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THREE DIMENSIONAL STRESS FIELDS IN CRACKED PLATES.(U)  
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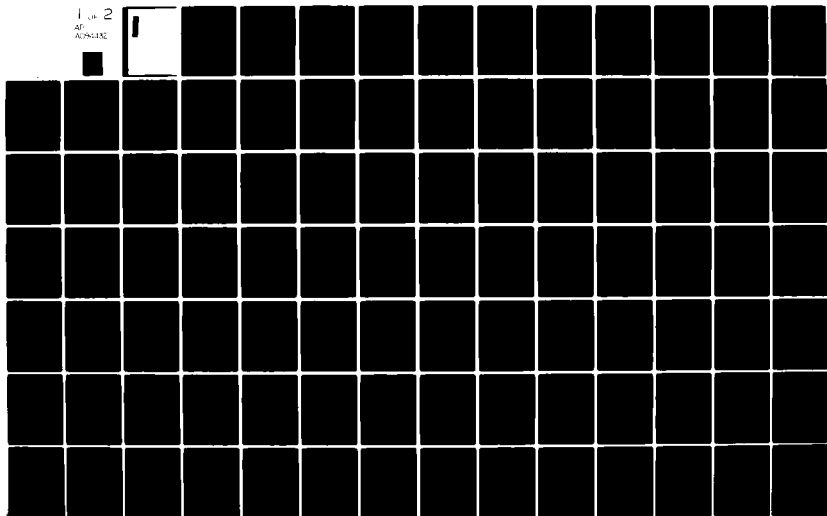
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A theoretical investigation was made for the determination of the three dimensional stress field of a cracked plate, of an arbitrary thickness, 2h, and subjected to a uniform external load of mode I. The displacement and stress fields are expressed in terms of the displacement V projected onto the plane containing the crack. In addition, the question of uniqueness is examined for a whole class of these three-dimensional crack problems. It is found that solutions to such problems in elastostatics are unique, provided they satisfy the condition of local finite energy everywhere. Finally, it is shown that			

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the solution is complete and it appears that at the corner, where the crack front meets the free surface of the plate, the solution is not separable either in spherical or cylindrical coordinates.

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THREE DIMENSIONAL STRESS  
FIELDS IN CRACKED PLATES

by

E. S. Folias  
November 1980

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PART I

THROUGH THE THICKNESS CRACK

## 1. Introduction.

In the field of Fracture Mechanics not very much theoretical work has been done in order to assess analytically the three-dimensional stress character which prevails at the base of a stationary crack. As a result, most of our current design criteria are based on already existing two-dimensional solutions and therefore are in general inadequate. For example, the common experimental observation of a change from ductile failure at the edge to brittle fracture at the center of a broken sheet material has so far defied analysis. Yet an orderly theoretical attack on the problem can provide important guidance to this and other phases of fracture research.

The mathematical difficulties, however, posed by three-dimensional fracture problems are substantially greater than those associated with plane stress or plane strain. Be that as it may, the author would like to investigate the subject further at least within the theory of linear elasticity. While he recognizes the fact that this theory cannot include the nonelastic behavior of the material at the crack tip per se, it can evince many characteristics of the actual behavior of a cracked plate, including those due to thickness. Thus the theory of elasticity is a logical fountainhead for detailed theoretical study.



## 2. Historical Development.

There exist in the literature very few analytical papers that deal specifically with the three-dimensional stress character at the base of a stationary crack. Moreover, in their present form these papers are not only incomplete but also contradictory. As a result, much controversy and many doubts have been raised. It is appropriate, therefore, to discuss these papers and their respective results in chronological order.

In 1972, Benthem, using the method of separation of variables\*, was able to solve for the stress distribution in the neighborhood of the corner point\*\* of a quarter plane crack. His results [2] show that the stresses there behave like  $\rho^{-\alpha}$ , where  $0.500 < \alpha < 0.709$ . In order to obtain the order of the singularity, Benthem had to truncate an infinite system which, in turn, he solved for the eigenvalues numerically. This approach, however, raises three important questions: One, is the solution really separable, particularly in  $\theta$  and  $\phi$ ? Two, is the solution thus obtained complete? Three, should the numerical determination of the singularity from a truncated system be trusted? Unfortunately, Benthem has provided no answers to any of the above important and difficult questions.

A few years later, Folias, using a method developed by Lur'e [3] and the application of Fourier Integral Transforms, was able to solve [4] Navier's equations for a more complicated problem, that of the

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\*This method was fully articulated by M.L. Williams [1] for classical planar elasticity in order to establish the singular behavior at re-entrant corners.

\*\*That is the point where the crack front meets the free surface of the half space.

