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C. B. Chang

Parameter Estimation Accuracy for Radar Targets Closely Spaced in Range

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This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Raymond L. Loiselle, Lt. Col., USAF
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

PARAMETER ESTIMATION ACCURACY FOR RADAR TARGETS CLOSELY SPACED IN RANGE

C. B. CHANG
Group 32

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LEXINGTON MASSACHUSETTS

ABSTRACT

In this report, we discuss the amplitude, location, and the relative separation estimation accuracy of radar targets closely spaced in range using the Cramer-Rao bound. It is assumed that the phases of successive signals are coherent and therefore contain relative line-of-sight location information. It is shown that this information can substantially reduce the estimation error when compared with the case where the relative signal phase is random.

CONTENTS

1. INTRODUCTION

Resolution of closely spaced signals has been a subject of interest for many years, see for example [1] - [14].* The traditional signal resolution problem deals with radar signals, [1] - [10], while more recent emphasis has been on optical signals, [10] - [14]. The main difference between the radar and optical signal is that the radar signal is a complex-valued process corrupted by complex-valued Gaussian white noise while the optical signal is a real-valued process corrupted by a real-valued non-Gaussian process and it is usually a random signal in colored noise problem.

There are two basic issues pertaining to the problem of signal resolution, i.e., the recognition of the existence of multiple signals (the problem of detection) and the estimation of pertinent signal parameters (the problem of estimation). The problem which has drawn more attention in the literature is the problem of detection. This is because the ordinary multiple array likelihood ratio test procedure is ill-conditioned for this problem (see [1] and [3]), and alternate information criteria must be employed (see [1] - [5] and [14]).

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*This list is not intended to be exhaustive. Ref. [1] has a short but concise review of the open literature.

As to the problem of estimation, the maximum likelihood estimator has been widely used due to its asymptotic optimality property [15], [16] and intimate tie with the detection algorithm. [1] - [10]. The difficulty of applying the maximum likelihood estimator is in the computational burden since it is generally a multi-dimensional optimization problem. Furthermore, there may exist many local maxima in the likelihood function for a certain class of signals. Once a sufficiently close initial guess is found however, an iterative optimization algorithm (e.g., the gradient algorithm) can usually find the optimal solution.

Another difficulty in the problem of estimation is performance evaluation. Due to the nonlinear nature of the problem an exact expression for the error covariance of the estimates cannot be obtained. Fortunately, the error covariance of the maximum likelihood estimates can be closely predicted using the Cramer-Rao bound when the signal-to-noise ratio is large. Furthermore, the maximum likelihood estimates are asymptotically Gaussian and approach the Cramer-Rao bound, [15] - [16]. For these reasons, Cramer-Rao bound analysis has gained significant popularity in recent years for predicting the estimation performance in the signal resolution problem, [7] - [10], [12] - [13].

The Cramer-Rao bound for parameter estimation of closely spaced radar signals was presented in [8] - [10]. Refs. [8] and [9] used some simplifying assumptions in which the amplitudes (and phases for [8]) of the signals were assumed known. Closed form expressions were obtained for the two-signal case which provided intuitive insights about the estimation accuracy. These bounds are unfortunately too optimistic. In Ref. [10], the Cramer-Rao bound for jointly estimating amplitudes, phases, and time delays was obtained. This provided a realistic bound for this problem and the close agreement with simulation results was also demonstrated. In this report, we extend the work of [10] to include the case where the phase angles of successive signal are coherent. There are many applications for which this condition is satisfied. For example, if one wishes to estimate the locations of scattering centers on a distributed target, the phase difference of two successive scatterers is proportional to their separation along the line of sight. Even if two point targets are completely resolved in terms of base band pulse shape, their relative separation is still contained in their phase difference given that the radar can process the returned signal coherently. This second example is indeed the basis of the phase-derived range technique illustrated in [18]. As will be shown later, this phase information is able to substantially reduce the estimation error.

This report is organized as follows. In the next section, we introduce the signal model and the Cramer-Rao bound. Equations for computing the Fisher information matrix and some numerical results are presented in Section 3. A summary and conclusions are stated in the last section. Three appendices are attached. In Appendix A, we discuss the Cramer-Rao bound for multiple radar pulses. In Appendix B, we give the formulation and structure of the maximum likelihood estimator for resolving closely spaced targets. In Appendix C, we re-state the Fisher information matrix of Ref. [10] for the random phase angle case for the purpose of comparison.

2. SIGNAL MODEL AND THE CRAMER-RAO BOUND

Let s(t) denote the complex low-pass transmitted waveform. The reflection from the ith point target (or scatterer) is $a_i\ s(t-\tau_i)$. The total received signal from a group of N targets may be expressed by

$$r(t) = \sum_{i=1}^{N} a_i\ s(t-\tau_i) + n(t) \tag{2.1}$$

where $(\tau_1,\ldots,\tau_N)^T \triangleq \underline{\tau}$ = locations of point targets in terms of relative range

$(a_1,\ldots,a_N)^T \triangleq \underline{a}$ = complex scattering amplitudes

$n(t)$ = receiver white noise which has a complex Gaussian distribution with variance $N_o/2$.

The log likelihood ratio of signals present versus signals absent is

$$\ln\Delta = \frac{-1}{N_o}\int\left|r(t) - \sum_{i=1}^{N} a_i s(t-\tau_i)\right|^2 dt + \frac{1}{N_o}\int|r(t)|^2 dt \tag{2.2}$$

Let
$$\gamma_i \triangleq \int s(t-\tau_i)r^*(t)dt$$
$$\rho_{ij} \triangleq \int s(t-\tau_i)s(t-\tau_j)dt$$
$$\underline{\gamma} \triangleq [\gamma_1,\ldots,\gamma_N]^T$$
$$P \triangleq [\rho_{ij}]$$

where the superscript "*" denotes the complex conjugate operation. The log likelihood ratio may be rewritten as

$$\ln\Lambda = \frac{1}{N_o}\left(\underline{a}^T\underline{\gamma} + \underline{a}^{*T}\underline{\gamma}^* - \underline{a}^T P\underline{a}^*\right) \tag{2.3}$$

The above constitutes the signal model used in [10]. The phase angles of $\underline{a}$ were assumed to be independent and uniformly distributed. To evaluate the Cramer-Rao bound on jointly estimating $\underline{\tau}$ and $\underline{a}$, one is required to evaluate the (3N x 3N) Fisher information matrix, [10].

In this report, we assume that the phase difference between two successive targets, $\Delta\Theta_{i,i+1}$, satisfies the following relation.

$$\Delta\Theta_{i,i+1} = \frac{4\pi}{\lambda}(\tau_{i+1} - \tau_i) \tag{2.4}$$

where λ is the wavelength. With the above assumption, one is only required to jointly estimate $\underline{\tau}$, the magnitude of the scattering amplitudes $|\underline{a}|$, and the phase angle of the first target, Θ_1. The associated Fisher information matrix has dimension ((2N+1)x(2N+1)). It may be written as

$$\mathscr{F} = \left[\begin{array}{c|c|c} -E\left[\frac{\partial^2 \ell n\Lambda}{\partial^2 |\underline{a}|}\right] & -E\left[\frac{\partial^2 \ell n\Lambda}{\partial |\underline{a}| \partial \underline{\tau}}\right] & -E\left[\frac{\partial^2 \ell n\Lambda}{\partial |\underline{a}| \partial \Theta_1}\right] \\ \hline \bullet & -E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau}^2}\right] & -E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau} \partial \Theta_1}\right] \\ \hline \bullet & \bullet & -E\left[\frac{\partial^2 \ell n\Lambda}{\partial \Theta_1^2}\right] \end{array}\right] \tag{2.5}$$

Notice that we have neglected the details of the lower half of $\mathscr{F}$ since it is a symmetric matrix. The inverse of $\mathscr{F}$ is the Cramer-Rao lower bound on the covariance of jointly estimating $|\underline{a}|$, $\underline{\tau}$, and Θ_1.

We note that the expectation, $E[\cdot]$ used in Equation (2.5) applies to all random variables of the likelihood function Λ. It was shown in [17] that a tighter bound can be obtained if the expectations in Eq. (2.5) are conditioned on the random parameters; and the expectation with respect to the random parameters is taken after taking the inverse of the Fisher information matrix. For example, the phase difference of two scatterers was assumed uniformly random in [10]. Let $\mathscr{F}$ denote the Fisher information matrix obtained with the expectation taken over all random variables and $\mathscr{F}_{\Delta\Theta}$ the Fisher's information matrix with conditional expectation upon $\Delta\Theta$, then

$$E_{\Delta\Theta}\left[\mathscr{F}_{\Delta\Theta}^{-1}\right] \geq \mathscr{F}^{-1} \tag{2.6}$$

where $E_{\Delta\Theta}[\ ]$ denotes the expectation with respect to $\Delta\Theta$. The above property was used in [10]. We will discuss its application to our case in the numerical result section.

3. THE CRAMER-RAO BOUND FOR A TWO-TARGET MODEL

In this section, we discuss the problem of the two-target case. The expression can be easily extended to the multiple-target case. Rewriting Eq. (2.2) explicitly for the two-target case one obtains

$$J \triangleq N_o \ln \Lambda$$

$$= e^{i\Theta_1}(\alpha_1\gamma_1+\alpha_2\gamma_2 e^{i\Delta\Theta}) + e^{-i\Theta_1}(\alpha_1\gamma_1^*+\alpha_2\gamma_2^* e^{-i\Delta\Theta})$$

$$-(\alpha_1^2+\alpha_2^2\ 2\alpha_1\alpha_2\rho \cos \Delta\Theta) \tag{3.1}$$

where $\alpha_i = |a_i|$,

$$\Delta\Theta \triangleq \Theta_2-\Theta_1 = \frac{4\pi}{\lambda}(\tau_2-\tau_1)$$

$$\rho(\tau_2,\tau_1) = \int s(t-\tau_1)s(t-\tau_2)dt$$

Notice that for most radar signals, $\rho(\tau_2,\tau_1) = \rho(\tau_2-\tau_1) \triangleq \rho(\tau)$. We use $\dot{\rho}$ and $\ddot{\rho}$ to denote the first and second derivatives of ρ with respect to τ, respectively. The root-mean-square (rms) bandwidth (the normalized second central moment of the signal spectrum) of s(t) is denoted by β and is equal to the square root of the negative of $\ddot{\rho}$ evaluated at $\tau=0$.*

With the above definitions, we now proceed to define the Fisher information matrix.

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*For a linear FM signal, β is approximately equal to $4\pi B/c\sqrt{12}$ where B is the LFM bandwidth in Hz, and c is the speed of light.

3.1. The Fisher Information Matrix

Using (2.5) and replacing $|\underline{a}|$ with $\underline{\alpha}$, we obtain these following submatrices.

