experience with specific cases of which a detailed report will be given separately. As (163) would suggest, (159) is usually convergent if the reciprocal decorrelation times and the modal frequency differences are sufficiently large. In fact, (159) is typically linearly convergent for all $I \geq 0$ if the modal density decreases rapidly enough with modal order and σ_1 falls below some bound determined by the other conditions of the problem.

In addition, numerical experience or inspection of (163) give rise to the following conjecture. With the partitioning:

$$\left.\begin{aligned} &\bar{\rho}_C = \begin{bmatrix} \bar{\rho}_D & \bar{\rho}_{CD} \\ \bar{\rho}_{CD}^H & \bar{\rho}_{CC} \end{bmatrix} \\ &\bar{\rho}_D \in C^{2N_D \times 2N_D} \end{aligned}\right\} \tag{164}$$

and with $\bar{\rho}_D$ fixed, there is an N_D large enough (and comparable to the order of the quasi-deterministic system which is not directly affected by the decorrelation times) that the sequence

$$\left.\begin{aligned} &\langle\bar{\rho}_{k+1}\rangle_{\ell m} = [\text{R H S of } (162.\text{a})]_{\bar{\rho}_D} \\ &\qquad m = 2N_D+1,\ldots,2N_C\ , \ \ell < m \\ &\{\bar{\rho}_{k+1}\}_{\ell} = [\text{R H S of } (162.\text{b})]_{\bar{\rho}_D} \\ &\qquad \ell = 2N_D+1,\ldots,2N_C \end{aligned}\right\} \tag{165}$$

is convergent. Here $[\ldots]_{\bar{\rho}_D}$ denotes $(\ldots)$ evaluated with $\bar{\rho}_D$ fixed.

If true, this conjecture suggests the following scheme for solution of (158) when N_C is large:

(1) To start: Set $\bar{\rho}_{CD}$, $\bar{\rho}_{CC} = 0$ in (164) and determine $\bar{\rho}_D$ by the methods of Section 3.6.

(2) With $\bar{\rho}_D$ as determined previously, compute $\bar{\rho}_{CD}$ and $\bar{\rho}_{CC}$ by use of (165).

(3) Given $\bar{\rho}_{CD}$ and $\bar{\rho}_{CC}$ as computed in step (2), calculate $\bar{\rho}_D$ from the upper left sub-block of (158) by either of the methods of Section 3.6. Then return to step (2) unless convergence is adequate.

Consequently, the generally convergent but more elaborate Newton method or perturbed weighting method is applied to a relatively low order stochastic Riccati equation. At the same time the large order sub-blocks $\bar{\rho}_{CD}$ and $\bar{\rho}_{CC}$ are handled with the conditionally convergent but much faster asymptotic refinement method. Thus, although general conditions for the convergence of this procedure remain a subject of further inquiry, it is seen that the special form of the stochastic Riccati equation admits the possibility of efficient numerical treatment for systems of rather large order.

4. SUMMARY AND CONCLUSIONS

This work has addressed two of the principal obstacles facing the application of modern control theory to structural vibration suppression: parameter uncertainties arising from the intrinsic inaccuracies of structural modelling and difficulties in the formulation of optimal control laws imposed by the large dimension of the system.

At the outset, the need for a design method which takes full advantage of the peculiarities of the system to be controlled in providing a statistical treatment of *a priori* parameter uncertainties was recognized. In addition, it was emphasized that a suitable method must employ limited data on parameter statistics, thereby eliminating the need for a complete probability model and reducing the number of required elementary measures of parameter variation to a manageable level.

A basic inspiration has been the notion that by consistent use of a system model incorporating limited parameter information one can so arrange matters that the computation required for formulation of an optimal control policy is correspondingly limited. Thus it was hoped that the obstacles of uncertainty and dimensionality could both be circumvented by a single design methodology.

To render this initial development tractable, we limited consideration to full state feedback regulation in the presence of uncertainties in the open-loop modal frequencies. In addition, the average of a standard quadratic functional over both disturbance and parameter ensembles was chosen as the performance measure. Within these restrictions, considerable progress in the direction outlined above can now be claimed.

Chapter 2 set forth the essential ideas of a new approach. This has been summarized in Section 2.4. To briefly recapitulate, the minimum set of parameter statistical data needed to preserve any fidelity in the overall model was first sought. The controlled or uncontrolled system was seen to be comprised of several more or less distinct qualitative regimes delimited according to the values of the "modal decorrelation times." These parameters constitute new measures of parameter uncertainty and are essential to an adequate description of second moment response. A measure of design conservatism is then achieved by constructing a full probability model of frequency uncertainties which is unconstrained save for prescribed values of the decorrelation times. The resulting white parameter model allows formulation of statistically closed equations determining the mean-square optimal control law.

At the close of Chapter 2, the problem was reduced to the solution of a stochastic Riccati equation which assumes the form of the standard Riccati equation when all parameters are deterministic. The specific properties of this modified Riccati equation were developed in Chapter 3. First, the situation with regard to the existence and uniqueness of steady-state control laws was found to be entirely analogous to that of deterministic plant regulation. Under the usual stabilizability and detectability restrictions, unique steady state solutions were found to exist for all levels of uncertainty in the modal frequencies - indicating the absence of an uncertainty threshold. In addition, for such constant gain controls, strong assurances of closed-loop stochastic stability were found in Section 3.5.

The asymptotic properties of steady state solutions for large uncertainties and high order modes were investigated - with significant implications for the dimensionality issue. In the limit as all uncertainties increase without bound the solution of the stochastic Riccati equation (in the eigen-basis of the nominal system matrix) reduces to a diagonal matrix whose elements are independent of modal frequency statistics. This asymptotic solution gives rise to a velocity feedback control law which is stable regardless of the values of modal frequencies or damping ratios. If all uncertainty levels are not small but do increase with increasing modal order, the solution approaches the asymptotic solution for high order modes but may resemble a deterministic plant control for very low order, well-known modes. In other words, an inherently robust velocity feedback control automatically emerges for the poorly known high-order modes.

This feature greatly reduces dimensionality problems in the solution of the stochastic Riccati equation as shown in Section 3.8. The convergent numerical procedures set forth in Section 3.6 need only be applied to the "coherent" portion of the system which consists of relatively few well-known modes, while the velocity feedback control for the high-order, "incoherent" portion can be determined with negligible computational burden. Which modes are to be included in the coherent system can be determined in advance of any burdensome calculations according to the accuracy desired in the determination of the control gain. Finally, even if the coherent system is inconveniently large, the computational schemes advanced in Section 3.9 show promise as efficient means of solution. We conclude that the stochastic Riccati equation, which results from inclusion of parameter uncertainties in the fundamental system model, is amenable to satisfactory numerical treatment for systems of arbitrary order. Detailed numerical results illustrating the above features will be given separately.

