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**ESTIMATING AVAILABILITY  
FOR SYSTEMS WITH REDUNDANCY,  
SPARES AND INSTALLATION TIMES**

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This paper shows how to compute expected steady state system availability for a system with parallel redundancy, when there is a pool of spare components. There may be several systems supported by this same pool, which is managed by a supplier following an (S,S-1) inventory policy. Failures are exponential, but non-exponential resupply and installation times are considered.		

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## 1. Introduction

This paper investigates the evaluation of steady state system availability for a system with parallel redundancy, when there is a pool of spare components. There may be several systems supported by this same pool. The pool is managed by a supplier following an (S,S-1) inventory policy under continuous review.

The basic sequence of events is: component fails; spare component is obtained, immediately if one is available; component is installed. Because of redundancy, the system need not be down while all this is occurring. When a component fails, a good component is made available to the pool manager a resupply time later. This time represents time to repair the component, or order it if it is not repairable.

Two alternative basic premises are considered:

"Cold Standby" - Component only fails when it is being used, and only one of the parallel components is used at a time.

"Warm Standby" - All of the parallel components are subject to failure at any given time, at the same rate.

Events are viewed as a Markov process and the steady state difference equation approach is used (c.f. Gross and Harris). The equations are presented in detail for our basic assumption set: one system is supported; there is warm standby; components can fail during installation. We indicate how to modify the difference equations to relax each of these assumptions. However, multiple systems supported cannot conveniently be handled unless there is warm standby and the spare pool manager follows a FIFO policy; i.e., backorders are eliminated in the order in which they occur.

It is assumed resupply and installation times are independently and identically distributed. In general, they must also both be exponential. However, in the second part of this report these cases are considered: general installation and resupply times but zero spares; deterministic installation times; exponential installation times but deterministic resupply times. Exact results are possible for the first two cases, an approximation for the deterministic resupply times.

The steady-state difference equations are not explicitly solved in this paper - there is no need as a simple computer program, for general number of components, bypasses this step.

## 2. Earlier Work

Surprisingly, considering the important practical applications, the problem addressed in this paper has received relatively little attention in the literature. Forry [5] reports on a model, ACCLOGTROM, with antecedents dating back to the 1960's, which not only evaluates system availability under more general system structures than are considered here, but optimizes inventory investment as well. However, this model relies on a simplification which can lead to very poor estimates in some cases. Similarly, Bein [2] considers evaluation of system availability in a system with redundancy and supply, and allows for partial degradation, but his results do not rest on a rigorous foundation.

We have learned that in the 1960's, Amster and Morra [1] at Bell Labs developed a working model based on the steady state difference approach, assuming exponential resupply, installation and repair times, and considering a single system supported by the pool manager. There are a number of rigorous published papers pertaining to the availability of a single system with redundancy and repair, but zero installation time. Among these, Natarajan and Rao [8] allow for deterioration of units in storage and derive steady state availability and expected time to system failure under exponential assumptions. Srinivasan and Gopalan [10] derive the mean time to system failure and the Laplace transform of the state probabilities,  $p^*(s)$ , for a two unit system with one repair channel and general repair distribution (by taking the limit as  $s$  goes to 0 of  $s p^*(s)$ , steady state availability can be determined). Gopalan\* [6] generalizes these results to  $n$  units with failure rate  $f(n)$  (but still one repair facility), thereby allowing for units in supply. Subramanian et al [11] allow for general failure distribution for an on-line unit, but exponential distributions for steady failures and repair times.

---

\* From a summary in International Abstracts in Operations Research.

### 3. Notation

- $\lambda$  - failure rate for  
    a system      Cold Standby  
    a component    Warm Standby
- $\mu$  - resupply rate, i.e. reciprocal of mean resupply time
- $\gamma$  - installation rate
- $n$  - number of systems supported by pool manager
- $c$  - number of redundant components per system
- $s$  - pool manager's stockage parameter
- $k$  - number of components in resupply
- $i$  - number of components being installed
- $P_{ik}$  - steady state probability of  $i, k$ .