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\alpha}^2}\right] = \frac{2}{N_o}\begin{bmatrix} 1 & \rho\cos\Delta\Theta \\ \rho\cos\Delta\Theta & 1 \end{bmatrix} \tag{3.2}$$

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\alpha}\partial \underline{\tau}}\right] = \frac{2}{N_o}\begin{bmatrix} \alpha\ \frac{4\pi}{2\lambda}\ \rho\ \sin\Delta\Theta & \alpha_2\dot{\rho}\ \cos\Delta\Theta \\ -\alpha_1\dot{\rho}\ \cos\Delta\Theta & 0 \end{bmatrix} \tag{3.3}$$

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\alpha}\partial \Theta_1}\right] = \frac{2}{N_o}\begin{bmatrix} -\alpha_2\rho\ \sin\Delta\Theta \\ \alpha_1\rho\ \sin\Delta\Theta \end{bmatrix} \tag{3.4}$$

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau}^2}\right] =$$

$$\frac{2}{N_o}\begin{bmatrix} \alpha_1^2\beta^2+\alpha_2^2(\frac{4\pi}{\lambda})^2-2(\frac{4\pi}{\lambda})\alpha_1\alpha_2\dot{\rho}\sin\Delta\Theta & -[\alpha_2^2(\frac{4\pi}{\lambda})^2+\alpha_1\alpha_2(\ddot{\rho}\cos\Delta\Theta-\frac{4\pi}{\lambda}\dot{\rho}\sin\Delta\Theta)] \\ -[\alpha_2^2(\frac{4\pi}{\lambda})^2+\alpha_1\alpha_2(\ddot{\rho}\cos\Delta\Theta-(\frac{4\pi}{\lambda})\dot{\rho}\sin\Delta\Theta)] & \alpha_2^2(\beta^2+(\frac{4\pi}{\lambda})^2) \end{bmatrix} \tag{3.5}$$

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau}\partial \Theta_1}\right] = \frac{2}{N_o}\begin{bmatrix} -[\alpha_2^2(\frac{4\pi}{\lambda}) + \alpha_1\alpha_2[(\frac{4\pi}{\lambda})\rho \cos \Delta\Theta - \dot{\rho} \sin \Delta\Theta] \\ \\ \alpha_2^2(\frac{4\pi}{\lambda}) + \alpha_1\alpha_2[(\frac{4\pi}{\lambda})\rho \cos \Delta\Theta + \dot{\rho} \sin \Delta\Theta] \end{bmatrix} \tag{3.6}$$

$$-E\left[\frac{\partial^2 \ell n\Lambda}{\partial \Theta_1^2}\right] = \frac{2}{N_o}(\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2 \rho \cos \Delta\Theta) \tag{3.7}$$

Notice that the above expressions do not depend upon Θ_1 , the initial phase. The relative phase $\Delta\Theta$ is a function of target separation. The expectation is taken with respect to the receiver noise only. The inverse of the above matrix is the lower bound on the covariance of jointly estimating α, $\underline{\tau}$, and $\underline{\Theta}_1$

For the purpose of comparison, the Fisher information matrix for the case of jointly estimating $|\underline{a}|$, $\underline{\tau}$, and $\underline{\Theta}$ is shown in Appendix C.

It is always interesting to know the case when the targets are disjoint in time, i.e., $\rho(\tau)=\dot{\rho}(\tau)=\ddot{\rho}(\tau)=0$. If we were to assume $\Delta\Theta$ random the information matrix would be diagonal (see Appendix C) but this is not true for a coherent phase relation. We discuss this case below.

3.2. Results for Two Resolved Targets

Substituting $\rho=\dot{\rho}=\ddot{\rho}=0$ into equations (3.2)-(3.7), one finds that the estimation of α_1 and α_2 are uncorrelated with the estimation of $\underline{\tau}$ and Θ_1. The lower bound on the variance of estimating α_i can be expressed as

$$\sigma_{\alpha_i}^{2} \geq \frac{N_o}{2} = \frac{\alpha_i^{2}}{SNR_i} \tag{3.8}$$

where $SNR_i = \frac{2\alpha_i^{2}}{N_o}$. This result is the same as the random phase case. The covariance of estimating $\underline{\tau}$ and Θ_1 is lower bounded by the inverse of the following matrix.

$$-E\left[\begin{array}{c|c} \frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau}^2} & \frac{\partial^2 \ell n\Lambda}{\partial \underline{\tau}\partial\Theta_1} \\ \hline \frac{\partial^2 \ell n\Lambda}{\partial\Theta_1\partial\underline{\tau}} & \frac{\partial^2 \ell n\Lambda}{\partial\Theta_1^{2}} \end{array}\right]$$

$$= \begin{bmatrix} \alpha_1^{2}\beta^{2}+\alpha_2^{2}(\frac{4\pi}{\lambda})^{2} & -\alpha_2^{2}(\frac{4\pi}{\lambda})^{2} & -\alpha_2^{2}(\frac{4\pi}{\lambda}) \\ -\alpha_2^{2}(\frac{4\pi}{\lambda})^{2} & \alpha_2^{2}\beta^{2}+\alpha_2^{2}(\frac{4\pi}{\lambda})^{2} & \alpha_2^{2}(\frac{4\pi}{\lambda}) \\ -\alpha_2^{2}(\frac{4\pi}{\lambda})^{2} & \alpha_2^{2}(\frac{4\tau}{\lambda}) & \alpha_1^{2}+\alpha_2^{2} \end{bmatrix} \tag{3.9}$$

After some tedious manipulation one finds that the bounds on the variance for τ_1, τ_2, and $\Delta\tau \triangleq \tau_2-\tau_1$ are

$$\sigma_{\tau_1}^2 \geq \frac{(\alpha_1^2+\alpha_2^2)\alpha_2^2\beta^2+\alpha_1^2\alpha_2^2(\frac{4\pi}{\lambda})^2}{(\alpha_1^2+\alpha_2^2)\alpha_1^2\alpha_2^2\beta^2[\beta^2+(\frac{4\pi}{\lambda})^2]} \tag{3.8}$$

$$\sigma_{\tau_2}^2 \geq \frac{(\alpha_1^2+\alpha_2^2)\alpha_1^2\beta^2+\alpha_1^2\alpha_2^2(\frac{4\pi}{\lambda})^2}{(\alpha_1^2+\alpha_2^2)\alpha_1^2\alpha_2^2\beta^2[\beta^2+(\frac{4\pi}{\lambda})^2]} \tag{3.9}$$

$$\sigma_{\Delta\tau}^2 \geq \frac{\beta^2(\alpha_1^2+\alpha_2^2)^2}{(\alpha_1^2+\alpha_2^2)\alpha_1^2\alpha_2^2\beta^2[\beta^2+(\frac{4\pi}{\lambda})^2]} \tag{3.10}$$

Letting $R \triangleq \frac{\alpha_2}{\alpha_1}$ = scatterer amplitude ratio and

$SNR_i \triangleq \frac{2\alpha_i^2}{N_o}$ = peak signal-to-noise ratio of i-th scatterer

one can rewrite the above expressions to a more familiar form.

$$\sigma_{\tau_1}^2 \geq \frac{1}{SNR_1\beta^2(1+R^2)}\left[1 + \frac{R^2}{1+(\frac{4\pi}{\lambda\beta})^2}\right] \tag{3.8a}$$

$$\sigma_{\tau_2}^2 \geq \frac{1}{SNR_2\beta^2(1+R^2)}\left[R^2 + \frac{1}{1+(\frac{4\pi}{\lambda\beta})^2}\right] \tag{3.9a}$$

$$\sigma_{\Delta\tau}^2 \geq \frac{1}{\beta^2[1+(\frac{4\pi}{\lambda\beta})^2]}\left[\frac{1}{SNR_1} + \frac{1}{SNR_2}\right] \tag{3.10a}$$

Comparing the above results with the case of random phase (see Appendix C), one notices that the difference is in the appearance of the wavelength dependent term $(4\pi/\lambda\beta)$. If this term does not appear, Eqs. (3.8a)-(3.10a) will be identical to the case when the relative phase is uniformly random.

3.3. Numerical Results

In this section, we present some numerical results and compare them with thosefor random phase. We assume that the signal autocorrelation function is represented by a Gaussian pulse shape

$$\rho(\tau) = e^{-\tau^2\beta^2/2} \tag{3.11}$$

Its first and second derivatives are

$$\dot{\rho}(\tau) = -\tau\beta^2 e^{-\tau^2\beta^2/2} \tag{3.12}$$

$$\ddot{\rho}(\tau) = -\beta^2(1-\tau^2\beta^2)e^{-\tau^2\beta^2/2} \tag{3.13}$$

In order to facilitate the presentation of our numerical results, we use the following change of variables.

(1) The Cramer-Rao bound will be evaluated as a function of target separation normalized with respect to the inverse of signal root-mean-square bandwidth, i.e.,

$$\begin{aligned} \ell &\triangleq \tau\beta \\ &= \text{normalized target separation} \end{aligned} \tag{3.14}$$

(2) The relative target phase can be rewritten in terms of ℓ as

$$\begin{aligned} \Delta\Theta &= \frac{4\pi\tau}{\lambda} \\ &= \frac{4\pi}{\lambda\beta}\,\ell \end{aligned} \tag{3.15}$$

(3) Examining the Fisher information matrix, it is evident that one can normalize it with respect to β. This results in changing all $4\pi/\lambda$ to $4\pi/\lambda\beta$.

(4) The final results will be shown as degradation factors, i. , the Cramer-Rao bound with closely spaced targets normalized by the Cramer-Rao bound with completely resolved targets. This eliminates the explicit dependence on the signal bandwidth and wavelength, rather, the results can be evaluated using the product of wavelength and bandwidth $\lambda\beta$, as a parameter.

Generally, the signal bandwidth in frequency units (e.g., the Linear Frequency Modulated (LFM) bandwidth) is at most equal to 10% of the center frequency. Let this bandwidth be denoted by

$$\begin{aligned} B &= kf \\ &= k\frac{c}{\lambda} \end{aligned} \qquad \text{(Hz)} \tag{3.16}$$

where k is a positive constant less than or equal to .1, f is the center frequency, and c is the speed of light.

The above bandwidth is related to the root-mean-square bandwidth $\beta(m^{-1})$ by the following equation

$$\beta = \frac{4\pi B}{c\sqrt{12}} \tag{3.17}$$

$$= \frac{4\pi}{c\sqrt{12}} \tag{3.18}$$

Equivalently, one has

$$\frac{4\pi}{\lambda\beta} = \frac{2\sqrt{3}}{k} \tag{3.19}$$

For example, if one considers a 10% bandwidth, one uses $k = .1$. This results in

$$\frac{4\pi}{\lambda\beta} = 20\sqrt{3} \tag{3.20}$$

A 5% bandwidth case will have

$$\frac{4\pi}{\lambda\beta} = 40\sqrt{3} \tag{3.21}$$

The degradation factors for estimating the first target amplitude, location, and relative target separation for the equal target amplitude case with 5% bandwidth are shown in Figs. 3.1-3.3, respectively. The corresponding results for the 10% bandwidth case are shown respectively in Figs. 3.4-3.6. Notice that these curves show a general decrease with increasing target separation with a superimposed ripple. Using Eqs. (3.19)-(3.21) and these figures, it is clear that the ripple period for 5% bandwidth is exactly half of that for of 10% bandwidth. Also traced are the results for the random phase angle. It is evident that the coherent phase information is able to substantially reduce the estimation error.