3-D Griffith crack (see Figure 1). The integrals were subsequently expanded asymptotically and the stress field, valid in the very inner layers\* of the plate, was recovered. From the results, one concludes that in the very inner layers of the plate:

- (1) the stresses possess the usual singularity,
- (2) the stresses possess the usual angular distribution,
- (3) the stress intensity factor  $K_I$  is a function of  $z$ ,
- (4) exact plane strain conditions exist only on the plane  $z = 0$ ,
- (5) a pseudo plane strain state exists and the equation

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

is satisfied,

- (6) as the plate thickness  $2h \rightarrow \infty$ , the plane solution is recovered,
- (7) as Poisson's ratio  $\nu \rightarrow 0$ , the plane stress solution is recovered.

Furthermore, he was able to show that at the corner the stresses are proportional to

$$\rho^{-(\frac{1}{2} + 2\nu)} f_{ij}(\theta, \phi).$$

In order to recover the value of the singularity, Folias solved analytically a difference-differential equation. Unfortunately, because of the enormous difficulties which the integral representations presented at the corner, he was unable at the time to recover the functions  $f_{ij}(\theta, \phi)$  explicitly.

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\*The reader should note that the asymptotic expansions are only valid for  $(z/h) \ll 1$  and for  $c/h \ll 1$ . This is because  $h$  was assumed to be very large so that a perturbation about the well-known plane-strain solution could be made.

It should be emphasized that Folias's main result at the corner should be interpreted as "the singularity at the corner can at most be of the order  $(\frac{1}{2} + 2\nu)$ ". This is because the functions  $f_{ij}(\theta, \phi)$  could very well be of the type that do vanish\* in the neighborhood of the corner point. Thus Folias's result may or may not be in contradiction with Benthem's.

Researchers in the field of Fracture Mechanics, however, were unwilling to accept the possibility of an infinite displacement field on the basis of physical intuition. Consequently, the results were considered highly controversial and the following two legitimate questions were raised\*\*: Is the solution really complete? Two, do the series representations converge? Unfortunately, Folias provided no answers to any of the above questions.

In 1976, Kawai [7], using the method of separation of variables was able to obtain an alternate solution to Benthem's problem. Although the method of approach is essentially the same as that of Benthem's, his results are definitely contradictory\*\*\*. His results show that at the corner the stresses behave like  $\rho^{-\alpha}$ , where  $\frac{1}{2} < \alpha < 1$ . In determining the singularity, Kawai used the collocation method in order to satisfy the three boundary condition on the free surface. Thus, as in Benthem's case, the same questions apply to this work also.

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\*The reader should note that this result was actually obtained by 'marching out' the solution from the inner to the outer layers, and as a result such a hypothesis may not be totally unreasonable. See also comments on p. 5.

\*\*See Discussion of paper by Benthem and Koiter [5] and author's Closure [6].

\*\*\*Mathematically, Kawai's method of construction of the solution is more systematic than that of Benthem's.

A few months later, Benthem discovered that his previously reported solution was incomplete and that his new results [8] now read

$$\sigma_{ij} \sim \rho^{-\alpha} \text{ with } 0 < \alpha < \frac{1}{2}.$$

Here again, the same questions raised during his previous work apply too.

Finally, in 1977 Kawai [9] reported an error in his previous analysis and although the correction affected slightly the value of  $\alpha$  the trend essentially remained the same.

In the meantime, Folias also discovered that his solution of the difference-differential equation was not quite complete either\*. The correction, however, does not directly alter the basic result at the corner.

It is interesting to note that Kawai does recover the same singularity that Folias reported. The singularity ( $-\frac{1}{2} - 2\nu$ ), however, disappears as he considers more and more terms in his collocation scheme but at the same time he experiences convergence problems. This observation strengthens, perhaps, the hypothesis that Folias's  $f_{ij}(\theta, \phi)$  functions do indeed vanish in the neighborhood of the corner point and that most likely are needed in the very inner layers of the plate. The later has also been observed by Newman [10] for  $(c/h)$  ratios less than one, which is comparable with the asymptotic expansion used by Folias.

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\*This is not to be confused with the question of completeness of the solution to Navier's equations, i.e. eqs. (52)-(54) ref. [4]. The corrected result to eq. (85) of reference [4] is given in Appendix I.

Be that as it may, the presence of a third solution obscured the issue even further and essentially raised more questions than gave answers. So the controversy still remains.

### 3. Purpose of Present Work.

In view of the preceding, it is evident that mathematical rigour becomes essential if one is to avoid any possible pitfalls. As a result, the author decided to seek the answers to the following two important questions first:

- (i) Is the solution of this notoriously difficult problem unique? And if so, under what conditions?
- (ii) Is the solution to Navier's equations as given by the author in reference [4], i.e. eqs. (52)-(54), general enough to represent the solution of this practical problem?

The answers to both of the above questions were given by Prof. Calvin Wilcox.

