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The present report attempts a systematics presentation and review of modern theories of the earth's rotation (precession, nutation, and polar motion) for a rigid earth, a purely elastic earth, and the Poincare model consisting of a rigid mantle and a liquid core. Emphasis is on the treatment of an elastic earth on the basis of Liouville's equation and on the consideration of the earth's rotation as an eigenvalue problem.

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FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technical University at Graz and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19628-79-C-0075, The Ohio State University Research Foundation, Project No. 711715, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, MA, with Mr. Bela Szabo, Project Scientist.

I N D E X

Introduction

1. The Tidal Potential 1

2. Rotation of a Rigid Body 7

3. The Liouville Equation 13

4. Free Polar Motion for an Elastic Earth 21

5. Influence of Sun and Moon on Polar Motions 27

6. Polar Motion : Other Axes 35

7. Precession and Nutation: Angular Momentum Axis 41

8. Precession and Nutation : Other Axes 49

9. Kinoshita's Theory of Precession and Nutation 62

10. Eigenvalue Problem for Rigid Body Rotation 72

11. Application to Nutation and Polar Motion 84

12. Poincaré's Liquid Core Model 100

13. Liquid Core Effects on Polar Motion and Nutation 118

References 143

INTRODUCTION

For geodetic, geodynamical, and astronomical purposes, two basic coordinate systems are needed: an inertial system and an earth-fixed system. These two systems are related through precession, nutation, and polar motion.

In order to relate these two systems in a precise fashion, we need not only highly sophisticated observation methods such as doppler, lunar laser, and VLBI observations, but also very accurate theories, which take into account the elasticity of the earth's mantle, as well as effects due to the liquid core. A clear understanding of this complex matter is a prerequisite also for practical work in this field.

The present report is intended as a systematic review which presents the basic principles in a rather detailed manner and is thus suitable as an introduction even for scientists with little or no previous knowledge of the field.

The report is restricted to those theories which regard the earth either as a rigid body, such as Kinoshita's recent theory, or as a purely elastic solid, such as McClure's work, or as a body consisting of a rigid mantle and a liquid core, the so-called Poincaré model. None of these models is fully realistic, but each contains important features which form a basis indispensable for understanding, and even for practically and numerically treating, a more realistic model consisting of an elastic mantle and a liquid core. The considerations of models of the latter kind, which is a complicated and difficult subject, will be deferred to another report.

Emphasis is on the treatment of an elastic earth on the basis of Liouville's equation, leading to a systematic theory of polar motion, precession, and nutation of various axes (rotation axis, angular momentum axis, figure axis, and the so-called celestial reference pole), and on the eigenvalue problem for rotation, leading to a similar theory for a rigid earth and for the Poincaré model.

1. THE TIDAL POTENTIAL

Precession, nutation, forced polar motion, and earth tides all have a common cause: the gravitational attraction of sun and moon. Therefore, the potential of this attraction, the tidal potential, plays a fundamental role in all these phenomena.

Consider the gravitational attraction of the moon (the sun can be treated analogously) at a point P on the earth's surface which, to an accuracy sufficient for the present purpose, can be represented by a sphere of radius a (Fig. 1.1). The potential of this attraction at P is

$$V = \frac{G\mu}{l} = G\mu \sum_{n=0}^{\infty} \frac{a^n}{d^{n+1}} P_n(\cos\psi) \quad , \tag{1-1}$$

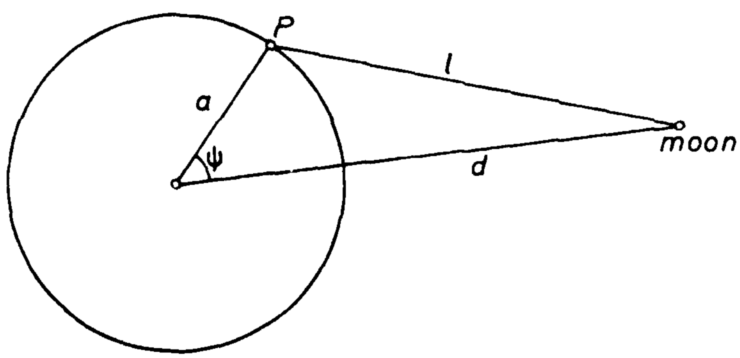


FIGURE 1.1. The tidal attraction

on expanding $1/l$ into a series of Legendre polynomials $P_n(\cos\psi)$ (Heiskanen and Moritz, 1967, p. 33). The notations are evident from Fig. 1.1; G is the gravitational constant and μ is the moon's mass. We follow (Moritz, 1980, sec. 55).

The zero and first degree terms ($n=0$ and 1) do not cause genuine deformations and are therefore omitted. The dominant term is $n=2$; higher-degree terms are small and will be neglected. Thus there remains

$$V = V_2 = G\mu\frac{a^2}{d^3}P_2(\cos\psi) \quad . \tag{1-2}$$

This spherical harmonic of second degree will, in the following, be considered as our tidal potential.

Let us now express $P_2(\cos\psi)$ in terms of the geocentric spherical coordinates of P and of the moon's center. In the usual earth-fixed equatorial system the point P has the coordinates (θ,λ) where $\theta = 90^\circ - \phi$ is the polar distance of P , ϕ denoting the geocentric latitude, and λ is the geocentric longitude. Similarly, the moon has the coordinates (p,h), where the polar distance is given by $p = 90^\circ - \delta$, δ being the declination of the moon, and h denotes the Greenwich hour angle of the moon, that is, the angle between the Greenwich meridian and the meridian passing through the moon's center. Contrary to astronomical usage, both λ and h are counted positively towards east (Fig. 1.2).

Then $P_2(\cos\psi)$ can be expressed by means of the decomposition formula for spherical harmonics (Heiskanen and Moritz, 1967, p. 33), and we obtain

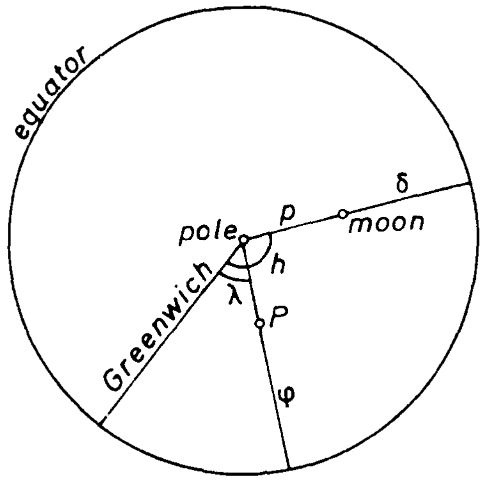


FIG. 1.2. Coordinates of P and of the moon.

$$V_2 = V_{20} + V_{21} + V_{22} \quad , \tag{1-3}$$

with the <u>zonal part</u>

$$V_{20} = G\mu \frac{a^2}{d^3} P_{20}(\cos\theta) P_{20}(\cos p) \quad , \tag{1-4}$$

the <u>tesseral part</u>

$$V_{21} = \frac{1}{3} G\mu \frac{a^2}{d^3} P_{21}(\cos\theta) P_{21}(\cos p) \cos(\lambda - h) \quad , \tag{1-5}$$

and the <u>sectorial part</u>

$$V_{22} = \frac{1}{12} G\mu \frac{a^2}{d^3} P_{22}(\cos\theta) P_{22}(\cos p) \cos 2(\lambda - h) \quad . \tag{1-6}$$

The Legendre functions are given by the well-known expressions:

$$P_{20}(\cos\theta) = P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} ,$$
$$P_{21}(\cos\theta) = 3\sin\theta\cos\theta, \quad P_{22}(\cos\theta) = 3\sin^2\theta . \tag{1-7}$$

The coordinates of the moon, p and h, are functions of the time since the moon moves along its orbit. Therefore, eqs. (1-4) to (1-6) can be considered spherical harmonics in θ and λ with coefficients that are functions of time. We expand these functions into trigonometric series and write the result in the form (Doodson, 1922; McClure, 1973, p. 92):

$$V = V_{20} + V_{21} + V_{22} ,$$
$$V_{nm} = \sum_{d=1}^{2} \frac{GM_d}{c_d} \left(\frac{a}{c_d}\right)^n P_{nm}(\cos\theta) \cdot \tag{1-8}$$
$$\cdot \sum_j A_{nmjd} \cos\left[\omega_{nmjd} t + \beta_{nmjd} + m\lambda + (n-m)\frac{\pi}{2}\right] .$$

Here $n=2$; $m=0,1,2$; the index d numbers moon ($d=1$) and sun ($d=2$), M_d denotes the masses of moon and sun, and c_d denotes the mean radii of the lunar and the solar orbit (considered with respect to the earth). As an example, we write (1-8) explicitly for the case $m=1$, which will be of particular importance:

$$V_{21} = -\sum_{d=1}^{2} \frac{GM_d}{c_d^3} a^2 P_{21}(\cos\theta) \cdot \tag{1-9}$$
$$\cdot \sum_j A_{21jd} \sin(\omega_{21jd} t + \beta_{21jd} + \lambda)$$

since $\cos(\alpha+\pi/2) = -\sin\alpha$. Omitting the subscripts $\cdot\cdot$ in the coefficients and considering the effect only of the moon (or only of the sun) we have

$$V_{21} = -\frac{G\mu_d}{c_d^3} a^2 P_{21}(\cos\theta) \sum_j A_j \sin(\omega_j . t + \beta_j + \lambda) \, . \qquad (1\text{-}10)$$

This simpler expression will frequently be used later.[1]

We remark that the arguments are linear combinations (with integer coefficients) of

- s ..... lunar mean longitude,
- h ..... solar mean longitude (not to be confused with the hour angle as used above),
- p ..... mean longitude of lunar perigee (same remark),
- N ..... longitude of the mean ascending node of the lunar orbit,
- p_s ..... mean longitude of solar perigee,
- τ ..... local mean lunar hour angle,

that is

$$\omega_{nmjd} t + \beta_{nmjd} + m\lambda = n_1\tau + n_2 s + n_3 h + \\ + n_4 p + n_5 N + n_6 p_s \qquad (1\text{-}11)$$

(Melchior, 1978, p. 33; McClure, 1973, p. 95).

---

1) Note that the sum $\sum_j$ is an infinite series!

Well known are the tidal developements of Doodson (1922) and Cartwright and Tayler (1971); see also (Cartwright and Edden, 1973). A new developement has been given by Heikkinen (1978).

2. ROTATION OF A RIGID BODY

The basic equation for the rotation of a rigid body is very simple:

$$\frac{d\underline{H}}{dt} = \underline{L} \ , \tag{2-1}$$

the derivative of the angular momentum $\underline{H}$ with respect to time t equals the torque $\underline{L}$; both $\underline{H}$ and $\underline{L}$ are vectors, which is indicated by underlining their symbols.

This equation holds in a nonrotating (inertial) coordinate system; in a body-fixed coordinate system (which rotates with the body), its equivalent is

$$\frac{\partial\underline{H}}{\partial t} + \underline{\omega} \times \underline{H} = \underline{L} \ . \tag{2-2}$$

Here $\partial/\partial t$ denotes the time derivative in the body-fixed system and $\underline{\omega}$ is the rotation vector whose direction coincides with the instantaneous axis of rotation and whose magnitude is the angular velocity ω of rotation; the cross $(\times)$ denotes the vector product of two vectors. These equations can be found in any text on mechanics; cf. (Synge, 1960).

Eq. (2-1) is fundamental for precession and nutation, which is the motion of the earth's axis in inertial space; and (2-2) is basic for polar motion, which is the motion of the earth's axis with respect to an earth-fixed coordinate system.

The velocity $\underline{v}$ of a point of the body is

$$\underline{v} = \underline{\omega} \times \underline{x} \quad , \tag{2-3}$$

$\underline{x}$ denoting the position vector of the point. This relation is substituted into the equation defining the angular momentum $\underline{H}$,

$$\underline{H} = \iiint \underline{x} \times \underline{v} dM \quad , \tag{2-4}$$

in which the integration is over the body, dM being the mass element. The result is

$$\underline{H} = \underline{C}\,\underline{\omega} \tag{2-5}$$

where $\underline{C}$ is a tensor (a 3x3 symmetric matrix), the <u>inertia tensor</u>. The elements C_{ij} of the matrix $\underline{C}$ are given by the formula

$$C_{ij} = \iiint (x_k x_k \delta_{ij} - x_i x_j) dM \quad , \tag{2-6}$$

using index notation: i,j,k run from 1 to 3 , $x_1 = x$, $x_2 = y$, $x_3 = z$ are coordinates in a body-fixed Cartesian System, $\delta_{ij} = 1$ if $i=j$ and 0 if $i \neq j$ (Kronecker delta), and summation over repeated subscripts in a product (k in the formula) is prescribed; cf. (Jeffreys, 1931).

Equivalent but somewhat more explicit is the form

$$\underline{C} = \begin{bmatrix} J_x & -D_{xy} & -D_{xz} \\ -D_{xy} & J_y & -D_{yz} \\ -D_{xz} & -D_{yz} & J_z \end{bmatrix} , \qquad (2-7)$$

where the diagonal elements are moments of inertia, e.g.,

$$J_x = \iiint (y^2 + z^2)\,dM \quad , \qquad (2-8)$$

and the off-diagonal terms are products of inertia, e.g.,

$$D_{xy} = \iiint xy\,dM \quad . \qquad (2-9)$$

If the principal axes of inertia are chosen as coordinate axes, then the tensor $\underline{C}$ assumes diagonal form:

$$\underline{C} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} , \qquad (2-10)$$

A,B,C being the principal moments of inertia.

In this case, (2-5) reduces to

$$H_{[illegible]} = A [illegible] ,$$
$$H_{[illegible]} = B [illegible] ,\qquad (2\text{-}11)$$
$$H_{[illegible]} = C [illegible] ,$$

and [illegible] gives Euler's equations

$$A [illegible] + (C-B) [illegible] = L_{[illegible]} ,$$
$$B [illegible] + (A-[illegible]) [illegible] = L_{[illegible]} ,\qquad (2\text{-}12)$$
$$[illegible] + (B-A) [illegible] = L_{[illegible]} ,$$

which are fundamental for gyroscopic motion; the time deriva- tive [illegible] is denoted by a dot.

Free rotation of a rigid earth. Let us give a very simple application of these equations to polar motion of the earth. Let us neglect the torque L exerted by sun and moon, and assume rotational symmetry so that $B = A$. Then (2-12) reduces to

$$A [illegible] + ([illegible]-A) [illegible] = 0 ,$$
$$A [illegible] - ([illegible]-A) [illegible] = [illegible] ,\qquad (2\text{-}13)$$
$$[illegible] = 0 ,$$

The third equation gives immediately [illegible] = const. [illegible] , and putting

$$[illegible] \; \frac{[illegible]-A}{A} [illegible] \qquad (2\text{-}14)$$

we transform the first and second equation into

$$\dot{\omega}_1 + \sigma_E \omega_2 = 0 \ ,$$
$$\dot{\omega}_2 - \sigma_E \omega_1 = 0 \ , \qquad (2\text{-}15)$$

of which the solution is

$$\omega_1 = \alpha \cos(\sigma_E t + \gamma) \ ,$$
$$\omega_2 = \alpha \sin(\sigma_E t + \gamma) \ , \qquad (2\text{-}16)$$

with constants α and γ. From (2-16) we have

$$\omega_1^2 + \omega_2^2 = \alpha^2 = \text{const.} \ , \qquad (2\text{-}17)$$

which is the equation of a circle. Together with ω_3=const. this means that the rotation axis describes a circular cone around the axis of symmetry (Fig. 2.1). The angle of aperture δ is about 0.2"; the period T is obtained from

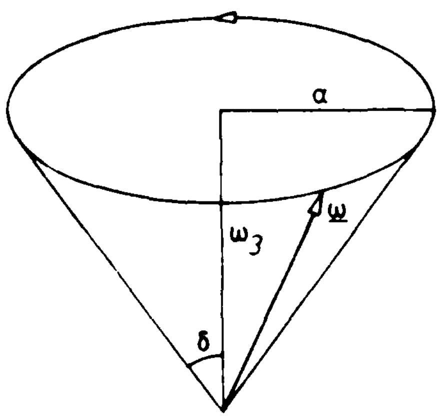


FIGURE 2.1. Free polar motion for a rigid earth.

(2-14) as

$$T = \frac{2\pi}{\sigma_E} = \frac{A}{C-A}\,\frac{2\pi}{\omega} \doteq 305 \text{ days} . \qquad (2\text{-}18)$$

This Euler period would hold if the earth was a rigid body. The fact that the actual period, the Chandler period, is about 430 days, indicates that the earth is not rigid.

Regarding terminology, the constant σ_E defined by (2-14) will be called the Euler frequency; it will play a basic role throughout the present report.

Strictly speaking, σ_E is an "angular frequency", whereas the name "frequency" is usually reserved for the quantity

$$\frac{\sigma}{2\pi} = \frac{1}{T} ;$$

however, we shall consistently speak of frequency in the sense of "angular frequency".

The coefficient a in (2-16) is called amplitude, and α is the phase.

3. THE LIOUVILLE EQUATION

The basic equation (2-1),

$$\frac{d\underline{H}}{dt} = \underline{L} , \tag{3-1}$$

is really quite general. It holds for the rotation of an arbitrary body, rigid or not (Truesdall and Toupin, 1960, p.531). The underlying coordinate system is an inertial system which we denote by XYZ or $X_1X_2X_3$.

In a rotating system xyz or $x_1x_2x_3$ which is attached to the rotating nonrigid body in a way to be explained later, the angular momentum equation takes again the form (2-2),

$$\frac{\partial\underline{H}}{\partial t} + \underline{\omega} \times \underline{H} = \underline{L} , \tag{3-2}$$

but (2-5) is generalized:

$$\underline{H} = \underline{C}\,\underline{\omega} + \underline{h} \tag{3-3}$$

where $\underline{h}$ is a relative angular momentum defined by

$$\underline{h} = \iiint \underline{x} \times \underline{u}\, dm . \tag{3-4}$$

Here $\underline{u}$ is the velocity with respect to the system $x_1x_2x_3$, which is related to the velocity $\underline{v}$ with respect to the inertial system $X_1 X_2 X_3$ by

$$\underline{v} = \underline{\omega} \times \underline{x} + \underline{u} \ . \tag{3-5}$$

Equations (3-3) and (3-4) are readily obtained by substituting (3-5) into the defining equation (2-4). The inertia tensor $\underline{C}$ is again given by (2-6).

The meaning of these equations is easily understood. If the earth (our rotating body will always be the earth) is not rigid, then there is no coordinate system at which all particles, of which the earth is composed, are at rest. Thus they move with respect to our system $x_1 x_2 x_3$ with velocity $\underline{u}$, which is considered small since it is zero for a rigid body. Thus (3-5) differs from (2-3) by a non-zero $\underline{u}$. This relative velocity $\underline{u}$ causes the relative angular momentum (3-4) to be, in general, different from zero.

By substituting (3-3) into (3-2) we get

$$\frac{\partial}{\partial t} (\underline{C}\,\underline{\omega} + \underline{h}) + \underline{\omega} \times (\underline{C}\,\underline{\omega} + \underline{h}) = \underline{L} \ , \tag{3-6}$$

which is called the Liouville equation (Munk and Macdonald, 1960, p.10). This equation will be fundamental for the mathematical description of polar motion for a nonrigid earth.

It is now of basic importance that the axes $x_1 x_2 x_3$ can be chosen such that $\underline{h}=0$. They have the property that

$$\iiint \underline{u}^2 dM = \text{minimum} \tag{3-7}$$

(Jeffreys, 1970, sec. 7.04) and are called Tisserand axes (Munk and Macdonald, 1960, p.10). Then the basic equations are formally the same as for a rigid body, eqs. (2-2) and (2-5), but note that now the inertia tensor $\underline{C}$, eq. (2-6), will be a function of time since the shape of the body will, in general, change with time. In the following we shall always use Tisserand axes.

This is convenient as long as one disregards relative motions such as ocean currents and winds, as we shall do. For the consideration of such effects see (Munk and Macdonald, 1960, p. 123; Lambeck and Cazenave, 1973, 1974; Capitaine, 1980; Lambeck, 1980).

Linearization. For Tisserand axes, the Liouville equation (3-6) may be written

$$\frac{\partial}{\partial t}(\underline{C}\,\underline{\omega}) + \underline{\omega} \times (\underline{C}\,\underline{\omega}) = \underline{L} . \tag{3-8}$$

We shall now linearize this equation as follows (Munk and Macdonald, 1960, p.37; McClure, 1973). The inertia tensor is written

$$\underline{C} = \underline{C}_0 + \underline{c} \tag{3-9}$$

where

$$\underline{C}_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}, \qquad \underline{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}. \tag{3-10}$$

Thus, $\underline{C}_0$ corresponds to the model of an undeformed earth whose principal axes of inertia coincide with the coordinate axes, which has rotational symmetry (B=A) and whose principal moments of inertia A and C are constant in time. The tensor $\underline{c}$ takes into account the deviation of the actual earth from this simplified model.

The rotation vector $\underline{\omega}$ is written as

$$\underline{\omega} = \underline{\omega}_0 + \delta\underline{\omega} \tag{3-11}$$

where

$$\underline{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \tag{3-12}$$

corresponds to a rotation with constant angular velocity Ω around the z-axis and

$$\delta\underline{\omega} = \Omega\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \tag{3-13}$$

expresses deviations of the rotation axis from the z-axis (m_1 and m_2) and variations of the rotational speed (m_3). Both $\underline{c}$ and $\underline{m}$ are considered small quantities whose squares, products and higher powers can be neglected.