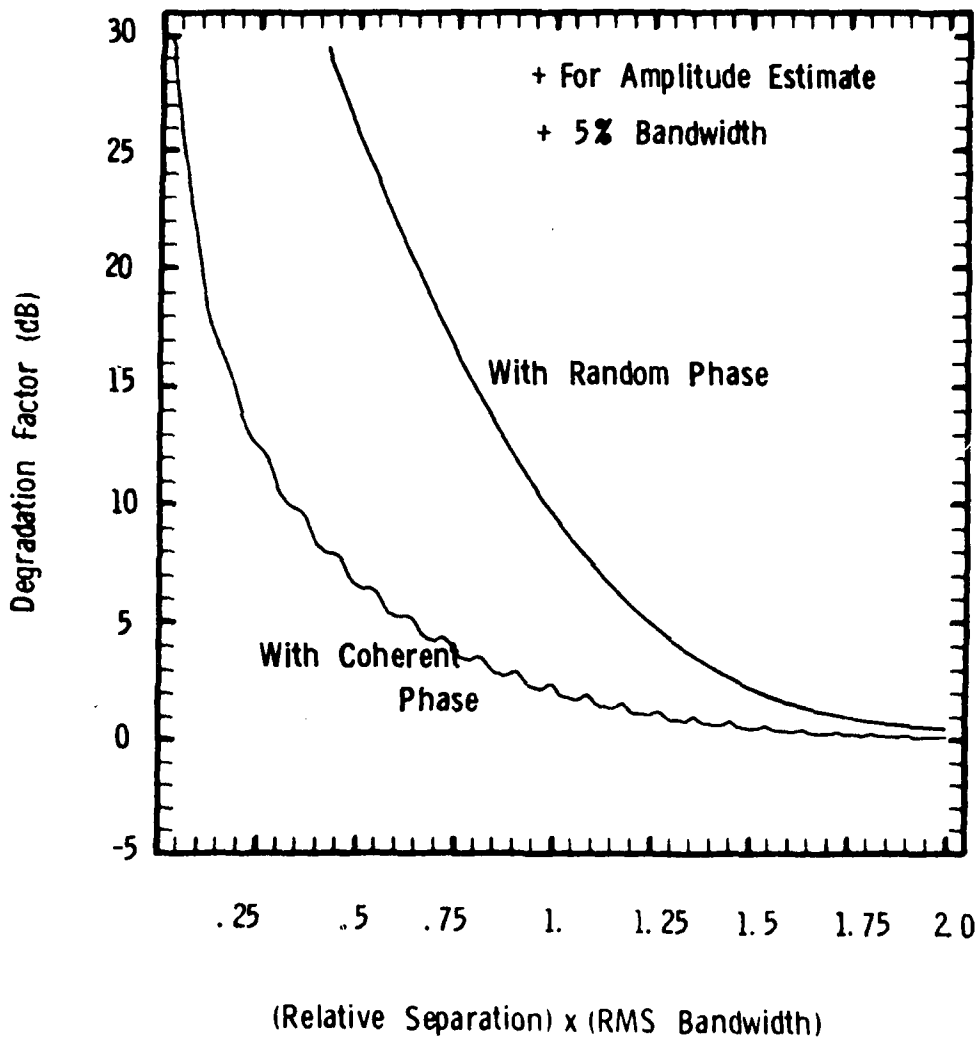

Fig. 3.1. Degradation factor for amplitude estimates, 5% bandwidth.

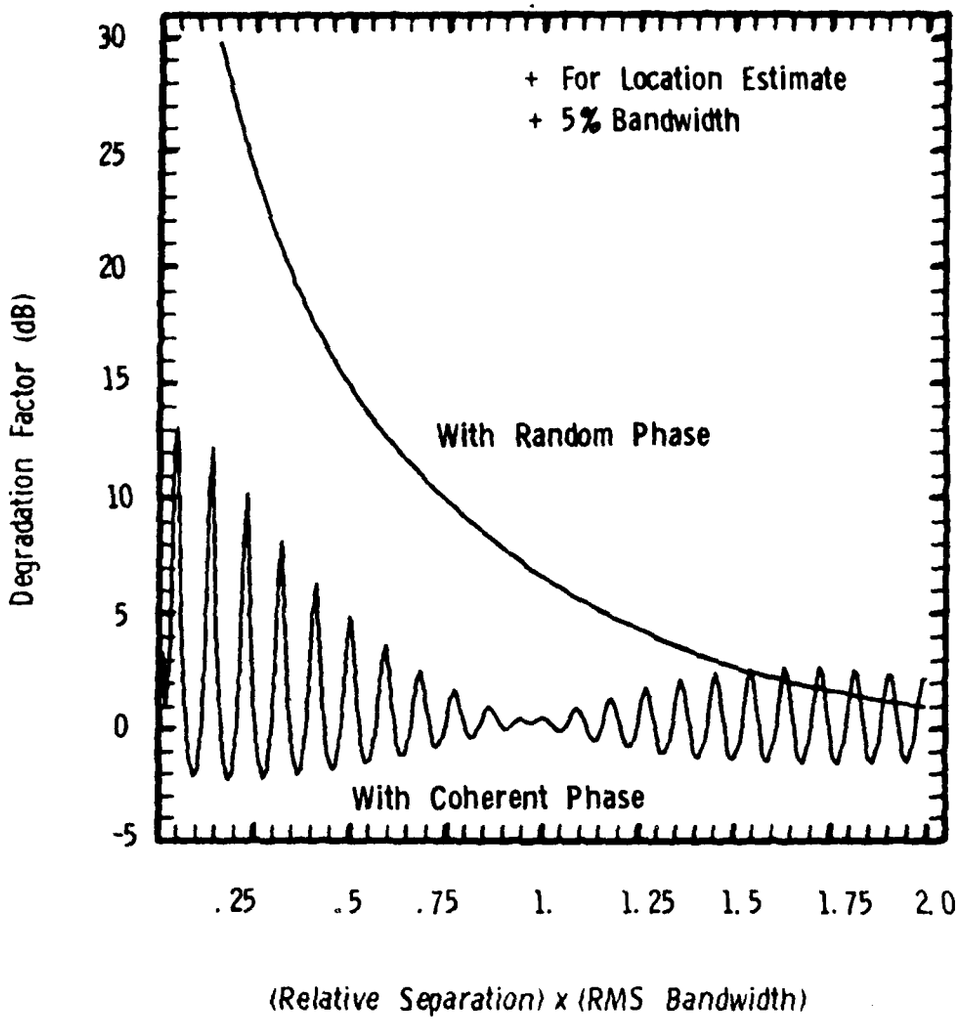

Fig. 3.2. Degradation factor for location estimates, 5% bandwidth.

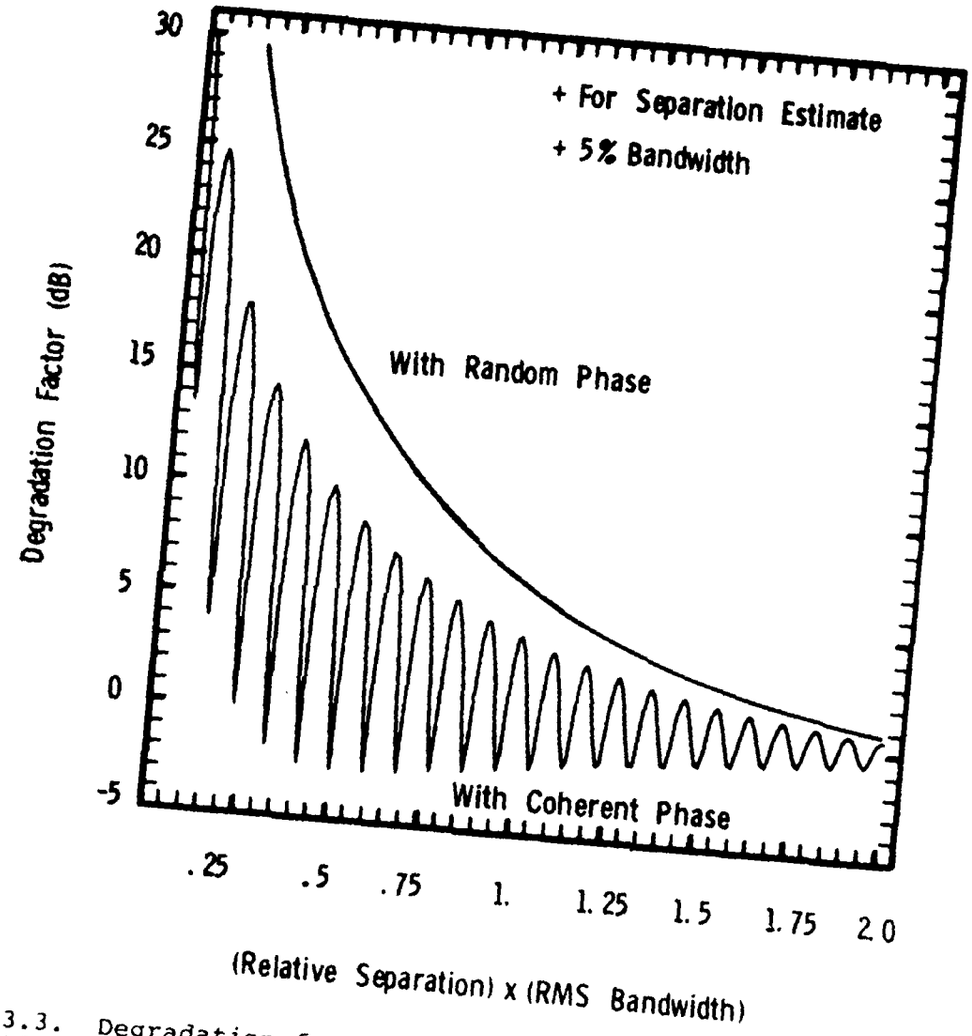

Fig. 3.3. Degradation factor for separation estimates, 5% bandwidth.

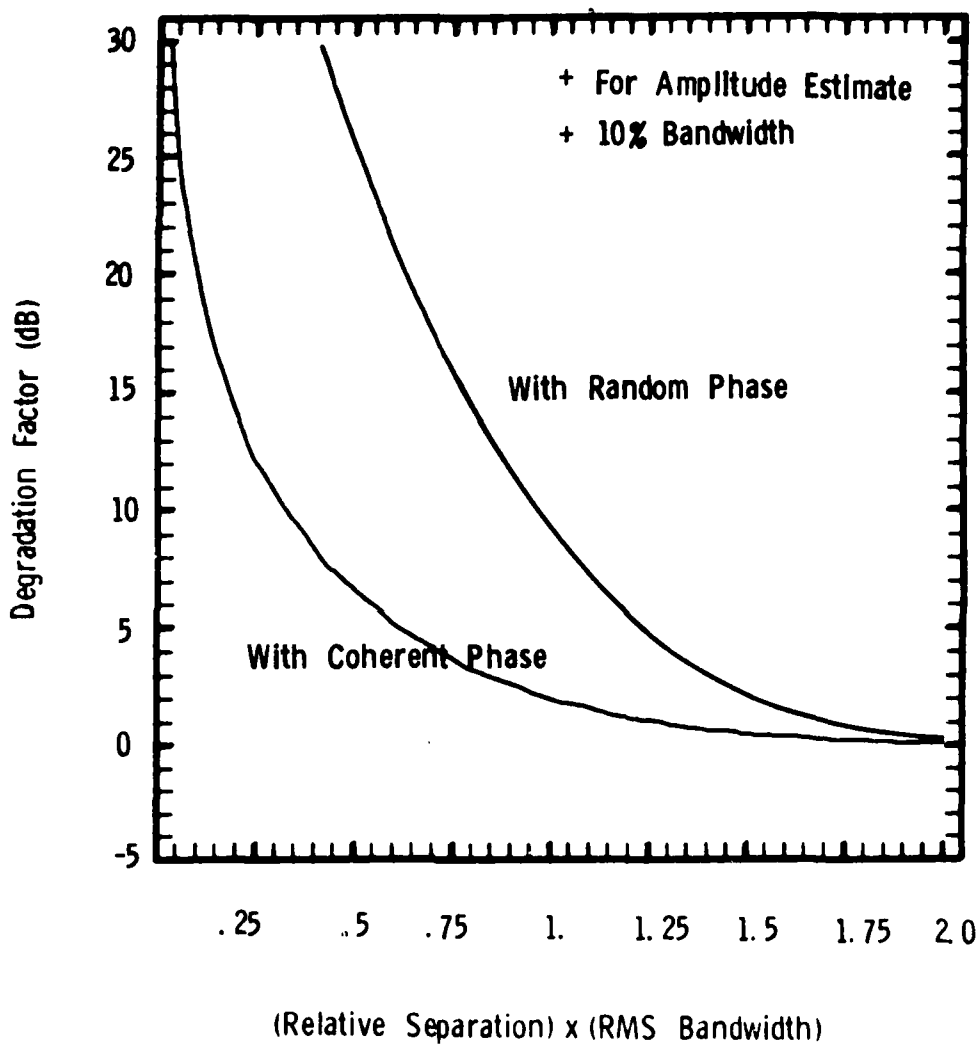

Fig. 3.4. Degradation factor for amplitude estimates, 10% bandwidth.