First of all, he was successful in proving\* [11] that a displacement field that satisfies the condition of local finite energy is unique. This of course is quite a departure from our traditional 2-D fracture mechanics thinking, for the displacements now can be allowed to be singular. Consequently, one may not a priori assume them to be finite as it is customarily done. In general, such an assumption makes the class of solutions too restrictive and, as a result, one may not find a solution to the problem. On the other hand, the solution could very well give finite displacements everywhere! Be that as it may, physical intuition should be used with extreme caution.

Second, he was able to show\*\* [12] that the Fourier integral expressions\*\*\* representing the general solution to Navier's equations are

\* See also part III.

\*\*\* See equations (52)-(54) of reference [4].

\*\* see also part IV.

complete and, furthermore, the 'symbolic method' used is justifiable. In order to prove this, he used a double Fourier integral transform in  $x$  and  $y$  and subsequently a contour integration to recover precisely the same expressions as those reported by Folias in reference [4].

Finally, it remains to determine explicitly the stress field ahead of the crack tip and throughout the thickness of the plate. In reference [4], the author, by the use of analytic continuation, attempted to 'march out' the solution from the inner to the outer layers of the plate. Although in principle this seems feasible, in practice it is very difficult and most of all tedious. Moreover, questions of convergence will inevitably be raised. As a result, in this paper we will use an alternate and more elegant approach in order to complete the problem.

By finding the biorthogonal relation for the eigenvectors, we will set up a double integral equation for the unknown function  $v$ , which, physically, represents the projection of the displacement  $v$  onto the  $xz$ -plane. The advantages of this new approach over that of reference [4] are:

- (i) we are now seeking the solution to one equation only,
- (ii) the unknown function is real and furthermore has physical meaning,
- (iii) the kernel of the integral equation is independent of the shape of the crack\*.

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\*In this analysis we restrict ourselves to planar and symmetric cracks subjected to mode I loadings.

#### 4. Formulation of the Problem.

Consider the equilibrium of a homogeneous, isotropic, elastic plate which occupies the space  $|x| < \infty$ ,  $|y| < \infty$ ,  $|z| < h$  and contains a plane crack in the  $x$ - $z$ -plane (see Figure 1). The crack faces, defined by  $|x| < c$ ,  $y = 0^{\pm}$ ,  $z \leq h$ , and the plate faces  $|z| = h$  are free of stress and constraint. Loading is applied on the periphery of the plate  $|x|$ ,  $|y| \rightarrow \infty$  and is given by

$$\sigma_x = \tau_{xy} = \tau_{yz} = 0, \quad \sigma_y = \bar{\sigma}_0.$$

In the absence of body forces, the coupled differential equations governing the displacement functions  $u$ ,  $v$ , and  $w$  are

$$\frac{m}{m-2} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) e + \nabla^2 (u, v, w) = 0 \quad (1)-(3)$$

where  $\nabla^2$  is the Laplacian operator,  $m \equiv 1/\nu$ ,  $\nu$  is Poisson's ratio,

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (4)$$

and the stress-displacement relations are given by Hooke's law as:

$$\sigma_x = 2G \left\{ \frac{\partial u}{\partial x} + \frac{e}{m-2} \right\}, \dots, \tau_{xy} = G \left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}, \dots \quad (5)-(10)$$

with  $G$  being the shear modulus.



As to boundary conditions, one must require that at:

$$|x| < c, y = 0^{\pm}, |z| \leq h: \tau_{xy} = \tau_{yz} = \sigma_y = 0 \quad (11)$$

$$|z| = h \quad : \tau_{xz} = \tau_{yz} = \sigma_z = 0 \quad (12)$$

$$|y| \rightarrow \infty \text{ and all } x \quad : \tau_{xy} = \tau_{yz} = 0, \sigma_y = \bar{\sigma}_0 \quad (13)$$

$$|x| \rightarrow \infty \quad : \sigma_x = \tau_{xy} = \tau_{zx} = 0. \quad (14)$$

It is found convenient to seek the solution to the crack plate problem in the form

$$u = u^{(P)} + u^{(C)} \text{ etc.}, \quad (15)$$

where the first component represents the usual "undisturbed" or "particular" solution of a plate without the presence of a crack. Such a particular solution can easily be constructed and for the particular problem at hand is

$$\begin{aligned} u^{(P)} &= - \frac{\bar{\sigma}_0}{2G\Delta} (m-2)^2 x, \\ v^{(P)} &= - [1 - (m-1)^2] \frac{\bar{\sigma}_0 (m-2)}{2G\Delta} y \\ w^{(P)} &= - (m-2)^2 \frac{\bar{\sigma}_0}{2G\Delta} z \end{aligned} \quad (16)$$

where

$$\Delta = (m-1)^3 - 3(m-1) + 2.$$

### 5. Method of Solution.

The complementary solution to Navier's equations, subject to the corresponding boundary conditions (12) and (13), is given by reference [4] as:

(i) complementary displacements\*

$$\begin{aligned}
 u^{(c)} = & \int_0^\infty \left\{ (P_1 + |y| Q_1 + \frac{1}{m+1} z^2 s Q_1) e^{-s|y|} \right. \\
 & - \frac{1}{m-2} \sum_{\nu=1}^\infty \Gamma_\nu \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos(\beta_\nu h) [(m-2 + m \cos^2(\beta_\nu h)) \cos(\beta_\nu z) - \\
 & \left. - m \beta_\nu z \sin(\beta_\nu z)] + \sum_{n=1}^\infty S_n e^{-\sqrt{s^2 + \alpha_n^2} |y|} \cos(\alpha_n z) \right\} \sin(xs) ds
 \end{aligned} \quad (17)$$

$$\begin{aligned}
 v^{(c)} = & \int_0^\infty \left\{ \left( -\frac{3m-1}{m+1} \frac{Q_1}{s} - P_1 - |y| Q_1 - \frac{1}{m+1} s z^2 Q_1 \right) e^{-s|y|} \right. \\
 & + \frac{1}{m-2} \sum_{\nu=1}^\infty \Gamma_\nu \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{s} \cos(\beta_\nu h) [(m-2 + m \cos^2(\beta_\nu h)) \cos(\beta_\nu z) - \\
 & \left. - m \beta_\nu z \sin(\beta_\nu z)] - \sum_{n=1}^\infty S_n \frac{s}{\sqrt{s^2 + \alpha_n^2}} e^{-\sqrt{s^2 + \alpha_n^2} |y|} \cos(\alpha_n z) \right\} \cos(xs) ds
 \end{aligned} \quad (18)$$

$$\begin{aligned}
 w^{(c)} = & \int_0^\infty \left\{ \left( \frac{2}{m+1} z Q_1 \right) e^{-s|y|} + \frac{1}{m-2} \sum_{\nu=1}^\infty \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{s \sqrt{s^2 + \beta_\nu^2}} \cos(\beta_\nu h) \right. \\
 & \left. \cdot [(2m - 2 - m \cos^2(\beta_\nu h)) \sin(\beta_\nu z) - m \beta_\nu z \cos(\beta_\nu z)] \right\} \cos(xs) ds
 \end{aligned} \quad (19)$$

---

\*These complementary displacements represent a 'general enough', or 'complete', solution for the satisfaction of the remaining boundary conditions. For a discussion of this, see reference [12].

with corresponding stress:

(ii) complementary stresses

$$\frac{a_z(c)}{2G} = \frac{m}{m-2} \int_0^\infty \sum_{\nu=1}^\infty \Gamma_\nu \beta_\nu^2 \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{s \sqrt{s^2 + \beta_\nu^2}} \cos(\beta_\nu h) [\sin^2(\beta_\nu h) \cos(\beta_\nu z) + \beta_\nu z \sin(\beta_\nu z)] \cdot \cos(xs) ds \quad (20)$$

$$\begin{aligned} \frac{\tau_{xz}(c)}{G} = & \int_0^\infty \left\{ \frac{2m}{m-2} \sum_{\nu=1}^\infty \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos(\beta_\nu h) [\cos^2(\beta_\nu h) \sin(\beta_\nu z) \right. \\ & \left. + \beta_\nu z \cos(\beta_\nu z)] - \sum_{n=1}^\infty S_n \alpha_n \frac{e^{-\sqrt{s^2 + \alpha_n^2} |y|}}{s} \sin(\alpha_n z) \right\} \sin(xs) ds. \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\tau_{yz}(c)}{G} = & \pm \int_0^\infty \left\{ \frac{2m}{m-2} \sum_{\nu=1}^\infty \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2 + \beta_\nu^2} |y|}}{s} \cos(\beta_\nu h) [\cos^2(\beta_\nu h) \sin(\beta_\nu z) \right. \\ & \left. + \beta_\nu z \cos(\beta_\nu z)] - \sum_{n=1}^\infty S_n \frac{\alpha_n s}{\sqrt{s^2 + \alpha_n^2}} \frac{e^{-\sqrt{s^2 + \alpha_n^2} |y|}}{s} \sin(\alpha_n z) \right\} \cos(xs) ds. \end{aligned} \quad (22)$$