Obviously, to be of practical value, this formulation must be extended beyond the restrictive assumptions initially adopted. First, more general types of parameter uncertainties must be accommodated. Secondly and most importantly, the assumption of full state feedback must be removed and the theory extended to treat fixed-order dynamic compensation in the presence of parameter uncertainties. These generalizations are the object of further investigation.

APPENDIX 1

Proof of Theorem 1

A. First, if $W(t_1,t_2)$ is differentiable almost everywhere then by direct substitution, it is seen that (16) satisfies:

$$\frac{\partial}{\partial t}\phi(t,\tau) = [\bar{\mu} + \nu(t) - \beta\kappa(t)]\phi(t,\tau) \tag{A1.1}$$

$$\phi(\tau,\tau) = I$$

almost everywhere and is the transition matrix of (11). If $W(t_1,t_2)$ is almost nowhere differentiable (see References 32, 21), we must re-interpret (A1.1) as

$$d\phi(t,\tau) = [\bar{\mu} - \beta\kappa(t)]\phi(t,\tau)dt + dW(t)\phi(t,\tau) \tag{A1.2}$$

where $dW(t)$ is the differential increment of W, where W now posesses independent increments. (16) may be shown to be the solution of (A1.2) in the sense that

$$\int_a^b d\phi(t,\tau) = \int_a^b [\bar{\mu} - \beta\kappa(t)]\phi(t,\tau)dt + \int_a^b dW(t)\phi(t,\tau) \tag{A1.3}$$

holds with probability 1 for all a,b.

To be consistent with the assumed almost everywhere continuity of W(t), we must take $\dot{W}(t)$ to be the limit of a band-limited process as the total power approaches infinity. This requires care in the definition of the stochastic integral:

$$G = \int_a^b dW(t)\phi(t,\tau) \tag{A1.4}$$

appearing in (A1.3). Following Stratonovich[33,34] and Wong and Zakai[35], the proper definition of (A1.4) is found to be

$$G \triangleq \lim \sum_{\nu} [W(t_{\nu+1})-W(t_{\nu})] \frac{\phi(t_{\nu},\tau)+\phi(t_{\nu+1},\tau)}{2} \tag{A1.5}$$

as $\underset{\nu}{\mathrm{MAX}}(t_{\nu+1}-t_{\nu})\to 0$, where $\{t_{\nu}\}$ is a partition of $[a,b]$. With this interpretation, (16) is seen to be the formal solution to (19). Moreover, $\phi(t,\tau)$ has the Itô stochastic differential with Stratonovich correction term of the form given in Eq. (20).

Next, the series of (16a) must be shown to be convergent. Since $\kappa(t)$ is bounded and $W(\tau,t)$ is almost everywhere bounded, there exist finite $M_{\kappa},M_W > 0$ such that for all $t \in [t_o,t_1]$, $\tau \in [t_o,t]$:

$$||\beta\kappa(t)|| < M_{\kappa}, \quad ||\phi_0(t,\tau)|| < M_W$$

where $||\ldots||$ denotes any matrix norm. Then it may be shown:

$$|\phi_k(t,\tau)| \underset{P.1}{<} M_W^{k+1} M_\kappa^k \frac{1}{k!} (t-\tau)^k$$

so that:

$$|\phi(t,\tau)| \underset{P.1}{<} M_W \exp[M_\kappa M_W(t-\tau)]$$

Therefore, the series (16a) is absolutely convergent and $\phi(t,\tau)$ exists and is bounded almost everywhere for all finite $(t-\tau)$.

B. Since $\kappa(t)$ is continuous and $W(t_1,t_2)$ is almost everywhere continuous, each $\phi_k (k = 0,\ldots\infty)$ shares this property as is evident from (16b,c). Furthermore (16a) is absolutely convergent, whence $\phi(t,\tau)$ is itself almost everywhere continuous in t and τ. This property immediately ensures continuity of the first and second moments of $\phi(t,\tau)$. To show differentiability of the mean of $\phi(t,\tau)$ we first average (16c) over the W ensemble. By virtue of (16b) it is clear that each element of $E[\phi_k(t,\tau)]$ is a functional of $\beta\kappa(\tau_\ell)$, $\ell = 1,\ldots k$ and the joint characteristic function of $W(\tau_1,t)$, $W(\tau_2,\tau_1)\ldots W(\tau,\tau_k)$. But this characteristic function is differentiable in all its arguments since increments (15) possess first-order absolute moments.[38] Moreover the arguments corresponding to $W(\tau_1,t)$, $W(\tau_2,\tau_1)\ldots W(\tau,\tau_k)$ are proportional to $(t-\tau_1)$, $(\tau_1-\tau_2)$ $\ldots(\tau_k-\tau)$ respectively. Hence, $E[\phi_k(t,\tau)]$ is a functional of continuous and differentiable functions of $t,\tau_1\ldots\tau_k$, and consequently $E[\phi(t,\tau)]$ is continuous and differentiable. An analogous argument shows this property with respect to second moments of $\phi(t,\tau)$.

C. and D. Direct substitution of (17) and (18) shows these expressions to be a formal solution of (14). The almost everywhere boundedness and continuity of $\phi(t,\tau)$ shown in the proof of part A and the assumed continuity and boundedness of $\beta\kappa$ ensure the same properties for $\psi(t,\tau)$. Consequently expression (17) exists, is bounded and continuous almost everywhere.

Substituting $\delta = \tau-t$ in (18) gives

$$\psi(t,\tau) = \phi^H(\tau,t)[\sigma_1 + \kappa^H(\tau)R_2\kappa(\tau)]\phi(\tau,t) \tag{A1.6}$$

Therefore, since $\sigma_1 \geq 0$ and $R_2 > 0$, $\psi(t,\tau)$ is positive semidefinite and so is $\rho(t)$.

Interpreting (14) in the sense of (A1.2) and (A1.3), and supposing that another solution distinct from (17) exists, we find that the difference obeys the homogeneous form of (14b,c). This has only the trivial solution, contrary to hypothesis and it follows that solution (17) is unique.

Finally, from (18), each element of $E[\psi(t,\tau)] \stackrel{\Delta}{=} \overline{\psi}(t,\tau)$ is given in terms of the second moments of $\phi(\tau,t)$. From part B. it follows that $\overline{\psi}(t,\tau)$ is a continuous and differentiable function of t. □

APPENDIX 2

Proof of Theorem 3

Here we derive the result of Theorem 3. First consider the discrete problem. The entropy may be defined using the joint statistics of the increments:

$$\left.\begin{array}{l} \chi_{\ell}^{k} \triangleq \delta_{k}\left((\ell-1)\frac{T}{N},\ \ell\frac{T}{N}\right) \\ \qquad \ell = 1,\ldots,N\ ;\ k = 1,\ldots,2n \end{array}\right\} \qquad \text{(A2.1)}$$

in place of increments (36). With

$$p_{\chi_1^1 \ldots \chi_N^{2n}}(X_1^1,\ldots,X_N^{2n})$$

denoting the joint density of increments (A2.1), where X_{ℓ}^{k} is the argument corresponding to χ_{ℓ}^{k}, we wish to determine the $p_{\chi_1^1 \ldots \chi_N^{2n}}$ which maximizes

$$H \triangleq -\int dX_1^1 \ldots dX_N^{2n}\ p_{\chi_1^1 \ldots \chi_N^{2n}}\ \ln p_{\chi_1^1, \ldots \chi_N^{2n}} \qquad \text{(A2.2)}$$

subject to conditions (a), (b) and (c), with (35.a) re-written in terms of the increments (A2.1).