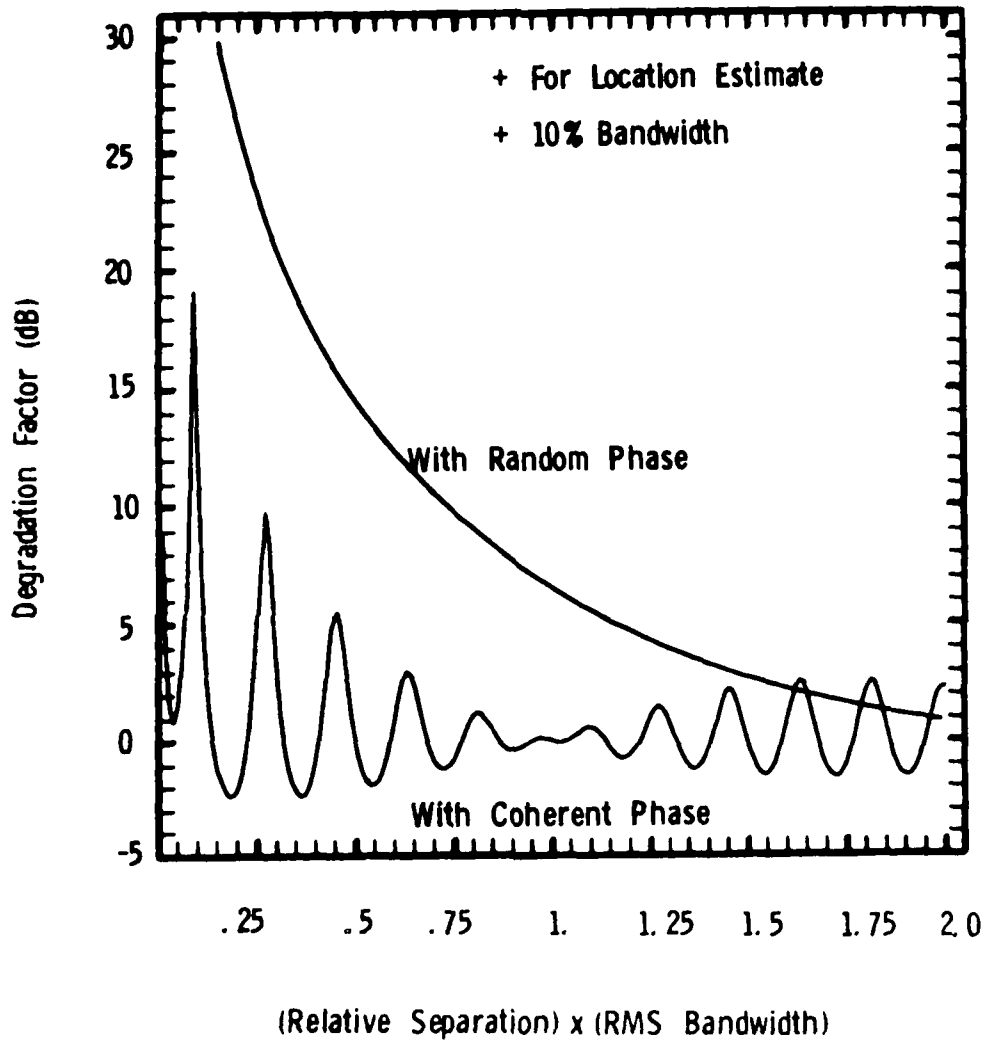

Fig. 3.5. Degradation factor for location estimates, 10% bandwidth.

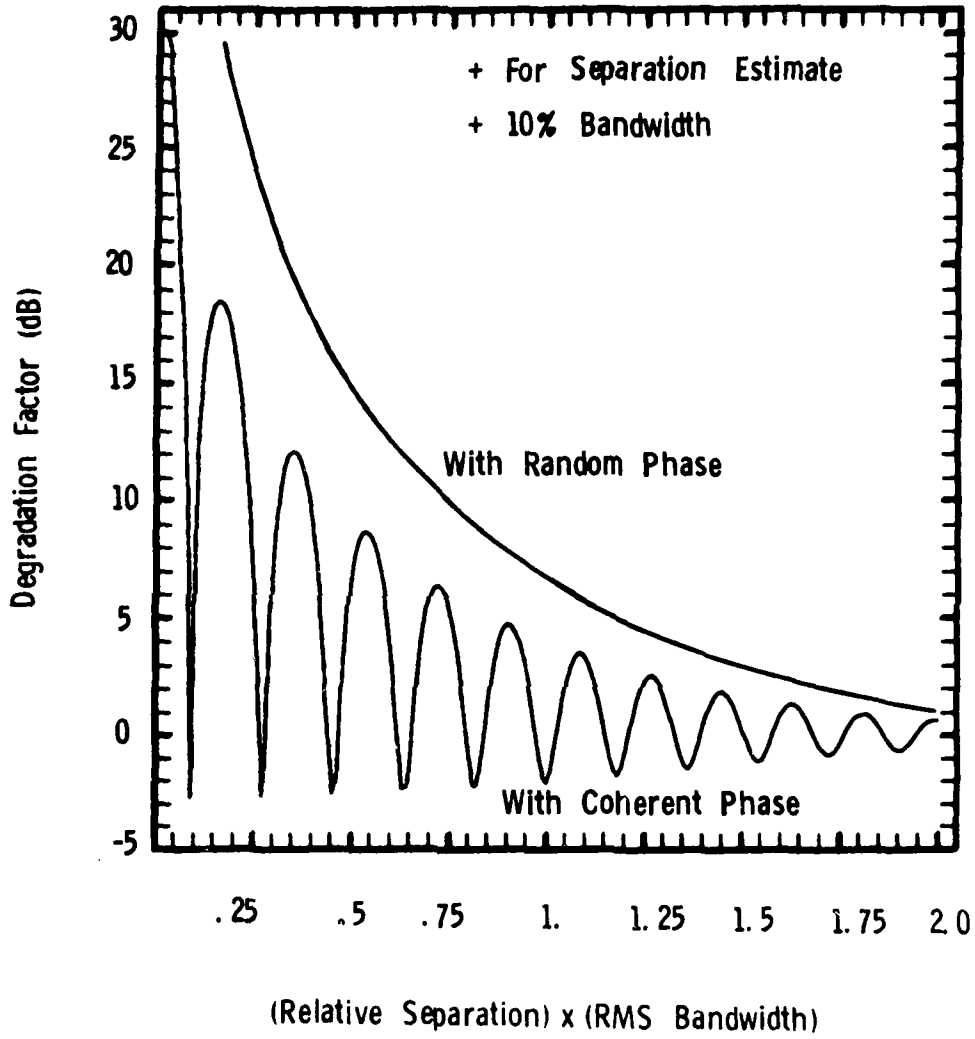

Fig. 3.6. Degradation factor for separation estimates, 10% bandwidth.

To get a feeling for the average estimation performance, we may treat the relative phase as a random variable uniformly distributed between 0 and 2π for the bound computation. In this case, we used the modified bound developed in [17] to take the expectation of the inverse of the Fisher information matrix with respect to $\Delta\Theta$. This step also eliminates the dependence on the percent bandwidth. The results for amplitude, location, and separation estimates are shown in Figs. 3.7-3.9, respectively.

There may be situations where even though the phase difference is coherent, the estimator does not make use of this knowledge. The Cramer-Rao bound for this situation is obtained by treating phase angles as unknown parameters (Ref. [10] or Appendix C) and not averaging over them. That is, the bound of Ref. [10] is evaluated for phase difference as a function of target separation. This step makes the bound depend upon the wavelength and bandwidth product. Results for 5% and 10% bandwidths are shown in Figs. 3.10-3.15. Notice that these results are extremely oscillatory. The 5% bandwidth case has a period exactly equal to half of 10% bandwidth. These results should be compared with the curves labelled "with random phase" of Figs. 3.1-3.6.

Comparing the above results, one concludes that using the information contained in the phase difference can substantially reduce the error of target amplitude and location estimation. The maximum likelihood estimator which asymptotically approaches these performance bounds is discussed in Appendix B.

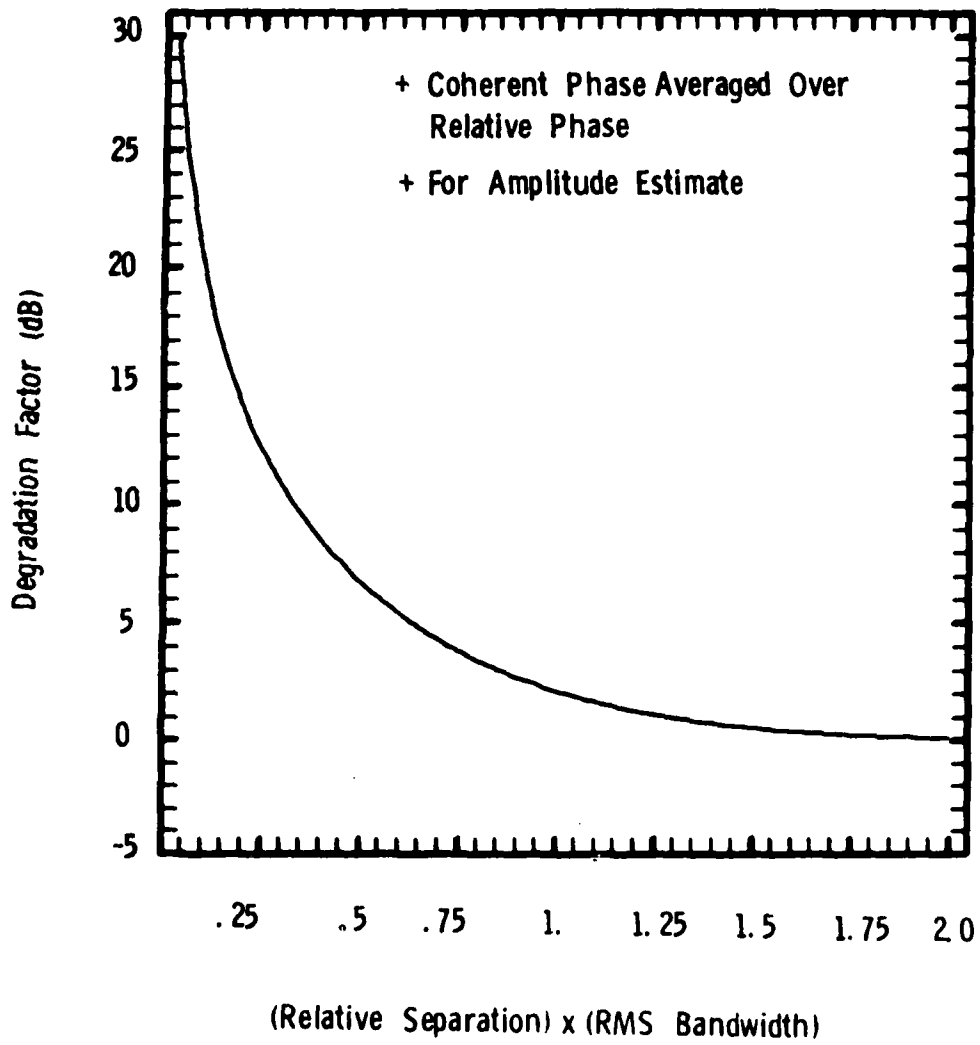

Fig. 3.7. Degradation factor for amplitude estimates, averaged over relative phase.