### 4. Steady State Equations

The steady state equations are given as Figure 1. The terms align with events this way:

- 1st Term: Instantaneous probability of leaving state.
- 2nd Term: State is entered because operating component fails.
- 3rd Term: In-installation component fails.
- 4th Term: Component is resupplied.
- 5th Term: Installation of a component is completed.

To facilitate comprehension of the equations we make a number of observations. So long as  $k \leq s$ , there are no components on backorder. Since each component fails with rate  $\lambda$ , total failure rate is  $(c)(\lambda)$ . This rate decomposes into failure rates of  $(c-i)(\lambda)$  for operating components and  $(i)(\lambda)$  for in-installation components. The resupply rate of  $(k)(\mu)$  is explained in standard texts and the same logic justifies use of  $(i)(\gamma)$  to represent the installation rate. A failure is instantaneously followed by the beginning of an installation (note modification when  $k = s$ ) so  $i$  does not change if an in-installation component fails.

Equations

for  $k = 0$  to  $s - 1$

$$0 = - [\lambda c + k\mu + \gamma i] P_{ik} + \lambda(c - i + 1) P_{i-1,k-1} \\ + (\lambda i) P_{i,k-1} + (\mu)(k+1) P_{i,k+1} + \gamma(i+1) P_{i+1,k}$$

for  $k = s$  change  $(\mu)(k+1) P_{i,k+1}$  in above to  $(\mu)(k+1) P_{i-1,k+1}$

for  $k = s+1$  to  $c + s$

$$0 = - [\lambda(c - k + s) + k\mu + \gamma i] P_{i,k} + \lambda[c - i - (k-1) + s] P_{i,k-1} \\ + \lambda(i+1) P_{i+1,k-1} + (\mu)(k+1) P_{i-1,k+1} + \gamma(i+1) P_{i+1,k}$$

FIGURE 1

STEADY STATE EQUATIONS



If  $k > s$ , a resupply eliminates a backorder and results in an additional in-installation component. Conversely, if an operating component fails when  $k > s$ , a backorder is created and  $i$  does not change. Total components which can fail are  $c - (k-s)$  since there are  $k-s$  backorders.

Modifications. If in-installation components cannot fail, then the probability of failure is  $(\lambda)(c-i)$  or  $(\lambda)(c - (k-s) - i)$  if  $k > s$ . The terms representing what will happen upon failure of an in-installation component are eliminated. Similarly, "cold" standby or "semi-warm" are modelled by more general forms for  $\lambda_g$ , the failure rate when there are  $g$  good components installed. If the pool manager is resupplied by a repair process with  $m$  channels, then the resupply rate becomes the minimum of  $(k\mu, m\mu)$ . Multiple systems are treated later.

#### Solution of Equations

Clearly, so long as  $\lambda, \mu$  are  $> 0$ , all feasible states communicate. Since the transition probabilities are homogeneous, a unique stationary distribution exists.

A FORTRAN computer program for solving the equations for general  $c$  constitutes Figure 2. The program generates the transition matrix of size  $z$  by  $z$  if there are  $z$  states. The row for leaving and entering state  $(0,0)$  is eliminated, and a row is added which represents the requirement that the stationary probabilities sum to 1.

In the computer program function ISTAF converts state index  $i,k$  into a unique single state number. Subroutine ASSIGN creates an entry in the transition matrix based on the coefficient fed it from the MAIN program. Subroutine LEQT2F solves matrix equations of the form  $A x = B$ , and replaces  $B$  with the solution  $x$ .

If a state does not exist ASSIGN detects this; e.g., in the equation for  $p_{i,k}$ ,  $p_{i-1,k}$  does not exist if  $i = 0$ . ASSIGN simply does not create an entry. Also  $k \leq c + s - i$  since (no. in resupply) + (no. operating) + (no. in-installation)  $\leq c + s$ .