The constraints imposed by the available information are seen to be independent of the joint statistics of any two sequences χ_ℓ^k; $\ell = 1,\ldots N$ and χ_ℓ^j; $\ell = 1,\ldots N$ for $k \neq j$. Moreover, H is maximized for $\chi_{\ell_1}^k$ and $\chi_{\ell_2}^j$; $j \neq k$ independent. We may conclude at once that

$$\left.\begin{aligned} p_{\chi_1^1 \ldots \chi_N^{2n}} &= \prod_{k=1}^{2n} p_{\chi_1^k \ldots \chi_N^k} \\ H &= \sum_k H^k \\ H^k &\triangleq -\int dX_1^k \ldots dX_N^k \, p_{\chi_1^k \ldots \chi_N^k} \, \ell n \, p_{\chi_1^k \ldots \chi_N^k} \end{aligned}\right\} \quad (A2.3)$$

and it remains to determine each $p_{\chi_1^k \ldots \chi_N^k}$ to maximize H^k separately.

Consider k fixed, and drop the k superscript. We may write:

$$\left.\begin{aligned} p_{\chi_1 \ldots \chi_N} &= p_{\chi_1} \, p_{\chi_2 \ldots \chi_N} \Gamma \\ \Gamma &\triangleq p_{\chi_2 \ldots \chi_N | \chi_1}(X_2 \ldots X_N | X_1) / p_{\chi_2 \ldots \chi_N}(X_2 \ldots X_N) \end{aligned}\right\} \quad (A2.4)$$

introducing Γ as a new independent function embracing the statistical dependence between χ_1 and $(\chi_2, \ldots, \chi_N)$. Clearly Γ must satisfy the constraints

$$\int dX_1 \ \Gamma \ p_{X_1} = 1$$

$$\int dX_2 \ldots dX_N \ \Gamma \ p_{X_2 \ldots X_N} = 1 \qquad \text{(A2.5)}$$

Now we may append (A2.5) to the original problem and seek Γ, p_{X_1} and $p_{X_2 \ldots X_N}$ to maximize H^k. If, for the moment, condition (a) is removed from consideration, it is easily seen that $\Gamma = 1$ maximizes the entropy. With this value, (35) may be written:

$$\frac{1}{N} \ |A_1|^2 [1 + \sum_{m=2}^{N} |A_{2m}|^2] = T_k/T \qquad \text{(A2.6)}$$

where

$$A_1 \triangleq \int dX_1 \ p_{X_1} \ e^{i\bar{\omega}_k X_1}$$

$$A_{2m} \triangleq \int dX_2 \ldots dX_N \ p_{X_2 \ldots X_N} \ e^{i\bar{\omega}_k \sum_{\ell=2}^{m} X_\ell} \qquad \text{(A2.7)}$$

and $\bar{T}_k/T \leq 1$ by (35b).

Since A_1 and A_{2m} are characteristic functions

$$|A_1|^2, \ |A_{2m}|^2 \ \varepsilon \ (0, 1)$$

so that there always exists a choice of p_{χ_1} and $p_{\chi_2 \ldots \chi_N}$ such that (A2.6) is satisfied. Consequently, H_k is maximized under all constraints by $\Gamma = 1$, and:

$$p_{\chi_1 \ldots \chi_N} = p_{\chi_1} \, p_{\chi_2 \ldots \chi_N}$$

But, by the stationary condition (b):

$$\begin{aligned} p_{\chi_2 \ldots \chi_N}(X_2 \ldots X_N) &= p_{\chi_1 \ldots \chi_{N-1}}(X_2 \ldots X_N) \\ &= p_{\chi_1}(X_2) p_{\chi_2 \ldots \chi_{N-1}}(X_3 \ldots X_N) \end{aligned}$$

Repetition of this argument (N-3) times shows that

$$p_{\chi_1 \ldots \chi_N} = \prod_{m=1}^{N} p_{\chi_1}(X_m)$$

Since this choice maximizes the entropy, we conclude that (A2.2) is maximized under conditions (35), (b) and (c) if the increments χ_ℓ^k are all mutually statistically independent and if for each k, χ_ℓ^k; $\ell = 1, \ldots N$ are identically distributed. This result holds for all N, $T \geq T_k$, $k = 1 \ldots 2n$.

Beyond this point, further conditions for entropy maximization need not be considered, because in the limits $N\uparrow\infty$ and $T\uparrow\infty$, (35) uniquely determines the statistical structure of the $\delta_k(t)$.

Considering the limit $N\uparrow\infty$, it is found that the $\delta_k(t_1)$, $t\,\varepsilon(0, T)$ are mutually independent and possess independent increments. Any increment $\delta_k(t_1,t_2)$ may be expressed as the limit of the sum of sub-increments defined over equally spaced, disjoint intervals. The sub-increments are independent and by conditions (b) and (c) are stationary and possess finite, non-zero variances. Thus, the conditions of the Lyapunov central limit theorem[38] hold and:

$$\delta_k(t_1,t_2) = N(0,\sigma_k(t_1,t_2)) \tag{A2.8}$$

$$k = 1,\ldots,2n$$

where σ_k denotes the variance. Furthermore, that $\delta_k(t)$ is zero mean and its increments are stationary and independent suffice to imply[47]

$$\sigma_k^2(t_1,t_2) = \beta_k|t_2 - t_1| \tag{A2.9}$$

where the β_k are positive constants.

With (A2.8) and (A2.9), (35) yields:

$$\frac{1}{\bar{\omega}_{N(k)}^2\beta_k}\left[1 - e^{-\bar{\omega}_{N(k)}^2\beta_k T}\right] = T_k \triangleq \frac{1}{\bar{\omega}_{N(k)}^2 I_k}$$

in the limit $N\uparrow\infty$. This determines β_k uniquely. In the limit $T\uparrow\infty$:

$$\beta_k = \frac{I_k}{\bar{\omega}_{N(k)}} \; ; \; k = 1,\ldots,2n \tag{A2.10}$$

and (39) follows.