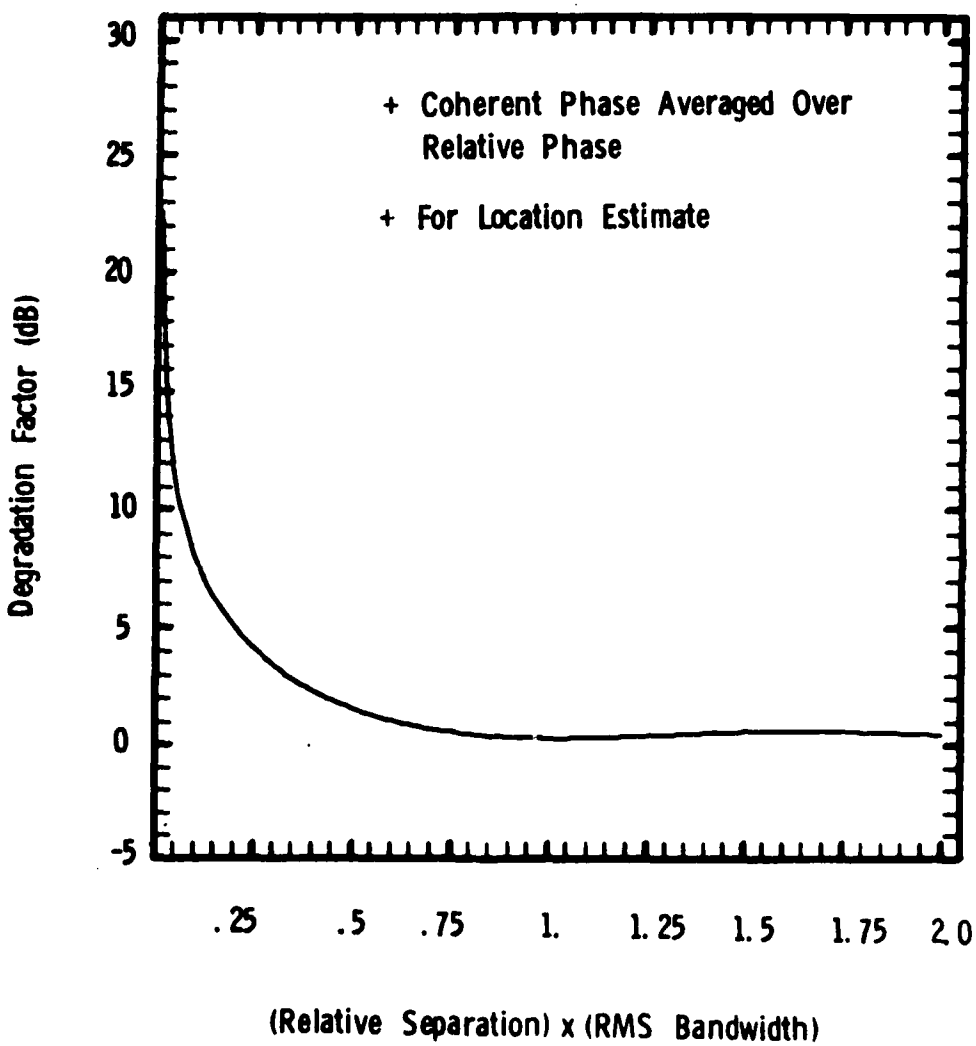

Fig. 3.8. Degradation factor for location estimates, averaged over relative phase.

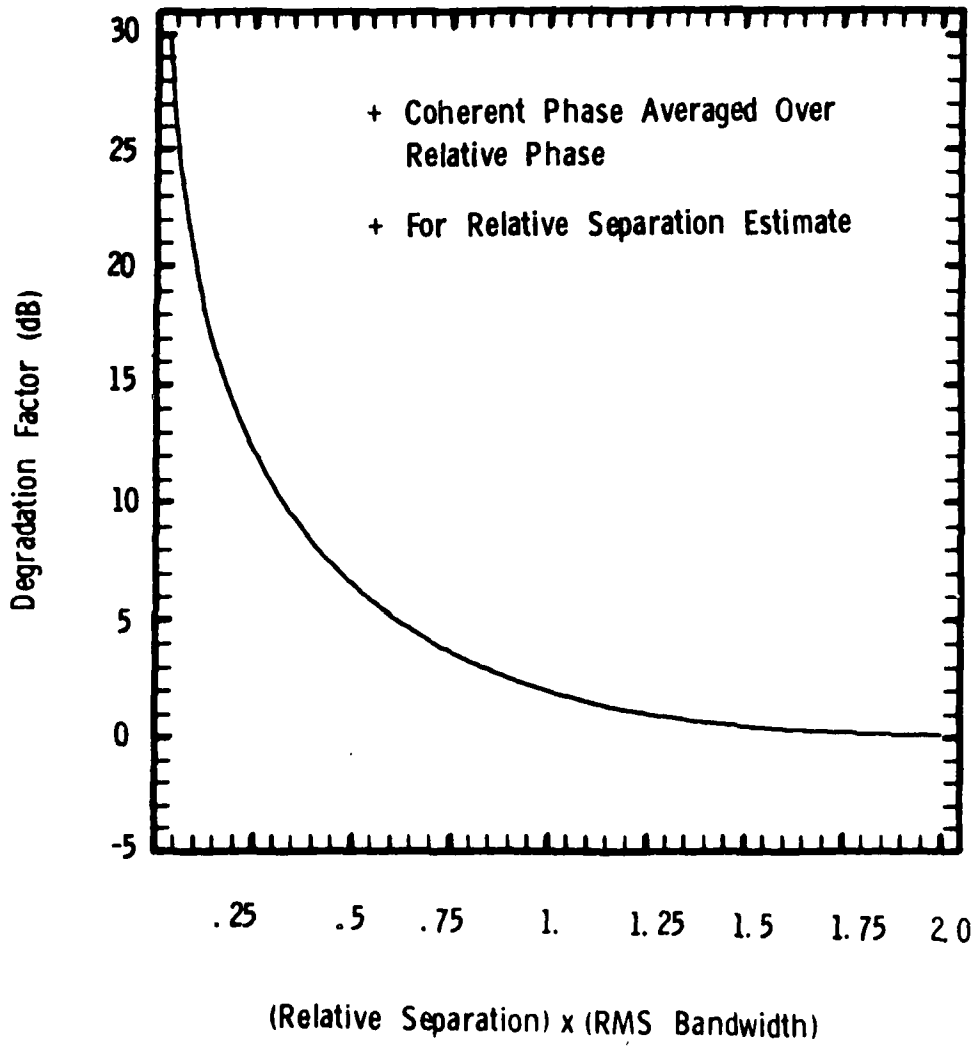

Fig. 3.9. Degradation factor for separation estimates, averaged over relative phase.

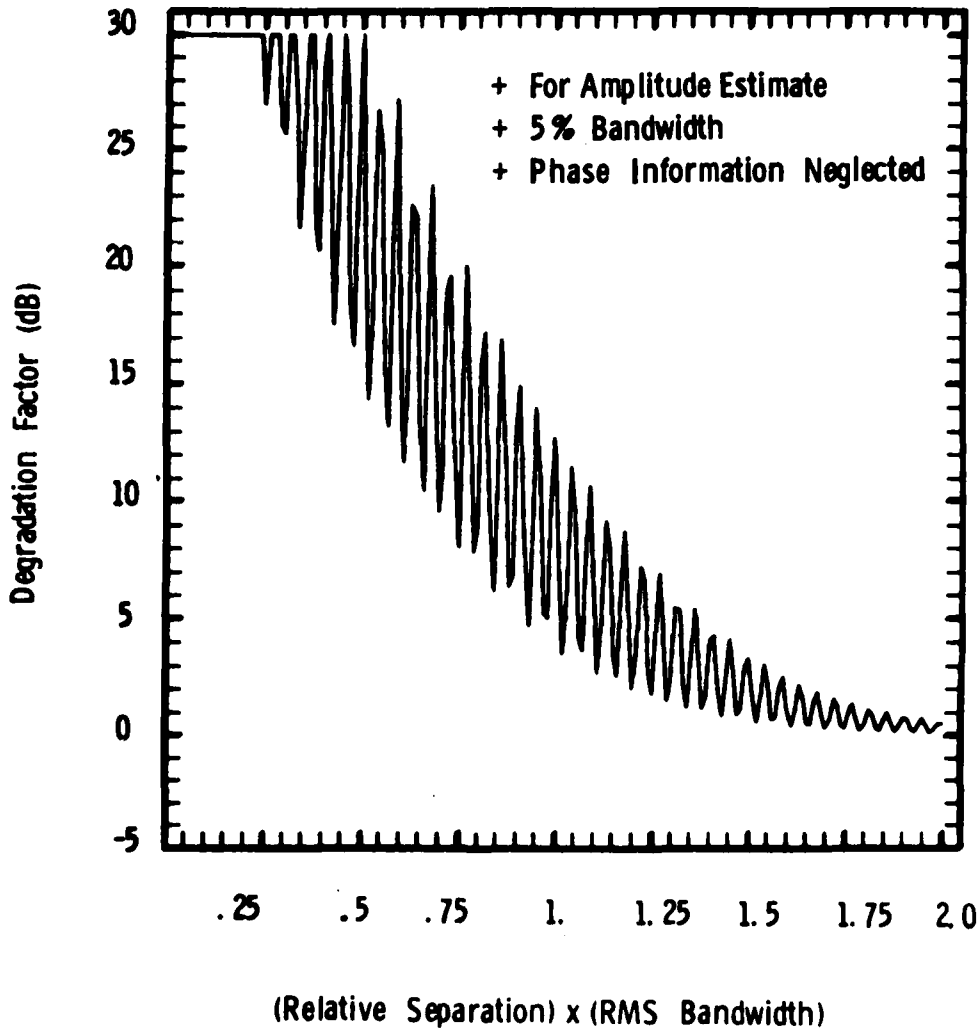

Fig. 3.10. Degradation factor for amplitude estimates by neglecting phase information, 5% bandwidth.