$$\begin{aligned}
\frac{\sigma_x(c)}{2G} &= \int_0^{\infty} \left\{ (sP_1 + |y|sQ_1 + \frac{1}{m+1} s^2 z^2 Q_1 - \frac{2}{m+1} Q_1) e^{-s|y|} \right. \\
&\quad + \frac{2}{m-2} \sum_{\nu=1}^{\infty} \beta_{\nu}^2 \Gamma_{\nu} \frac{e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|}}{s \sqrt{s^2 + \beta_{\nu}^2}} \cos(\beta_{\nu} h) \cos(\beta_{\nu} z) \\
&\quad - \frac{1}{m-2} \sum_{\nu=1}^{\infty} \Gamma_{\nu} \frac{e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|}}{\sqrt{s^2 + \beta_{\nu}^2}} \cos(\beta_{\nu} h) [(m-2+m \cos^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - m\beta_{\nu} z \sin(\beta_{\nu} z)] . \\
&\quad \left. + \sum_{n=1}^{\infty} S_n s e^{-\sqrt{s^2 + \alpha_n^2} |y|} \cos(\alpha_n z) \right\} \cos(xs) ds
\end{aligned} \tag{23}$$

$$\begin{aligned}
\frac{\sigma_y(c)}{2G} &= \int_0^{\infty} \left\{ \left( -\frac{2m}{m+1} Q_1 - sP_1 - |y|sQ_1 - \frac{1}{m+1} s^2 z^2 Q_1 \right) e^{-s|y|} \right. \\
&\quad + \frac{2}{m-2} \sum_{\nu=1}^{\infty} \beta_{\nu}^2 \Gamma_{\nu} \frac{e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|}}{s \sqrt{s^2 + \beta_{\nu}^2}} \cos(\beta_{\nu} h) \cos(\beta_{\nu} z) \\
&\quad + \frac{1}{m-2} \sum_{\nu=1}^{\infty} \Gamma_{\nu} \frac{\sqrt{s^2 + \beta_{\nu}^2}}{s} e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|} \cos(\beta_{\nu} h) [(m-2+m \cos^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - m\beta_{\nu} z \sin(\beta_{\nu} z)] \\
&\quad \left. - \sum_{n=1}^{\infty} S_n s e^{-\sqrt{s^2 + \alpha_n^2} |y|} \cos(\alpha_n z) \right\} \cos(xs) ds
\end{aligned} \tag{24}$$

$$\begin{aligned}
\frac{\tau_{xy}(c)}{G} = & \mp \int_0^{\infty} \left\{ \left( 2 \frac{m-1}{m+1} \right) Q_1 + 2sP_1 + \frac{2}{m+1} z^2 s^2 Q_1 + 2|y|sQ_1 \right\} e^{-s|y|} \\
& - \frac{2}{m-2} \sum_{v=1}^{\infty} \Gamma_v e^{-\sqrt{s^2 + \beta_v^2} |y|} \cos(\beta_v h) \left[ (m-2+m \cos^2(\beta_v h)) \cos(\beta_v z) - \right. \\
& \quad \left. - m\beta_v z \sin(\beta_v z) \right] \\
& + \sum_{n=1}^{\infty} S_n \frac{2s^2 + \alpha_n^2}{\sqrt{s^2 + \alpha_n^2}} e^{-\sqrt{s^2 + \alpha_n^2} |y|} \cos(\alpha_n z) \sin(xs) ds,
\end{aligned} \tag{25}$$

where the  $\pm$  signs refer to  $y > 0$  and  $y < 0$  respectively and the constants  $P_1$ ,  $Q_1$ ,  $\Gamma_v$  and  $S_n$  are to be determined from the remaining boundary conditions. Moreover,  $\alpha_n = \frac{n\pi}{h}$  ( $n=1,2,3\dots$ ), and  $\beta_v$  are the roots of the equation

$$\sin(2\beta_v h) = -(2\beta_v h) : \tag{26}$$

This equation has an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are chosen to be the complex conjugate pairs with positive real parts. The only real root  $\beta_v = 0$  must be ignored\*.

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\*The first few roots are tabulated in Appendix II.

By direct substitution, it can easily be ascertained that the above complementary displacements satisfy Navier's equations and furthermore the corresponding stresses  $\sigma_z^{(c)}$ ,  $\tau_{zx}^{(c)}$ ,  $\tau_{yz}^{(c)}$  do vanish at the plate faces  $z = \pm h$ .