APPENDIX 3

Proof of Corollary 2

First, it is advantageous to rearrange the state vector so that modal coordinates are the first n states and modal velocities are the second n states. In other words, perform the coordinate transformations

$$X = \begin{bmatrix} E_o \\ E_e \end{bmatrix} X' \ , \ \bar{\xi} = \begin{bmatrix} E_o \\ E_e \end{bmatrix} \xi' \qquad \text{(A3.1)}$$

where the m^{th} row $(m = 1,\ldots,n)$ of E_o is e^T_{2m-1} while the m^{th} row of E_e is e^T_{2m} where e_i denotes the standard unit basis vector.

Under this transformation, the various matrices appearing in (48) become:

$$\left.\begin{aligned}
\bar{\mu} &= \begin{bmatrix} \nu & o \\ o & \nu^* \end{bmatrix} \ ; \ \nu \triangleq \mathrm{diag}(i\bar{\omega}_1 - \eta_1\bar{\omega}_1, i\bar{\omega}_2 - \eta_2\bar{\omega}_2, \ldots) \\
I &= \begin{bmatrix} J & o \\ o & J \end{bmatrix} \ ; \ J \triangleq \mathrm{diag}(\bar{\omega}_1 I_2, \bar{\omega}_2 I_4, \ldots) \\
\sigma_1 &= \begin{bmatrix} s_1 & s_{12} \\ s^*_{12} & s^*_1 \end{bmatrix} \ ; \ s_1 = s^H_1 \ , \ s_{12} = s^T_{12} \\
\sigma_2 &= \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}
\end{aligned}\right\} \qquad \text{(A3.2)}$$

where C is real. Consistent with the above partitionings, the matrix of $\overline{\rho}$ under transformation (A3.1) may be written

$$\overline{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^H & \rho_{22} \end{bmatrix} \tag{A3.3}$$

so that:

$$I\{\overline{\rho}\} = \begin{bmatrix} J\{\rho_{11}\} & 0 \\ 0 & J\{\rho_{22}\} \end{bmatrix} \tag{A3.4}$$

Using (A3.2), (A3.3) and (A3.4) we may expand (48) into its four sub-blocks. Manipulation of these relations yields:

$$\left.\begin{aligned} -\dot{z}_1 &= (\nu-\tfrac{1}{2}J)z_1 + z_1(\nu-\tfrac{1}{2}J) - z_1C\rho_{11} - \rho_{12}^H C\, z_2 + z_1 C\, z_2 \\ &\quad + z_1 C\, \rho_{12}^H + \rho_{12}^H C\, z_1 - z_1 C\, z_1 - z_2^* C\rho_{11} + \rho_{11}^* C\, z_2 \\ &\quad + z_2^* C\, \rho_{12}^H - \rho_{11}^* C\, z_1 \\ z_1(t_1) &= 0 \end{aligned}\right\} \tag{A3.5}$$

$$\left.\begin{aligned}
-\dot{z}_2 &= (\nu^* - \tfrac{1}{2}J)z_2 + z_2(\nu - \tfrac{1}{2}J) + J\{z_2\} \\
&\quad -z_2C\,\rho_{11} - \rho_{11}C\,z_2 + z_2C\,z_2 + z_2C\,\rho_{12}^H + \rho_{11}C\,z_1 - z_2C\,z_1 \\
&\quad +z_1^HC\,\rho_{11} + \rho_{12}C\,z_2 - z_1^HC\,z_2 - z_1^HC\,\rho_{12}^H - \rho_{12}C\,z_1^H + z_1^HC\,z_1^H \\
z_2(t_1) &= 0
\end{aligned}\right\} \quad \text{(A3.6)}$$

where:

$$\begin{aligned}
z_1 &\triangleq \rho_{12}^H - \rho_{12}^* \\
z_2 &\triangleq \rho_{11} - \rho_{22}^*
\end{aligned} \qquad \text{(A3.7)}$$

From the homogeneity of (A3.5) and (A3.6) in z_1 and z_2 we conclude:

$$z_1 = z_2 = 0 \;;\; t\,\varepsilon[t_o,t_1] \qquad \text{(A3.8)}$$

so that any hermitian solution of (48) assumes the form:

$$\left.\begin{aligned}
&\bar{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{11}^* \end{bmatrix} && \text{a.} \\
&\rho_{11} = \rho_{11}^H \;,\; \rho_{12} = \rho_{12}^T && \text{b., c.}
\end{aligned}\right\} \quad \text{(A3.9)}$$

Under transformation (A3.1), the definition (8) produces:

$$\Phi = \begin{bmatrix} I & I \\ i\Omega & -i\Omega \end{bmatrix} \tag{A3.10}$$

$$\Omega \triangleq \mathrm{diag}(\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_n)$$

so that

$$\bar{P} \triangleq \Phi^{-1} H_\rho \Phi^{-1}$$

$$= \tfrac{1}{2} \begin{bmatrix} \mathrm{Re}[\rho_{11}+\rho_{12}] & \mathrm{Im}[\rho_{11}-\rho_{12}]\Omega^{-1} \\ -\Omega^{-1}\mathrm{Im}[\rho_{11}+\rho_{12}] & \Omega^{-1}\mathrm{Re}[\rho_{11}-\rho_{12}]\Omega^{-1} \end{bmatrix} \tag{A3.11}$$

This shows that $\bar{P}$ is real and, by virtue of (A3.9b,c) is symmetric. □

APPENDIX 4

Proof of Lemma 1

A. With $(\bar{\mu}_m, \sigma_1^{\frac{1}{2}})$ detectable, the rigid body modes are necessarily reconstructible. Furthermore since damping in all elastic modes is assumed non-zero, only states corresponding to elastic modes are contained in the unreconstructible subspace. Now $\bar{\mu}_e - \frac{1}{2}I_e$ is diagonal and its non-zero elements distinct since the $\bar{\omega}_k$, $k = 1,\ldots,n_e$ were assumed distinct in Section 3.1. Thus the unreconstructible subspace is spanned by a set of unit basis vectors (in the eigen-basis of A) corresponding to the distinct unreconstructible poles of $\bar{\mu}_e - \frac{1}{2}I_e$. With an analogous result for the reconstructible subspace, it follows that the system equations may be put into a reconstructibility canonical form merely by rearrangement of the state vector as in (79). The forms given by (80) for $\bar{\mu}_m$ and σ_1 then follow at once.

B. First we show that the solution to (64) exists. The right side of (64a) is analytic in $\bar{\rho}_{kj}, \forall(k,j)$, whence $\bar{\rho}(\tau)$ is continuously differentiable at τ when $\bar{\rho}(\tau)$ exists. Hence $\bar{\rho}(\tau)$ also exists on $[\tau - \varepsilon, \tau]$ for some $\varepsilon > 0$ by analytic continuation[48]. Then sirce $\bar{\rho}(t_1)$ is given, $\bar{\rho}(t)$ exists on $(-\infty, t_1]$.