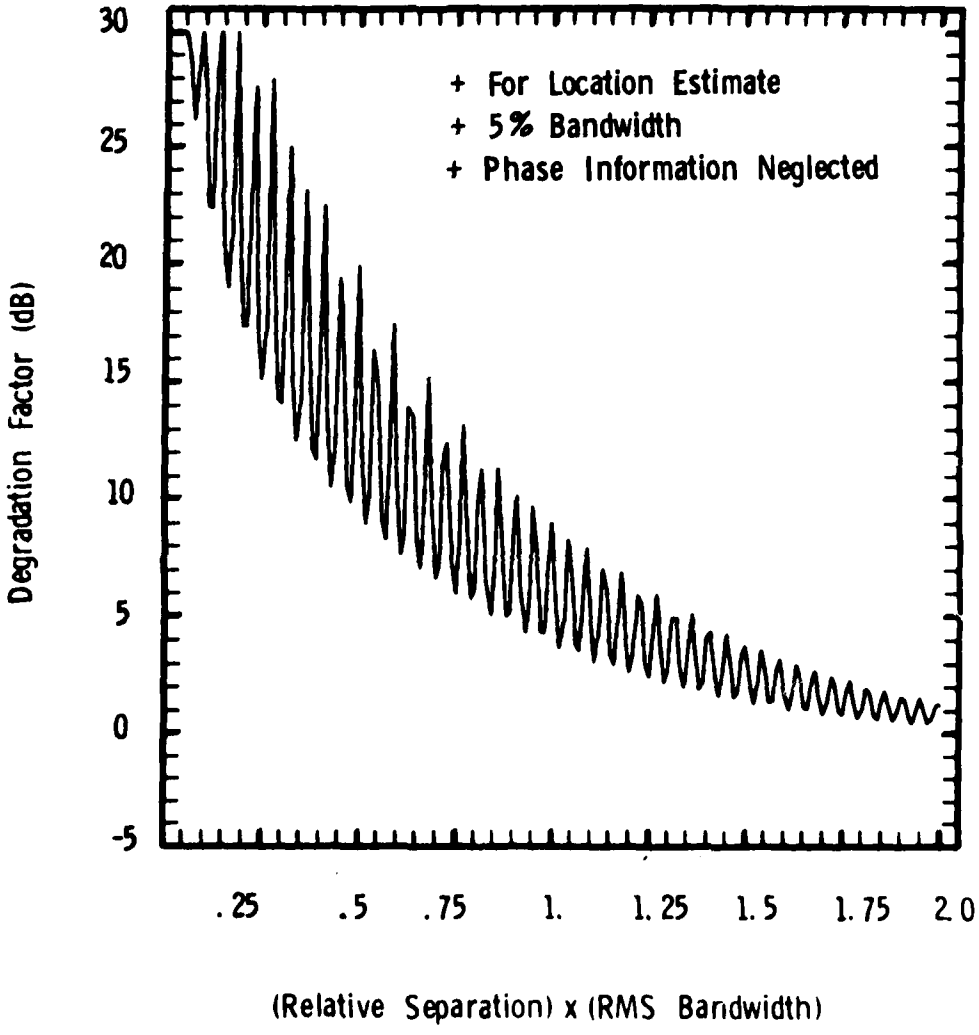

Fig. 3.11. Degradation factor for location estimates by neglecting phase information, 5% bandwidth.

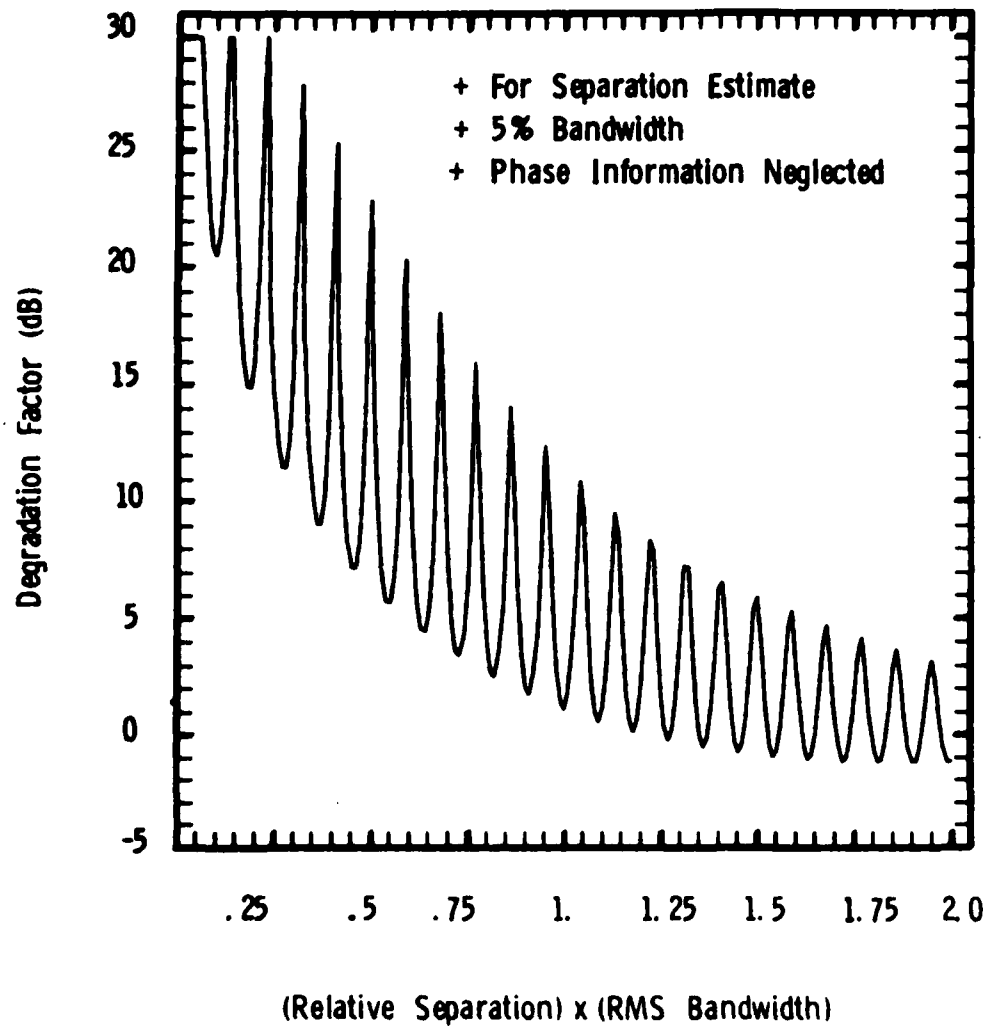

Fig. 3.12. Degradation factor for separation estimates by neglecting phase information, 5% bandwidth.

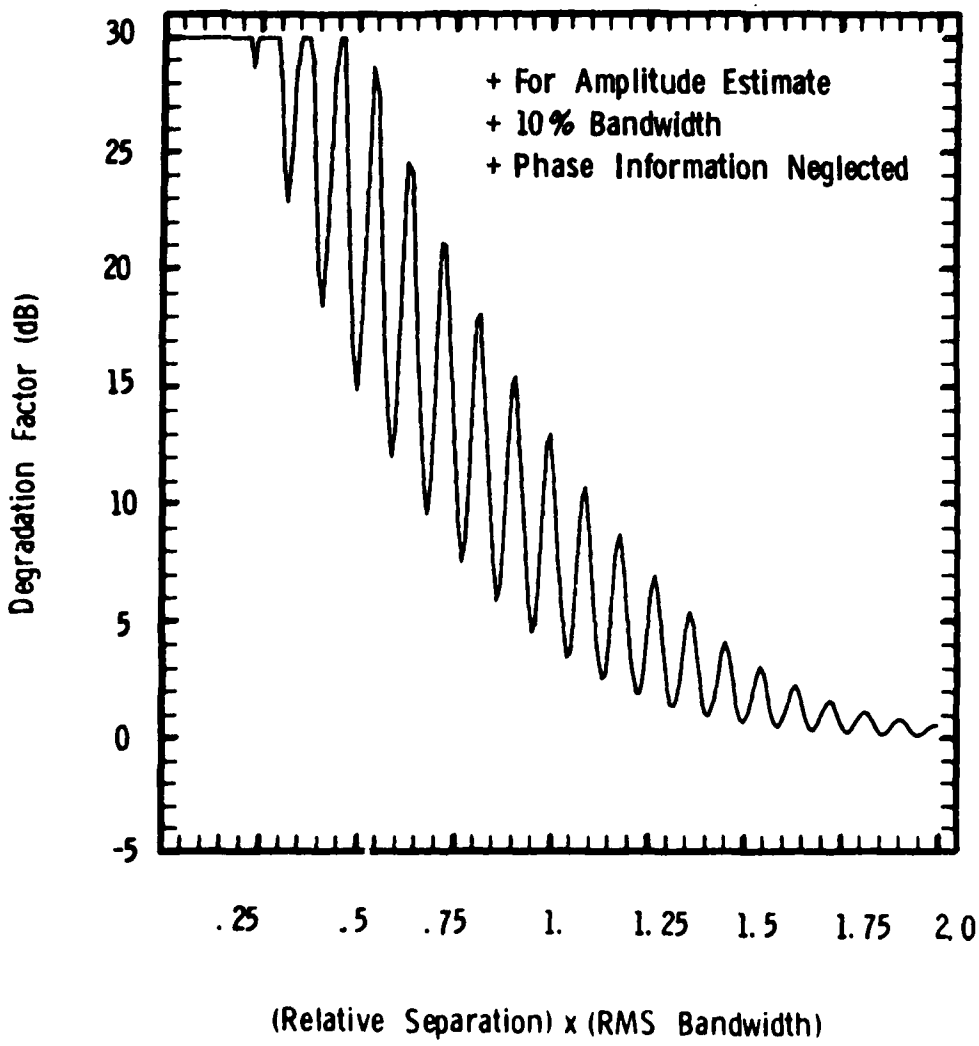

Fig. 3.13. Degradation factor for amplitude estimates by neglecting phase information, 10% bandwidth.