We substitute (3-9) and (3-11) into (3-8) and retain linear terms only. The result is

$$\begin{aligned}
L_1 &= A\Omega\dot{m}_1 + (C-A)\Omega^2 m_2 + \Omega\dot{c}_{13} - \Omega^2 c_{23} \ , \\
L_2 &= A\Omega\dot{m}_2 - (C-A)\Omega^2 m_1 + \Omega\dot{c}_{23} + \Omega^2 c_{13} \ , \\
L_3 &= C\Omega\dot{m}_3 + \dot{c}_{33}\Omega \ ,
\end{aligned} \tag{3-14}$$

the dot denoting a time derivative $\partial/\partial t$.

It will be very convenient to combine the first two equations by using complex notation, putting

$$\begin{aligned}
m &= m_1 + im_2 \ , \\
c &= c_{13} + ic_{23} \ , \\
L &= L_1 + iL_2
\end{aligned} \tag{3-15}$$

where $i^2 = -1$ (note the difference between the complex number L and the three-vector $\underline{L}$, and between the complex number c and the tensor $\underline{c}$!). The result is

$$L = A\Omega\dot{m} - i(C-A)\Omega^2 m + \Omega\dot{c} + i\Omega^2 c \ ; \tag{3-16}$$

in fact, by substituting (3-15), performing the complex multiplications and separating real and imaginary parts, we get back the first two equations of (3-14).

Eq. (3-16) can be written in the form

$$\dot{m} = i\sigma_E(m-\psi) \tag{3-17}$$

where

$$\sigma_E = \frac{C-A}{A}\,\Omega \tag{3-18}$$

is again the Euler frequency (2-14) (we may identify the constant ω in sec. 2 with the present Ω); and

$$\psi = \frac{iL}{(C-A)\Omega^2} + \frac{c}{C-A} - \frac{i\dot{c}}{(C-A)\Omega} . \tag{3-19}$$

The quantity ψ is called the <u>excitation function</u> because it causes a deviation of polar motion from the simple case of free rotation of a rigid earth considered in sec. 2, for which $\psi=0$.

If the function ψ is known, then (3-17) can be solved for m, obtaining the components m_1 and m_2 characterizing a deviation of the earth's rotation axis from the z-axis, that is, <u>polar motion</u>. Similarly, the third equation of (3-14) which can be written as

$$\dot{m}_3 = \frac{L_3}{C\Omega} - \frac{\dot{c}_{33}}{C} , \tag{3-20}$$

can be solved for m_3 to get variations of the speed of rotation, or variations <u>of the length of day</u>.

In other terms, polar motion is the variation of the direction of the rotation vector $\underline{\omega}$, and m_3 characterizes variations in the length of $\underline{\omega}$. It is very remarkable that both phenomena are separated in the linear approximation. In the sequel we shall restrict ourselves to polar motion which is more interesting and less simple.

We shall also linearize the expression (3-3), which for $\underline{h}=0$ reduces to

$$\underline{H} = \underline{C}\,\underline{\omega} \; . \tag{3-21}$$

The substitution of (3-9) and (3-11) yields, on neglecting terms of second and higher order,

$$H = A\Omega m + \Omega c \; , \tag{3-22}$$

where

$$H = H_1 + iH_2 \; , \tag{3-23}$$

and

$$H_3 = C\Omega + C\Omega m_3 + \Omega c_{33} \; . \tag{3-24}$$

The division of the equatorial components H_1 and H_2 of the vector $\underline{H}$ by its length, which approximately is $C\Omega$, gives the equatorial components h_1 and h_2 of the unit vector $\underline{H}/\underline{H}$:

$$h_1 = \frac{H_1}{C\Omega}, \qquad h_2 = \frac{H_2}{C\Omega}, \qquad h = h_1 + ih_2 . \tag{3-25}$$

These quantities define the deviation of the <u>angular momentum axis</u> (the direction of $\underline{H}$) from the z-axis, in the same way as m_1 and m_2 define the deviation of the <u>rotation</u> axis from the z-axis.[1]

Finally, we consider the <u>figure axis</u> which is the axis of maximum inertia for the deformed earth, that is, a principal axis of the tensor $\underline{C}$ (the z-axis as a principal axis of the tensor $\underline{C}_0$). The determination of the principal axes of a symmetric matrix is straightforward; for the tensor (3-9) we get

$$f_1 = \frac{c_{13}}{C-A}, \qquad f_2 = \frac{c_{23}}{C-A}, \tag{3-26}$$

or in complex notation

$$f = f_1 + if_2 = \frac{c}{C-A} \tag{3-27}$$

(McClure, 1973, Appendix E). The quantities f_1 and f_2 are equatorial components of the unit vector of the figure axis.

1) Since the relative angular momentum vector (3-4) is zero, the letter h is free for our present new use.

4. FREE POLAR MOTION FOR AN ELASTIC EARTH

If the earth rotates about an axis which deviates from the axis of symmetry, there occur centrifugal forces which tend to distort it, and an elastic earth yields to this distortion (this is similar to distorting forces acting on an unbalanced wheel). It is well known that this distortion introduces products of inertia c_{13} and c_{23} which are proportional to the deviation of the rotation axis, m_1 and m_2 (Jeffreys, 1970, sec. 7.04; Munk and Macdonald, 1960, p. 38). In complex notation, using (3-15), we have

$$c = \frac{k}{k_s}(C-A)m \ , \tag{4-1}$$

where k is a Love number well known from elasticity theory, and k_s is an abbreviation

$$k_s = \frac{3G(C-A)}{a^5\Omega^2} \tag{4-2}$$

and bears the somewhat unfortunate name of secular Love number (Munk and Macdonald, 1960, p. 26). Here G denotes the gravitational constant and a is the earth's equatorial radius or, to the same accuracy, its mean radius: $a \doteq R = 6371$ km. More about the constants k and k_s will be said at the end of this section.

This quantity (4-1) is to be used in the excitation function (3-19). In the case of free rotation to be considered in this section, there is no effect of sun and moon, that is, no external torque L. Thus (3-19) reduces to

$$\psi = \frac{c}{C-A} - \frac{i\dot{c}}{(C-A)\Omega} = \frac{k}{k_s}(m - \frac{i\dot{m}}{\Omega}) = \psi_{RD} \ , \tag{4-3}$$

the subscript "RD" denoting "rotational deformation", and (3-17) becomes

$$\dot{m} = i\sigma_C m \qquad \text{(elastic earth)} . \tag{4-4}$$

Here

$$\sigma_C = \sigma_E \frac{1 - \frac{k}{k_s}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} \tag{4-5}$$

is the Chandler frequency, the name will be explained below; note that σ_E is the Euler frequency (2-14).

The complex equation (4-4) splits up into two real ones:

$$\begin{aligned} \dot{m}_1 + \sigma_C m_2 &= 0 \ , \\ \dot{m}_2 - \sigma_C m_1 &= 0 \ , \end{aligned} \tag{4-6}$$

which have the same form as (2-15) which, in our present notation, could be written as

$$\dot{m} = i\sigma_E m \qquad \text{(rigid earth)} . \qquad (4\text{-}7)$$

With the numerical values

$$k \doteq 0.30, \qquad k_s = 0.96 \text{ (dimensionless)} \qquad (4\text{-}8)$$

we get from (4-5)

$$\sigma_C \doteq 0.7\sigma_E , \qquad (4\text{-}9)$$

which means that the period (2-18) is lengthened by the factor $1/0.7 \doteq 1.4$. The multiplication by this factor brings the Euler period of 305 days (which would hold for a rigid earth and corresponds to the Euler frequency σ_E) close to the actual Chandler period of about 430 days, which corresponds to the Chandler frequency σ_C .

The solution of (4-4) may be written in complex form as

$$m = m_0 e^{i\sigma_C t} \qquad (4\text{-}10)$$

with an arbitrary (complex) constant m_0. This is immediately verified by substitution into (4-4), which again shows the advantage of complex notation. Of course, (4-10) is equivalent to (2-16), with σ_E replaced by σ_C .

Remark on Love numbers. The usual Love number k characterizes the elastic response of the earth in the sense that an external disturbing potential V (which is supposed to be a spherical harmonic function of second degree) causes an elastic distortion of the earth and thereby a change of its gravitational potential ΔV by

$$\Delta V = kV ; \qquad (4-11)$$

it is also called a *tidal-effective Love number* (Munk and Macdonald, 1960, p. 27).

The secular Love number k_s as defined by (4-2), on the other hand, "can be interpreted as a measure of the earth's yield to centrifugal deformation in the cause of its developement during the last five billion years or so" (*ibid.*, p. 25). This is a permanent deformation which is typically *nonelastic*; it corresponds to the rotational deformation of a *fluid* earth, being numerically equal to the "fluid Love number" k_f (*ibid.*, p. 26).

The difference between k and k_s according to (4-3) may be explained by a number of hypotheses, but it is not known which of these (if any) is correct (*ibid.*, p. 27). The usual approach is to use the elastic Love number k for all elastic deformations, whether they are caused by centrifugal or tidal disturbances and whether they represent constant or temporally variable deformations (cf. Melchior, 1978). Recently, however, some authors, following McClure (1973), are using the usual Love number k for temporally variable deformations and the secular Love number k for constant elastic deformations.

In the opinion of the present author, such an approach is inappropriate. The theory of deformations of an elastic earth makes no difference between deformations which are constant in time and for those which are temporally variable: for both, the Love number k is relevant. Even for earth models with a liquid core, for which is a slight dependence of k on frequency, give for constant deformations a k that is close to 0.3; cf. (Wahr, 1979). It would require an earth model of a completely different rheology to get for constant deformations a k close to 1, if this is at all possible and physically meaningful. No such model has been given so far.

Since geodynamical computations should be based on a well-defined meaningful model, it is strongly advocated to use a k following from such a model (Molodensky, 1961; Wahr, 1979). No physically observable error is introduced in this way: even if the secular Love number were "true" for constant deformations, this would only imply a reference model of a slightly different flattening but not change at all the observable physical situation.

Therefore we shall use k to characterize the elastic response of the earth also for the constant part of the deformation, restricting k_s purely to its use as the abbreviation (4-2) without attempting a physical interpretation of k_s in the sense of an elastic or nonelastic response.

We finally note that, in a rigid body, there are no elastic deformations and the shape of the body and its gravitational potential V do not change. Hence $\Delta V=0$, so that (4-11) implies

$$k = 0 \qquad \text{for a rigid body.} \tag{4-12}$$

Therefore, formulas for a rigid earth can be obtained from those for an elastic earth by simply putting $k=0$.

5. INFLUENCE OF SUN AND MOON ON POLAR MOTION

The basic equations are (3-1[illegible]) and (3-19):

$$\dot{m} = i\sigma_E(m-\phi) \; , \qquad 5-1$$

$$\phi = \frac{iL}{(C-A)\Omega} + \frac{c}{C-A} - \frac{i\dot{c}}{(C-A)\Omega} \; . \qquad 5-2$$

The excitation function may be split up as follows:

$$\phi = \phi_{RD} + \phi_L + \phi_{TD} \; , \qquad 5-3$$

of which the terms represent:

ϕ_{RD} ... rotational deformation because of centrifugal distortions,

ϕ_L ... effect of the torque L exerted by sun and moon,

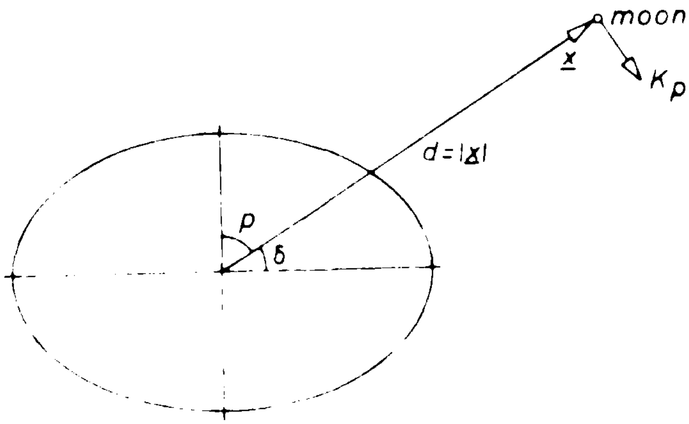
ϕ_{TD} ... tidal deformation of the earth.

The rotational deformation has already been considered in the preceding section, ϕ_{RD} being given by (4-[illegible]). The remaining two terms will be treated now.

Lunisolar torque. The torque exerted by the moon on the earth is equal, but with opposite sign, to the torque exerted by the earth on the moon (Kaula, 1968, sec. 4.3; Melchior, 1978, p. 39). The la ter is equal to $\underline{x} \times \underline{K}$, where $\underline{x}$ is the position vector of the moon (Fig. 5.1) and $\underline{K}$ is the force of attraction of the moon by the earth. By the definition of the vector product, and in view of the rotational symmetry of the earth, this torque has the magnitude $d K_p$, where K_p is the meridian component of $\underline{K}$:

$$\text{torque} = d \cdot K_p = d \cdot \frac{1}{d}\frac{\partial V}{\partial p} = \frac{\partial V}{\partial p} \qquad (5\text{-}4)$$

[illegible] normal to the meridian plane; here [illegible] the [illegible] of the moon and [illegible] the gravitational potential [illegible]



[illegible]

the earth, which for rotational symmetry takes the well-known form

$$V = \frac{GM}{r}\left[1 - J_2\left(\frac{a}{r}\right)^2 P_2(\cos\theta) \ldots\right] . \qquad (5-6)$$

Cf. (Heiskanen and Moritz, 1967, p. 73); we restrict ourselves to second-order spherical harmonics as usual. Here G is the gravitational constant and M is the earth's mass, and

$$J_2 = \frac{C-A}{Ma^2}$$

is the dynamical form factor for the earth, a [illegible] being its equatorial radius. [illegible]

[illegible]

[illegible]

$$L_1 = GM_m \frac{a^2}{d^3} J_2 P_{21}(\cos p)\sin\eta \,,$$
$$L_2 = -\, GM_m \frac{a^2}{d^3} J_2 P_{21}(\cos p)\cos\eta \,. \tag{5-9}$$

here η is defined by Fig. 1.2; note that we have $\sin\eta$ and $-\cos\eta$ instead of $\cos\eta$ and $\sin\eta$ since the torque is normal to the meridian plane.

This equation gives the equatorial components of the torque exerted by the moon on the earth; the influence of the sun is expressed by an analogous formula.

The comparison of (5-9) with (1-5) now shows immediately that

$$L_1 = [illegible] = 90° \,, \quad L_2 = - [illegible] = 0 \,, \tag{5-10}$$

[illegible] represents (1-5) on putting [illegible] = 90°, [illegible].

[illegible] the [illegible] torque [illegible] directly related [illegible] potential. This basic fact [illegible] by [illegible] (197[illegible]).

[illegible] (1-1[illegible]) we [illegible]

$$L_1 = -3GM_j MJ_2 \frac{a^2}{c_j^3} \sum A_j \cos [illegible] ,$$

$$L_2 = 3GM_j MJ_2 \frac{a^2}{c_j^3} \sum A_j \sin [illegible] , \qquad (5-11)$$

where the subscript j distinguishes sun and moon, and finally the simpler complex form

$$L = -3GM_j MJ_2 \frac{a^2}{c_j^3} \sum A_j e^{[illegible]} ; \qquad (5-12)$$

as usual, $L = L_1 + iL_2$.

Now (5-2) gives m_L, using (5-6):

$$m_L = \frac{iL}{(C-A)[illegible]} = i \sum B_j e^{-i(\omega_j t + \beta_j)} \qquad (5-13)$$

where the dimensionless coefficients B_j are defined by

$$B_j = -3 \frac{GM_j}{[illegible] c_j^3} A_j . \qquad (5-14)$$

<u>Tidal deformation.</u> The products of inertia, C_{13} and C_{23}, are well known to be related to the tesseral term (n=2, m=1) of the spherical-harmonic expansion of the gravitational potential V; cf. (Heiskanen and Moritz, 1967, p. [illegible]). In the present case, where c_{13} and c_{23} represent the disturbances of the inertia tensor due to the tidal deformation, the relevant potential is

$$\Delta V = kV , \qquad (5-15)$$

which is the change of the gravitational potential [illegible] due to the deformation of the earth by the tidal potential V; cf. (4-11). In V we thus need only the tesseral part (1-10), and using (5-14) we get in a rather straightforward way (McClure, 1973, p. 22):

$$c_1 = [illegible] \frac{a}{C} [illegible] A \sin([illegible] t + [illegible]) ,$$
$$c_2 = [illegible] \frac{a}{C} [illegible] A \cos([illegible] t + [illegible]) , \qquad (5\text{-}16)$$

which again can be simplified by complex notation

$$c = - i [illegible] (C-A) [illegible] B e^{[illegible]} ; \qquad (5\text{-}17)$$

here we have used (4-2).

Tidal deformation causes a change of products of inertia, so (5-2) reduces to

$$[illegible] = \frac{c}{C-A} - \frac{i\dot{c}}{(C-A)[illegible]} , \qquad (5\text{-}18)$$

since [illegible] has been considered in (5-13). The differentiation of (5-17) gives $\dot{c}$, which together with (5-17) is substituted into (5-18). The result can be written as

$$[illegible] B e^{[illegible]} , \qquad (5\text{-}19)$$

where

$$\Delta\omega_j = \omega_j - \Omega \tag{5-20}$$

and k_s is defined by (4-2); B_j is given by (5-14).

The physical meaning of ψ_L and ψ_{TD} should be carefully kept in mind. The function ψ_L represents the torque exerted by sun and moon. It is the same for a rigid and a nonrigid earth. The function ψ_{TD} is caused by elastic deformation; it therefore depends on the Love number k and is zero for a rigid body, in accordance with (4-12). Thus ψ_L represents the "direct tidal effect" and ψ_{TD} the "indirect tidal effect", and the sum

$$\psi_T = \psi_L + \psi_{TD} = i\sum_j \left(1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}\right) B_j e^{-i(\omega_j t + \beta_j)} \tag{5-21}$$

expresses the total effect of sun and moon.

<u>Solution of the basic equation.</u> In view of (5-3) and (5-21), the basic equation (5-1) may be written

$$\dot{m} - i\sigma_E(m - \psi_{RD}) = -i\sigma_E\psi_T . \tag{5-22}$$

Using (4-3) and (4-5), this can be brought into the form

$$\begin{aligned} \dot{m} - i\sigma_C m &= -i\frac{\sigma_E}{1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}}\psi_T \\ &= \frac{\sigma_E}{1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}}\sum_j \left(1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}\right) B_j e^{-i(\omega_j t + \beta_j)} \end{aligned} \tag{5-23}$$

The solution of this equation is given by

$$m = m_0 e^{i\sigma_C t} + i\sum_j \frac{\sigma_E}{\omega_j+\sigma_C}\,\frac{1+\frac{k}{k_s}\frac{\omega_j}{\sigma_E}}{1+\frac{k}{k_s}\frac{\sigma_C}{\sigma_E}}\,B_j e^{-i(\omega_j t+\beta_j)} , \qquad (5\text{-}24)$$

as can be verified by substitution. It contains two constants, which are the real and the imaginary part of the complex constant m_0, and is thus the general solution of (5-23).

The solution (5-24) consists of the solution of the homogeneous equation (4-4), corresponding to free motion without external forces, and a term representing the effect of lunisolar perturbations.

6. POLAR MOTION : OTHER AXES

Eq. (5-24) describes the movement of the instantaneous rotation axis, which is characterized by the complex number $m = m_1 + im_2$ where m_1 and m_2 are the equatorial components of the unit vector of the rotation axis. Similarly, the angular momentum axis is characterized by the complex number h given by (3-25), and the figure axis is described by the complex number f given by (3-27).

Eq. (3-27) gives the figure axis:

$$f = \frac{c}{C-A} , \tag{6-1}$$

where $c = c_{13} + ic_{23}$ is a complex combination of products of inertia. By (3-22), (3-23), and (3-25) we have

$$h = \frac{A}{C} m + \frac{1}{C} c . \tag{6-2}$$

The combination of (6-1) and (6-2) finally gives

$$h = \frac{A}{C} m + \frac{C-A}{C} f \tag{6-3}$$

for the angular momentum axis.