Next, let $q \in C^{2n}$ be defined on $(-\infty, t_1]$ by

$$\dot{q} = (\bar{\mu}_m - \tfrac{1}{2}\sigma_2\bar{\rho})q \; ; \; q(t_1) = q_1$$

for any $q_1 \in C^{2n}$. Differentiation yields:

$$\frac{d}{dt}(q^H\bar{\rho}q) = -q^H[I\{\bar{\rho}\} + \sigma_1]q$$

so that:

$$q^H(t)\bar{\rho}(t)q(t) = \int_t^{t_1} d\tau \; q^H(\tau)[\mathcal{I}\{\bar{\rho}(\tau)\} + \sigma_1]q(\tau)$$

and it thus follows:

$$q^H(t)\bar{\rho}(t)q(t) \geq 0 \;, \; \forall \; t \leq t_1$$

Note that $q(t)$ can be selected as any vector in C^{2n} because $q(t) = \Phi(t - t_1)q_1$ and state transition matrix Φ is nonsingular. Thus $\bar{\rho}(t)$ is positive semi-definite on $(-\infty, t_1]$.

Finally we show that $\bar{\rho}$ reduces to the form given by (81) and (82). Partition $\bar{\rho}$ in a manner consistent with (79):

$$\bar{\rho} \triangleq \begin{bmatrix} \hat{\bar{\rho}} & \bar{\rho}_{12} \\ \bar{\rho}_{12}^H & \bar{\rho}_2 \end{bmatrix} \qquad \text{(A4.1)}$$

Then with (80) and (83a) the (1,2) and (2,2) sub-blocks of (64) may be written:

$$\left.\begin{aligned}
-\dot{\bar{\rho}}_{12} &= \hat{\bar{\mu}}_m^H \bar{\rho}_{12} + \bar{\rho}_{12} \bar{\mu}_{mu} - \hat{\bar{\rho}}[\hat{\sigma}_2 \bar{\rho}_{12} + \hat{\beta} R_2^{-1} \beta_u^H \bar{\rho}_2] \\
&\qquad - \bar{\rho}_{12}[\beta_u R_2^{-1} \hat{\beta}^H \bar{\rho}_{12} + \beta_u R_2^{-1} \beta_u^H \bar{\rho}_2] \\
-\dot{\bar{\rho}}_2 &= \bar{\mu}_{mu}^H \bar{\rho}_2 + \bar{\rho}_2 \bar{\mu}_{mu} + \mathcal{I}_u\{\bar{\rho}_2\} - \bar{\rho}_{12}^H[\hat{\sigma}_2 \bar{\rho}_{12} + \hat{\beta} R_2^{-1} \beta_u^H \bar{\rho}_2] \\
&\qquad - \bar{\rho}_2[\beta_u R_2^{-1} \hat{\beta}^H \bar{\rho}_{12} + \beta_u R_2^{-1} \beta_u^H \bar{\rho}_2] \\
\bar{\rho}_{12}(t_1) &= \bar{\rho}_2(t_1) = 0
\end{aligned}\right\} \quad (A4.2)$$

It is seen from the above that $\bar{\rho}_{12}$ and $\bar{\rho}_2$ are continuously differentiable at $t = t_1$ and, moreover, all derivatives vanish. Thus by analytic continuation:

$$\bar{\rho}_{12} = \bar{\rho}_2 = 0 \; ; \; \forall \, t \leq t_1$$

and (81) is justified. Substitution of (81) into (64) produces (82) as the only non-zero sub-block. □

APPENDIX 5

Proof of Theorem 8

First, note that by virtue of Lemma 2 the steady state solution, $\bar{\rho}$, of (64) is given by

$$\bar{\rho} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$$

Next define Q(t) by (69) with v = 0 and Q_o any positive semi-definite matrix. Partitioning Q(t) consistently with (79):

$$Q = \begin{bmatrix} \hat{Q} & Q_{12} \\ Q_{12}^H & Q_2 \end{bmatrix} \tag{A5.1}$$

and substituting this into (69) along with (80) and (83), one obtains:

$$\left.\begin{aligned} \dot{\hat{Q}} &= (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)\hat{Q} + \hat{Q}(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H + \hat{I}\{\hat{Q}\} \\ \dot{Q}_{12} &= (\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)Q_{12} + Q_{12}\,\bar{\mu}_{mu}^H - \hat{Q}\,\Lambda\,\hat{\beta}\,R_2^{-1}\,\beta_u^H \\ \dot{Q}_2 &= \bar{\mu}_{mu}Q_2 + Q_2\,\bar{\mu}_{mu}^H + I_u\{Q_2\} \\ &\qquad -\beta_u\,R_2^{-1}\,\hat{\beta}^H\,\Lambda\,Q_{12} - Q_{12}^H\,\Lambda\,\hat{\beta}\,R_2^{-1}\,\beta_u^H \end{aligned}\right\} \tag{A5.2}$$

By assumption, $\Delta_{\hat{Q}}[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)Q + Q(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H + \hat{I}\{Q\}]$ is asymptotically stable. Thus, from (A5.2a), $\hat{Q}$ converges exponentially to zero and there exist real $\alpha_o, \beta_o > 0$ such that:

$$||\hat{Q}||_2 \le \beta_o \, e^{-\alpha_o(t-t_o)} \tag{A5.3}$$

Note that the contribution of $\hat{I}\{Q\}$ to $\Delta_{\hat{Q}}$ is of the form of a diagonal matrix, each diagonal element being either zero or equal to one of the diagonal elements of $\hat{I}$. Consequently:

$$\lambda_R\{\Delta_{\hat{Q}}[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)\hat{Q} + \hat{Q}(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H]\}$$

$$\le \lambda_R\{\Delta_{\hat{Q}}[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)\hat{Q} + \hat{Q}(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H + \hat{I}\{Q\}]\}$$

where $\lambda_R\{\ldots\}$ denotes the real part of the eigenvalue. Since, by assumption the right side above is negative, $\Delta_{\hat{Q}}[(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)\hat{Q} + \hat{Q}(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)^H]$ is asymptotically stable; and this implies that $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda$ is asymptotically stable.