Finally, if we consider the following two combinations to vanish

$$\begin{aligned} & \frac{2m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} \cos(\beta_{\nu} h) [\sin^2(\beta_{\nu} h) \cos(\beta_{\nu} z) + \beta_{\nu} z \sin(\beta_{\nu} z)] \\ & + \sum_{n=1}^{\infty} \frac{s}{\sqrt{s^2 + \alpha_n^2}} \frac{S_n}{2} \cos(\alpha_n z) + \frac{4m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} = 0 \end{aligned} \quad (27)$$

and

$$\begin{aligned} & \frac{2}{m-2} \sum_{\nu=1}^{\infty} \Gamma_{\nu} \cos(\beta_{\nu} h) [(m-2+m \cos^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) - m\beta_{\nu} z \sin(\beta_{\nu} z)] \\ & - \sum_{n=1}^{\infty} \frac{2s^2 + \alpha_n^2}{\sqrt{s^2 + \alpha_n^2}} S_n \cos(\alpha_n z) - \frac{2}{1+m} s^2 z^2 Q_1 - 2sP_1 - 2 \frac{m-1}{m+1} Q_1 = 0 \end{aligned} \quad (28)$$

for all  $|z| \leq h$ , then two of the remaining stress boundary conditions are satisfied automatically, i.e.

$$\tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0 \quad \text{for all } x, |z| \leq h \text{ and } y = 0.$$

We will suppress for the time being the satisfaction of the last boundary condition and will focus our attention to the continuity conditions.

As it can easily be seen, all continuity conditions are satisfied if one considers the following two combinations to vanish

\*Notice that the derivative of eq. (27) with respect to  $z$  leads to the integrand of eq. (22).

$$\begin{aligned}
& \int_0^{\infty} \left\{ -\frac{3m-1}{m+1} \frac{Q_1}{s} - P_1 - \frac{1}{m+1} s z^2 Q_1 \right. \\
& \quad + \frac{1}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} \cos(\beta_{\nu} h) [(2m-2+m \sin^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) \\
& \quad \left. + m \beta_{\nu} z \sin(\beta_{\nu} z)] + \frac{4m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} \right\} \cos(xs) ds = 0 ; \\
& \quad |x| > c, \forall |z| < h .
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
& \int_0^{\infty} \left\{ -\frac{4m}{m+1} Q_1 + \frac{2m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s^2} \beta_{\nu}^2 [(1 + \cos^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) \right. \\
& \quad \left. - \beta_{\nu} z \sin(\beta_{\nu} z)] \right\} \sin(xs) ds = 0 ; |x| > c, \forall |z| < h .
\end{aligned} \tag{30}$$

which by Fourier inversion lead to:

$$\begin{aligned}
& -\frac{3m-1}{m+1} \frac{Q_1}{s} - P_1 - \frac{1}{m+1} s z^2 Q_1 + \frac{1}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} \cos(\beta_{\nu} h) \cdot \\
& \quad \cdot [(2m-2+m \sin^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) + m \beta_{\nu} z \sin(\beta_{\nu} z)] \tag{31} \\
& \quad + \frac{4m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s} = \frac{2}{\pi} \int_0^c v(\xi, 0, z) \cos(s\xi) d\xi
\end{aligned}$$

and \*

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\*The reader should note that eqs (31) and (32) automatically satisfy eq. (28).

$$\begin{aligned}
& - \frac{4m}{m+1} Q_1 + \frac{2m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s^2} \beta_{\nu}^2 \cos(\beta_{\nu} h) [(1 + \cos^2(\beta_{\nu} h)) \cos(\beta_{\nu} z) \\
& - \beta_{\nu} z \sin(\beta_{\nu} z)] = \mp \frac{2}{\pi} \int_0^c \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial \xi} \right)_{y=0} \sin(s\xi) d\xi \\
& = \pm \frac{2}{\pi} \int_0^c 2 \left( \frac{\partial v}{\partial \xi} \right)_{y=0} \sin(s\xi) d\xi \\
& = \mp \frac{4s}{\pi} \int_0^c v(\xi, 0, z) \cos(s\xi) d\xi .
\end{aligned} \tag{32}$$

Adopting next the following definitions

$$Z_{\nu}^{(1)}(z) \equiv (\beta_{\nu} h)^2 [\beta_{\nu} h \sin(\beta_{\nu} h) \cos(\beta_{\nu} z) - \beta_{\nu} z \cos(\beta_{\nu} h) \sin(\beta_{\nu} z)] \tag{33}$$

$$\begin{aligned}
Z_{\nu}^{(3)}(z) \equiv & -(\beta_{\nu} h)^2 [\beta_{\nu} h \sin(\beta_{\nu} h) \cos(\beta_{\nu} z) - \beta_{\nu} z \cos(\beta_{\nu} h) \sin(\beta_{\nu} z)] \\
& - 2(\beta_{\nu} h)^2 \cos(\beta_{\nu} h) \cos(\beta_{\nu} z)
\end{aligned} \tag{34}$$

$$f^{(1)}(z) \equiv \mp \frac{2h^2}{\pi s} \left( \frac{m-2}{m-1} \right) \int_0^c \left\{ \frac{\partial^2 v}{\partial z^2} - \frac{1}{m} s^2 v \right\} \cos(s\xi) d\xi \tag{35}$$