For the resulting formulas we shall recall the notations for a few constants and shall introduce a new one : the secular Love number (4-2)

$$k_s = \frac{3G(C-A)}{a^5\Omega^2} , \tag{6-4}$$

the Eulerian frequency (3-18)

$$\sigma_E = \frac{C-A}{A}\,\Omega \quad , \tag{6-5}$$

the Chandlerian frequency (4-5)

$$\sigma_C = \frac{1-\frac{k}{k_s}}{1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}}\,\sigma_E \quad , \tag{6-6}$$

the nutational frequency (5-20)

$$\Delta\omega_j = \omega_j - \Omega \tag{6-7}$$

(the name will become clear in sec. 7), and the factor

$$\gamma_j = \frac{\Omega}{\omega_j+\sigma_C}\,\frac{1+\frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}}{1+\frac{k}{k_s}\frac{\sigma_E}{\Omega}} = \left\{\frac{\frac{\omega_j}{\Omega}}{1+\frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}} + \frac{\sigma_E}{\Omega}\right\}^{-1} . \tag{6-8}$$

Using these notations, we can represent the polar motion p of various axes. From (5-24) we get

$$p_R = m = m_0\, e^{i\sigma_C t} + i\frac{C-A}{A}\sum_j \gamma_j B_j e^{-i(\omega_j t+\beta_j)} \tag{6-9}$$

for the polar motion of the <u>rotation axis</u> R . The <u>figure axis</u> F is obtained from (6-1), where c is the sum of

(4-1) for rotational deformation and (5-17) for tidal deformation:

$$p_F = f = \frac{k}{k_s} m_0 e^{i\sigma_C t} - i\frac{k}{k_s} \sum_j \left(1 - \frac{\sigma_E}{\Omega}\kappa_j\right) B_j e^{-i(\omega_j t+\beta_j)} . \tag{6-10}$$

Finally, these two equations are linearly combined by (6-3) to getthe polar motion of the <u>angular momentum axis</u>:

$$p_H = h = h_0 e^{i\sigma_C t} + i\,\frac{C-A}{C} \sum_j \frac{1-\dfrac{k}{k_s}}{1+\dfrac{k}{k_s}\dfrac{\sigma_E}{\Omega}}\,\kappa_j B_j e^{-i(\omega_j t+\beta_j)} \tag{6-11}$$

with

$$h_0 = \left(\frac{A}{C} + \frac{C-A}{C}\,\frac{k}{k_s}\right) m_0 . \tag{6-12}$$

These formulas are illustrated by Fig. 6.1, which shows the plane tangent to the terrestrial sphere at the point 0 representing the z-axis and, at the same time, a mean position of all three axes. The other points designate the rotation axis R , the angular momentum axis H , the figure axis F , and their force-free counterparts R_0, H_0 and F_0, as given by the first term on the right-hand side of (6-9), (6-11), and (6-10) . In order to get a feeling for the orders of magnitude, we note that

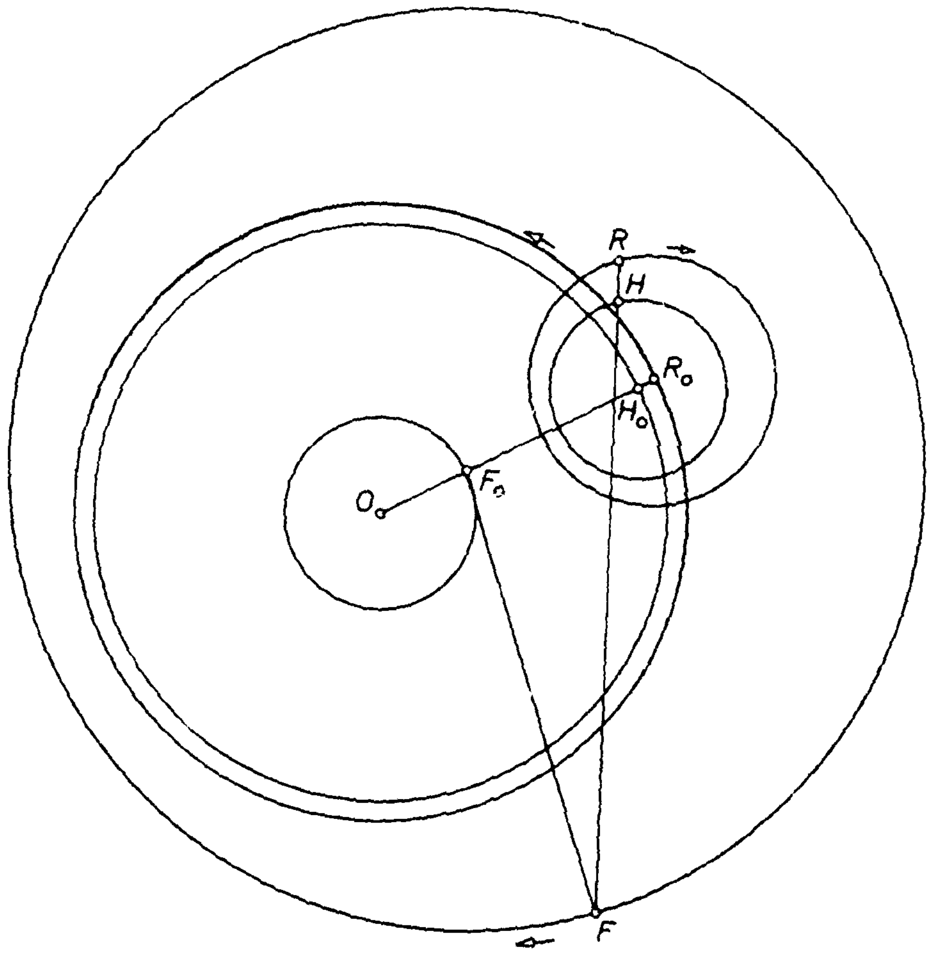


FIGURE 6.1. Polar motion for an elastic eart'

$$\kappa_j \doteq 1 - \frac{\sigma E}{\Omega}\kappa_j \doteq \frac{\kappa_j}{1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\omega}} \doteq \frac{A}{C} \doteq 1 \quad ,$$

$$\frac{C-A}{C} \doteq \frac{C-A}{A} \doteq 0.003 \quad , \tag{6-13}$$

$$\frac{k}{k_s} \doteq 0.3 \quad .$$

Let us first consider the <u>free motion</u> only. The points R_0 and H_0 describe concentric circles whose radius is on the order of 0.2" which corresponds to 6m . Both points are very close to each other since $h_0 \doteq m_0$ by (6-12) and (6-13); there is $H_0R_0 \doteq 2$cm. The point 0 corresponds to the figure axis in the undisturbed case. Since the axis of rotation does not coincide with the figure axis, the rotation of the earth produces a nonsymmetric deformation (sec. 4) which causes the axis of maximum inertia to shift to F_0 . The first term on the right-hand side of (6-10) shows that $OF \doteq 2$m since $k/k_s \doteq 0.3$. The prints R_0, H_0, and F_0 lie on the same radius and slowly rotate together around 0; the period is the Chandler period of about 430 days.

So much for the free motion. The attraction of sun and moon causes <u>forced motions</u> which are represented by the second term on the right-hand sides of (6-9), (6-10), and (6-11). The instantaneous pole of rotation, R , describes a near-circular closed curve around R_0; we have $R_0R \doteq 60$ cm. A similar curve is described by the angular momentum pole H around H_0;

since $1 - k/k_s \doteq 0.7$, we have $H_0H \doteq 40$ cm. Especially remarkable is the motion of the pole F of the figure axis: it describes a quasi-circular motion around F_0 whose radius is by $(k/k_s)A/(C-A) \doteq 100$ times larger than OR_0, namely about 60 m ! From (6-3) it is clear that R, H, and F lie on a straight line.

Since $\omega_j \doteq \Omega$, the period of these forced motions is on the order of 1 day; we therefore speak of diurnal polar motions.

We also point out the evident fact that the motion of the various axis for the case of a rigid earth can be obtained by putting the Love number $k = 0$ in these formulas. The main change in Fig. 6.1 is that, for a rigid earth, the points F and F_0 will coincide with the origin O and the radii R_0R and H_0H will be almost equal.

A final remark is in order. The actual free polar motion is much more complicated than the simple circular model considered here, for a variety of reasons, not all of which are well understood. The free motion cannot, therefore, be adequately predicted by an analytical model and can only be determined by observation (International Latitude Service, International Polar Motion Service, Doppler, Laser, VLBI). On the other hand it appears that the lunisolar (forced) motion of the pole can be predicted well.

7. PRECESSION AND NUTATION OF ANGULAR MOMENTUM AXIS

The starting point is our basic equation (2-1):

$$\frac{d\underline{H}}{dt} = \underline{L} \quad . \tag{7-1}$$

This equation, which states that the time derivative of the angular momentum $\underline{H}$ equals the torque $\underline{L}$, is valid for a nonrotating (inertial) system $X_1X_2X_3$. This system is specified as follows (Fig. 7.1). The X_1X_2-plane is the ecliptic

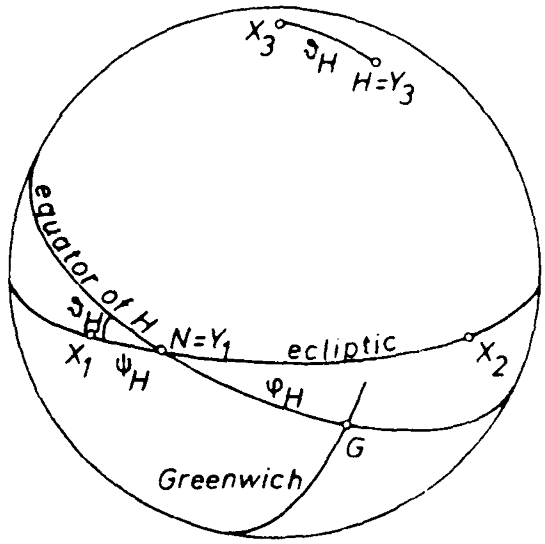


FIGURE 7.1. The angular momentum axis in space illustrated by means of a unit sphere.

[illegible] angular momentum vector $\underline{H}$, whose [illegible] representation [illegible] the point H.

It is clear that the position of the point H can be fixed by giving the two angles ϕ_H and θ_H shown in Fig. 7.1. Here ϕ_H is the longitude of the node and θ_H is very close to the obliquity of the ecliptic, about 23.5°. The components of the angular momentum vector are then

$$\underline{H} = H \begin{bmatrix} -\sin\theta_H \sin\phi_H \\ \sin\theta_H \cos\phi_H \\ \cos\theta_H \end{bmatrix} . \qquad (7\text{-}2)$$

This equation is differentiated with respect to time. We treat H as a constant since, to a high accuracy,

$$H = C\omega = \text{const.}, \qquad (7\text{-}3)$$

cf. the third equation of (2-11). The first two components of (7-1) are then

$$C\omega(-\cos\theta_H \sin\phi_H \dot{\theta}_H - \sin\theta_H \cos\phi_H \dot{\phi}_H) = L_{X1} ,$$
$$C\omega(\cos\theta_H \cos\phi_H \dot{\theta}_H - \sin\theta_H \sin\phi_H \dot{\phi}_H) = L_{X2} , \qquad (7\text{-}4)$$

where L_{X1} and L_{X2} are the X_1 and X_2 components of the torque $\underline{L}$. The solution with respect to θ_H and ϕ_H gives

[illegible]

wh [illegible]

[illegible]

and [illegible]

that [illegible]

easily [illegible]

Y, Y, Y are given [illegible]

[illegible]

[illegible]

knowing that [illegible]

(Leick, 1978, p. 1[illegible]). Thus [illegible]

[illegible]

[illegible]

These are the well-known Poisson equations [illegible]

the angular momentum axis, for a [illegible]

(Woolard, 1953).

where

$$\Delta\omega_j = \omega_j - \omega \qquad \text{(7-1[illegible])}$$

as before (6-7). Using (5-13), we thus get from (7-9)

$$\dot{\theta}_H + i\dot{\psi}_H \sin\theta_H = -i\,\frac{C-A}{C}\sum_j B_j e^{-i(\Delta\omega_j t + \text{[illegible]})} \,. \qquad \text{(7-13)}$$

Before integrating this equation, we must distinguish the cases $\Delta\omega_j = 0$ and $\Delta\omega_j \neq 0$. The first case also occurs since ω is a frequency that appears in the developement of the tidal potential (1-8). Let us number the frequencies ω_j in such a way that $j = 0$ corresponds to the frequency ω:

$$\omega_0 = \omega\,, \qquad \Delta\omega_0 = \omega - \omega = 0\,.$$

Then (7-13) can be split up by distinguishing the cases $j = 0$ and $j \neq 0$ (it may be shown that $\beta_0 = 0$):

$$\dot{\theta}_H + i\dot{\psi}_H \sin\theta_H = -i\,\frac{C-A}{C}\,B_0 - i\,\frac{C-A}{C}\sum_{j\neq 0} B_j e^{-i(\Delta\omega_j t + \text{[illegible]})} \,.$$

This equation can be immediately integrated since we may assume $\sin\theta_H$ on the left-hand side to be constant without loosing accuracy. This gives

$$[\text{illegible}] + i [\text{illegible}] \sin [\text{illegible}] = -i \frac{C-A}{C} B [\text{illegible}] t +$$

$$+ \frac{C-A}{C} [\text{illegible}] B [\text{illegible}] e^{[\text{illegible}]} + C_1 + iC_2 \; ,$$

where $C_1 + iC_2$ is a complex constant of integration. On putting

$$[\text{illegible}]_H - C_1 = [\text{illegible}]_H , \quad [\text{illegible}]_H \sin [\text{illegible}]_H - C_2 = [\text{illegible}]_H \sin [\text{illegible}]_H \; ,$$

this takes the final form

$$[\text{illegible}]_H + i [\text{illegible}]_H \sin [\text{illegible}]_H = -i \frac{C-A}{C} B [\text{illegible}] t +$$

$$+ \frac{C-A}{C} \sum_{j \neq 0} [\text{illegible}] B_j e^{-i [\text{illegible}]} \; . \qquad (7-14)$$

The first term on the left-hand side increases linearly with time; it is a secular term expressing precession. The second term is periodic (or rather quasiperiodic) and expresses nutation. Real expressions for $[\text{illegible}]_H$ and $[\text{illegible}]_H$ can be obtained by separating this equation into a real and an imaginary part. This immediately shows that there is no secular part in $[\text{illegible}]_H$: precession affects only $[\text{illegible}]_H$. On the other hand, nutation affects both $[\text{illegible}]_H$ (nutation in longitude) and $[\text{illegible}]_H$ (in obliquity).

These equations also show that the nutation frequency [illegible] is obtained from the tidal frequency [illegible] by subtracting the sidereal frequency (rotational speed) [illegible]. This fact, expressed by (7-1[illegible]), is also intuitively evident since the tidal potential refers to an earth-fixed coordinate system $x_1 x_2 x_3$, and precession and nutation refer to an inertial system ...; both systems are rotating relatively to each other with ... velocity Ω .

Since the angular motions of sun and moon are ... smaller than the rotation of the earth, we have

$$\Delta\omega_j \ll \omega , \qquad \omega_j \doteq \omega . \qquad (7-15)$$

We also know (Melchior, 1978, p.33) that the tidal spectrum is symmetric with respect to ... . Let us use positive and negative subscripts in such a way that j and $-j$ always denote a symmetric frequency pair:

$$\Delta\omega_{-j} = -\Delta\omega_j . \qquad (7-16)$$

Then, two symmetric frequencies ω_j and ω_{-j} give the same nutational frequency $\Delta\omega_j$ (only the sign of the coefficients will be different).

It is clear that the sidereal frequency $\omega = \Omega$ produces precession. The principal nutation component comes from an ω_j that corresponds to the motion of the lunar node. It has a period of 18.66 years and amplitudes in longitude (i.e., in ...) of 17.2" and in obliquity (i.e., in $\Delta\epsilon$) of 9.2" . The tidal effect of this motion is insignificant, but in view of the smallness of $\Delta\omega$, the corresponding nutational coefficient, proportional to $B_j/\Delta\omega_j$ by (7-14), is greatly magnified. Other periods are a year (sun) and a month (moon) and their multiples.

Generally, we see that the coefficients in precession and nutation figuring in (7-14) are simply related to the coefficients of the tidal potential, namely through (5-14). The fact that precession (as well as nutation) is proportional to the constant

$$H = \frac{C-A}{C} \tag{7-17}$$

makes it possible to determine this constant. It is called dynamical ellipticity and is of basic importance for physical geodesy (cf. Heiskanen and Moritz, 1967, p.339).

The relation between precession, nutation and tidal potential has been studied in particular detail by Melchior (1971, 1978).

Finally we point out that precession and nutation of the angular momentum axis depend only on the lunisolar torque and not on the Love number k. Therefore, the formulas (7-14) are the same for a rigid and an elastic earth; cf. also (Fedorov, 1963, p. 16).

8. PRECESSION AND NUTATION : OTHER AXES

Of the three axes considered in sec.6 : the instantaneous rotation axis, the angular momentum axis, and the figure axis, we have treated the spatial motion of the angular momentum axis in the preceding section. The spatial motion of the remaining two axes will be studied now. As we have seen in sec.6 , however, the instantaneous figure axis performs, in the case of an elastic earth, such a large daily motion with respect to the earth's body (about 60 m) that it is of little practical use. Much more useful is the z-axis which corresponds to the figure axis of an undeformed earth and, for a rigid earth, coincides with the figure axis. So we shall study the motion of the z-axis and of the instantaneous rotation axis, as well as the so-called celestial pole.

It will turn out that the precession of different axes is the same, and the nutation nearly so. The very small differences between the nutation of the angular momentum axis and that of other axes are sometimes called Oppolzer terms (Woolard, 1953; Kinoshita, 1977).

Motion of the z-axis. We again employ our usual two coordinate systems : a "space-fixed" inertial coordinate system $X_1X_2X_3$ as specified in the beginning of sec.7 (cf. Fig. 7.1) and an "earth-fixed" system $x_1x_2x_3$ introduced in sec.3. In the latter system, the axis $x_3 = z$ corresponds to the figure axis of the undeformed earth, and it is a Liouville axis, the equation of motion taking the simple form (3-8). The z-axis is, therefore, also called a <u>mean Tisserand figure axis</u>. The X_1X_2 plane is the ecliptic, and the X_1 axis represents the vernal equinox (both at a fixed epoch). The x_1x_2 plane represents the equator (more precisely, the mean equator of figure), and the x_1 axis corresponds to the Greenwich meridian (more precisely, to a conventionally assumed fixed direction close to the Greenwich meridian).

Since we are concerned only with directions and rotations, the origins are of no interest here; we can for the present purpose consider both systems geocentric.

The relative orientation of the $x_1x_2x_3$ system relative to the $X_1X_2X_3$ system can be given by the three Euler angles

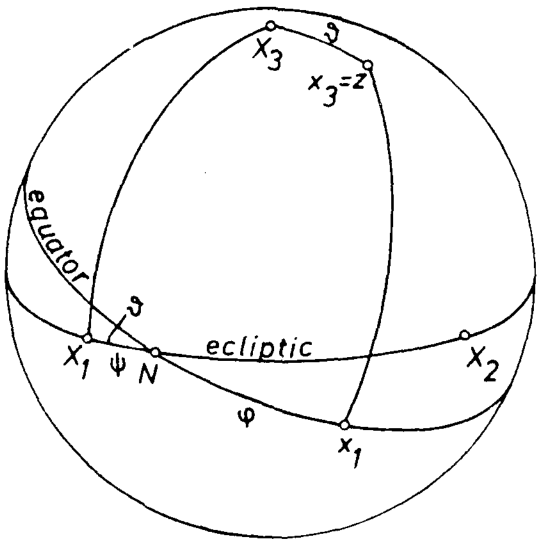


FIGURE 8.1. The basic Euler angles.

ϕ, θ, ψ defined as in Fig. 8.1. The quantities ϕ, θ, ψ are similar, but not identical, to the angles ϕ_H, θ_H, ψ_H of Fig. 7.1 : now the pole is the z-axis and not the angular momentum axis. We get the system $x_1x_2x_3$ by rotating the system $X_1X_2X_3$ first about the 3-axis by the angle ψ, until it coincides with the node N (which is, of course, not exactly the same point as N in Fig. 7.1), then about the nodal axis by the angle θ, and finally about the 3-axis by the angle ϕ.

The angle ψ is the longitude of the node, θ represents the obliquity of the ecliptic, and ϕ is an angle that measures the rotation of the earth. These Euler angles are frequently used in physics and astronomy; note, however, that almost every author uses a different definition of them. We follow (Plummer, 1918) and (McClure, 1973).

The components of the rotation vector $\underline{\omega}$ (which does not in general coincide with the $x_3 = z$ axis!) are denoted by $\omega_1, \omega_2, \omega_3$. Compared to (3-11), (3-12) and (3-13) we have

$$\omega_1 = \Omega m_1 , \qquad \omega_2 = \Omega m_2 , \qquad \omega_3 = \Omega(1+m_3) . \tag{8-1}$$

They are connected to the time derivatives of the Euler angles by Euler's well-known kinematical equations:

$$\begin{aligned} \omega_1 &= -\dot{\theta}\cos\phi - \dot{\psi}\sin\theta\sin\phi , \\ \omega_2 &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi , \\ \omega_3 &= \dot{\psi}\cos\theta + \dot{\phi} , \end{aligned} \tag{8-2}$$

cf. (Synge, 1960, p.28), but note the different definition of Euler angles.

The complex combination of the first two equations gives

$$\omega_1 + i\omega_2 = -e^{-i\phi}(\dot{\theta} + i\dot{\psi}\sin\theta) \ , \qquad (8\text{-}3)$$

whence

$$\begin{aligned} \dot{\theta} + i\dot{\psi}\sin\theta &= -e^{i\phi}(\omega_1 + i\omega_2) \\ &= -e^{i\phi}\Omega(m_1 + im_2) \ , \end{aligned}$$

by (8-1) , or briefly, using (3-15)

$$\dot{\theta} + i\dot{\psi}\sin\theta = -\Omega m e^{i\phi} \ . \qquad (8\text{-}4)$$

This equation is of basic importance: it relates precession and nutation, as expressed by the Euler angles ψ and θ , to the polar motion of the rotation axis, m .