By the detectability assumption of Lemma 1 and the condition of non-zero damping on all elastic modes, $\bar{\mu}^H_{mu}$ is also asymptotically stable, whence (A5.2b) has the unique solution:

$$Q_{12} = e^{(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)(t-t_o)} \, Q_{12}(t_o) \, e^{\bar{\mu}^H_{mu}(t-t_o)}$$

$$- \int_{t_o}^{t} d\tau \, e^{(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)(t-\tau)} \, \hat{Q} \, \Lambda \, \hat{\beta} \, R_2^{-1} \, \beta_u^H \, e^{\bar{\mu}^H_{mu}(t-\tau)}$$

The stability of $\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda$ and $\bar{\mu}_{mu}^H$ ensures exponential convergence to zero of the first term above. Moreover, the bound (A5.3) and the stability of $(\hat{\bar{\mu}}_m - \hat{\sigma}_2\Lambda)$ and $\bar{\mu}_{mu}$ imply exponential convergence for the second term as well. Therefore, there exist real and positive α_1, β_1 such that:

$$||Q_{12}||_2 \leq \beta_1 e^{-\alpha_1(t-t_o)} \tag{A5.4}$$

Next we show that $\Delta_{Q2}[\bar{\mu}_{mu}Q_2 + Q_2\bar{\mu}_{mu}^H + I_u\{Q_2\}]$ is asymptotically stable for all $I_u \geq 0$. Consider

$$\left.\begin{aligned} \dot{Q}_2 &= \bar{\mu}_{mu}Q_2 + Q_2\bar{\mu}_{mu}^H + I_u\{Q_2\} \\ Q_2(t_o) &= Q_{20} \end{aligned}\right\} \tag{A5.5}$$

for any hermitian $Q_{2o} \geq 0$. Recalling that $\bar{\mu}_{mu} \triangleq \bar{\mu}_u - \frac{1}{2}I_u$, we have from (A5.5):

$$\frac{d}{dt} \operatorname{tr} Q_2 = 2 \operatorname{tr}[(\operatorname{Re} \bar{\mu}_u)Q_2] \leq 0$$

where the equality on the right holds only if $||Q_2||_2 = 0$. Moreover, as $[\operatorname{tr} Q_2] > 0$ for $t \in [t_o,\infty)$, $\operatorname{tr} Q_2$ is seen to converge exponentially to zero as $t\uparrow\infty$. This implies exponential convergence for $\{Q_2\}$ also since $||\{Q_2\}||_2 \leq \operatorname{tr} Q_2$. From (A5.5), $Q_2(t)$ satisfies

$$Q_2(t) = e^{\bar{\mu}_{mu}(t-t_o)} Q_{2o}\, e^{\bar{\mu}^H_{mu}(t-t_o)}$$

$$+ \int_{t_o}^{t} d\tau\, e^{\bar{\mu}_{mu}(t-\tau)} \mathcal{I}_u\{Q_2\}\, e^{\bar{\mu}^H_{mu}(t-\tau)}$$

Stability of $\bar{\mu}_{mu}$ and exponential convergence of $\{Q_2\}$ imply exponential convergence of $Q_2(t)$ as defined by (A5.5). Therefore $\Delta_{Q2}[\bar{\mu}_{mu}Q_2 + Q_2\bar{\mu}^H_{mu} + \mathcal{I}_u\{Q_2\}]$ is asymptotically stable. Finally, from (A5.2c) this property together with the norm bound (A5.4) and implies asymptotic convergence of $||Q_2||_2$.

Combining the above results, one sees from (A5.1) that

$$\lim_{t-t_o \uparrow \infty} Q = 0$$

so that $\Delta_Q[(\bar{\mu}_m - \sigma_2\bar{\rho})Q + Q(\bar{\mu}_m - \sigma_2\bar{\rho})^H + \mathcal{I}\{Q\}]$ is asymptotically stable. □

APPENDIX 6

Proof of Lemma 6

To prove the first statement, consider:

$$0 = \hat{\bar{\mu}}^H Y + Y\hat{\bar{\mu}} + s_1 - Y\,\hat{\sigma}_2\,Y$$

$$s_1 \geq 0, \; (\hat{\bar{\mu}},\, s_1^{\frac{1}{2}}) \text{ completely reconstructible} \tag{A6.1}$$

The conditions on s_1 and the assumed stabilizability of $(\hat{\bar{\mu}},\hat{\beta})$ ensure, by Lemma 5 that (A6.1) has a unique positive definite solution and that $(\hat{\bar{\mu}} - \hat{\sigma}_2 Y)$ is asymptotically stable.

Now, with the control gain $R_2^{-1}\hat{\beta}^H Y$, the reconstructible state, $\hat{\xi}$, is given by:

$$\dot{\hat{\xi}} = (\hat{\bar{\mu}} - \hat{\sigma}_2 Y) + \sum_k B_k \hat{\delta}_k(t)\,\hat{\xi} \tag{A6.2}$$

where B_k is zero except for the (k,k) element which equals $i\,\mathrm{Im}\hat{\bar{\mu}}_k$. Clearly B_k is skew-hermitian. This fact, together with the stability of $\hat{\bar{\mu}} - \hat{\sigma}_2 Y$ imply[46] that the null solution of (A6.2) is P^{th} mean exponentially stable for any P (in particular, P = 2). Since Δ_Λ is the hermitian transpose of the equivalent coefficient matrix of the second moment matrix of (A6.2), Δ_Λ is asymptotically stable.

In view of the stability properties established above for (A6.2) we can show the result for the case in which the system lacks rigid body modes (i.e., $\bar{\mu} = \bar{\mu}_e$), by demonstrating the asymptotic stability of $\hat{\bar{\mu}} - \hat{\sigma}_2 Y$ for Y positive and diagonal. Thus, consider:

$$\dot{\xi} = (\hat{\bar{\mu}} - \hat{\sigma}_2 Y)\ \xi$$
$$\xi(t_o) = \xi_o \tag{A6.3}$$

for any $\xi_o \neq 0$ and positive, diagonal Y. Defining

$$\mathcal{L} \triangleq \tfrac{1}{2}\xi^H Y\ \xi \tag{A6.4}$$

we obtain by (A6.3):

$$\frac{d}{dt}\mathcal{L} = (Y^{\frac{1}{2}}\xi)^H [\mathrm{Re}\hat{\bar{\mu}} - Y^{\frac{1}{2}}\hat{\sigma}_2 Y^{\frac{1}{2}}](Y^{\frac{1}{2}}\xi) \tag{A6.5}$$

The right side is negative for all $\xi \neq 0$ since damping is assumed non-zero for all elastic modes. Therefore $\mathcal{L}$ is a Lyapunov function for (A6.3) and $\hat{\bar{\mu}} - \hat{\sigma}_2 Y$ is asymptotically stable. By the argument following (A6.2), the asymptotic stability of Δ_Λ is established. □

APPENDIX 7

Proof of Theorem 11

First, $(-\frac{1}{2}I\overline{\rho} + I\{\overline{\rho}\} - \frac{1}{2}\overline{\rho}I)$ is shown to be negative definite. Let W be a real diagonal matrix of zero-mean, statistically independent, Gaussian random variables such that:

$$E[W_{kk}^2] = I_{kk} \tag{A7.1}$$

Then, defining:

$$\Gamma \triangleq \overline{\rho} - e^{iW\varepsilon}\overline{\rho}\, e^{-iW\varepsilon}, \qquad \varepsilon > 0 \tag{A7.2}$$

one obtains the identity:

$$-\frac{1}{2}I\overline{\rho} + I\{\overline{\rho}\} - \frac{1}{2}\overline{\rho}I = -\underset{\varepsilon \downarrow 0}{\mathrm{LIM}}\, \frac{1}{\varepsilon^2}\, E[\Gamma] \tag{A7.3}$$