$$f^{(3)}(z) \equiv \pm \frac{2sh^2}{\pi} \left( \frac{m-2}{m} \right) \int_0^c v \cos(s\xi) d\xi - 2 \left( \frac{m-2}{m+1} \right) Q_1 h^2 , \tag{36}$$

equation (32) and the second derivative of equation (31) with respect to  $z$  become

$$\begin{aligned}
- \frac{m}{m-2} \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s^2} Z_{\nu}^{(3)}(z) & = \mp \frac{2sh^2}{\pi} \int_0^c v(\xi, 0, z) \cos(s\xi) d\xi \\
& + \frac{2m}{m+1} Q_1 h^2
\end{aligned} \tag{37}$$



and\*

$$\begin{aligned}
 - \left(\frac{m-1}{m-2}\right) \sum_{\nu=1}^{\infty} \frac{\Gamma_{\nu}}{s^2} \{Z_{\nu}^{(1)}(z) + Z_{\nu}^{(3)}(z)\} = \\
 = \pm \frac{2h^2}{\pi s} \int_0^c \left\{ \frac{\partial^2}{\partial z^2} - s^2_{\nu} \right\}_{y=0} \cos(s\xi) d\xi \\
 + 2 \left(\frac{m-1}{m+1}\right) Q_1 h^2
 \end{aligned} \tag{38}$$

respectively, which upon simplifying one has

$$\sum_{\nu=1}^{\infty} \left(\frac{\Gamma_{\nu}}{s^2}\right) \begin{bmatrix} Z_{\nu}^{(3)}(z) \\ Z_{\nu}^{(1)}(z) \end{bmatrix} = \begin{bmatrix} f^{(3)}(z) \\ f^{(1)}(z) \end{bmatrix} \tag{39}$$

Next, following reference [13], we can construct the biorthogonal relations

$$W_{\nu}^{(4)}(z) \equiv -\beta_{\nu}^* z \cos(\beta_{\nu}^* h) \sin(\beta_{\nu}^* z) + \beta_{\nu}^* h \sin(\beta_{\nu}^* h) \cos(\beta_{\nu}^* z) \tag{40}$$

and

$$\begin{aligned}
 W_{\nu}^{(2)}(z) \equiv -\beta_{\nu}^* z \cos(\beta_{\nu}^* h) \sin(\beta_{\nu}^* z) + [\beta_{\nu}^* h \sin(\beta_{\nu}^* h) \\
 - 2 \cos(\beta_{\nu}^* h)] \cos(\beta_{\nu}^* z),
 \end{aligned} \tag{41}$$

where  $\beta_{\nu}^*$  stands for the complex conjugate of the  $\beta_{\nu}$  roots. The orthogonality condition now reads

---

\*Notice that the continuity conditions are to be satisfied in the interior of the plate only.

$$\sum_{\nu=1}^{\infty} \left(\frac{\Gamma_{\nu}}{s^2}\right) \frac{1}{h} \int_{-h}^h [W_k^{*(4)}(\eta) W_k^{*(2)}(\eta)] \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{\nu}^{(1)}(\eta) \\ Z_{\nu}^{(3)}(\eta) \end{bmatrix} d\eta \quad (42)$$

$$= \frac{1}{h} \int_{-h}^h [W_k^{*(4)}(\eta) W_k^{*(2)}(\eta)] \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} f^{(1)}(\eta) \\ f^{(3)}(\eta) \end{bmatrix} d\eta ,$$

or

$$\left(\frac{\Gamma_{\nu}}{s^2}\right) e^{K_{\nu}} = \frac{1}{h} \int_{-h}^h \{W_{\nu}^{*(4)}(\eta) f^{(1)}(\eta) + [2W_{\nu}^{*(4)}(\eta) - W_{\nu}^{*(2)}(\eta)] \cdot f^{(3)}(\eta)\} d\eta \quad (43)$$

where for simplicity we have defined

$$e^{K_{\nu}} \equiv \frac{1}{h} \int_{-h}^h \{W_{\nu}^{*(4)}(\eta) Z_{\nu}^{(1)}(\eta) + [2W_{\nu}^{*(4)}(\eta) - W_{\nu}^{*(2)}(\eta)] \cdot Z_{\nu}^{(3)}(\eta)\} d\eta . \quad (44)$$