The angle ϕ measures the earth's rotation. Since the changes of θ and ψ are very small, it is a sufficient approximation to put, similarly to (7-10),

$$\phi = \Omega t \ . \qquad (8\text{-}5)$$

Then we can substitute m from (6-9) into (8-4) to obtain

$$\dot{\theta} + i\dot{\psi}\sin\theta = -\Omega m_0 e^{i(\Omega+\sigma_C)t} - i\,\frac{C-A}{A}\,\Omega\sum_j \kappa_j B_j e^{-i(\omega_j t+\beta_j)} \quad . \qquad (8\text{-}6)$$

This equation is very similar to (7-13) and is integrated in the same way, with the result

$$\Delta\theta + i\Delta\psi\sin\theta = i\,\frac{\Omega}{\Omega+\sigma_C}\,m_0 e^{i(\Omega+\sigma_C)t} - i\,\frac{C-A}{C}\,\Omega B_0\,t + \frac{C-A}{A}\sum_{j\neq 0}\frac{\Omega}{\omega_j}\,\kappa_j B_j e^{-i(\omega_j t+\beta_j)} \quad . \qquad (8\text{-}7)$$

where, in analogy to (7-14), $\Delta\theta$ and $\Delta\psi$ are the difference between the instantaneous values of θ and ψ and some reference values.

This equation is very similar to (7-14). In fact, the precession term is the same, and the nutation terms only differ by the factor κ_j which is very close to unity by (6-8) and (6-13). An important difference is the presence of the first term on the right-hand side of (8-7), which originates from free polar motion.

This free motion term has a nearly diurnal frequency, since $\Omega + \sigma_C$ is very close to the sidereal frequency. The forced motion (lunisolar) term, on the other hand, is long-periodic, the principal periods being 18.66 years, a year, and a month, as we have mentioned in sec.7.

This is in marked contrast to polar motion where the forced part is nearly diurnal and the free motion being long-periodic with the Chandler period of 430 days.

To repeat, $\Delta\psi$ and $\Delta\varepsilon$ in (8-7) represent precession and nutation of the mean Tisserand figure axis.

Motion of the rotation axis. Let ϑ_R and ψ_R be the Euler angles of the rotation axis, defined in analogy to the definition of ϑ_H and ψ_H for the angular momentum axis in sec.7. This definition is shown by Fig. 8.2, where R denotes the rotation axis and "equator of R " corresponds to a plane normal to the rotation axis. We also introduce the differences

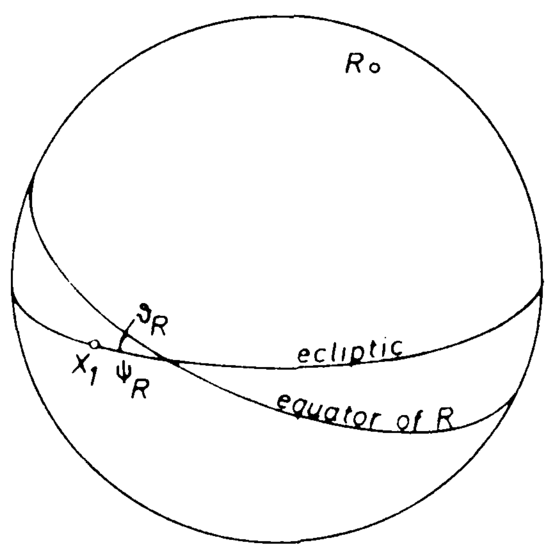


FIGURE 8.2. The rotation axis R in space illustrated on the unit sphere.

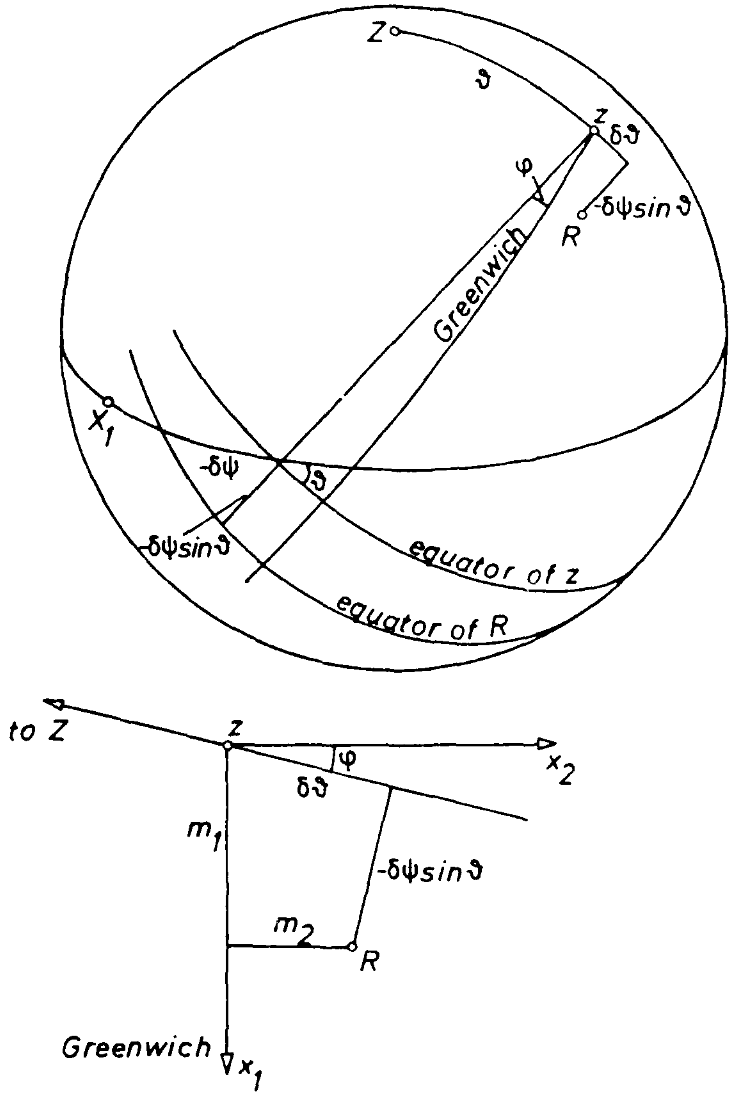


FIGURE 8.3. Direction difference between rotation axis R and x_3 axis z, illustrated on the unit sphere (above) and on the tangent plane at z (below).

$$\Delta\psi = \psi_R - \psi , \qquad \Delta\theta = \theta_R - \theta \tag{8-8}$$

between the Euler angles for the rotation axis and those for the z-axis as considered in (8-7).

Fig. 8.3 shows that $\Delta\theta$ and $(-\Delta\psi \sin\theta)$ are related to the polar motion components m_1 and m_2 by a plane rotation, which is best written in complex form:

$$\Delta\theta + i\Delta\psi\sin\theta = -ime^{i\Phi} = -ime^{i\Omega t} \tag{8-9}$$

where $m = m_1 + im_2$ as usual. Again we substitute (6-9) and obtain

$$\Delta\theta + i\Delta\psi\sin\theta = -im_0 e^{i(\Omega+\sigma_C)t} + \\ + \frac{C-A}{A} \sum_j \kappa_j B_j e^{-i(\Delta\omega_j t+\beta_j)} . \tag{8-10}$$

This is added to (8-7) and gives

$$\Delta\theta_R + i\Delta\psi_R \sin\theta = -i \frac{\sigma_C}{\Omega+\sigma_C} m_0 e^{i(\Omega+\sigma_C)t} + \\ -i \frac{C-A}{C} B_0 \Omega t + \frac{C-A}{C} B_0 + \\ + \frac{C-A}{A} \sum_{j\neq 0} \frac{\Omega}{\Delta\omega_j} \left(1 + \frac{\Delta\omega_j}{\Omega}\right) \kappa_j B_j e^{-i(\Delta\omega_j+\beta_j)} . \tag{8-11}$$

The equation is very similar to (7-14) and (8-7). Again, the precession term is the same, and the nutation coefficients nearly so. Again, there is a term due to free polar motion (the first on the right-hand side) which, however, is much smaller tnan the free motion term in (8-7).

The celestial pole. Free polar motion cannot be adequately modeled in a simple mathematical way; the circle

$$m_0 e^{i\sigma_c t}$$

around 0 with radius OR_0 (Fig. 6.1) is valid only for an ideally elastic earth. In reality, the polar motion curve is rather irreqular and can only be determined empirically; cf. (Mueller, 1969, p.83).

On the other hand, lunisolar effects can be predicted well, both in polar motion and in nutation. It is, therefore, appropriate to refer calculation of precession and nutation to an axis which is not affected by free polar motion. This excludes the rotation axis (8-11) and the z-axis (8-7), but leaves the angular momentum axis which does not contain a free motion term in nutation by (7-14).

An inspection of Fig. 6.1. shows, however, that the instantaneous angular momentum axis H shows a forced nearly diurnal polar motion of radius 40 cm around H_0. If we wisn to have an axis which not only is a suitable reference for precession and nutation, but also is relatively stable with respect [illegible] will be a better [illegible] called the *Celes-tial pole* [illegible] below.

By (6-11), the polar motion [illegible] described by the complex number

$$n_{[illegible]} = n\,e^{[illegible]} \quad [illegible] - n_{LS}\,, \qquad (8\text{-}12)$$

where n_{LS} denotes the lunisolar effect given by the last term of (6-11). Denoting by

$$[illegible]_{HC} \quad [illegible]_{H}\,, \qquad [illegible]_{HC} \quad [illegible]_{H} \qquad (8\text{-}13)$$

the differences between the Euler angles of H and C = H[illegible], we get by an application of (8-9) to the present situation (with $-h_{LS}$ replacing m):

$$[illegible]_{HC} + i\,[illegible]_{HC}\sin[illegible] = +ih_{LS}e^{i[illegible]}$$

$$[illegible]\frac{C-A}{C}\;\frac{1-\frac{k}{k_s}}{1+\frac{k}{k_s}}\;[illegible]\,B\,e^{-i[illegible]}\;[illegible] \;. \qquad (8\text{-}14)$$

This [illegible] (8-14) to give, after some algebra,

$$[illegible]\,n\;[illegible]\frac{C-A}{C}\,[illegible] + [illegible]\frac{A(C-A)}{C^{[illegible]}}\left(1-\frac{k}{k_s}\right)B\;+$$

$$+\frac{C-A}{A}\;[illegible]\,B\,e^{-i[illegible]}\;[illegible] \;. \qquad (8\text{-}15)$$

Thus, by (8-12), the polar motion of C contains no forced part and, by (8-15), its nutation contains no free part. Both missing parts would have had a nearly diurnal period as we have pointed out in above. Both motions are thus optimally smooth.

Furthermore, $C = H$ shares with the angular momentum axis H the property that its nutation is analytically predictable since it does not contain a free term which would be accessible only afterwards by observation.

For these and other reasons (Leick, 1978; Leick and Mueller, 1979), it appears that C is the best candidate for an appropriate definition of a celestial reference pole. At its General Assembly in Montreal in 1979, the International Astronomical Union has, in fact, adopted C as the official celestial pole of reference.

We finally note that, for $m = 0$, the expressions (8-7) and (8-15) coincide. Eq. (6-10) shows that for a rigid body (k=0), also $f = 0$ in this case, so that the figure axis remains at the origin of polar motion. Thus, for the case of a rigid earth and in the absence of polar motion (but not generally!), the celestial pole coincides with the pole F of the figure axis. Therefore, the direction of the celestial pole has also been called "figure axis". This unfortunate terminology (Atkinson, 1975) has caused considerable confusion, as described by Mueller (1980).

If the earth is elastic (even with a liquid core) and polar motion is absent, then we can only say that C coincides with the origin O in Fig. 6.1, which is seen to be different from the actual figure [illegible] axis [illegible] moment, the [illegible] motion, [illegible] coincides with the [illegible] The nutations computed by Wahr (1979, [illegible]) [illegible] an axis and hence to the celestial pole.

Case of a rigid earth. Eqs. (8-7), (8-11) [illegible] contain the elastic constant k only through [illegible], where its influence is not very pronounced. Therefore, the lunisolar nutations for a purely elastic earth do not differ strongly from those of a rigid earth given by [illegible]; the nutation of the angular momentum axis is even the same in both cases as we have seen in the preceding section (cf. also Capitaine, 1980). An essential effect on nutation is, however, produced by the liquid core, as we shall see in [illegible] 13.

Remark on method. The theory of precession and nutation described in secs. 7 and 8 seems to be the most systematic, transparent and complete approach to the problem. Systematic, because it is directly related to the theory of tides and polar motion; transparent because of its relative simplicity; and complete because it is valid, not only for a rigid but also for an elastic earth. It contains, however, some simplifications which, for a rigid body, make it slightly less accurate than Kinoshita's theory.

The treatment of polar motion, precession and nutation presented in secs. [illegible] through [illegible] is based on the fundamental work by McClure (19[illegible]), which has been further elaborated by [illegible] (1978) and Capitaine (19[illegible]). [illegible] (195[illegible]) theory of nutation has some affinity to the present method, especially in the distinction between angular momentum axis, figure axis, and rotation axis; this relation has been pointed out by McClure (1973, sec.8). It is restricted to a rigid earth and presents a less systematic and complete treatment of all aspects, but numerical details such as the neglect of terms are more carefully considered (although several authors have pointed out minor errors in Woolard's treatment). Woolard's theory has been a breakthrough and a classic. It has served for decades as the official reference for precession and nutation of the International Astronomical Union. Now, however, it has been superseded by Kinoshita's theory for a rigid earth and by other theories for elastic and liquid core models.

9. KINOSHITA'S THEORY OF PRECESSION AND NUTATION.

The most elegant formulation of analytical mechanics is Hamilton's theory. It can be summarized as follows, cf. (Arnold, 1978) or (Synge, 1960). Let a conservative mecnanical system be described by n independent variables q_k which are called "generalized coordinates", let its kinetic energy be T and its potential energy U. Define the "generalized impulses" p_i by

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \tag{9-1}$$

where $\dot{q}_i = dq_i/dt$, and define the Hamiltonian function by

$$H = T + U , \tag{9-2}$$

equal to the total energy. Then the equations of motion for the dynamical system under consideration are Hamilton's canonical equations

$$\begin{aligned} \frac{dq_k}{dt} &= \frac{\partial H}{\partial p_k} , \\ \frac{dp_k}{dt} &= -\frac{\partial H}{\partial q_k} . \end{aligned} \tag{9-3}$$

The quantities p_k and q_k are called canonical variables.

Because of its formal simplicity, Hamilton's theory possesses considerable theoretical advantages. Canonical variables are, therefore, frequently used in celestial mechanics; cf. (Brouwer and Clemence, 1961, chapter XVII). They have been introduced into the theory of precession and nutation by H. Andoyer in 1911; cf. (Andoyer, 1923, 1926). Recently, Kinoshita (1977) has used Andoyer variables to derive the most accurate theory of precession and nutation available for a rigid earth. In this section we shall present the theoretical foundation of Kinoshita's theory which is very simple whereas the details are enormously complicated.

Andoyer's variables are denoted by

$$\begin{aligned} q_1 &= l \,, & p_1 &= L \,, \\ q_2 &= g \,, & p_2 &= G \,, \\ q_3 &= h \,, & p_3 &= H \,, \end{aligned} \tag{9-4}$$

so that coordinates and corresponding impulses are denoted by the same letter. Their definition is as explained by means of Fig. 9.1.

We use the two systems $X_1X_2X_3$ (inertial) and $x_1x_2x_3$ (earth-fixed) as before (cf. sec.8). The ecliptic (fixed at an epoch t_0) corresponds to the X_1X_2 plane, and the (instantaneous) equator to the x_1x_2 plane. The point x_3 denotes the figure axis which, in fact, for a rigid body coincides with the $z = x_3$ axis. The point M denotes the angular momentum vector $\underline{M}$. Note the change in notation for this vector which

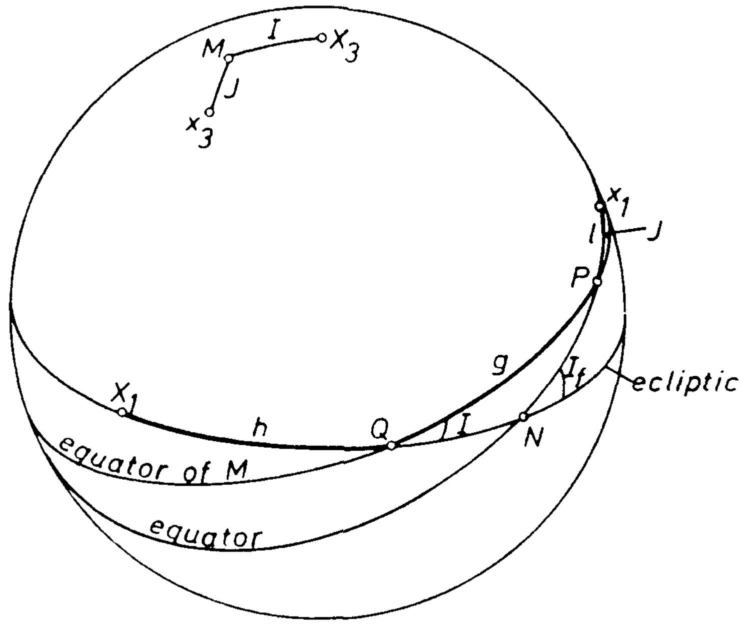


FIGURE 9.1. The Andoyer variables h, g, l . The point M denotes the angular momentum vector.

elsewhere in this report is denoted by $\underline{H}$. This change is necessary since we wish to retain Kinoshita's notation (9-4) where H is a canonical variable. The equation of M corresponds to a plane which is normal to the angular momentum vector $\underline{M}$.

Fig. 9.1 can be considered a superposition of Figs. 7.1. and 8.1 , with the important difference that now we look, so to speak, at the back of the unit sphere : the nodes N and

Q shown in Fig. 9.1 differ from the corresponding nodes in the previous figures by 180°.

Apart from this difference, the Andoyer variable h corresponds to ψ_H in Fig. 7.1 and the new variable l is seen essentially to be the argument of polar motion ($\sigma_E t + \cdot$ in eq. (2-16)). The sum g + l is nearly equal to the Euler angle $\phi + 180°$. The angles I and I in Fig. 9.1 are equivalent to θ_H and θ in the previous figures.

Kinoshita also introduces the spherical distances

$$X_1 N = h_\varepsilon \quad , \qquad\qquad N x_1 = \phi \quad , \tag{9-5}$$

corresponding, in Fig. 8.1., to ψ and ϕ, respectively. Hence h_ε, I_ε, ϕ are the Euler angles (apart from 180°) in Kinoshita's notation.

Having thus introduced the canonical coordinates l, g, h, we can easily find the corresponding canonical impulses L, G, H by (9-1). First we have to find the kinetic energy T. From classical mechanics (Arnold, 1978, p. 137) we know the relation

$$T = \frac{1}{2}\, \underline{\omega}^T \underline{C}\, \underline{\omega} \tag{9-6}$$

where $\underline{C}$ is the inertia tensor (2-10), $\underline{\omega}$ is the rotation vector, and $\underline{\omega}^T$ its transpose. The vector $\underline{\omega}$ can now be written as follows:

$$\underline{\omega} = \dot{h}\underline{e}_h + \dot{g}\underline{e}_g + \dot{l}\underline{e}_l \ . \tag{9-7}$$

This means that the total rotation is split up into a rotation about the X_3 axis with speed $\dot{h} = dh/dt$, into a rotation about the angular momentum axis with speed $\dot{g}$, and a rotation about the X_3 axis with speed $\dot{l}$. The vectors $\underline{e}_h$, $\underline{e}_g$, and $\underline{e}_l$ are the unit vectors of the directions around which the rotations are performed: $\underline{e}_h$ is the unit vector of the X_3 axis (the angle h is counted in the plane normal to this axis), $\underline{e}_g$ is the unit vector of the angular momentum vector $\underline{M}$ (the angle g is counted in the plane normal to $\underline{M}$), and $\underline{e}_l$ is the unit vector of the x_3 axis (for a similar reason).

The differentiation of (9-6) with respect to $\dot{l}$ yields:

$$\frac{\partial T}{\partial \dot{l}} = \frac{1}{2}\,\underline{\omega}^T \underline{C}\,\frac{\partial \underline{\omega}}{\partial \dot{l}} + \frac{1}{2}\,\frac{\partial \underline{\omega}^T}{\partial \dot{l}}\,\underline{C}\,\underline{\omega} = \underline{\omega}^T \underline{C}\,\frac{\partial \underline{\omega}}{\partial \dot{l}} = \underline{M}^T \underline{e}_l$$

$$= \underline{M} \cdot \underline{e}_l \quad , \qquad (9\text{-}8)$$

because of the symmetry of $\underline{C}$, using (2-5) with $\underline{M}$ instead of $\underline{H}$ and (9-7), and denoting the inner product by a dot as usual. Other derivatives are obtained in the same way. The canonical impulses are now given by (9-1) with (9-4) :

$$L = \frac{\partial T}{\partial \dot{l}} = \underline{M} \cdot \underline{e}_l \ ,$$

$$G = \frac{\partial T}{\partial \dot{g}} = \underline{M} \cdot \underline{e}_g \ , \qquad (9\text{-}9)$$

$$H = \frac{\partial T}{\partial \dot{h}} = \underline{M} \cdot \underline{e}_h \ .$$

Thus, L is the x_3 component of $\underline{M}$, G is the magnitude $|\underline{M}|$ of the angular momentum vector itself since the unit vector $\underline{e}_g$ and $\underline{M}$ have the same direction, and H is the X_3 component of $\underline{M}$.

Briefly we may say that a canonical impulse (L, G, or H) is the component of $\underline{M}$ normal to the plane along which the corresponding canonical coordinate (l, g, or h) is counted.