For the system to be unavailable, there must be 0 operating components; i.e., either  $i + k = c + s$ , or  $i = c$ . Probabilities of states which satisfy either of those conditions are summed. Solutions generated by the computer programs have been verified by simulation, and by analytic means for special cases.

```

PROGRAM STATE (INPUT-OUTPUT)
C BEGIN (MSL7)
COMMON IC-IS-A(75,75)
DIMENSION B(75)
DIMENSION WKAREA(1771)
1 PRINT *, 'IC, XLAM, MU, TAU, S', WKAREA, IER)
READ *, IC, XLAM, YMU, TAU, IS
IF (IC .EQ. 0) STOP
IZERO = 0
IROW = 0
DO 20 I=1,75
DO 20 J=1,75
20 A(I,J)=0.
C DEVELOP CO MATRIX
DO 100 I=IZERO,IC
LIM=IC+IS-I
DO 80 K=IZERO,LIM
C NEXT ELIMINATES FLOW EON FOR
1 STATE
IF (K .EQ. 0) GO TO 80
IROW=IROW+1
IF (K .GE. IS+1) GO TO 70
IF (IROW .GE. 75) STOP 2
CO = -(XLAM*(IC+K)*YMU+ITAU)
CALL ASIN(IROW,I,K,CO)
CO=XLAM*(IC-I+1)
CALL ASIN(IROW,I-1,K-1,CO)
CO=XLAM*(I)
CALL ASIN(IROW,I,K-1,CO)
CO=YMU*(K+1)
IF I
IF (K .GE. IS+1-I-1)
CALL ASIN(IROW,I,K+1,CO)
CO=TAU*(I+1)
CALL ASIN(IROW,I+1,K,CO)
GO TO 80
70 CO = -(XLAM*(IC-K+IS)+K*YMU
+ITAU*I)
CALL ASIN(IROW,I,K,CO)
CO=XLAM*(IC-K+IS-I+1)
CALL ASIN(IROW,I,K-1,CO)
CO=XLAM*(I+1)
CALL ASIN(IROW,I+1,K-1,CO)
CO=YMU*(K+1)
CALL ASIN(IROW,I-1,K+1,CO)
CO=TAU*(I+1)
CALL ASIN(IROW,I+1,K,CO)
90 CONTINUE
100 CONTINUE
IROW=IROW+1
DO 120 I=1,IROW
120 A(IROW,I)=1
C DEVELOP B MATRIX
LIM=IROW-1

```

```

DO 130 I=1,LIM
B(I)=0.
B(IROW)=1.
IER=0
CALL LEOT2F(A,1-IROW,75,0.7
-WKAREA,IER)
IF (IER .EQ. 0) GO TO 250
PRINT *, 'IER', IER
DO 200 I=1,IROW
200 PRINT 400,I,(A(I,J),J=1,IROW
)
400 FORMAT(' ROW = ',I4,' 15F4.0 /
F8.0,14F4.0)
PRINT *, (B(I),I=1,IROW)
250 CONTINUE
UNAU=0
ICM=IC-1
DO 255 I=IZERO,ICM
ISTATE=ISTAF(I,IS+IC-I)
255 UNAU=UNAU+B(ISTATE)
DO 240 K=IZERO,IS
ISTATE=ISTAF(IC,K)
240 UNAU=UNAU+B(ISTATE)
PRINT *, 'UNAU', UNAU
GO TO 1
END
SUBROUTINE ASIN(IROW,I,K,CO)

```

```

COMMON IC-IS-A(75,75)
IF (I .LT. 0) RETURN
IF (I .GT. IC) RETURN
IF (K .GT. IC+IS-I) RETURN
IF (K .LT. 0) RETURN
ISTATE=ISTAF(I,K)
A(IROW,ISTATE)=CO
RETURN
END
FUNCTION ISTAF(I,K)
COMMON IC-IS
NO=0
IF (I .EQ. 0) GO TO 50
DO 10 J=1,I
10 NO=NO+IC+IS+1 -(I-1)
50 ISTATE=K+1+NO
RETURN
END

```

FIGURE 2  
COMPUTER ALGORITHM

Note that the total number of states is  $(c+1)(c/2 + s+1)$  as:

for  $k = 0$  to  $s$ ,  $i = 0$  to  $c$  for a total of  $(s+1)(c+1)$  states

for  $k = s + 1$  to  $c + s$ ,  $i = 0$  to  $c - (k-s)$  for a total of  $(c)(c+1)/2$  states.