Employing the reconstructibility decomposition afforded by Lemma 1, Γ assumes the form

$$\Gamma = \begin{bmatrix} \hat{\Gamma} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\Gamma} = \Lambda - e^{i\hat{W}\varepsilon}\Lambda\, e^{-i\hat{W}\varepsilon} \tag{A7.4}$$

where Λ is the positive definite solution to (88). Letting $\tilde{W} \triangleq \Lambda^{\frac{1}{2}}\hat{W}\Lambda^{-\frac{1}{2}}$ and $\tilde{\xi} \triangleq \Lambda^{\frac{1}{2}}\xi$ for any $\xi \neq 0$, we have:

$$\begin{aligned} \xi^H\hat{\Gamma}\xi &= \tilde{\xi}^H[I - e^{i\tilde{W}^H\varepsilon}e^{-i\tilde{W}\varepsilon}]\tilde{\xi} \\ &= ||\tilde{\xi}||_2 - ||e^{-i\tilde{W}\varepsilon}\tilde{\xi}||_2 \end{aligned} \tag{A7.5}$$

Since:

$$\begin{aligned} ||e^{i\tilde{W}\varepsilon}\tilde{\xi}||_2 &\leq ||\tilde{\xi}||_2||e^{i\tilde{W}\varepsilon}||_2 \\ &\leq ||\tilde{\xi}||_2 \end{aligned}$$

where the last line follows because $i\tilde{W}$ is skew-hermitian, (A7.5) yields:

$$\xi^H\hat{\Gamma}\xi \geq 0 \;,\; \forall\, \xi \neq 0$$

Thus $\hat{\Gamma}$ and therefore Γ is positive semi-definite for each realization of W over its statistical ensemble. Clearly $E[\Gamma]$ is also positive semi-definite and is of order ε^2 by definition. Consequently, the right side of (A7.3) exists as a negative semi-definite matrix. This suffices to show:

$$S \leq \sigma_1 \tag{A7.6}$$

Next, rearranging (102), $\bar{\rho}$ is seen to be a positive semi-definite solution of

$$0 = \bar{\mu}^H \bar{\rho} + \bar{\rho}\,\bar{\mu} - \bar{\rho}\,\sigma_2\,\bar{\rho} + S \tag{A7.7}$$

Because $\lambda_R\{\bar{\mu}\} \leq 0$ and $\bar{\rho} \geq 0$, $(\bar{\mu}^H \bar{\rho} + \bar{\rho}\,\bar{\mu})$ is negative semi-definite. Consequently:

$$S = -(\bar{\mu}^H \bar{\rho} + \bar{\rho}\,\bar{\mu}) + \bar{\rho}\,\sigma_2\,\bar{\rho} \geq 0 \tag{A7.8}$$

Finally, we show that detectability is preserved despite the presence of the stochastic terms in (107). Denoting $(-\frac{1}{2}I\bar{\rho} + I\{\bar{\rho}\} - \frac{1}{2}\bar{\rho}I)$ by α and partitioning in accordance with (59), one obtains

$$\alpha = \begin{bmatrix} 0 & \alpha_{re} \\ \alpha_{re}^H & \alpha_e \end{bmatrix} \tag{A7.9}$$

by virtue of (63a). Denoting the "r" sub-block of σ_1 by σ_{1r}, it follows that $(\sigma_1 + \alpha)^{\frac{1}{2}}$ assumes the form

$$(\sigma_1 + \alpha)^{\frac{1}{2}} = (\sigma_{1r}^{\frac{1}{2}}, \tilde{s}) \tag{A7.10}$$

i.e., the first $2n_r$ columns of $(\sigma_1 + \alpha)^{\frac{1}{2}}$ and of $\sigma_1^{\frac{1}{2}}$ are identical.

Now, with $\bar{\mu}_r$ of the form (60), the necessary and sufficient condition for complete reconstructibility of the rigid body modes is (see p.45 of Reference (44)).

$$\text{rank}[\bar{\mu}_r^H, (\sigma_1 + \alpha)_r^{\frac{1}{2}H}] = n_r \tag{A7.11}$$

where $(\sigma_1 + \alpha)_r^{\frac{1}{2}}$ denotes the "r" sub-block of $(\sigma_1 + \alpha)^{\frac{1}{2}}$, and is merely $\sigma_{1r}^{\frac{1}{2}}$ by (A7.10). Thus (A7.11), which becomes:

$$\text{rank}[\bar{\mu}_r^H, \sigma_{1r}^{\frac{1}{2}H}] = n_r \tag{A7.12}$$

is satisfied since $(\bar{\mu}, \sigma_1^{\frac{1}{2}})$ was assumed detectable. Therefore $(\bar{\mu}, S^{\frac{1}{2}})$ is detectable. □

APPENDIX 8

Proof of Theorem 17

The boundedness conditions on η_k, B and R_2 may be stated as:

$$\left.\begin{array}{ll} b_{kj} < \bar{b} & \text{a.} \\ R_{2kk} > r & \text{b.} \\ \underline{\eta} \leq \eta_k \leq \bar{\eta} & \text{c.} \end{array}\right\} \tag{A8.1}$$

where $\bar{b}$, r, $\underline{\eta}$ and $\bar{\eta}$ are positive and the b_{kj} are non-zero elements of B as given by (3). From (3), (8) and (A8.1a,b) it follows:

$$\sigma_{2kk} < \frac{1}{4r} \frac{\bar{b}^2}{\bar{\omega}^2_{N(k)}} \tag{A8.2}$$

This, together with condition (2) of the Theorem and (123) yield:

$$|\langle\bar{\rho}\rangle_{kj}| \leq \frac{4\bar{\sigma}_1 \bar{\omega}_{N(k)} \; \bar{\omega}_{N(j)}}{I_k \bar{\omega}_{N(k)} + I_j \bar{\omega}_{N(j)}} \tag{A8.3}$$

With use of (A8.2) and (A8.3), manipulation of (124a) produces:

$$|B_k| \le \frac{2}{r}\,\bar{b}^2\,\bar{\sigma}_1 f_k \tag{A8.4}$$

Similarly, (124b) becomes:

$$|C_k| \le \frac{4}{r}\,\bar{\sigma}_1^{-2}\,\bar{\omega}_{N(k)}^{-2}\,\bar{b}^2 f_k^2 \tag{A8.5}$$

where

$$f_k \triangleq \sum_{\ell=1}^{2n} \frac{1}{I_k\bar{\omega}_{N(k)} + I_\ell\bar{\omega}_{N(\ell)}} \tag{A8.6}$$