Using the angles I and J in Fig. 9.1, we may write

$$G = |\underline{M}| , \qquad L = G\cos J , \qquad H = G\cos I . \tag{9-10}$$

If we know the canonical variables (l, g, h, L, G, H), then the Euler angles can easily be computed: by (9-10) we have

$$\cos I = \frac{H}{G} , \qquad \cos J = \frac{L}{G} ; \tag{9-11}$$

the solution of the spherical triangle NPQ then gives QN and NP, and finally

$$\begin{aligned} h_E &= X_1N = h + QN \\ \phi &= Nx_1 = NP + l . \end{aligned} \tag{9-12}$$

We now need an expression of the kinetic energy T in terms of canonical variables. In the body-fixed system $x_1x_2x_3$, formed by the principal axis of inertia, the expression (9-6) becomes, using (2-10) and (2-11),

$$T = \frac{1}{2} \underline{M} \cdot \underline{\omega} = \frac{1}{2} (M_1 \omega_1 + M_2 \omega_2 + M_3 \omega_3) =$$

$$= \frac{1}{2} \left(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C} \right) ; \qquad (9\text{-}13)$$

we do not presuppose rotational symmetry so that the principal equatorial moments of inertia, A and B, may be different.

Now, $M_3 = L$ by (9-9). Further, the projection of $\underline{M}$ onto the $x_1 x_2$ plane is

$$\sqrt{M_1^2 + M_2^2} = \sqrt{G^2 - L^2} , \qquad (9\text{-}14)$$

and forms with the x_1 axis the angle $270^\circ - l$, so that

$$M_1 = - \sqrt{G^2 - L^2} \sin l ,$$
$$M_2 = - \sqrt{G^2 - L^2} \cos l . \qquad (9\text{-}15)$$

Hence, (9-13) becomes

$$T = \frac{1}{2} \left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) (G^2 - L^2) + \frac{L^2}{2C} , \qquad (9\text{-}16)$$

which expresses the kinetic energy in terms of the canonical variables.

Now we may write the Hamiltonian equations (9-3) for our problem:

$$\frac{dl}{dt} = \frac{\partial \mathcal{H}}{\partial L} , \qquad \frac{dL}{dt} = - \frac{\partial \mathcal{H}}{\partial l} ,$$

$$\frac{dg}{dt} = \frac{\partial \mathcal{H}}{\partial G} , \qquad \frac{dG}{dt} = - \frac{\partial \mathcal{H}}{\partial g} , \tag{9-17}$$

$$\frac{dh}{dt} = \frac{\partial \mathcal{H}}{\partial H} , \qquad \frac{dH}{dt} = - \frac{\partial \mathcal{H}}{\partial h} .$$

By (9-2) we have

$$\mathcal{H} = T + U ,$$

where T is given by (9-16) and the potential energy U comes from the attraction of sun and moon.

Equations (9-17) may be solved by standard perturbation methods of celestial mechanics (Brouwer and Clemence, 1961, chapter XVII), using, as a first approximation, free motion with $U = 0$. This free motion is discussed in detail, e.g. in (Arkhangelsky, 1977). Andoyer (1926) used simple variation of constants, whereas Kinoshita employs a considerably more sophisticated technique (method of Hori, cf. (Schneider, 1979, p.409)).

We cannot go into the details of this solution of the forced motion, which are enormously envolved, and refer the reader to (Kinoshita, 1977). The results are series for precession and nutation of the angular momentum axis, the figure axis (for a rigid body coinciding with the $z = x_3$ axis), and the rotation axis which have the form of the series derived in secs. 7 and 8 . In fact, $\Delta h = \Delta\psi_H$ and $\Delta I = \Delta\varepsilon_H$ are nutation in longitude and obliquity for the angular momentum axis, and similarity for the other axes.

Kinoshita's results are accurate to 0.0001" , corresponding to 3 mm in position. They represent the most complete and precise theory of precession and nutation available for a rigid earth, especially because he developes a very accurate expression for the lunisolar potential, and are thus a progress with respect to (Woolard, 1953). Kinoshita's theory is also somewhat more accurate than the method described in secs. 7 and 8, but this latter method is valid also for an elastic earth whereas Kinoshita's method is restricted to a rigid body.

The best way to compute nutations for an elastic earth seems to apply the formulas of sec. 8 to compute differences

$$\Delta\psi_{elastic} - \Delta\psi_{rigid} ,$$

$$\Delta\varepsilon_{rigid} - \Delta\varepsilon_{rigid}$$

by applying these formulas first for the actual Love number k and then for $k = 0$ (which gives rigid body results). Since these differences are very small, they can be computed quite precisely by the theory of sec. 8 . These differences are then added to Kinoshita's rigid-body results to give

$\Delta\psi_{elastic}$ and $\Delta\theta_{elastic}$.

A similar procedure is advocated by Wahr (1979,1980) to take into account effects of the liquid core. In fact, such effects are considerably larger and more important than effects of elasticity. Therefore, the remaining part of the report will be devoted to the influence of the liquid core.

10. EIGENVALUE PROBLEM FOR RIGID BODY ROTATION

The consideration of the free rotation of a rigid body as an eigenvalue problem is of basic theoretical importance and will serve as a preparation for a detailed understanding of liquid core effects to be treated later in this report. We shall restrict ourselves to a rotationally symmetric earth in which the principal equatorial moments of inertia are equal:

$$A = B \; ; \tag{10-1}$$

the general case $A \neq B$ can be treated in a similar way.

<u>Eigenvalues for Euler's Equations.</u> By (2-13) we have

$$\begin{aligned} A\dot{\omega}_1 + (C-A)\omega_2\omega_3 &= 0 \;, \\ A\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= 0 \;, \\ C\dot{\omega}_3 &= 0 \;. \end{aligned} \tag{10-2}$$

Here C is the maximum (polar) moment of inertia, and

$$[illegible] \;, \tag{10-3}$$

which is the instantaneous rotation vector referred to the body-fixed system of principal axes of inertia, which we denote as usual by $x_1x_2x_3$, or briefly, by x_i.

As in (3-11) we put

$$\underline{\omega} = \underline{\omega}_0 + \Delta\underline{\omega} = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} , \tag{10-4}$$

where Ω represents a constant value for the average speed of rotation and $\omega_1, \omega_2, \omega_3$ are very small as compared to Ω, so that their squares and higher-order powers can be neglected.

Thus we may put $\omega_3 \doteq \Omega$ in the first two equations of (10-2), so that Euler's equations become linear:

$$\begin{aligned} \dot{\omega}_1 + \sigma_E \omega_2 &= 0 , \\ \dot{\omega}_2 - \sigma_E \omega_1 &= 0 , \\ \dot{\omega}_3 &= 0 , \end{aligned} \tag{10-5}$$

where

$$\sigma_E = \frac{C - A}{A}\,\Omega \tag{10-6}$$

is the Euler frequency (2-14). (We are just looking at the problem of sec. 2 from a somewhat different angle.)

As before, it will be convenient to use complex quantities. We put

$$u = \omega_1 + i\omega_2 \qquad (i = \sqrt{-1}) . \tag{10-7}$$

Then the system (10-5) becomes

$$\dot{u} - i\omega_E u = 0 ,$$
$$\dot{v} = 0 . \qquad (10\text{-}8)$$

This is a system of homogeneous linear differential equations with constant coefficients, whose solution can be represented in the usual exponential form:

$$u = ae^{i\omega t} , \qquad v = be^{i\omega t} . \qquad (10\text{-}9)$$

Here ω is an eigenfrequency (proper frequency) to be determined, and a and b are complex constants. The corresponding kinds of rotation are called <u>proper modes</u>.

The substitution of (10-9) into (10-8) leads to a system of algebraic linear equations for a and b:

$$i\omega a - i\omega_E a = 0 ,$$
$$i\omega b = 0 ,$$

or

$$(\omega - \omega_E) a = 0 ,$$
$$\omega b = 0 . \qquad (10\text{-}10)$$

This homogeneous system has a solution only if the determinant [illegible]

$$\begin{vmatrix} \sigma - \sigma_E & 0 \\ 0 & \sigma \end{vmatrix} = \sigma(\sigma - \sigma_E) = 0 . \qquad (10\text{-}11)$$

This is a very simple quadratic equation for σ, of which the solutions obviously are

$$\sigma_1 = 0 , \qquad (10\text{-}12)$$

$$\sigma_2 = \sigma_E = \frac{C - A}{A} \Omega . \qquad (10\text{-}13)$$

These two values constitute the two eigenfrequencies (eigenvalues, proper values) for Eulerian motion considered as an eigenvalue problem.

For the first eigenvalue $\sigma_1 = 0$, eq. (10-10) gives $a = 0$ and

$$\omega_3 = b = \text{const.}$$

but otherwise arbitrary. Thus also

$$\delta\omega_3 = \omega_3 - \Omega = \text{const.} , \qquad (10\text{-}14)$$

which gives an arbitrary constant increment to the angular velocity of rotation without changing the direction of the rotation axis. This particular proper mode of rotation is the axial spin mode (ASM) of (Smith, 1977).

More important is the second eigenvalue $\sigma_2 = \sigma_E$, which, of course, is the Eulerian frequency of polar motion for the rigid earth as discussed in sec. 2 . In fact, now (10-10) gives a arbitrary and $b = 0$, which means that the angular velocity remains unchanged but the rotation axis undergoes a periodic motion according to

$$u = a e^{i\sigma_E t} , \tag{10-15}$$

which is equivalent to (2-16). In the terminology of Smith (1977), this proper mode is <u>Chandler wobble</u> (CW) , Eulerian motion being Chandler motion for a rigid earth and "wobble" being a synonym for polar motion.

<u>Eigenvalues for Spatial Position.</u> The integration of Euler's equations gives $\omega_1, \omega_2, \omega_3$, which define the position of the rotation axis with respect to a body-fixed coordinate system. The position of the rotation axis with respect to an inertial system -- that is, its orientation as defined, e.g., by the Euler angles θ, ϕ, ψ -- requires another integration, as we have seen before, cf. (8-4). Let us see how this affects our eigenvalues.

The Euler angles relate the body frame x_i to the inertial system X_i . For the present purpose it will be more convenient to relate the body-fixed system $x_1 x_2 x_3$ to another rectangular system $x_1' x_2' x_3'$ which is connected to the inertial system X_i in a prescribed simple way. If, furthermore, the system x_i^0 is close to the body system x_i, then the rotation from one system to the other can be effected by an "infinitesimal rotation" described by small quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3$, rather than by the Euler angles which can be large.

For the x_3^0 axis we take a constant direction in inertial space which is close to the figure axis x_3; this is possible since the x_3 axis has an almost constant direction. Let the system $x_1^0 x_2^0 x_3^0$ rotate with respect to the inertial system with constant angular velocity Ω, such that the axes x_1 and x_2 never deviate much from their uniformly rotating counterparts x_1^0 and x_2^0.

Thus the system $x_1^0 x_2^0 x_3^0$ is indeed related to the inertial system $X_1 X_2 X_3$ in a prescribed simple way and can equally well be used as a reference for the motion of the body in space.

In the auxiliary system x_i^0, its rotation vector $\underline{\omega}^0$ with respect to inertial space has, by its definition, the components

$$\underline{\omega}^0 = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} . \qquad (10\text{-}16)$$

Since the deviation of the frame x_i from the system x_i^0 is small, the transformation from one to the other can be effected by a rotation matrix that is close to the unit matrix I:

$$\underline{x} = (I + \varepsilon)\, \underline{x}^0 , \qquad (10\text{-}17)$$

if the components of the vectors $\underline{x}$ and $\underline{x}$ represent the coordinates of the same point in the respective systems.

The small matrix , representing an "infinitesimal rotation", is skew-symmetric and may be expressed as

$$\underline{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} . \tag{10-18}$$

Introducing the vector

$$\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} , \tag{10-19}$$

we may write (10-17) also in the form

$$\underline{x} = \underline{x}^{0} - \underline{\omega} \times \underline{x}^{0} , \tag{10-20}$$

where the cross denotes the vector product as usual.

Eq. (10-20) holds for any vector and may be used to transform vectors from the x_i^{0} to the x_i system. In particular we have

$$[illegible] . \tag{10-21}$$

represents the earth's actual instantaneous rotation vector;

it has, in the body frame x_i , the components $\omega_1, \omega_2, \omega_3$ entering in Euler's equations (10-2). The vector $\underline{\omega}^0$ comprises the components of the same vector in the system x_i^0 ; it should therefore not be confused with the vector $\underline{\omega}_0$ (10-16).

In fact, we have

$$\underline{\omega}^0 = \underline{\omega}_0 + \dot{\underline{\theta}} \quad . \qquad (10\text{-}22)$$

This can be seen in the following way. Consider a point at rest in the body frame so that $\underline{x}$ = const. Then the differentiation of (10-20) gives

$$0 = \dot{\underline{x}}_0 - \dot{\underline{\theta}} \times \underline{x}_0 - \underline{\theta} \times \dot{\underline{x}}_0 \quad .$$

Since the vector $\underline{\theta}$ is very small ("infinitesimal"), we shall consistently neglect second and higher powers of it, retaining only linear terms. Then the last term of (10-18) is readily seen to be of second order and will be neglected. Thus the last equation becomes

$$\dot{\underline{x}}_0 = \dot{\underline{\theta}} \times \underline{x}_0 \quad . \qquad (10\text{-}23)$$

The comparison of (10-23) with (2-3) shows that $\dot{\underline{\theta}}$ is the angular velocity vector of the rotation of the system x_i with respect to the system x_i^0 . Since the vector (10-16) describes the rotation of x_i^0 with respect to the inertial system, the sum of these two rotation vectors gives the rotation of the body frame with respect to the inertial system, that is the earth's actual rotation vector. This proves (10-22).

The combination of (10-21) and (10-22) now yields

$$\underline{\omega} = \underline{\omega}_E + \dot{\underline{\theta}} - \underline{\theta} \times \underline{\omega}_E , \tag{10-24}$$

as usual up to second-order terms. In terms of components this is

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 - \omega\theta_2 , \\ \omega_2 &= \dot{\theta}_2 + \omega\theta_1 , \\ \omega_3 &= \dot{\theta}_3 + \omega . \end{aligned} \tag{10-25}$$

Let us now substitute these expressions in Euler's equations (10-5). The result is

$$\begin{aligned} \ddot{\theta}_1 - (\omega - \sigma_E)\dot{\theta}_2 + \omega\sigma_E\theta_1 &= 0 , \\ \ddot{\theta}_2 + (\omega - \sigma_E)\dot{\theta}_1 + \omega\sigma_E\theta_2 &= 0 , \\ \ddot{\theta}_3 &= 0 . \end{aligned} \tag{10-26}$$

This is again a system of homogeneous linear differential equations with constant coefficients, which again can be simplified by using complex quantities. Putting

$$w = \theta_1 + i\theta_2 \tag{10-27}$$

we get

$$\ddot{w} + i(\Omega - \sigma_E)\dot{w} + \Omega\sigma_E w = 0 \, , \qquad \ddot{\theta}_3 = 0 \, . \tag{10-28}$$

The solution will again have exponential form:

$$w = \alpha e^{i\sigma t} \, , \qquad \theta_3 = \beta e^{i\sigma t} \, , \tag{10-29}$$

and the substitution into (10-28) gives

$$-\sigma^2\alpha - (\Omega - \sigma_E)\sigma\alpha + \Omega\sigma_E\alpha = 0 \, ,$$
$$-\sigma^2\beta = 0 \, ,$$

or

$$[\sigma^2 + (\Omega - \sigma_E)\sigma - \Omega\sigma_E]\alpha = 0 \, , \qquad \sigma^2\beta = 0 \, . \tag{10-30}$$

The condition of solution is the vanishing of the determinant, giving the equation

$$\sigma^2[\sigma^2 + (\Omega - \sigma_E)\sigma - \Omega\sigma_E] = 0 \, , \tag{10-31}$$

whose roots are

$$\sigma_1 = 0 \quad , \tag{10-32}$$

$$\sigma_2 = \sigma_E = \frac{C - A}{A}\,\Omega \quad , \tag{10-33}$$

$$\sigma_3 = -\Omega \quad . \tag{10-34}$$

The first two roots are the same as in the case of Euler's equations, namely (10-12) and (10-13), defining the axial spin mode (ASM) and Chandler wobble (CW), respectively.

Now the third eigenvalue is $\sigma_3 = -\Omega$. The corresponding proper mode is by (10-29) :

$$w = \alpha e^{-i\Omega t} \quad , \qquad \omega_3 = 0 \quad , \tag{10-35}$$

whereas (10-10) gives for $\sigma = \Omega$ only the trivial solution $a = 0 = b$, so that by (10-7) and (10-9)

$$\omega_1 = \omega_2 = \omega_3 = 0 \ .$$

In this mode, therefore, the rotation axis <u>within the body</u> (as described by $\omega_1, \omega_2, \omega_3$) remains unchanged, as well as the speed of rotation, but there is a nonvanishing w , that is, ϑ and φ differ from zero, which corresponds to a tilt of the rotation axis <u>in space</u>.

This is the <u>tilt-over mode</u> (TOM) of (Smith, 1977). It corresponds to a tilt of the whole earth (with the rotation axis invariably fixed to it) in space, so that the earth rotates with the same speed around a slightly different axis.

It follows that the TOM does not affect polar motion (the axis within the earth does not change) but affects nutation (the axis changes periodically in space according to (10-35)). It is thus clear that the TOM does not show up in Euler's equation but appears only in the equations (10-26) describing spatial orientation.

Since any body (regardless of its internal constitution) freely rotating around a certain axis, can also rotate if the axis (with the body invariably attached to) is tilted in space, the TOM must exist for an arbitrary body, rigid, elastic, liquid, even inhomogeneous. For a fluid earth model and models with a liquid core, this mode has been pointed out already by Poincaré (1910, pp. 497, 508, 513).

These proper modes, especially CW and TOM, will play a basic role in the following sections.

11. APPLICATION TO NUTATION AND POLAR MOTION

The eigenvalue theory described in the preceding section allows an elegant treatment of precession and nutation. We shall apply methods outlined by Smith (1977) and extended by Wahr (1979, 1980).

We are using the same two coordinate systems as in sec. 10 : $x_1 x_2 x_3$ is the body-fixed system of principal axes of inertia, and $x_1^0 x_2^0 x_3^0$ is the uniformly rotating auxiliary system. The two systems are related by

$$\underline{x}^0 = \underline{x} + \underline{\theta} \times \underline{x} = \underline{x} + \underline{\theta} \times \underline{x}^0 \tag{11-1}$$

(up to terms of second order in $\underline{\theta}$ as usual). The vector $\underline{\theta}$ describes the infinitesimal rotation by which our two systems differ.

The rotation of the earth is described by

$$\frac{dx^0}{dt} = [illegible] \times x [illegible] ; \tag{11-2}$$

all vectors refer to the system x_i^0. By (10-16) and (10-22), the rotation vector is

$$[illegible] + \dot{[illegible]} = e_3 + \dot{[illegible]} , \tag{11-3}$$

$\underline{e}_3$ being the unit vector of the x_3 axis.

Since we do not consider changes in the speed of rotation, we have $\omega_3 = 0$ ... (that is, we disregard the axial spin mode, ASM, see sec. 10). Thus

$$\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ 0 \end{bmatrix} . \tag{11-4}$$

Complex notation will again be convenient. We put

$$\omega = \omega_1 + i\omega_2 , \tag{11-5}$$

using ω as a symbol for the quantity that has been denoted by w in (10-27).

We shall use this complex notation simultaneously with three-dimensional vector notation. We put

$$\underline{e}_2 = i\underline{e}_1 , \tag{11-6}$$

that is, <u>the number</u> i <u>represents rotation around the</u> x_3 <u>axis by the angle of</u> $+\pi/2$; $\underline{e}_1$ and $\underline{e}_2$ denote the unit vectors of the x_1 and x_2 axes. Thus, i can simply be interpreted as a rotation matrix; cf. (Duschek and Hochrainer, 1961, p. 222).

This is the only convention needed; everything else follows automatically. In particular,

$$\underline{[illegible]} = [illegible]\,\underline{e}_{[illegible]} \qquad (11\text{-}7)$$

gives the connection between the complex number (11-5) and the corresponding vector (11-4). In fact, by (11-6),

$$\underline{[illegible]} = [illegible]\,\underline{e}_{[illegible]} = ([illegible] + i[illegible])\,\underline{e}_{[illegible]}$$

$$= [illegible]\,\underline{e}_{[illegible]} + [illegible]\,i\,\underline{e}_{[illegible]} = [illegible]\,\underline{e}_{[illegible]} + [illegible]\,\underline{e}_{[illegible]} \;, \qquad (11\text{-}8)$$

identical to (11-4). A relation of the form (11-7) holds for any vector <u>which has no component along the</u> [illegible] <u>axis</u>. We shall always use the same letter for two quantities related in this way: the vector is underlined, the corresponding complex number is not underlined.

Assume now [illegible] to be an exponential

$$[illegible] = [illegible]\,e^{i[illegible]t+[illegible]} \qquad (11\text{-}9)$$

as in (10-29), [illegible] and [illegible] being constant complex numbers. Then

$$[illegible] = i[illegible]\,e^{i[illegible]} = i[illegible] \;, \qquad (11\text{-}10)$$

so that differentiation is equivalent to multiplication by $i[illegible]$ (a fact of basic usefulness, well known from spectral analysis). By (11-7),

$$[illegible] = [illegible]\,\underline{e}_{[illegible]} = i[illegible] \;, \qquad (11\text{-}11)$$

so that the same relation also holds for vectors.