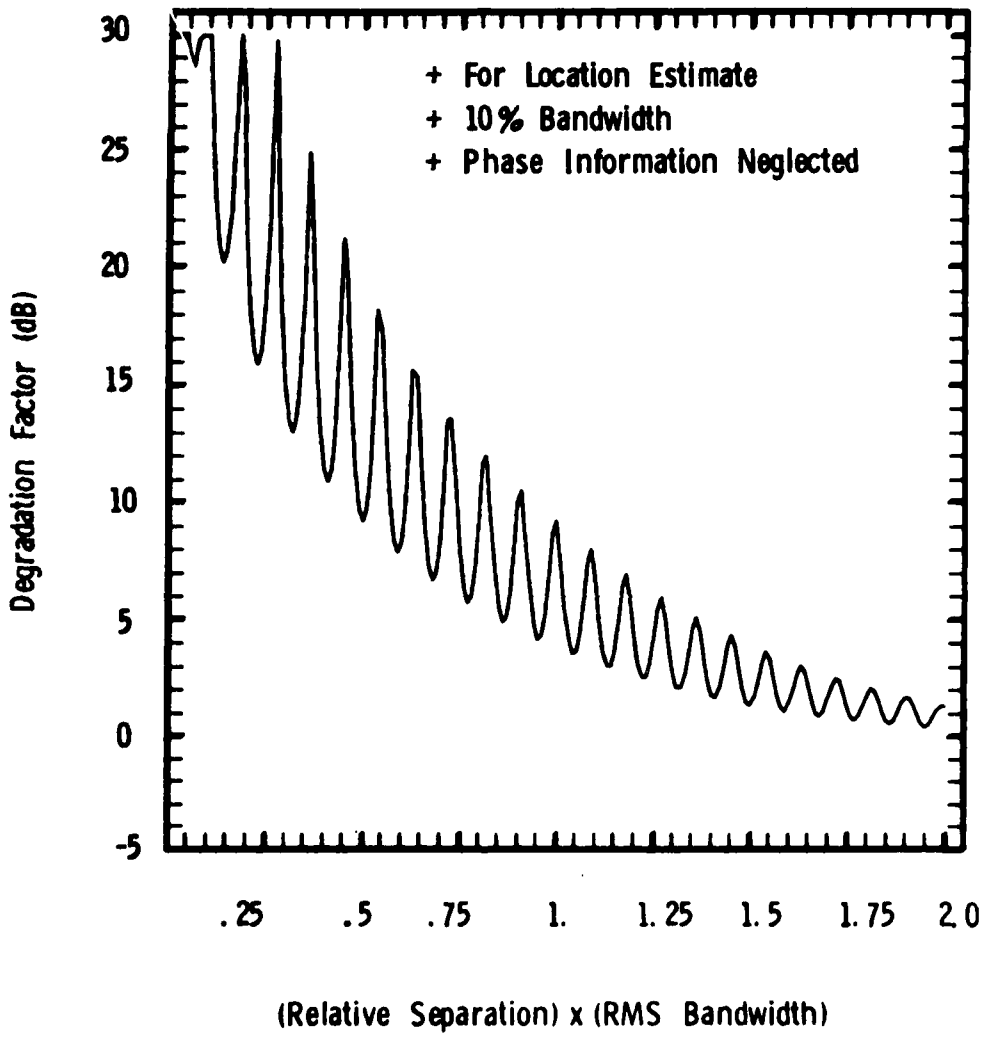

Fig. 3.14. Degradation factor for location estimates by neglecting phase information, 10% bandwidth.

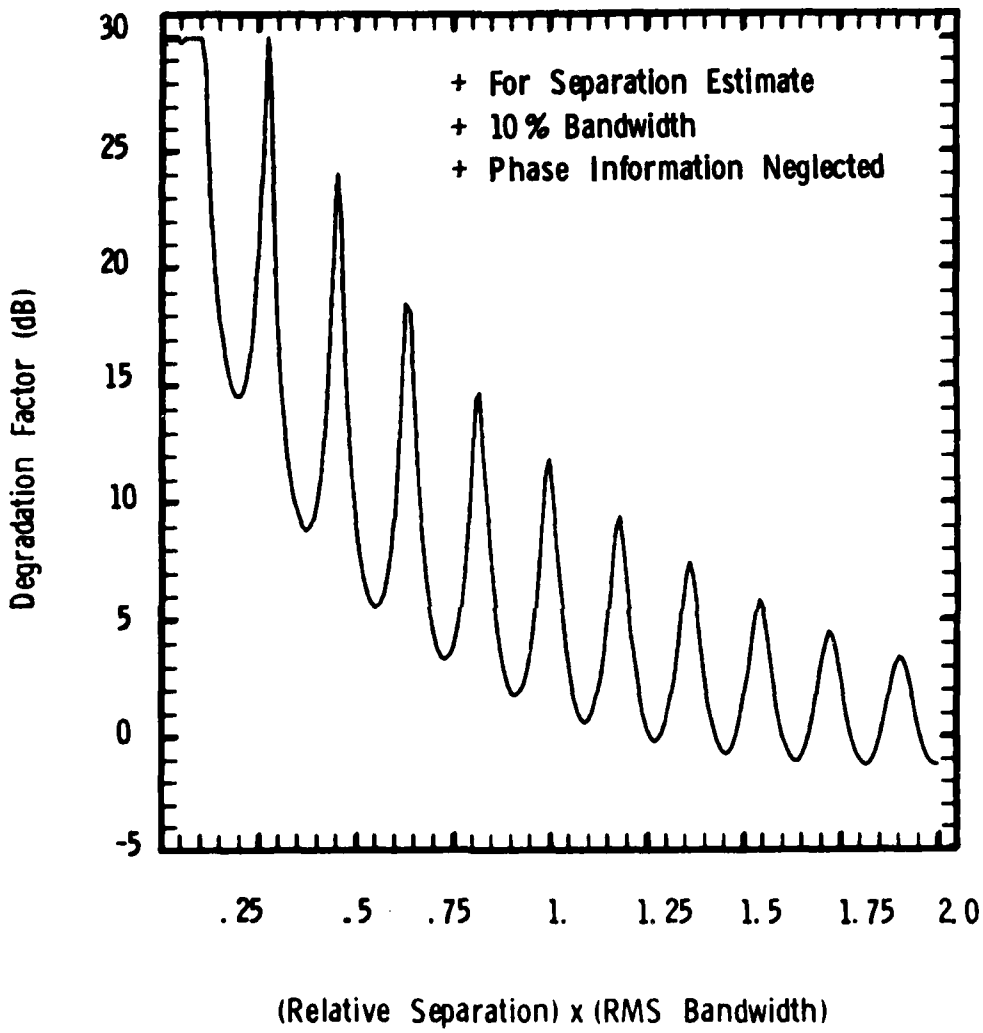

Fig. 3.15. Degradation factor for separation estimates by neglecting phase information, 10% bandwidth.

4. SUMMARY

In this report, we have discussed the accuracy of parameter estimation of radar signals closely spaced in range using the Cramer-Rao bound analysis. Specifically, we extended the result of [10] to include the case when the phase angle from successive signals are coherent. Use of this additional information is shown to substantially reduce the estimation error.

A comparison of the performance of these estimators measured by the target separation required to achieve a 3 dB degradation is summarized in Table 4.1. Notice that because the degradation curves are oscillatory, nominal or a range of values is given for some cases.

Although there are no simulation results presented in this report, it is known that the maximum likelihood estimator can achieve these performance bounds when the signal-to-noise ratio is high, [10], [15], [16]. For this reason, the maximum likelihood estimator equations for the radar signal resolution problem are presented in Appendix B.

TABLE 4.1

SUMMARY OF PERFORMANCE COMPARISON

Parameter / Phase Relation	Normalized* Target Separation for 3 dB Degradation		
	Amplitude	Location	Separation
Coherent	0.9	~0.35	~0.8
Random	1.45	1.45	1.45
Coherent but Ignored	1.15-1.65	1.2-1.7	0.8-2.0

*Separation is in units of 1/RMS Bandwidth or equivalently 0.55 x nominal range resolution (C/2B). Rayleigh criterion corresponds to normalized separation of 1.8.

APPENDIX A:

The Cramer-Rao Bound With Multiple Pulses

The results of this report are based upon measurement with a single radar pulse. In this Appendix, we extend the Cramer-Rao bound formulation to include multiple pulses. Let $\{r_m(t),\ 0 \le t \le T\}$ denote the m-th received pulse and assume that r_m and $r_{m'}$ are independent for $m \neq m'$, then one has the following log likelihood function

$$\ell n\Lambda = \frac{-1}{N_o} \sum_{m=1}^{M} \int \left| r_m(t) - \sum_{i=1}^{N} a_i s(t-\hat{\tau}_i) \right|^2 dt$$

$$+ \frac{1}{N_o} \sum_{m=1}^{M} \int \left| r_m(t) \right|^2 dt \qquad \text{(A.1)}$$

Let

$$\underline{\gamma}_m = \begin{bmatrix} \int r_m^* s(t-\hat{\tau}_1)dt \\ \vdots \\ \int r_m^* s(t-\hat{\tau}_N)dt \end{bmatrix}$$

one has

$$J = N_o \ell n\Lambda = \underline{a}^T \left(\sum_{m=1}^{M} \underline{\gamma}_m \right) + \underline{a}^{*T} \left(\sum_{m=1}^{M} \underline{\gamma}_m^* \right) - M\underline{a}^T P \underline{a}^* \qquad \text{(A.2)}$$

Let $\underline{u} = \sum_{m=1}^{M} \underline{\gamma}_m$ then the ith component of $\underline{u}$ is

$$u_i = M \int s(t-\hat{\tau}_i) \left[\sum_{j=1}^{N} a_j^* s(t-\hat{\tau}_j) \right] dt + \sum_{m=1}^{M} \int s(t-\hat{\tau}_i) n_m^*(t) dt = M q_i \tag{A.3}$$

where $q_i = \int s(t-\hat{\tau}_i) \left[\sum_{j=1}^{N} a_j^* s(t-\hat{\tau}_j) \right] dt + \frac{1}{M} \sum_{m=1}^{M} \int s(t-\hat{\tau}_i) n_m^*(t) dt$

Notice that the q_i above is similar to the γ_i defined in (2.2). Their first terms are the same and their second terms have the same variance $N_o/2$.

Using the above notation in (A.2) yields

$$J = N_o \ell n \Lambda = M[\underline{a}^T \underline{q} + \underline{a}^{*T} \underline{q}^* - \underline{a}^T P \underline{a}^*] \tag{A.4}$$

From the above derivation it is evident that the Cramer-Rao bound for measurements containing m pulses is equal to the Cramer-Rao bound for a single pulse divided by M.

We note that the above derivation uses the assumption that all pulses can be aligned properly for the required summation. It also assumes that the relative position and orientation of all scatterers are preserved for M pulses.

APPENDIX B:

The Maximum Likelihood Estimator

The maximum likelihood estimates are those parameter values maximizing the log likelihood function. The log likelihood function for our case is

$$\ell n\Lambda = \frac{1}{N_o}(\underline{a}^T\underline{\gamma} + \underline{a}^{*T}\underline{\gamma}^* - \underline{a}^T P\underline{a}^*) \tag{B.1}$$

In the following, we will discuss the cases of targets with random phase and coherent phase individually. In the last subsection, we will extend our results to the case of multiple measurements.

B.1. Targets with Random Phase

In this case, one first maximizes (B.1) with respect to $\underline{a}$ this yields the estimate of $\underline{a}$

$$\hat{\underline{a}} = P^{-1}\underline{\gamma}^* \tag{B.2}$$

Substituting $\hat{\underline{a}}$ into (B.1) yields

$$\ell n\hat{\Lambda} = \frac{1}{N_o}\underline{\gamma}^T P^{-1}\underline{\gamma}^* \tag{B.3}$$

The $\underline{\tau}$ estimate is therefore the vector $\hat{\underline{\tau}}$ which maximizes (B.2). For the two target case, the above expression becomes

$$\ell n\hat{\Lambda} = \frac{[\gamma_1\gamma_1^*+\gamma_2\gamma_2^*-\rho(\hat{\tau}_2-\hat{\tau}_1)(\gamma_1\gamma_2^*+\gamma_2\gamma_1^*)]}{N_o(1-\rho^2(\hat{\tau}_2-\hat{\tau}_1))} \tag{B.4}$$

The estimates $(\hat{\tau}_1,\hat{\tau}_2)$ are obtained by searching through the set

of two-target (after matched filtering) time samples which maximizes the above expression. The estimator structure is illustrated in Fig. B.1.

The above algorithm assumes that the received signal is analog. If only sampled data is available, one may first use the available samples to find an approximate solution and refine this solution by interpolation. One may also devise algorithm working directly with the sampled data. One such algorithm is defined in Ref. [14]. We will not discuss this further here.