##### 5. Multiple Systems Supported

Basic Assumption Set. Under warm standby and FIFO issue policy, by the pool manager, the state probabilities are no different for a problem with  $n$  systems and  $c$  components than they are for a problem with 1 system and  $(n)(c)$  components. Furthermore, let there be  $B$  non-operating components. Choose a subset of  $c$  components from the total of  $(n)(c)$ . Then the conditional probability that all  $c$  are not operating does not depend on how the  $c$  were chosen from the  $nc$  provided only that status was not a basis of selection.

In particular, if  $B_I$  are the number of components not operating in weapon system  $I$ ,  $I$  chosen at random,

$$\Pr(B_I = c | B = j) = \prod_{i=1}^c \frac{j - (i-1)}{nc - (i-1)}$$

This is an example of sampling without replacement. The probability the first component of system  $I$  is not operating given  $B = j$  is  $j/(n)(c)$ . The probability the second component is not operating is  $(j-1)/(nc-1)$ , and so on.

Alternative Assumption Sets. To handle alternative assumption sets, the state space must be greatly expanded so that component status is specified by system, number up as well as number in installation. Under warm standby this is required for any issue policy which considers system status; e.g. when there are backorders and a unit is received from resupply, issue to the system with most non-operating components. Under cold standby the expanded state space is always required because the total failure generation rate depends on the distribution of non-operating components. If  $n = 2$  and  $c = 2$  and 2 components are backordered, the failure rate will be lower if both backorders are on the same system ( $\lambda$  rather than  $2\lambda$ ).

## 6. Zero Spares, General Distribution

If the pool manager has no spares, we redefine the state space, dropping  $i$  and letting  $p_k$  be the probability that  $k$  units are in "service", where service encompasses both obtaining the spare and then installing it. Our problem reduces to a state dependent Poisson arrival process - the component failures - with general service distribution. Brumelle [3,4]<sup>\*</sup> proves that the steady state number of components in service is then independent of the service distribution, depending only on mean service time.

Clearly then, steady state system availability is independent of service distribution; i.e., it is determined by distribution of operating components which in turn is determined by distribution of components in service. Thus, the results found using the steady state equations, correct for the exponential, must be generally applicable.

If components can fail during installation, these failures should be considered as prolonging the service time, and should not be included in the failure rate used to get the  $p_k$ . Kaplan showed how to get expected service time in this case for general resupply distribution, and exponential or deterministic installation time. For convenience, the relevant material is reprinted as Appendix 2.

## 7. Deterministic Installation Time ( $\tau$ )

Given failures during installation, system availability for any resupply time distribution can be determined by building on the state equation approach. Let the states be defined only in terms of number in resupply  $k$ . Then availability at time  $t + \tau$  depends only on the value of  $k$  at time  $t$  and the number of distinct components which fail in the interval  $(t, t + \tau]$ . The rationale is that any component which is either operating or in-installation at time  $t$  will be operating by time  $t + \tau$  provided it does not fail in the interim. Once a component fails in  $(t, t + \tau]$  it cannot be made to operate by time  $t + \tau$ .

---

\* Second reference is an abbreviated but more widely circulated version of first reference. Brumelle's work carries forward work by Sherbrooke [9] and others.

Thus, defining  $v(t)$  as the number of components in resupply at time  $t$ , expected unavailability at time  $t + \tau$  is

$$\sum_{k=0}^s \text{pr}(v(t) = k) (1 - e^{-\lambda\tau})^c +$$

$$\sum_{k=s+1}^{s+c} \text{pr}(v(t) = k) (1 - e^{-\lambda\tau})^{c-(k-s)}$$

For steady state unavailability, we substitute the steady state probabilities of  $k$  in resupply. These can be determined from the difference equations for general resupply distribution and are [7]

$$p_k = \pi \frac{\lambda_{i-1}}{\mu_i} p_0$$

$$p_0 = 1 / \sum_{k=0}^{c+s} \pi \frac{\lambda_{i-1}}{\mu_i}$$

$$\lambda_i = (\lambda) [c - \max(0, i - s)]$$

$$\mu_i = (\mu)(i)$$

## 8. General Resupply Distribution

There is a rationale for using the exponential to model other resupply distributions. Consider a simplified version of the state space in which only the value of "k" is of concern; i.e., we are not concerned with the relative proportion of items which are operating versus in-installation. We assume failure can occur during installation so that the failure rate depends only on the number in resupply and not on the breakdown between operating and in-installation components. Then the Brumelle result cited in section 6 applies to the number in resupply; i.e. service time is equated to resupply time, and it is as if there were no installation time.