Now the rotation vector (11-3) becomes

$$\underline{\omega} = [illegible]\,\underline{e}_{[illegible]} + [illegible] = [illegible]\,\underline{e}_{[illegible]} + i[illegible] \quad (11\text{-}[illegible])$$

This is the actual instantaneous rotation vector; the [illegible] ding unit vector

$$\underline{e}_R = [illegible] = \underline{e}_{[illegible]} + i[illegible] \quad (11\text{-}[illegible])$$

represents the instantaneous rotation axis. It [illegible] indeed a unit vector since $\underline{e}_R$ [illegible] 1 + [illegible].

The figure axis [illegible], the coordinate [illegible]. Its vector $\underline{x}$ is therefore the unit vector [illegible]. In the x_i system we have by (11-1

$$\underline{e}_F = \underline{e}_{[illegible]} + [illegible]\,\underline{e}_{[illegible]} . \quad (11\text{-}14)$$

Now the vector product

$$\underline{\omega} \times \underline{e}_3 = \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} [illegible] \\ -[illegible] \\ 0 \end{bmatrix} = -i[illegible] . \quad (11\text{-}15)$$

This is immediately verified, using $i^2 = -1$:

$$-i\omega = -i(\ldots + \ldots)e \ldots -ie \ldots + e \ldots$$ [illegible]

$$= \ldots e \ldots - \ldots e \ldots - \ldots$$ [illegible]

$$\ldots 0 \ldots$$ [illegible]

Hence the unit vector of the figure axis F in the x_i system is simply

$$e_F = e \ldots - i \ldots \qquad (11\text{-}16)$$

Finally we consider the angular momentum axis. The angular momentum vector in the body system x_i is

$$\ldots = C \ldots ; \qquad (11\text{-}17)$$

here the inertia tensor C has the diagonal form

$$\ldots = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \ldots \end{bmatrix} , \qquad (11\text{-}18)$$

and [illegible] is given by (10-24). Thus

$$\ldots = \ldots + \ldots - \ldots \qquad (11\text{-}19)$$

[illegible] $\ldots$, we have by (11-15)

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$$\underline{\theta} \times \underline{\omega}_0 = -i\Omega\underline{\theta} \quad , \tag{11-20}$$

and $\dot{\theta}$ is (11-11). Thus (11-19) becomes

$$\underline{H} = \underline{C}\,\underline{\omega}_0 + i(\sigma + \Omega)\ \underline{C}\,\underline{\theta} \tag{11-21}$$

or

$$\underline{H} = C\Omega\underline{e}_3 + iA(\sigma + \Omega)\ \underline{\theta} \quad . \tag{11-22}$$

This is in the body frame. In the $x_i^{\,0}$ system we have by (11-1)

$$\underline{H}^o = \underline{H} + \underline{\theta} \times \underline{H} \quad . \tag{11-23}$$

To first order, by (11-15),

$$\underline{\theta} \times \underline{H} = \underline{\theta} \times C\Omega\underline{e}_3 = -iC\Omega\underline{\theta} \quad . \tag{11-24}$$

From the last three equations there follows

$$\begin{aligned} \underline{H}^o &= C\Omega\underline{e}_3 + i(A\sigma + A\Omega - C\Omega)\ \underline{\theta} \\ &= C\Omega\underline{e}_3 + iA(\sigma - \sigma_E)\ \underline{\theta} \quad , \end{aligned} \tag{11-25}$$

using the Euler frequency (10-6). The corresponding unit vector finally is

$$\underline{e}_H = \underline{e}_3 + i\,\frac{A(\sigma - \sigma_E)}{C\Omega}\,\underline{\theta} \quad . \tag{11-26}$$

<u>Relation between $\underline{\theta}$ and torque $\underline{L}$</u>. The infinitesimal rotation $\underline{\theta}$ can easily be related to the torque $\underline{L}$ of the lunisolar attraction. The basic equation is

$$\dot{\underline{H}}^0 + \underline{\omega}_0 \times \underline{H}^0 = \underline{L}^0 \quad , \tag{11-27}$$

all quantities referring to the x_i^0 system rotating with uniform velocity $\underline{\omega}_0 = \Omega\underline{e}_3$.

Deriving (11-25) with respect to time gives

$$\dot{\underline{H}}^0 = iA(\sigma - \sigma_E)\dot{\underline{\theta}} = -A\sigma(\sigma - \sigma_E)\underline{\theta}$$

using (11-11), and

$$\begin{aligned}
\underline{\omega}_0 \times \underline{H}^0 &= \Omega\underline{e}_3 \times \underline{H}^0 \\
&= iA\Omega(\sigma - \sigma_E)\underline{e}_3 \times \underline{\theta} \\
&= -iA\Omega(\sigma - \sigma_E)\underline{\theta} \times \underline{e}_3 \\
&= -A\Omega(\sigma - \sigma_E)\underline{\theta}
\end{aligned}$$

using (11-15). Thus (11-27) becomes

$$-A(\sigma + \Omega)(\sigma - \sigma_E)\underline{\theta} = \underline{L}^0 \quad . \tag{11-28}$$

We assume $\underline{L}$ to have the form

$$\underline{L}^{0} = \begin{bmatrix} L_1 \\ L_2 \\ 0 \end{bmatrix} = L\underline{e}_1 \tag{11-29}$$

where

$$L = L_1 + iL_2 \tag{11-30}$$

as before, cf. (3-15), and L is an exponential

$$L = \beta e^{i\sigma t + \gamma} . \tag{11-31}$$

Then the complex numbers θ and L are related by

$$\theta = - \frac{L}{A(\sigma+\Omega)(\sigma-\sigma_E)} . \tag{11-32}$$

This is equivalent to

$$\theta = \frac{1}{C\Omega} \left(\frac{1}{\sigma+\Omega} - \frac{1}{\sigma-\sigma_E} \right) L , \tag{11-33}$$

as is readily verified by computation.

This latter form shows very well the resonance at the proper frequencies $-\Omega$ and σ_E : if the external moment L has a frequency σ equal to either of the two proper frequencies, then the expression (11-33) will have a singularity. For lunisolar effects, whose frequencies are grouped around the sidereal frequency Ω , the relevant resonance is at $\sigma = -\Omega$, which gives precession; cf. sec. 7, especially eq. (7-14) (no lunisolar effect would have a frequency σ_E).

Eq. (11-32) or (11-33) provide the possibility to express all our quantities in terms of the external torque L . For instance, (11-25) and (11-32) give

$$\underline{H}^{\circ} = C\Omega\underline{e}_3 - \frac{iL}{\sigma + \Omega}\underline{e}_1 , \tag{11-34}$$

which clearly shows the resonance at $\sigma = -\Omega$. This shows the importance of the "tilt-over mode" (TOM, cf. sec. 10) for precession and nutation.

Nutation and polar motion. Nutation is the periodic motion of any of the three axes: angular momentum axis H , figure axis F , rotation axis R , with respect to a fixed reference axis for which it is natural to take the axis x_3° that is fixed in space; it has the unit vector $\underline{e}_3$. Thus the nutation vector $\underline{n}$ of any of these is obtained from the corresponding unit vector $\underline{e}$ by subtracting $\underline{e}_3$:

$$\begin{aligned} \underline{n}_R &= \underline{e}_R - \underline{e}_3 , \\ \underline{n}_F &= \underline{e}_F - \underline{e}_3 , \\ \underline{n}_H &= \underline{e}_H - \underline{e}_3 , \end{aligned} \tag{11-35}$$

the unit vectors being given by (11-13), (11-16), and (11-26). Using the complex number n corresponding to the vector $\underline{n}$

by

$$\underline{n} = n\underline{e}_1 \ , \tag{11-36}$$

we thus get

$$\begin{aligned} n_R &= i\,\frac{\sigma}{\Omega}\,\theta \ , \\ n_F &= -i\theta \ , \\ n_H &= i\,\frac{A(\sigma-\sigma_E)}{C\Omega}\,\theta \ . \end{aligned} \tag{11-37}$$

Polar motion is treated in the same way. It represents motion around the figure axis $\underline{e}_F$. Therefore, polar motion is defined by the vectors

$$\begin{aligned} \underline{p}_R &= \underline{e}_R - \underline{e}_F \ , \\ \underline{p}_H &= \underline{e}_H - \underline{e}_F \ , \end{aligned} \tag{11-38}$$

in analogy to (11-35) (of course, $p_F = 0$). The expressions (11-13), (11-16), and (11-26) give for the corresponding complex numbers

$$\begin{aligned} p_R &= i\,\frac{\sigma+\Omega}{\Omega}\,\theta \ , \\ p_H &= i\,\frac{A}{C}\,\frac{\sigma+\Omega}{\Omega}\,\theta \ . \end{aligned} \tag{11-39}$$

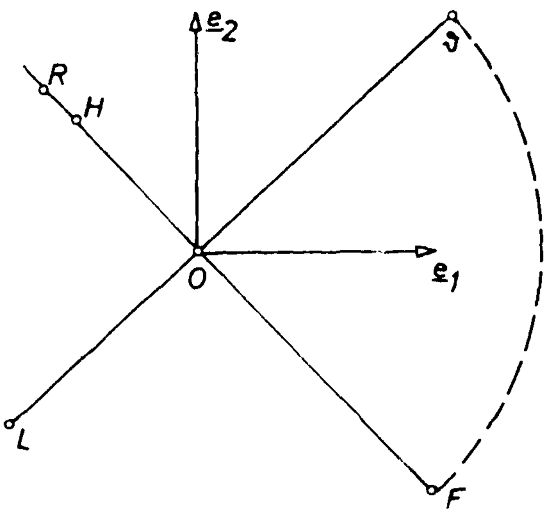
Their expression in terms of L by (11-32),

$$p_R = - \frac{iL}{A\Omega(\sigma - \sigma_E)} ,$$

$$p_H = - \frac{iL}{C\Omega(\sigma - \sigma_E)} , \qquad (11\text{-}39')$$

shows resonance only at the Chandler (or rather Euler) frequency σ_E , as it is natural for polar motion. The lunisolar (tidally-induced) effect on polar motion is not resonant.

These relations are illustrated in Fig. 11.1, in which the origin O corresponds to the x_3^0 axis.



$n_R = OR \qquad p_R = FR$
$n_F = OF$
$n_H = OH \qquad p_H = FH$

FIGURE 11.1 Forced nutation and polar motion of the rotation axis R, the figure axis F, and the angular momentum axis H .

<u>Comparison with previous formulas.</u> In (5-13) we have put

$$L = (C - A)\Omega^2 \sum_j B_j e^{-i(\omega_j t+\beta_j)} \ . \tag{11-40}$$

We write this as

$$L = \sum_j L_j \tag{11-41}$$

with

$$L_j = (C - A)\Omega^2 B_j e^{-i(\omega_j t+\beta_j)} \ . \tag{11-42}$$

We shall apply the preceding results to each frequency separately and only at the end sum over all frequencies.

Thus we shall identify L in (11-31) with an L_j as given by (11-42). The comparison shows that

$$\sigma = -\omega_j \quad ; \tag{11-43}$$

of course, $\gamma = -\beta_j$ but this we shall not need.

Then (11-39') gives

$$\begin{aligned} p_{R,j} &= \frac{iL_j}{A\Omega(\omega_j+\sigma_E)} \ , \\ p_{H,j} &= \frac{iL_j}{C\Omega(\omega_j+\sigma_E)} \ , \end{aligned} \tag{11-44}$$

and using (11-42) and summing over j we obtain

$$p_R = i \sum_j \frac{\sigma_E}{\omega_j + \sigma_E} B_j e^{-i(\omega_j t + \delta_j)} \quad , \tag{11-45}$$

$$p_H = \frac{A}{C} p_R \quad , \tag{11-46}$$

which are identical to the forced part of (5-24) and (6-11) for a rigid earth with $k = 0$.

For free Eulerian motion we have

$$L = 0 \quad , \qquad \sigma = \sigma_E \quad . \tag{11-47}$$

Equations such as (11-32) or (11-39') give 0/0 and cannot be used, but (11-37) and (11-39) remain valid with

$$\theta = \varepsilon e^{i\sigma_E t} \tag{11-48}$$

for the proper mode CW according to (10-29) (θ is the same as w). For free polar motion, (11-39) gives

$$p_R = i \frac{\sigma_E + \Omega}{\Omega} \theta \quad , \tag{11-49}$$

and the comparison with (5-24) shows that

$$m_0 e^{i\sigma_E t} = i \frac{\Omega + \sigma_E}{\Omega} \theta \quad ,$$

so that

$$\theta = -i \frac{\Omega}{\Omega + \sigma_E} m_0 e^{i\sigma_E t} \tag{11-50}$$

is the relation between free polar motion m_0 and the corresponding infinitesimal rotation θ. With this θ, the second equation of (11-39) gives the free term in (6-11) with $k = 0$.

Nutation is handled in the same way. With (11-43), eqs. (11-37) give

$$\begin{aligned} n_{R,j} &= -i \frac{\omega_j}{\Omega} \theta_j , \\ n_{F,j} &= -i\theta_j , \\ n_{H,j} &= -i \frac{A(\omega_j+\sigma_E)}{C\Omega} \theta_j , \end{aligned} \tag{11-51}$$

where, by (11-32),

$$\theta_j = - \frac{L_j}{A\Delta\omega_j(\omega_j+\sigma_E)} \tag{11-52}$$

since

$$\sigma + \Omega = -\omega_j + \Omega = -\Delta\omega_j$$

where

$$\Delta\omega_j = \omega_j - \Omega \tag{11-53}$$

as usual. Substitution of (11-42) and summation over j gives

$$n_R = i \sum_j \frac{\omega_j \sigma_E}{\Delta\omega_j(\omega_j + \sigma_E)} B_j e^{-i(\omega_j t + \beta_j)} ,$$

$$n_F = i \sum_j \frac{\Omega \sigma_E}{\Delta\omega_j(\omega_j + \sigma_E)} B_j e^{-i(\omega_j t + \beta_j)} , \tag{11-54}$$

$$n_H = i \sum_j \frac{\Omega \sigma_E}{\Delta\omega_j(\Omega + \sigma_E)} B_j e^{-i(\omega_j t + \beta_j)} .$$

To these forced terms, the corresponding free terms (11-37), with $\sigma = \sigma_E$ and ν from (11-50), must be added. Then the comparison with (7-14), (8-7), and (8-11) shows that

$$\Delta\epsilon + i\Delta\psi\sin\epsilon = -ine^{i\Omega t} \tag{11-55}$$

for any of the axes R, F, or H; the geometry of this correspondence between polar motion and nutation is the same as in (8-9).

The factor $-i$ expresses a rotation by $-90°$ (or $+270°$) which has no deeper significance as in characterizes only the choice of coordinate axes. The factor $e^{i\Omega t}$ expresses, of course, the uniform rotation of the $x_1{}^0 x_2{}^0 x_3{}^0$ system, to which n refers, with respect to the inertial system, to which

. and ~ refer.

In fact, the present method offers the simplest and most direct derivation of polar motion and nutation for a rigid earth. For this particular case we get, in a considerably simpler way, the same results as with the approach of secs. 3 through 8. This latter approach, however, holds for an elastic earth and thus is more general.

In sec. 13 we shall extend the present method to an earth model with a liquid core.

12. POINCARE'S LIQUID CORE MODEL

Neither the rigid nor the elastic earth model are capable of adequately describing earth tides and nutation. The effect of the liquid core must be taken into account.

The oscillations of a rotating ellipsoidal shell containing a homogeneous liquid were first treated simultaneously by Sloudzky and by Hough (1895). The most elegant treatment is by Poincaré (1910). His paper is so frequently used and quoted that it has become customary to speak of the Poincaré model.

Since Poincarés method is treated in easily accessible textbooks (Lamb, 1932, p. 724; Melchior, 1978, p. 122) we shall here only describe it in general terms, rather than deriving it step by step.

Let us refer the ellipsoidal shell to principal axes xyz , then the inner ellipsoidal surface, which encloses the liquid-filled cavity, has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad . \tag{12-1}$$

By the change of variables

$$x' = \frac{x}{a} , \qquad y' = \frac{y}{b} , \qquad z' = \frac{z}{c} , \tag{12-2}$$

this surface is transformed into the unit sphere

$$x'^2 + y'^2 + z'^2 = 1 \quad . \tag{12-3}$$

Poincaré considers a motion of the liquid such that, by the transformation (12-2), it is transformed into a rotation of the sphere (12-3). Thus, by (2-3), the velocity in the auxiliary $x'y'z'$ system is

$$\underline{\chi} \times \underline{x} = \begin{bmatrix} \chi_2 z' - \chi_3 y' \\ \chi_3 x' - \chi_1 z' \\ \chi_1 y' - \chi_2 x' \end{bmatrix} \tag{12-4}$$

if the corresponding rotation vector is $\underline{\chi}$. Going back to the real system xyz by(12.2) and adding the actual rotation $\underline{\omega}$, we obtain

$$\begin{aligned} \dot{x} &= \frac{a}{c}\chi_2 z - \frac{a}{b}\chi_3 y + \omega_2 z - \omega_3 y \ , \\ \dot{y} &= \frac{b}{a}\chi_3 x - \frac{b}{c}\chi_1 z + \omega_3 x - \omega_1 z \ , \\ \dot{z} &= \frac{c}{b}\chi_1 y - \frac{c}{a}\chi_2 x + \omega_1 y - \omega_2 x \ . \end{aligned} \tag{12-5}$$

Here $\underline{\omega}$ represents the rotation of the earth with respect to the inertial system, and $\underline{\chi}$ expresses a rotation of the fluid core with respect to the earth. (The latter, of course, is a strict rotation only after the formal transformation (12-2) to the auxiliary $x'y'z'$ system, but for a nearly spherical earth, this holds approximately also in the actual xyz system.)

The motion described by the velocity components (12-5) represents the simplest possible motion of an ideal fluid. The velocity is linear in the coordinates x, y, z . This is quite natural, considering that such a linear dependence somehow corresponds to a quadratic potential, and the luni-solar potential in the usual treatment (sec.1) is indeed quadratic, namely a spherical harmonic of second degree.

The kinetic energy T is found by summing (i.e. integrating) the square of the velocity over all mass elements dm :

$$2T = \iiint_{\text{earth}} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, dm \ .$$

The substitution of (12-5) and integration over the whole earth (liquid core plus ellipsoidal shell) yields

$$\begin{aligned} 2T = {} & A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + \\ & + A_c\chi_1^2 + B_c\chi_2^2 + C_c\chi_3^2 + \\ & + 2F\omega_1\chi_1 + 2G\omega_2\chi_2 + 2H\omega_3\chi_3 \ . \end{aligned} \qquad (12\text{-}6)$$

Here A, B, C are the principal moments of inertia for the whole body, and

$$
\begin{aligned}
A_c &= \tfrac{1}{5}M_c(b^2 + c^2) , & F &= \tfrac{2}{5}M_c bc , \\
B_c &= \tfrac{1}{5}M_c(c^2 + a^2) , & G &= \tfrac{2}{5}M_c ca , \\
C_c &= \tfrac{1}{5}M_c(a^2 + b^2) , & H &= \tfrac{2}{5}M_c ab ,
\end{aligned}
\tag{12-7}
$$

M_c denoting the total mass of the liquid core.

The equations of motion may be made plausible in the following way. Assume that the liquid core is absent and the earth is a simple rigid body. Then the equations of motion may be written

$$
\frac{d}{dt}\frac{\partial T}{\partial \omega_1} - \omega_3\frac{\partial T}{\partial \omega_2} + \omega_2\frac{\partial T}{\partial \omega_3} = L_1 \tag{12-8}
$$

plus two other equations resulting from cyclic permutation of subscripts. In fact, in this limiting case we have $A_c = B_c = C_c = F = G = H = 0$, and the substitution of (12-6) into (12-8) immediately gives Euler's equations (2-12).

It will now be assumed that (12-8) also holds for the general case of a liquid core. Since the vector $\underline{\chi}$ plays a role analogous to $\underline{\omega}$, we may guess that the following equations also hold:

$$
\frac{d}{dt}\frac{\partial T}{\partial \chi_1} + \chi_3\frac{\partial T}{\partial \chi_2} - \chi_2\frac{\partial T}{\partial \chi_3} = 0 \tag{12-9}
$$

and two cyclically permuted equations. The change in sign in these two equations are due to the fact that $\underline{\chi}$, expressing rotation of the core with respect to the body, has a similar character as $-\underline{\omega}$, which describes rotation of inertial space with respect to the body. On the right-hand side of (12-9) there is zero since external forces do not affect the relative motion of the core.

A rigorous derivation of (12-8) even in the presence of a liquid core is not difficult since it is simply equivalent to the moment equation (2-2); cf. (Lamb, 1932, p. 724). On the other hand, (12-9) may be derived using Helmholtz' vorticity equation, which means going rather deep into fluid mechanics; cf. (Lamb, 1932, p. 725) or (Melchior, 1978, p.124).

Our plausibility reasoning using arguments of symmetry between $\underline{\omega}$ and $\underline{\chi}$ can, however, be made rigorous. This has been done already by Poincaré (1910, p. 484), using a theorem on dynamical systems with groups of symmetry, earlier given also by Poincaré (1901). This very elegant theorem is unfortunately not found in standard treatises on analytical mechanics, with the exception of (Whittaker , 1937, p. 43), (Loomis and Sternberg, 1968, p. 541), and (Abraham and Marsden, 1978, sec. 4.4), where similar theorems are presented; cf. also the remark in (Klein and Sommerfeld, 1910, p. 162).