In view of conditions (1) and (3), f_k possesses an upper bound proportional to the Riemann zeta function, $\zeta(s)$, with $s > 1$. Therefore f_k is convergent as $n\uparrow\infty$ and, moreover, both f_k and $\underset{n\uparrow\infty}{LIM}\, f_k$ are monotone decreasing with increasing k. Consequently, there exists an $\tilde{N}$ sufficiently large that:

$$\left.\begin{array}{lr} \{\sigma_1\}_k - C_k > 0 & \text{a.} \\[2ex] 1 - \dfrac{\underline{\sigma}_1\bar{b}^2}{\underline{\eta} r\bar{\omega}_{N(k)}}\, f_k > 0 & \text{b.} \end{array}\right\} \quad \text{(A8.7)}$$

for all $k > 2\tilde{N}$, and we may restrict attention, in the following to the behavior of ρ_{CI} and ρ_I for $N_c > \tilde{N}$.

Condition (A8.7a) permits the use of part (b) of Theorem (15) in the evaluation of an upper bound to $|\{\bar{\rho}\}_k - \{\rho^*\}_k|$. But first, bounds must be determined for $\{\rho^*\}_k$. Note that (122) can be written in the form:

$$\left.\begin{aligned} \{\rho^*\}_k &= \frac{\sigma_{1kk}}{|\mathrm{Re}\bar{\mu}_k|}\left\{\frac{1}{x}[-1+\sqrt{1+x}]\right\} && \text{a.} \\ x &\triangleq \frac{\sigma_{1kk}\sigma_{2kk}}{(\mathrm{Re}\,\bar{\mu}_k)^2} \geq 0 && \text{b.} \end{aligned}\right\} \quad \text{(A8.8)}$$

Obviously:

$$\{\rho^*\}_k \leq \frac{1}{2}\frac{\sigma_{1kk}}{|\mathrm{Re}\bar{\mu}_k|}, \ \forall\, k,\ x \geq 0$$

By virtue of (A8.2), (A8.1c) and condition (2) of the Theorem, x decreases with k increasing. Consequently, as (A8.8a) reveals, for given N there is an $M > 1$ such that:

$$\{\rho^*\}_k \geq \frac{1}{2M}\frac{\sigma_{1kk}}{|\mathrm{Re}\bar{\mu}_k|}, \ \forall\, k > 2N$$

In the following discussion, N is considered fixed with $N < \tilde{N}$ so that M is fixed. With this proviso, we have:

$$\left.\begin{array}{c}\frac{1}{2M}\,\frac{\underline{\sigma}_1}{\bar{\eta}}\,\bar{\omega}_{N(k)} \le \{\rho^*\}_k \le \frac{1}{2}\,\frac{\bar{\sigma}_1\bar{\omega}_{N(k)}}{\underline{\eta}} \\ \forall\; k > 2\tilde{N}\end{array}\right\} \qquad \text{(A8.9)}$$

Using (A8.1), (A8.4), (A8.5), (A8.9) and conditions (1) through (3) of the Theorem, (126) may be made to assume the form:

$$\frac{|\{\bar{\rho}\}_k - \{\rho^*\}_k|}{\{\rho^*\}_k} \le \frac{\bar{\eta}}{\underline{\eta}}\,\frac{M}{r}\,\bar{\sigma}_1\bar{b}^2\,\frac{\left[4f_k^2 + \frac{1}{\underline{\eta}\;\bar{\omega}_{N(k)}}f_k\right]}{\left[1 - \frac{\underline{\sigma}_1\bar{b}^2}{\underline{\eta}r\;\bar{\omega}_{N(k)}}f_k\right]} \qquad \text{(A8.10)}$$

Since f_k is monotone decreasing with k and in view of (A8.7b), the right side of (A8.10) is a positive monotone decreasing function of k for $k > 2\tilde{N}$. Thus, given $\tilde{\varepsilon} > 0$, there is an $\tilde{N}_c$ such that:

$$\frac{1}{\{\rho^*\}_k}\,|\{\bar{\rho}\}_k - \{\rho^*\}_k| \le \tilde{\varepsilon}\;;\; k \ge \tilde{N}_c \qquad \text{(A8.11)}$$

In particular, let $\tilde{N}_{c1}$ correspond to $\tilde{\varepsilon} < 1$ in (A8.11). Then:

$$\{\bar{\rho}\}_k \ge \{\rho^*\}_k(1 - \tilde{\varepsilon}) > 0\;;\; k \ge \tilde{N}_{c1} \qquad \text{(A8.12)}$$

Using this, (A8.3) yields:

$$\left.\begin{aligned} |\langle \check{\rho} \rangle_{kj}| &\le \left(\frac{2M\bar{\eta}}{\{\bar{\rho}\}_j (1-\tilde{e}) \underline{\sigma}_1} \right)^{\frac{1}{2}} \frac{4\,\bar{\sigma}_1\,\bar{\omega}_{N(k)}\,\bar{\omega}_{N(j)}}{I_k \bar{\omega}_{N(k)} + I_j \bar{\omega}_{N(j)}} \\ k &\ge \tilde{N}_{c1} \end{aligned}\right\} \qquad (A8.13)$$

The right side of the above inequality is a positive, monotone decreasing function of k for $k \ge \tilde{N}_{c1}$. Therefore, from (A8.11) and (A8.13), it is possible to determine a $N_c \ge \tilde{N}_{c1}$ such that both:

$$\left.\begin{aligned} &\frac{1}{\{\rho^*\}_k} \left| \{\bar{\rho}\}_k - \{\rho^*\}_k \right| \le \varepsilon \ , \ \forall\, k \ge N_c \\ \text{and} \\ &|\langle \check{\rho} \rangle_{kj}| \le \varepsilon \ ; \ \forall\, j \ , \ \forall\, k \ge N_c \end{aligned}\right\} \qquad (A8.14)$$

These relations suffice to show (152) directly. □

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Difficulties arising from inherent inaccuracies in structural modelling and from the high dimensionality of the dynamical system force a re-examination of the problem of optimal control of large flexible structures within the context of stochastic system theory. This work specifically addresses the mean-square optimal control of a linear mechanical system with random open-loop modal frequencies. Recognizing that a complete probabilistic description of modal parameters can never be provided, we first identify the minimum set of a-priori data needed to preserve any measure of modelling fidelity. Acknowledging this data as available, a complete probability assignment is then induced by a maximum entropy principle. The resulting mean-square optimization problem is consequently reduced to the solution of a modified Riccati equation. Various properties of this equation are explored. In particular, it is shown that the proposed approach guarantees stochastic stability, directly produces inherently robust controls for highly uncertain systems and significantly reduces the burden of design computation.

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