What this argument shows is that in our original problem  $p_k = \sum_i p_{ik}$  found using exponential assumptions holds for general resupply distribution. It suggests that results for system availability based on the exponential may be acceptable approximations for other resupply distributions. In the table following, simulation results for deterministic resupply times are compared to theoretical results based on the exponential. In all examples shown,  $c$  was set to 2,  $\mu$  to 1 and  $s$  to 1. Other examples tried gave similar results.

A text book approach [cg Kleinrock] to treating a general resupply distribution would be to model it as a sum and/or mixture of exponentials. Deterministic resupply time would be approximated by an Erlangian of order  $n$ , the approximation improving as  $n$  is increased. The disadvantage is that the state space must be greatly expanded to specify the  $n$  possible stages for each component in resupply.

A different approach, based on Brumelle's findings for state dependent queues is discussed in the Appendix. It is applicable to deterministic resupply, which is of practical interest, either because it represents ship time from supplier to pool manager, or because component repair time is quite predictable. Preliminary results with this different approach are only partially encouraging, but the derivation may also be of academic interest, for the insights it offers to the implications of Brumelle's work.

#### 9. Conclusion

While this paper provides a number of useful results, it highlights two areas for further research: issue policies for a pool manager supporting multiple systems, and better approximations or exact solutions for general resupply distributions.

$\lambda$	$\tau$	Simulated	Exponential
.06	1	0.45%	0.41%
.06	2	0.16%	0.14%
.06	.5	1.30%	1.28%
.25	1	6.29%	6.12%
.25	2	2.95%	2.81%
.25	.5	13.8%	13.7%
.03	2	0.037%	0.033%
.03	.25	1.18%	1.19%

TABLE 1

UNAVAILABILITY

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## APPENDIX I

### A HEURISTIC FOR DETERMINISTIC RESUPPLY TIMES

This Appendix is organized so that the reader need only delve as deeply as his interest takes him. We begin with results, then the basic concept, followed by the mathematics and an example.

Results. The table following is a reprint of the table in the text, with heuristic projections also shown. Values in parenthesis are the estimated standard deviations of the simulation percent results. One case was added, for inputs for which it was suspected the heuristic might do particularly badly.

The heuristic helps for cases in which the exponential error, as a % of the true value, is highest (cases 1,2,7). It sometimes makes things worse, but the error is never a large % of the simulated value.

Concept. There are three steps:

(1) Calculate all  $p_{ik}$  and instantaneous transition probabilities as if supply were exponential.

(2) For each state in which the system would be down, reestimate expected time in that state assuming that the values calculated in Step (1) hold for all other states, and using additional results of Brumelle.

(3) Set  $p'_{ik} = p_{ik} \left( \frac{\text{RESTIME}}{\text{EXPTIME}} \right)$  where

RESTIME is the reestimated time from Step (2) and EXPTIME is the time found in Step (1) under the exponential assumption.

In using results of Brumelle, we will refer to the "reduced" system. This is the system as discussed in Section 8 for which his results are applicable, in which we do not distinguish between components in installation or operational. Recall also that in this context resupply and service times are synonymous.

A more involved version of the heuristic would execute Step 3 for all states, normalize so that probabilities sum to 1, and possibly iterate.