From (12-8) and (12-9), using (12-6), we derive immediately

$$\frac{d}{dt}(A\omega_1 + F\chi_1) - \omega_3(B\omega_2 + G\chi_2) + \omega_2(C\omega_3 + H\chi_3) = L_1 \ ,$$

$$\frac{d}{dt}(B\omega_2 + G\chi_2) - \omega_1(C\omega_3 + H\chi_3) + \omega_3(A\omega_1 + F\chi_1) = L_2 ,$$
$$(12\text{-}10)$$
$$\frac{d}{dt}(C\omega_3 + H\chi_3) - \omega_2(A\omega_1 + F\chi_1) + \omega_1(B\omega_2 + G\chi_2) = L_3 ;$$

$$\frac{d}{dt}(F\omega_1 + A_c\chi_1) + \chi_3(G\omega_2 + B_c\chi_2) - \chi_2(H\omega_3 + C_c\chi_3) = 0 ,$$

$$\frac{d}{dt}(G\omega_2 + B_c\chi_2) + \chi_1(H\omega_3 + C_c\chi_3) - \chi_3(F\omega_1 + A_c\chi_1) = 0 ,$$
$$(12\text{-}11)$$
$$\frac{d}{dt}(H\omega_3 + C_c\chi_3) + \chi_2(F\omega_1 + A_c\chi_1) - \chi_1(G\omega_2 + B_c\chi_2) = 0 .$$

For an ellipsoid of revolution we have

$$a = B , \qquad A = B , \qquad A_c = B_c , \qquad F = G \qquad (12\text{-}12)$$

and furthermore, by (12-7),

$$H = C_c . \qquad (12\text{-}13)$$

Then the third equations of (12-10) and (12-11) give

$$\frac{d}{dt}(C\omega_3 + C_c\chi_3) + F(\omega_1\chi_2 - \omega_2\chi_1) = L_3 ,$$
$$(12\text{-}14)$$

$$\frac{d}{dt}(C\,[illegible] + C\,[illegible]) + F([illegible] - \omega\,[illegible]) = 0\ . \tag{12-14}$$

As usual, we disregard L_3 which causes variation of rotational speed but not polar motion (sec. 3). Thus we put $L_3 = 0$ and subtract both equations (12-14). The result is (the dot denotes d/dt) :

$$[illegible] = 0\ , \qquad [illegible] = \text{const.} = [illegible]\ . \tag{12-15}$$

Then the second equation of (12-14) becomes

$$C\,[illegible] + F([illegible] - [illegible]) = 0\ . \tag{12-16}$$

Now [illegible] and [illegible] have a similar small order of magnitude as [illegible] and [illegible], so that [illegible] is a quantity of second order, which we shall consistently neglect in the sequel, as we did in the preceding sections. To this accuracy, (12-16) reduces to

$$[illegible] = 0\ , \qquad [illegible] = \text{const.}\ ; \tag{12-17}$$

we take

$$[illegible]\ 0\ . \tag{12-18}$$

Using (12-1[illegible]), (12-13), (12-15), and (12-18), we may write the first two equations of (12-10) and (12-11) in the form :

$$A\dot{\omega}_1 + F\dot{\chi}_1 + (C - A)\Omega\omega_2 - F\Omega\chi_2 = L_1 \ ,$$
$$A\dot{\omega}_2 + F\dot{\chi}_2 - (C - A)\Omega\omega_1 + F\Omega\chi_1 = L_2 \ ; \tag{12-19}$$

$$F\dot{\omega}_1 + A_c\dot{\chi}_1 - C_c\Omega\chi_2 = 0 \ ,$$
$$F\dot{\omega}_2 + A_c\dot{\chi}_2 + C_c\Omega\chi_1 = 0 \ . \tag{12-20}$$

Again, complex notation will be convenient. We put

$$u = \omega_1 + i\omega_2 \ ,$$
$$v = \chi_1 + i\chi_2 \ , \tag{12-21}$$
$$L = L_1 + iL_2 \ ,$$

so that (12-19) and (12-20) take the simple form:

$$A\dot{u} + F\dot{v} - i(C - A)\Omega u + iF\Omega v = L \ ,$$
$$F\dot{u} + A_c\dot{v} + iC_c\Omega v = 0 \ . \tag{12-22}$$

These formulas generalize Euler's equations to the case of a liquid core.

<u>Eigenvalue problem.</u> We proceed similarly as in sec. 10. We put

$$u = \alpha e^{i\sigma t} \ , \qquad v = \beta e^{i\sigma t} \tag{12-23}$$

and

$$L = 0 \ .$$

Then (12-22) reduces to

$$\begin{aligned} A\sigma\alpha + F\sigma\beta - A\sigma_E\alpha + F\Omega\beta &= 0 \ , \\ F\sigma\alpha + A_C\sigma\beta + C_C\Omega\beta &= 0 \ , \end{aligned} \tag{12-24}$$

using

$$\sigma_E = \frac{C - A}{A}\,\Omega \tag{12-25}$$

as usual. The determinant must be zero:

$$\begin{vmatrix} A(\sigma-\sigma_E) & F(\sigma+\Omega) \\ F\sigma & A_C\sigma + C_C\Omega \end{vmatrix} = 0 \ ,$$

which gives

$$A(\sigma - \sigma_E)(A_C\sigma + C_C\Omega) - F^2\sigma(\sigma + \Omega) = 0 \ . \tag{12-26}$$

With

$$F^2 = 2A_C C_C - C_C^2 = C_C(2A_C - C_C) \tag{12-27}$$

which follows from (12-7), this becomes

$$A(\sigma - \sigma_E)(A_c\sigma + C_c\Omega) = C_c(2A_c - C_c)\Omega(\sigma + \Omega) \; . \qquad (12\text{-}28)$$

We now introduce

$$\varepsilon = \frac{C_c - A_c}{C_c} \; , \qquad (12\text{-}29)$$

the dynamical ellipticity of the core. By (12-7) this is also equal to the core's geometric flattening (apart from negligible second-order terms):

$$\varepsilon = \frac{a^2 - c^2}{2a^2} = \frac{(a-c)(a+c)}{2a^2} = \frac{a - c}{a} + O(\varepsilon^2) \; ; \qquad (12\text{-}30)$$

$O(\varepsilon^2)$ denoting terms of order ε^2 as usual.

Then

$$C_c = A_c(1 + \varepsilon) \; , \qquad (12\text{-}31)$$

and

$$F^2 = C_c(2A_c - C_c) = A_c^2(1 + \varepsilon)(1 - \varepsilon)$$

$$= A_c^2(1 - \varepsilon^2) = A_c^2 \qquad (12\text{-}32)$$

since terms of second order in ϵ will now consistently be neglected.

Thus (12-28) becomes

$$AA_c(\sigma - \sigma_E)(\sigma + \Omega + \epsilon\Omega) = A_c^2\sigma(\sigma + \Omega)$$

or

$$[A\sigma - (C - A)\Omega](\sigma + \Omega + \epsilon\Omega) = A_c\sigma(\sigma + \Omega) \quad . \qquad (12\text{-}33)$$

The introduction of the principal moments of inertia for the mantle

$$A_m = A - A_c \quad , \qquad C_m = C - C_c \quad , \qquad (12\text{-}34)$$

allows a reduction to the form

$$[A_m\sigma - (C - A)\Omega](\sigma + \Omega) = -[A\sigma - (C - A)\Omega]\Omega\epsilon \quad . \qquad (12\text{-}35)$$

This equation is solved by successive approximations. We first put $\epsilon = 0$. Then the right-hand side is zero, and the equation has the two roots

$$\begin{aligned} \sigma_1^0 &= \frac{C - A}{A_m}\Omega \quad , \\ \sigma_2^0 &= -\Omega \quad . \end{aligned} \qquad (12\text{-}36)$$

As a second approximation we put

$$\sigma_1 = \sigma_1^0 + \varepsilon\tau_1 \; , \qquad \sigma_2 = \sigma_2^0 + \varepsilon\tau_2 \; . \tag{12-37}$$

For $\sigma = \sigma_1$ we have

$$A_m\sigma_1 - (C - A)\Omega = A_m\sigma_1^0 + A_m\varepsilon\tau_1 - (C - A)\Omega = \varepsilon\tau_1 A_m \; ,$$

so that (12-35) becomes

$$\varepsilon\tau_1 A_m(\sigma_1^0 + \Omega) = - A(\sigma_1^0 - \sigma_E)\Omega\varepsilon \; .$$

Now ε is the core flattening (12-29). Both σ_1^0, by (12-36) and σ_E, by (12-25), have the same order of magnitude as $\varepsilon\Omega$. Thus the right-hand side has actually the order of ε^3 and can be neglected. Hence, to our usual linear approximation we have $\tau_1 = 0$ and

$$\sigma_1 = \frac{C - A}{A_m}\Omega . \tag{12-38}$$

This is the frequency of the Chandler wobble (CW) for the present case. It differs from CW for the rigid body, (10-6), by the replacement of A in the denominator by A_m, the principal moment of inertia for the mantle.

A new feature is brought into the picture by the first root. For $\sigma = \sigma_2$ we have

$$[\text{illegible}] + [\text{illegible}] = [\text{illegible}] + [\text{illegible}] + [\text{illegible}] = [\text{illegible}] \ ,$$

so that (12-35) becomes

$$A_m [\text{illegible}] - (C - A)[\text{illegible}] = - [\text{illegible}] A\sigma_2 [\text{illegible}] - (C - A)[\text{illegible}]\varepsilon \ ,$$

or with (12-36),

$$- (A_m + C - A)[\text{illegible}] = (A + C - A)[\text{illegible}] \ ,$$

with the solution

$$[\text{illegible}] = - \frac{C}{A_m + C - A} [\text{illegible}] \ . \qquad (12\text{-}39)$$

Now, using (12-31) and (12-34)

$$\begin{aligned} A_m + C - A &= C - A_c = C - C_c(1 - \varepsilon) \\ &= C - C_c + O(\varepsilon) = C_m + O(\varepsilon) \ . \end{aligned}$$

Since [illegible] is multiplied by ε in (12-37), the term $O(\varepsilon)$ is multiplied with ε to give a negligible second-order term. Thus (12-39) becomes

$$[\text{illegible}] = - \frac{C}{C_m} [\text{illegible}] \qquad (12\text{-}40)$$

and

$$\sigma_2 = -\Omega(1 + \varepsilon \frac{C}{C_m}) \quad . \tag{12-41}$$

This root has no equivalent in the rigid-body rotation. In particular, it is not the direct equivalent of the tilt-over mode although it is numerically very close. We shall come back to this question at the end of the present section.

To give an idea of the order of magnitude, we take (Melchior, 1978, pp. 129)

$$\frac{A_c}{A} \doteq 0.11 \doteq \frac{C_c}{C} \quad ,$$

$$\varepsilon \doteq \frac{1}{400} \quad . \tag{12-42}$$

Then

$$\frac{C}{C_m} = \frac{1}{1 - C_c/C} \doteq 1.12 \quad ,$$

$$\varepsilon \frac{C}{C_m} = \frac{1.12}{400} = 0.0028 \quad ,$$

so that

$$\sigma_2 = - 1.0028\Omega \quad . \tag{12-43}$$

This is the eigenvalue for the nearly diurnal free wobble (NDFW) according to the terminology of (Smith, 1977) ([illegible] would represent a diurnal period).

The comparison of the present treatment with sec. 10 shows one comparable (though not numerically equal) eigenvalue, namely the CW value [illegible]. Our present [illegible], for the NDFW, has no equivalent in the case of Euler's equations; on the other hand, we have not yet obtained the eigenvalue 0 corresponding to the axial spin mode (ASM). Its physical interpretation -- rotation about the same axis with a slightly different speed of rotation -- shows that it must be quite general and not restricted to a solid body. In fact, we have it also in the Poincaré model. The equation (12-15)

$$[illegible] = 0$$

has the exponential solution

$$[illegible] = [illegible]\, e^{[illegible]}$$

with

$$i [illegible]\, e^{[illegible]} = 0 \,,$$

which gives

$$[illegible] = 0 \qquad (12\text{-}44)$$

as another eigenvalue. This ASM value corresponds to in sec. 10. In the present problem, it could be obtained also from (12-17).

Spatial position. So far, our exposition has corresponded to the treatment of Euler's equation in sec. 10, so that we have not yet obtained the tilt-over mode TOM, which should, however, occur in the present model as well, for similar physical reasons as the ASM.

We therefore proceed as in sec. 10, considering spatial position. Since the mantle is rigid, the reasoning leading from (10-17) to (10-25) holds unchanged for the Poincaré model, and the first two equations of (10-25) can be combined in complex notation to give

$$u = \dot{w} + i\Omega w , \tag{12-45}$$

using (10-27) and (12-21). The substitution of this into (12-22), and of the third equation of (10-25) into (12-15), gives

$$\begin{aligned}
&A(\ddot{w} + i\Omega\dot{w}) - i(C - A)\omega(\dot{w} + i\Omega w) + F(\dot{v} + i\Omega v) = L , \\
&F(\ddot{w} + i\Omega\dot{w}) + A_c\dot{v} + iC_c\Omega v = 0 , \\
&\ddot{\theta}_3 = 0 .
\end{aligned} \tag{12-46}$$

The exponential form (proportional to $e^{i\sigma t}$) for w, v, and θ_3 leads, as usual, to a system of homogeneous linear equations (if we put $L = 0$). The condition of vanishing determinant may be brought to the form

$$A(\,[illegible]\, - \,[illegible]_E)(A_{[illegible]}\,[illegible] + C_{[illegible]}\,[illegible]) - F\,[illegible]\,(\,[illegible] + [illegible])\,[illegible]\,(\,[illegible] + [illegible]) = 0 \quad (12\text{-}47)$$

The solutions of this equation are

$$\begin{aligned} &[illegible] = 0 && \text{(ASM)} , \\ &[illegible] = \frac{C - A}{A_m}\,[illegible] && \text{(CW)} , \\ &[illegible] = -[illegible](1 + [illegible]\frac{C}{C_m}) && \text{(NDFW)} , \\ &[illegible] = -[illegible] && \text{(TOM)} . \end{aligned} \quad (12\text{-}48)$$

For [illegible] and [illegible] this is immediately obvious, and [illegible] and [illegible] are seen to be the same as (12-38) and (12-41) by noting that the expression between parentheses is identical to (12-26).

Although the modes TOM and NDFW are conceptually completely different, their numerical closeness is so striking that it is tempting to look for a physical interpretation of NDFW in terms of TOM. A hint is provided by noting that for a strictly spherical core, with [illegible] = 0 , we have [illegible] = −[illegible] = [illegible]. Thus, for [illegible] = 0 , NDFW coincides with TOM.

Now the tilt-over mode characterizes a tilt of the body with respect to some external reference. In TOM in the proper sense, such an external reference is inertial space. An ideally fluid spherical core is mechanically completely independent from the mantle ("decoupled") since, in the case of spherical symmetry, a coupling could only be effected by drag of friction, which is absent with an ideal fluid. Thus the core, being independent of the mantle, can serve as an external reference for TOM into which thus NDFW degenerates for a spherical core.

This decoupling is no longer true if the core is elliptical. Due to the unsymmetry, there is now a mechanical coupling : the inertia of the core resists a rotation of the shell. Thus we have an inertial coupling, or Poincaré coupling, between core and shell which is zero for $\varepsilon = 0$ and can be expected to be proportional to ε by small ε . This is indeed borne out by (12-48) : the deviation of σ_3 from σ_4 is proportional to ε .

A detailed study of the mechanical situation from a somewhat different angle is found in (Toomre, 1974).

Proper modes are also called resonant. The presence of two different but almost equal eigenvalues σ_2 and σ_3 causes a significant deviation of the rotational behavior of an earth with a liquid core from that of a rigid body, which by Poincaré (1910) has been called double resonance.

13. LIQUID CORE EFFECTS ON POLAR MOTION AND NUTATION

Let us first introduce a convenient terminology:

$u = \omega_1 + i\omega_2$	body-referred rotation,
$v = \chi_1 + i\chi_2$	core rotation,
$w = \theta_1 + i\theta_2 = \psi$	space rotation.

In fact, (ω_1, ω_2) express the position of the actual rotation axis with respect to the body-fixed system of figure axes; (χ_1, χ_2) are a measure of the rotation of the core with respect to this body frame; and (θ_1, θ_2) characterize the rotation of the body with respect to inertial space. The quantity $\theta_1 + i\theta_2$ has been denoted by ψ in secs. 10 and 11 and by w in sec. 12; we shall continue to use this notation.

The body-referred rotation u characterizes polar motion. In fact, the complex number m describing *polar motion* is related to u by

$$u = \Omega m \; ; \qquad (13\text{-}1)$$

cf. (3-13) and (3-15).

The quantity u is a solution of the basic equation (12-22). The eigenvalues of this equation have been found to be (12-38),

$$\sigma_1 = \frac{C - A}{A_m}\,\Omega \; ; \qquad (13\text{-}2)$$

for CW, and

$$\sigma_2 = -(1 + \varepsilon \frac{C}{C_m})\Omega \tag{13-3}$$

for NDFW.

Thus the general free solution (without external forces) has the form

$$u = \alpha_1 e^{i\sigma_1 t} + \alpha_2 e^{i\sigma_2 t} \quad . \tag{13-4}$$

The first term constitutes the usual Chandler wobble, the second term is the nearly diurnal free wobble. The principal contribution to polar motion, of course, comes from CW, and it is an open question whether, for the real earth, the NDFW has a coefficient α_2 which is significantly large to be observable at present (Rochester et al., 1974; Yatskiv, 1980).

Practically much more important is the effect of the NDFW on forced motion. Consider the inhomogeneous equation (12-22), the torque L being given by the exponential (11-31),

$$L = \beta e^{i(\sigma t + \gamma)} \quad , \tag{13-5}$$

which represents a typical term in an expansion such as (5-12) with $\sigma = -\omega_j$. We put

$$\beta e^{i\gamma} = iK \; , \qquad L = iKe^{i\sigma t} \tag{13-6}$$

and

$$u = Ue^{i\sigma t} , \qquad v = Ve^{i\sigma t} . \tag{13-7}$$

The substitution into (12-22) with $F \doteq A_c$ by (12-32), then leads to

$$\begin{aligned} iA\sigma U + iA_c\sigma V - i(C - A)\Omega U + iA_c\Omega V &= iK , \\ iA_c\sigma U + iA_c\sigma V + iC_c\Omega V &= 0 \end{aligned}$$

or

$$\begin{aligned} [A\sigma - (C - A)\Omega]U + A_c(\sigma + \Omega)V &= K , \\ A_c\sigma U + (A_c\sigma + C_c\Omega)V &= 0 . \end{aligned} \tag{13-8}$$

The solution by means of determinants gives

$$U = \frac{A_c\sigma + C_c\Omega}{\Delta} K , \tag{13-9}$$

$$V = -\frac{A_c\sigma}{\Delta} K , \tag{13-10}$$

where Δ is the determinant

$$\Delta = [A\sigma - (C - A)\Omega](A_c\sigma + C_c\Omega) - A_c^2\sigma(\sigma + \Omega) . \tag{13-11}$$

Of principal interest is U which gives polar motion. Let us compare U with the value U_0 which would correspond to the same moment K if the earth were rigid. Then Euler's equations or simply (13-8) for $A_c = 0 = C_c$ give

$$U_0 = \frac{K}{A\sigma - (C-A)\Omega} \quad . \tag{13-12}$$

From (13-9), (13-11), and (13-12) we thus get

$$\begin{aligned}\frac{U_0}{U} &= \frac{[A\sigma - (C-A)\Omega](A_c\sigma + C_c\Omega) - A_c^2\sigma(\sigma + \Omega)}{[A\sigma - (C-A)\Omega](A_c\sigma + C_c\Omega)} \\ &= 1 - \frac{A_c^2\sigma(\sigma + \Omega)}{[A\sigma - (C-A)\Omega](A_c\sigma + C_c\Omega)} \quad .\end{aligned} \tag{13-13}$$

This can also be brought into the form

$$\frac{U_0}{U} = 1 - \frac{A_c\sigma(\sigma + \Omega)}{A(\sigma - \sigma_E)(\sigma + \Omega + \varepsilon\Omega)} \quad , \tag{13-14}$$

where σ_E is (12-25) as usual.

If we change over to our usual notation for polar motion, precession, and nutation, we must put

$$\begin{aligned}\sigma &= -\omega_j = -\Delta\omega_j - \Omega \ , \\ \Delta\omega_j &= \omega_j - \Omega = -(\sigma + \Omega) \ .\end{aligned} \tag{13-15}$$

Then (13-14), on putting $q = U_0/U$, becomes

$$q = \frac{U}{U_0} = \left[1 - \frac{A_c \omega_j \Delta\omega_j}{A(\omega_j + \sigma_E)(\Delta\omega_j - \Delta\sigma)} \right]^{-1} , \qquad (13\text{-}16)$$

identical to eq. (6-43) of (Melchior, 1978, p. 128); note that Melchior's $\Delta\omega_j$ has *opposite sign.*

Thus the amplitudes of forced polar motion, computed for a rigid earth, must be multiplied by this factor to get the corresponding amplitudes for Poincarés model. Later in this section we shall see that the factor U/U_0 also holds for the amplitudes of nutation of rotation and figure axes (the nutation of the angular momentum axis is the same as for a rigid earth).