B.2. Targets with Coherent Phase

In this case, one does not have to estimate all complex target amplitudes, but, rather the first phase angle and the magnitude of all target amplitudes.

Let

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{i\Theta_{2,1}} & 0 & \cdots & 0 \\ 0 & 0 & e^{i\Theta_{3,1}} & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & e^{i\Theta_{N,1}} \end{bmatrix}$$

where $\Theta_{j,1} = \Theta_j - \Theta_1 = \frac{4\pi}{\lambda}(\tau_j - \tau_1)$. Then $\underline{a} = e^{i\Theta_1}\, \underline{\Phi}\, \underline{\alpha}$

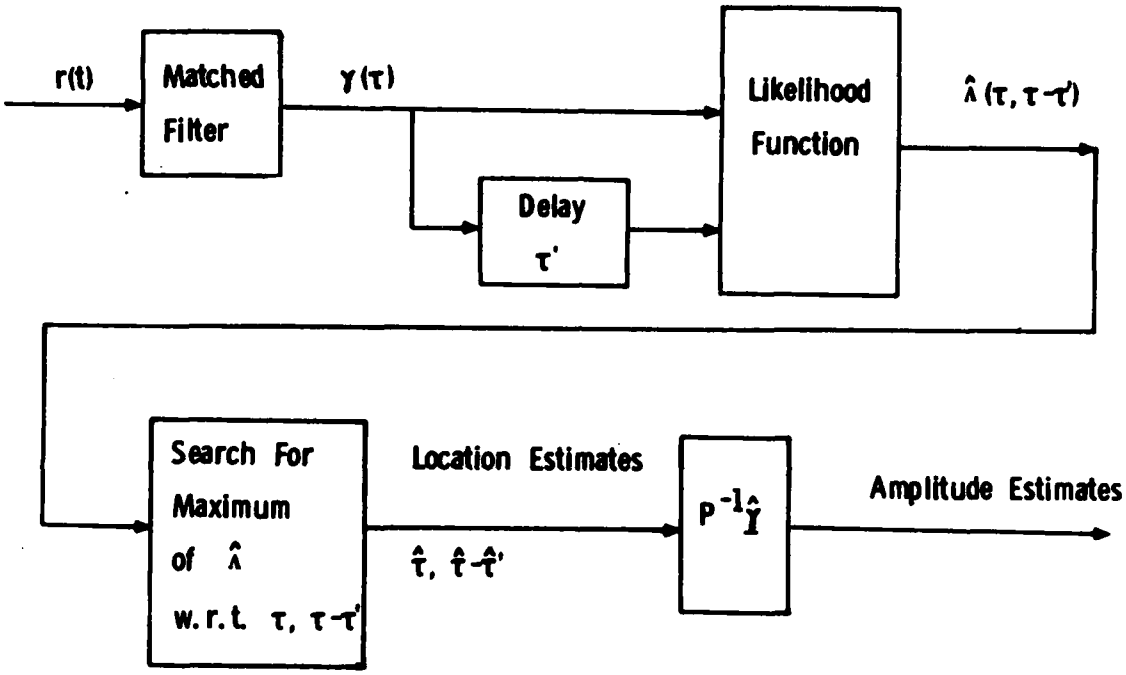

Fig. B.1. Estimator structure.

The log likelihood ratio becomes

$$J = e^{i\Theta_1}\underline{\alpha}^T\Phi\underline{\gamma} + e^{-i\Theta_1}\underline{\alpha}^T\Phi^*\underline{\gamma}^* - \underline{\alpha}^T\Phi P\Phi^*\underline{\alpha} \tag{B.5}$$

One may maximize (B.5) with respect to $\underline{\alpha}$, this yields

$$\hat{\underline{\alpha}} = [\Phi P\Phi^* + \Phi^* P\Phi]^{-1}[e^{i\Theta_1}\Phi\underline{\gamma} + e^{-i\Theta_1}\Phi^*\underline{\gamma}^*] \tag{B.6}$$

Let

$$\Sigma = \Phi P\Phi^* + \Phi^* P\Phi \tag{B.7}$$

Substituting (B.6) into (B.5) and using (B.7), one has

$$\hat{J} = [\underline{\gamma}^T\Phi e^{i\Theta_1} + \underline{\gamma}^{*T}\Phi^* e^{-i\Theta_1}][\Sigma^{-1} - \Sigma^{-1}\Phi P\Phi^*\Sigma^{-1}] \tag{B.8}$$

$$\cdot\ [\Phi\underline{\gamma}e^{i\Theta_1} + \Phi^*\underline{\gamma}^* e^{-i\Theta_1}]$$

The maximum likelihood estimates of $\Theta_1, \tau_1, \ldots, \tau_N$ are those values which maximize (B.8).

For the two-target case, one obtains the following expression after some tedious manipulations.

$$\hat{J} = \frac{[d_1^2 + d_2^2 - 2d_1 d_2 \rho(\hat{\tau}_2 - \hat{\tau}_1)\cos\Delta\Theta]}{N_o(1-\rho^2(\hat{\tau}_2-\hat{\tau}_1)\cos\Delta\Theta)} \tag{B.9}$$

where $\Delta\Theta = \frac{4\pi}{\lambda}(\hat{\tau}_2 - \hat{\tau}_1)$

$d_1 = |\gamma_1|\cos(\hat{\Theta}_1 + \delta_1)$

$d_2 = |\gamma_2|\cos(\hat{\Theta}_1 + \Delta\Theta + \delta_2)$

δ_i = phase angle of γ_i

The estimator structure for the above problem is similar to that of the previous subsection. The difference is that one has to search for three parameters Θ_1, τ_1, and τ_2 instead of two.

B.3. With Multiple Pulses

Similar to the case for the Cramer-Rao bound, the log likelihood ratio becomes

$$J = \underline{a}^T\left(\sum_{m=1}^{M} \underline{Y}_m\right) + \underline{a}^{*T}\left(\sum_{m=1}^{M} \underline{Y}_m^*\right) - M\cdot\underline{a}^T P\underline{a}^* \qquad \text{(B.10)}$$

Maximizing J with repect to $\underline{a}$ yields

$$\hat{\underline{a}} = P^{-1}\left(\frac{1}{M}\sum_{m=1}^{M} \underline{Y}_m\right) \qquad \text{(B.11)}$$

Substituting (B.11) into (B.10) yields

$$\hat{J} = \frac{1}{M}\left(\sum_{m=1}^{M} \underline{Y}_m^T\right) P^{-1}\left(\sum_{m=1}^{M} \underline{Y}_m\right) \qquad \text{(B.12)}$$

The optimal estimate $\hat{\underline{\tau}}$ is obtained by maximizing $\hat{J}$ with respect to $\underline{\tau}$.

Notice that the above estimator first requires a summation of all pulses. This assumes that all pulses can be aligned properly. This assumption may not be true in general due to the performance limit of the realtime tracking algorithm. One alternative is to process each pulse individually then average the results (e.g., for the target separation estimate).

This procedure does not maximize the log likelihood function, but if the signal-to-noise ratio is high and the maximum likelihood estimator is unbiased, then this alternate procedure should achieve near optimum performance.

APPENDIX C:

The Fisher Information Matrix for a Two-Target Model With Random Phase Angles

For the purpose of comparison, we state the Fisher information matrix for a two-target model with random phase angles in this appendix. This case was first discussed in Ref. [10].

Using (2.3) and (3.1) one has

$$J = N_o \ln\Lambda = \alpha_1\gamma_1 e^{i\Theta_1} + \alpha_2\gamma_2 e^{i\Theta_2} + \alpha_1\gamma_1^* e^{-i\Theta_1} + \alpha_2\gamma_2^* e^{-i\Theta_2} - \alpha_1^2 - \alpha_2^2 - 2\alpha_1\alpha_2 \cos(\Theta_2-\Theta_1)\rho(\tau_2-\tau_1) \tag{C.1}$$

The Fisher information matrix has the following terms.

$$-E\left[\frac{\delta^2 J}{\delta \underline{\alpha}^2}\right] = 2\begin{bmatrix} 1 & \rho\cos\Delta\Theta \\ \rho\cos\Delta\Theta & 1 \end{bmatrix}$$

$$-E\left[\frac{\partial^2 J}{\partial \underline{\tau}^2}\right] = 2\begin{bmatrix} \alpha_1^2\beta^2 & -\alpha_1\alpha_2\ddot{\rho}\cos\Delta\Theta \\ -\alpha_1\alpha_2\ddot{\rho}\cos\Delta\Theta & \alpha_2^2\beta^2 \end{bmatrix}$$

$$-E\left[\frac{\partial^2 J}{\partial \underline{\Theta}^2}\right] = 2\begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2\cos\Delta\Theta\ \rho(\tau) \\ \alpha_1\alpha_2\cos\Delta\Theta\ \rho(\tau) & \alpha_2^2 \end{bmatrix}$$

$$-E\left[\frac{\partial^2 J}{\partial \underline{\alpha}\partial \underline{\tau}}\right] = 2\begin{bmatrix} 0 & \alpha_2\cos\Delta\Theta\ \dot{\rho}(\tau) \\ -\alpha_1\cos\Delta\Theta\ \dot{\rho} & 0 \end{bmatrix}$$

$$-E\left[\frac{\partial^2 J}{\partial \underline{\alpha}\partial \underline{\Theta}}\right] = 2\begin{bmatrix} 0 & -\alpha_2 \sin \Delta\Theta \ \rho(\tau) \\ \alpha_1 \sin \Delta\Theta \ \rho(\tau) & 0 \end{bmatrix}$$

$$-E\left[\frac{\partial^2 J}{\partial \underline{\tau}\partial \underline{\Theta}}\right] = 2\begin{bmatrix} 0 & \alpha_1\alpha_2 \sin \Delta\Theta \ \dot{\rho}(\tau) \\ \alpha_1\alpha_2 \sin \Delta\Theta \ \dot{\rho}(\tau) & 0 \end{bmatrix}$$

Notice that if two targets are completely resolved so that $\rho(\tau) = \dot{\rho}(\tau) = 0$, then the above matrix becomes a diagonal matrix and one obtains the following familiar equations

$$\sigma_{\Theta_i}^2 \geq \frac{1}{\frac{2\alpha_i^2}{N_o}} = \frac{1}{SNR_i} \tag{C.2}$$

$$\sigma_{\tau_i}^2 \geq \frac{1}{\frac{2\alpha_i^2\beta^2}{N_o}} = \frac{1}{SNR_i^2\beta^2} \tag{C.3}$$

$$\sigma_{\alpha_i}^2 \geq \frac{1}{\frac{2}{N_o}} = \frac{\alpha_i^2}{SNR_i} \tag{C.4}$$

These equations hold for an arbitrary number of targets. The variance on the estimate of relative delay $\tau=\tau_2-\tau_1$ has the following lower bound

$$\sigma_\tau^2 \geq \frac{1}{\beta^2}\left[\frac{1}{SNR_1} + \frac{1}{SNR_2}\right] \tag{C.5}$$

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In this report, we discuss the amplitude, location, and the relative separation estimation accuracy of radar targets closely spaced in range using the Cramer-Rao bound. It is assumed that the phases of successive signals are coherent and therefore contain relative line-of-sight location information. It is shown that this information can substantially reduce the estimation error when compared with the case where the relative signal phase is random.

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