Expected Time in State (ET). Let  $G(t)$  be the probability the state is not left by time  $t$ . (Subscript  $i,k$  is omitted) It is well known that expectation can be calculated as:

$$(A1) \quad ET = \int_0^{\infty} G(t) dt$$

Case	$\lambda$	$\tau$	Simulated	Exponential	Heuristic
1	.06	1	0.45%/(.006)	0.41%	.44%
2	.06	2	0.16%/(.003)	0.14%	.15%
3	.06	.5	1.30%/(.013)	1.28%	1.35%
4	.25	1	6.29%/(.031)	6.12%	6.35%
5	.25	2	2.95%/(.026)	2.81%	2.91%
6	.25	.5	13.8%/(.07)	13.7%	14.1%
7	.03	2	0.037%/(.0009)	0.033%	.035%
8	.03	.25	1.18%/(.010)	1.19%	1.22%
9	.03	.125	3.78%/(.033)	3.80%	3.84%

A state can be left through resupply, failure or installation. These events are independent so if  $G_1(t)$  pertains to probability of no resupply and  $G_2(t)$  pertains to the other 2 events (no failure or installation)

$$(A2) \quad ET = \int_0^{\infty} G_1(t) G_2(t) dt$$

Given Poisson failure and installation rates, and given  $i + k = c + s$  (all units are either in installation or in resupply so the system is not operating)

$$(A3) \quad G_2(t) = e^{-1(\lambda+\gamma)t}$$

$G_1(t)$  depends on what state  $(i,k)$  was entered from. From Brumelle [3], p.2, when there is a departure from the reduced system - i.e. a service = resupply action is completed (cf Section 8) - service time remaining for each open resupply action is independently and identically distributed with distribution  $H^*(t)$ , the equilibrium resupply distribution.

Hence, if state  $(i,k)$  was reached by a resupply action

$$(A4) \quad G_1(t) = [1 - H^*(t)]^k$$

The system found by an arrival, i.e. a component failure, is like that found by a departure [3]. However, this pertains to the  $k-1$  units already in resupply, not the arrival which will not result in a completed resupply action until a full resupply time later. Hence, if state  $(i,k)$  was entered from state  $(j,k-1)$ , then

$$(A5) \quad G_1(t) = [1-H^*(t)]^{k-1} (1-H(t))$$

Defining  $ET_1$  and  $ET_2$  as the conditional expected time in a state given that it was entered from a state with  $k+1$  and  $k-1$  components in resupply, respectively:

$$(A6a) \quad ET_1 = \int_0^{\infty} [1-H^*(t)]^k e^{-1(\lambda+\gamma)t} dt$$

$$(A6b) \quad ET_2 = \int_0^{\infty} [1-H^*(t)]^{k-1} (1-H(t)) e^{-i(\lambda+\gamma)t} dt$$

For deterministic resupply time, equal to  $1/u$ :

$$(A7) \quad H^*(x) = u \int_0^x [1-H(t)] dt = u \int_0^x 1 dt = u x \quad x \leq 1/u$$

$$= 1 \quad x > 1/u$$

or

$$1 - H^*(t) = 1 - ut \quad t \leq 1/u$$

$$= 0 \quad t > 1/u$$

Also,

$$1 - H(t) = 1 \quad t < 1/u$$

To get ET from  $ET_1$  and  $ET_2$ , the conditional expectations are weighted by the relative probabilities of entering  $(i,k)$  from states with number in resupply of  $k+1$ ,  $k-1$  respectively.

Comments. Even for the reduced system, the expected time in a state depends on the resupply distribution and possibly the state from which it was entered, notwithstanding that the steady state probabilities of being in a state are independent of resupply distribution.

For exponential resupply, it is well known that

$$ET = 1 / \{ \lambda [c - \max(0, k-s)] + (u)(k) \}$$

As a check some examples were tried and ET derived by the procedure just described, weighting  $ET_1$  and  $ET_2$ , reduced to the correct answer when the exponential equilibrium distribution was used in A7.

The use made of Brumelle's results is not entirely correct because in getting to state  $i,k$  from state  $j, k+1$ ,  $j$  is implied by  $i$ ; hence we know more than that we left some state  $\bar{p}_{-,k+1}$  and conditional on this knowledge the equilibrium results may not entirely hold.

Example: Let  $s = 1$  and  $c = 2$ . We wish to calculate expected time in state  $(1,2)$ ; i.e., 1 in installation, 2 in resupply.