Numerical values are seen from the following table which is taken from (Melchior, 1978, p. 129).

	$\Delta\omega_j$	$q = U_0/U$
Precession	0	1
Principal nutation	$+\omega/6800$	0.994
(18.7 years)	$-\omega/6800$	1.007
Annual nutation	$+\omega/365$	3.458
	$-\omega/365$	1.062
Semiannual nutation	$-\omega/183$	1.260
	$-\omega/133$	1.083

Fortnightly nutation	+[illegible]/13.7	1.1[illegible]1
	−[illegible]/13.7	1.1[illegible]1

This factor thus remains 1 for precession, but it becomes [illegible] for the proper frequencies [illegible] and [illegible] as will be shown later.

<u>Angular momentum and torque.</u> The treatment of [illegible] 11 is easily extended to the Poincaré model since the mantle is still rigid in this model and the model is axially symmetric. Therefore the kinematics, up to (11-16), remains unchanged. What changes, is the dynamics.

The angular momentum expression (11-17) is now replaced by

$$\underline{H} = \underline{C}\,\underline{[illegible]} + \underline{D}\,\underline{[illegible]} \,, \qquad (13\text{-}17)$$

where $\underline{C}$ is given by (11-18) and $\underline{D}$ is the matrix

$$\underline{D} = \begin{bmatrix} F & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & H \end{bmatrix} . \qquad (13\text{-}18)$$

The quantity $\underline{D}\,\underline{[illegible]}$ is an additional angular momentum due to the motion of the liquid core (Lamb, 1932, p. 724, eq. ([illegible])).

(In fact, the first equation of (12-22) is nothing but the basic equation $\dot{H} + \underline{\omega} \times \underline{H} = \underline{L}$ referred to body axes.)

Since [illegible] $= 0$ by (12-18) and $F = G$ in view of rotational symmetry, we have

$$\underline{D}\,[illegible] = \begin{bmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & H \end{bmatrix} \begin{bmatrix} [illegible] \\ [illegible] \\ 0 \end{bmatrix} = \begin{bmatrix} F[illegible] \\ F[illegible] \\ 0 \end{bmatrix}$$

$$= F[illegible]\,\underline{e}_1 + F[illegible]\,\underline{e}_2 = F[illegible]\,\underline{e}_1 + F[illegible]\,i\underline{e}_2$$

$$= F([illegible] + i[illegible])\underline{e} = Fv\underline{e}\ ; \qquad (13\text{-}19)$$

here we have used (11-6) and (12-21). This term has to be added to (11-22), giving

$$\underline{H} = [illegible]\,e + iA([illegible] + [illegible]) + Fve \qquad (13\text{-}20)$$

in the body frame. In the [illegible] system this becomes by (11-25)

$$\underline{H} = H[illegible] + [illegible]H$$

$$= [illegible]\,e + iA[illegible] + [illegible] + Fv[illegible]\ , \qquad (13\text{-}2[illegible])$$

[illegible]

[illegible]

$$\underline{e}_H = \underline{e}_3 + i\frac{A}{C_{..}}(\ldots - \ldots_{E})\ldots + \frac{A}{C_{..}} v e_{..} \;; \qquad (13\text{-}22)$$

it is indeed a unit vector since e_H differs from [illegible] only by small terms of second order in [illegible] and v .

To find the relation between [illegible] and the angular momentum $\underline{L}$, we may proceed as follows. Equation (13-45) for exponentials

$$u = Ue^{i\ldots} , \qquad w = We^{i\ldots} \qquad [illegible]$$

becomes

$$u = [illegible] + [illegible] w , \qquad w = [illegible] \qquad [illegible]$$

Thus,

$$w = [illegible] Ue^{i\ldots}$$

[illegible]

[illegible]

Thus, by (11-32),

$$w = qw \quad = -\, q \,\frac{L}{A(\;\; + \;\;)(\;\; - \;\;_E)} \quad . \qquad (13\text{-}25)$$

This is the desired relation between and L. It shows that the factor q, eq. (13-16) which converts u from the rigid case to the Poincaré model, likewise converts w.

It could appear that w becomes singular if $= \;_E$ although this is not an eigenvalue. Such a singularity is not real, however, since $U = 0$ for $= \;_E$. This becomes clear if we express u by (13-9) ; by the aid of (13-6) and (1[illegible]-23) this becomes

$$u = -i \,\frac{A \;\; + C}{\;\;}\, L \quad . \qquad (13\text{-}26)$$

The determinant can obviously be written in the form

[illegible] (13-27)

[illegible] the eigenvalues [illegible] and [illegible]. [illegible]

[illegible] (13-28)

and (13-24) gives

$$w = -\frac{A_c \sigma + C_c \Omega}{A_m A_c(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma + \Omega)} L \quad . \qquad (13-29)$$

An expansion into partial fractions gives expressions of the form

$$u = \frac{\alpha_1}{\sigma - \sigma_1} + \frac{\alpha_2}{\sigma - \sigma_2} \; ,$$

$$w = \frac{\beta_1}{\sigma - \sigma_1} + \frac{\beta_2}{\sigma - \sigma_2} + \frac{\beta_3}{\sigma - \sigma_3} \; , \qquad (13-30)$$

where α_i and β_i are constants and the eigenvalues σ_1, σ_2, σ_3 are given by (12-48). Thus, polar motion u is resonant for CW and NDFW, and space rotation w is resonant for CW, NDFW, and TOM, as it should be.

The relation between $\underline{H}$ and $\underline{L}$ is found as follows. With

$$\underline{\theta} = \theta \underline{e}_3 = w \underline{e}_3 = -\frac{iu}{\sigma + \Omega} \underline{e}_3 \; , \qquad (13-31)$$

by (13-24), we may transform (13-21) as follows

$$\underline{H} = C \underline{e}_3 + \frac{A(\text{[illegible]})}{\text{[illegible]}} \text{[illegible]}\, u + \text{[illegible]}\, v \; \underline{e}_3 \; . \qquad (13-32)$$

Now the first equation of (13-[illegible]), on multiplying by $e^{[illegible]}$ and taking into account (13-6) and (13-7), may be written

$$A([illegible] - \sigma_E)u + A_{[illegible]}([illegible] + [illegible])v = -iL \; . \qquad (13\text{-}33)$$

Combining (13-32) and (13-33) and noting $F = A_{[illegible]}$, we get

$$H = C\,e_{[illegible]} - \frac{iL}{[illegible] + [illegible]}\,e_{[illegible]} \; , \qquad (13\text{-}34)$$

which is the same relation as for a rigid body. This confirms the independence of the relation between angular momentum H and torque $\underline{L}$ from the internal structure of the body; cf. the concluding remark of sec. [illegible].

<u>Nutation and polar motion.</u> Now it is straightforward to extend the formulas for precession (11-37) and polar motion (11-[illegible]) to the present case. Since the first two relations are purely kinematical, they continue to hold :

$$\eta_R = i[illegible] = i[illegible]w \; ,$$

$$\eta_F = -i[illegible] = -iw \; . \qquad (13\text{-}35)$$

This gives the nutation of the rotation axis R and of the figure axis F. The present w, of course, differs from the rigid body value w by the factor q, according to (13-25).

Thus the amplitudes of nutation for the rotation axis and the figure axis differ from the corresponding rigid-body values by the factor q .

This is not true for the nutation of the angular momentum axis. From (13-22) and (11-35) we get

$$n_H = i \frac{A(\sigma - \sigma_E)}{C\Omega} w + \frac{A_c}{C} v \quad , \tag{13-36}$$

which differs from (11-37) by a term due to core motion. Expressed in terms of L , we even get exactly the same nutation as in the rigid body case :

$$n_H = - \frac{iL}{C\Omega(\sigma + \Omega)} \tag{13-37}$$

by (13-34).

The polar motion of the rotation axis, being purely kinematical, remains the same as (11-39) :

$$p_R = i \frac{\sigma + \Omega}{\Omega} w = \frac{u}{\Omega} \quad , \tag{13-38}$$

and consequently differs from the rigid-body case by the factor q .

The polar motion of the angular momentum axis is different from (11-39) :

$$p_H = n_H - n_F$$

$$= i\frac{A}{C}\,\frac{[illegible]}{[illegible]}\,w + \frac{A_{[illegible]}}{C_{[illegible]}}\,v \qquad (13\text{-}39)$$

or, by (13-38)

$$p_H = \frac{A}{C}\,p_R + \frac{A_{[illegible]}}{C_{[illegible]}}\,v \quad . \qquad (13\text{-}40)$$

Here v obtained from (13-10), which by (13-6) and (13-7) becomes:

$$v = i\,\frac{A_{[illegible]}}{[illegible]}\,L \quad . \qquad (13\text{-}41)$$

These relations are illustrated by Fig. 13.1, which is the extension of Fig. 11.1 to the Poincaré model. It shows that the rotation axis R is no longer close to the angular momentum axis H .

<u>Free motion.</u> In the absence of external forces, for $L = 0$, the basic equations (12-22) have the solution:

$$\begin{aligned} u &= [illegible]\,e^{i[illegible]t} + [illegible]\,e^{i[illegible]t} \ , \\ v &= [illegible]\,e^{i[illegible]t} + [illegible]\,e^{i[illegible]t} \ ; \end{aligned} \qquad (13\text{-}42)$$

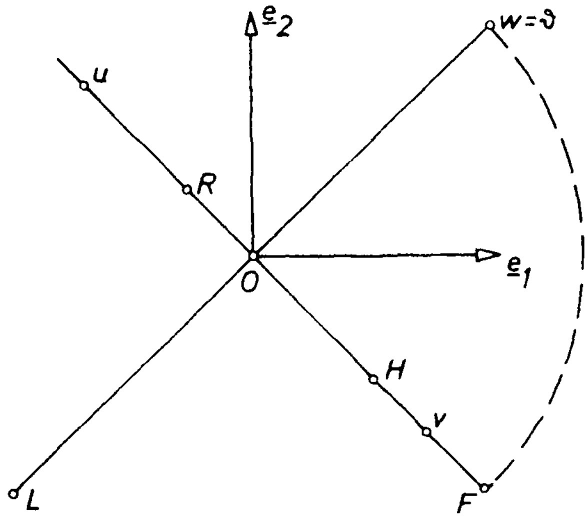


FIGURE 13.1 Forced nutation and polar motion of the rotation axis R , the figure axis F , and the angular momentum axis H .

cf. (13-4) and (12-23) . The coefficients κ_i are related to the corresponding coefficients α_i $(i = 1,2)$ by (12-24) with $\sigma = \sigma_i$. Since the determinant vanishes, we may take either of the two equations of (12-24). We take the second, with $F = A_c$:

$$A_c \sigma_i \alpha_i + (A_c \sigma_i + C_c \Omega) \beta_i = 0 \qquad (13\text{-}43)$$

or

$$\begin{aligned} \kappa_1 \alpha_1 + (1 + \kappa_1)\beta_1 &= 0 \quad , \\ \kappa_2 \alpha_2 + (1 + \kappa_2)\beta_2 &= 0 \end{aligned} \qquad (13\text{-}44)$$

with

$$\kappa_1 = \frac{A_c \sigma_1}{C_c \Omega} \quad , \qquad \kappa_2 = \frac{A_c \sigma_2}{C_c \Omega} \quad . \qquad (13\text{-}45)$$

Eqs. (13-44) are satisfied if

$$\begin{aligned} \alpha_1 &= (1 + \kappa_1)\lambda \ , \qquad \beta_1 = -\kappa_1 \lambda \ , \\ \alpha_2 &= (1 + \kappa_2)\mu \ , \qquad \beta_2 = -\kappa_2 \mu \ , \end{aligned} \qquad (13\text{-}46)$$

with arbitrary complex constants λ and μ ; this is easily seen to be the general solution. Thus

$$\begin{aligned} u &= (1 + \kappa_1)\lambda e^{i\sigma_1 t} + (1 + \kappa_2)\mu e^{i\sigma_2 t} \ , \\ v &= -\kappa_1 \lambda e^{i\sigma_1 t} - \kappa_2 \mu e^{i\sigma_2 t} \end{aligned} \qquad (13\text{-}47)$$

represents the general free solution.

Important is the order of magnitude. From (13-45) with (13-2) and (12-42) we get

$$\kappa_1 \doteq \frac{\sigma_1}{\Omega} = \frac{C - A}{A_m} = \frac{C - A}{A - A_c} \doteq \frac{C - A}{A} \doteq \frac{1}{300} = O(\epsilon) \quad ,$$

and similarly using (13-3) and (12-31)

$$\kappa_2 = \frac{1}{1 + \epsilon} \frac{\sigma_2}{\Omega} \doteq - \frac{1 + \epsilon C/C_m}{1 + \epsilon} =$$

$$= - \left(1 + \epsilon \frac{C}{C_m} - \epsilon\right) = -1 - \frac{C_c}{C_m} \epsilon \quad . \qquad (13\text{-}48)$$

Hence, approximately,

$$u \doteq \lambda e^{i\sigma_1 t} + O(\epsilon) \mu e^{i\sigma_2 t} \, ,$$

$$v \doteq O(\epsilon) \lambda e^{i\sigma_1 t} + \mu e^{i\sigma_2 t} \quad . \qquad (13\text{-}49)$$

This shows that polar motion, (u) comes principally from CW (frequency σ_1) but core motion (v) is caused mainly by NDFW (frequency σ_2).

The corresponding free value of spatial rotation is given by (13-24) where $\sigma = \sigma_1$ or σ_2 , depending on the mode. Thus from (13-47) we get

$$iw = \frac{1+\kappa_1}{\sigma_1+\Omega}\, A e^{i\sigma_1 t} + \frac{1+\kappa_2}{\sigma_2+\Omega}\, B e^{i\sigma_2 t} \quad . \tag{13-50}$$

Now free nutation can be computed by (13-35) and (13-36) using (13-47) and (13-50), and free polar motion is obtained in the same way from (13-38) and (13-40).

It is very instructive to consider the NDFW only. Then, putting

$$B e^{i\sigma_2 t} = h \tag{13-51}$$

and using (13-48) and (13-3) we get from (13-47) and (13-50) :

$$\begin{aligned} u &= -\frac{C_c}{C_m}\,\varepsilon h \;, \\ v &= \left(1 + \frac{C_c}{C_m}\,\varepsilon\right) h \;, \\ iw &= \frac{C_c}{C_m}\, h \quad . \end{aligned} \tag{13-52}$$

We recall that the core flattening

$$\varepsilon_c = \frac{C_c - A_c}{C_c} \doteq \frac{1}{400}$$

and that for the moments of inertia of core and mantle we have

$$\frac{C_c}{C} \doteq 0.1 \,, \qquad \frac{C_m}{C} = \frac{C - C_c}{C} \doteq 0.9 \,.$$

This shows that the ratios

$$\frac{v}{u} = -\left(1 + \frac{C_c}{C_m}\varepsilon\right)\frac{C_m}{C_c\varepsilon}$$

$$\doteq -\frac{C_m}{C_c}\frac{1}{\varepsilon} \,,$$

$$\frac{i\Omega w}{u} = -\frac{C_m}{C}\frac{1}{\varepsilon}$$

are both very large.

In particular we have for this mode by (13-38) and (13-40)

$$\frac{p_H}{p_R} = \frac{A}{C} + \frac{A_c}{C\Omega}\frac{v}{p_R} = \frac{A}{C} + \frac{A_c}{C}\frac{v}{u}$$

$$\doteq \frac{A}{C} - \frac{A_c}{C}\frac{C_m}{C_c}\frac{1}{\varepsilon} \,.$$

The first term on the right-hand side is close to 1 and can be neglected with respect to the very large second term. Furthermore, $A_c/C_c = 1 + O(\epsilon)$. Thus

$$\frac{p_H}{p_R} = -\frac{C_m}{C_c} \doteq -400 . \tag{13-53}$$

Using (13-3) this can also be written

$$\frac{p_H}{p_R} = \frac{\Omega}{[illegible] + \Omega} . \tag{13-54}$$

This simple relation permits a kinematical interpretation in terms of body- and space-fixed cones according to Poinsot (Rochester et al., 1974).

The nearly diurnal free wobble is illustrated by Fig. 13.2, which in view of (13-53) shows that the angle γ between the angular momentum vector and the figure axis is about 400 times larger than the angle β between the rotation axis (vector $\underline{\omega}$) and the figure axis. Cf. also (Toomre, 1974), (Rochester et al., 1974) and (Yatskiv, 1980); Rochester et al. have

$$\frac{\beta}{\gamma} = \frac{A_m}{A_c} , \tag{13-55}$$

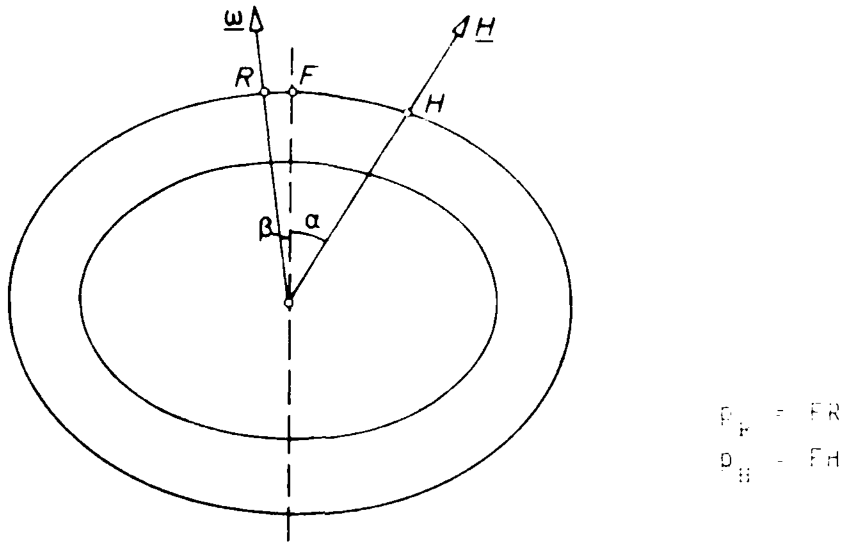


FIGURE 13.2 The nearly-diurnal free wobble.

which differs from (13-53) only by $O(\varepsilon)$ which we have disregarded.

The same phenomenon may also be looked at from a slightly different angle. In the absence of external forces, the angular momentum $\underline{H}$ retains its position in space ; the nutation of the angular momentum axis is zero :

$$n_H = 0 \ . \tag{13-56}$$

This is evident from (13-37) since the denominator differs from zero if $\sigma = \sigma_1$ or σ_2 , and the numerator is zero if $L = 0$.

To the nearly diurnal free wobble there exists a corresponding nutation which is about 400 times larger. Such a nutation has not yet been observed, which is another indication that the amplitude of the NDFW for the earth must be very small.

Compilation of formulas. The basic quantities for free motion are body-referred rotation u , core rotation v , and space rotation w , which we collect first:

(13-59)

Then the free polar motion of the rotation axis R and the angular momentum axis H become

[illegible]

The free [illegible]
by (13-3[illegible]). Converting [illegible]

[illegible]

we have

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad \text{(II-64)}$$

[illegible] given [illegible] II-4[illegible], [illegible] the outer part, we

$$[illegible] \qquad \text{(II-65)}$$

[illegible]

[illegible]

[illegible]

For the angular momentum axis, (13-40) gives

$$p_{H}^{free} = \frac{A}{C}\, p_{F}^{free} + \frac{A}{C} \ldots \text{[illegible]}$$

Forced motion for the angular momentum axis [illegible] the same as for the rigid-body case; for the other axes we have the factor q_i as (13-35) shows. Thus the rigid-body formulas of sec. 11, eqs. (11-54) and (11-55), give

$$(\ldots + i \ldots \sin \ldots)_{H}^{forced} = \frac{A \ldots}{C \ldots}\, B\, e^{\ldots} \text{[illegible]} ,$$

$$(\ldots + i \ldots \sin \ldots)_{R}^{forced} = \frac{\ldots}{\ldots(\ldots + \ldots_{E})}\, q_i\, B\, e^{\ldots} \text{[illegible]} \qquad (13-\ldots)$$

$$(\ldots + i \ldots \sin \ldots)_{F}^{forced} = \frac{\ldots_{E}}{\ldots(\ldots + \ldots_{E})}\, q_i\, B\, e^{\ldots} \text{[illegible]} .$$

The total polar motions and nutations are obtained by adding the corresponding free and forced terms. A check is provided by the fact that, for [illegible] $= \ldots_{E}$, [illegible] $= 0$, [illegible], and $q_i = 1$, must give the same rigid-body formulas as eqs. (5-24), (6-11), (7-14), (8-[illegible]), and (8-11) for [illegible] $= 0$. The precessional part proportional to B [illegible] in these formulas remains the same also for the present case.

As regards the celestial pole C (sec. [illegible]), its polar [illegible] $p_{?}$ equals the free part of p_H :

$$p_{?} = p_H^{[illegible]} , \qquad (13-70)$$

whereas its nutation equals the forced nutation of the [illegible] axis :

$$[illegible] (\Delta\epsilon + i\Delta\psi.\cos\epsilon)_F^{f\ [illegible]} . \qquad (13-71)$$

Concluding remark. An adequate earth model must take into account the elasticity of the mantle, as well as a core which is not an ideal fluid. This complex and difficult topic [illegible] to Poincaré (1910) and has been extensively studied during the last decades, highlights being (Jeffreys and Vicente, [illegible]), (Molodensky, 1961), and (Wahr, 1979). These methods will be the subject of another report.

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8