Referring to (A6) and using (A7) and (A3)

$$(A8) \quad E_1 = \int_0^{1/u} (1-ut)^2 e^{-(\lambda+\gamma)t} dt =$$

$$\frac{1}{\lambda+\tau} - \frac{2u^2}{(\lambda+\tau)^3} \left[ \frac{\lambda+\gamma}{u} - 1 + e^{-(\lambda+\gamma)/u} \right]$$

$$E_2 = \int_0^{1/u} (1-ut) e^{-(\lambda+\gamma)t} dt =$$

$$\frac{u}{(\lambda+\gamma)^2} \left[ \frac{\lambda+\gamma}{u} - 1 + e^{-(\lambda+\gamma)/u} \right]$$

State (1,2) can be reached from states (0,3), (2,1) or (1,1) with instantaneous transition probabilities of  $3u$ ,  $2\lambda$  and  $\lambda$  respectively (recall we assume transition probabilities based on exponential are valid). Note that if we are in state (1,1) there is an operating component, a component in installation and one in resupply. To go from (1,1) to (1,2) the operating component must fail, hence the transition probability of  $\lambda$ .

Hence

$$E_{12} = \frac{p_{0,3}(3u)E_1 + [p_{21}(2\lambda) + p_{11}(\lambda)] E_2}{p_{0,3}(3u) + p_{21}(2\lambda) + p_{1,1}(\lambda)}$$

where the  $p_{ik}$  are gotten from step 1.

APPENDIX 2

FAILURES DURING INSTALLATION

Notation

p - probability of failure during installation

t - expected time to failure, during installation, given a failure during installation

MTTI - mean time to install

TMTTI - expected total mean time to install including time to get another component if first fails during installation

LDT - mean logistic down time (time to get a component)

MTBF - mean time between failure

r - 1/MTTI

f - 1/MTBF

Case 1: Installation time (given no failure) is deterministic. Failures are exponential.

Then

$$TMTTI = (1-p)MTTI + p [t + LDT + TMTTI]$$

$$TMTTI = MTTI + \frac{p[t + LDT]}{1-p}$$

and

$$p = 1 - e^{-(f)(MTTI)}$$

$$t = \frac{1}{p} \int_0^{MTTI} t f e^{-ft} dt$$

Case 2: Installation Time and Failures are Exponential.

Then

Probability of no installation by time x is  $e^{-rx}$ , so

$$\begin{aligned} p &= \int_0^{\infty} f e^{-fx} (e^{-rx}) dx \\ &= \frac{f}{f+r} \int_0^{\infty} (f+r) e^{-(f+r)x} dx = \frac{f}{f+r} \end{aligned}$$

$$t = \frac{\int_0^{\infty} x f e^{-fx} e^{-rx} dx}{p} = \frac{1}{p} \frac{f}{f+r} \int_0^{\infty} (x) (f+r) e^{-(f+r)x} dx$$

$$= \frac{1}{p} \frac{f}{f+r} \frac{1}{f+r} = \frac{1}{f+r}$$

When installation times are exponential, the fact that installation was completed before a failure occurred, tells us something about how long the installation took. If

$$\text{MTTIC} \equiv \text{Ex(Installation Time | Installation occurs before failure)}$$

Then, by a derivation analogous to derivation of  $t$ ,

$$\text{MTTIC} = \frac{\int_0^{\infty} x r e^{-fx} e^{-rx} dx}{1-p} = \frac{1}{f+r}$$

This equals  $t$ . In other words, something will happen in average time of  $(f+r)$ . With probability  $p$  it is a failure and probability  $(1-p)$  it is an installation.

$$\text{TMTTI} = (1-p) (\text{MTTIC}) + p[t + \text{LDT} + \text{TMTTI}]$$

and by algebra

$$\begin{aligned} \text{TMTTI} &= \text{MTTIC} + \left(\frac{p}{1-p}\right) [t + \text{LDT}] \\ &= \frac{1}{f+r} + \frac{f}{r} \left(\frac{1}{f+r} + \text{LDT}\right) \\ &= \frac{1}{f+r} \left(1 + \frac{f}{r}\right) + \frac{f}{r} (\text{LDT}) \\ &= \frac{1}{r} + \frac{f}{R} (\text{LDT}) \end{aligned}$$

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