Finally, in view of equations (33)-(36), (40)-(41) and (43)-(44), one finds after some simple calculations that

$$e^{K_{\nu}} = -4 (\beta_{\nu} h)^2 \cos^4(\beta_{\nu} h) \quad (45)$$

and

$$\begin{aligned}
e^{K_{\nu}} \left(\frac{\Gamma_{\nu}}{s}\right) &= \pm \frac{2}{\pi \operatorname{sh}} \left(\frac{m-2}{m-1}\right) \int_0^c \int_{-h}^h v(\xi, 0, \eta) \cos(s\xi) \\
&\cdot \{ (hs)^2 [\beta_{\nu} h \sin(\beta_{\nu} h) \cos(\beta_{\nu} \eta) - \beta_{\nu} \eta \cos(\beta_{\nu} h) \sin(\beta_{\nu} \eta) \\
&+ 2 \left(\frac{m-1}{m}\right) \cos(\beta_{\nu} h) \cos(\beta_{\nu} \eta)] \\
&+ (h\beta_{\nu})^2 [\beta_{\nu} h \sin(\beta_{\nu} h) \cos(\beta_{\nu} \eta) - \beta_{\nu} \eta \cos(\beta_{\nu} h) \sin(\beta_{\nu} \eta) \\
&+ 2 \cos(\beta_{\nu} h) \cos(\beta_{\nu} \eta)] \} d\eta d\xi .
\end{aligned} \tag{46}$$

Similarly, from equations (27) and (32), we find that

$$\frac{s_{\eta}}{\sqrt{s^2 + \alpha_{\eta}^2}} (\alpha_{\eta} h)^2 = \mp \frac{4sh^2}{\pi h} \int_0^c \int_{-h}^h v(\xi, 0, \eta) \cos(s\xi) \cos(\alpha_{\eta} \eta) d\eta d\xi \tag{47a}$$

and

$$Q_1 = \pm \frac{m+1}{2m} \frac{(sh)}{\pi} \int_0^c \int_{-h}^h v(\xi, 0, \eta) \cos(s\xi) d\eta d\xi . \tag{47b}$$

Returning now to the last boundary condition, we require that\*

---

\*Where we have made use of eq. (28).

$$\begin{aligned}
& \int_0^{\infty} \left\{ -Q_1 + \frac{m}{m-2} \sum_{v=1}^{\infty} \frac{\Gamma_v}{s^2} \frac{s\beta_v^2}{\sqrt{s^2+\beta_v^2}} \cos(\beta_v h) [(1 + \cos^2(\beta_v h)) \cos(\beta_v z) \right. \\
& \left. - \beta_v z \sin(\beta_v z)] \right. \\
& + \frac{1}{m-2} \sum_{v=1}^{\infty} \frac{\Gamma_v}{s^2} \left[ \frac{s^3}{\sqrt{s^2+\beta_v^2}} - s^2 \right] \cos(\beta_v h) [(m-2 + m \cos^2(\beta_v h)) \cos(\beta_v z) \\
& \left. - m \beta_v z \sin(\beta_v z)] \right. \\
& \left. - \sum_{n=1}^{\infty} \frac{S_n}{\sqrt{s^2+\alpha_n^2}} \left[ \frac{s(s^2+\alpha_n^2)}{\sqrt{s^2+\alpha_n^2}} - (s^2 + \frac{\alpha_n^2}{2}) \right] \cos(\alpha_n z) \right\}
\end{aligned} \tag{48}$$

$$\cos(xs) ds = -\frac{\sigma_0}{2G}; \quad |z| < h, \quad |x| < c$$

which, upon using the relations (46)-(47) and interchanging the order of integration, can also be written in the form of a double integral equation i.e.,

$$\begin{aligned}
& \frac{1}{\pi h} \sum_{v=1}^{\infty} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} \frac{\partial^2}{\partial x \partial z} H_1[|x-\xi|; \eta, z] d\eta d\xi \\
& + \frac{1}{\pi h} \sum_{n=1}^{\infty} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} \frac{\partial^2}{\partial x \partial z} H_2[|x-\xi|; \eta, z] d\eta d\xi \\
& - \frac{1}{\pi h} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} \left( \frac{m+1}{4m} \right) \frac{\partial^2}{\partial x \partial z} \left[ \frac{z}{x-\xi} \right] d\eta d\xi \\
& = -\frac{\sigma_0}{2G}; \quad |z| < h, \quad |x| < c.
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
 H_1 [ |x-\xi| ; \eta, z ] &= \frac{m}{m-1} \frac{\beta_\nu h^2}{e K_\nu} [\cos(\beta_\nu h) \sin(\beta_\nu z) \\
 &+ \beta_\nu h \sin(\beta_\nu h) \sin(\beta_\nu z) + \beta_\nu z \cos(\beta_\nu h) \cos(\beta_\nu z)] \cdot \\
 &\cdot \{ \beta_\nu \frac{|x-\xi|}{(x-\xi)} K_1 [\beta_\nu |x-\xi| ] [\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu \eta) - \beta_\nu \eta \cos(\beta_\nu h) \sin(\beta_\nu \eta) \\
 &+ 2 \frac{(m-1)}{m} \cos(\beta_\nu h) \cos(\beta_\nu \eta)] \\
 &+ \beta_\nu^2 \int_0^{x-\xi} K_0 [\beta_\nu |x'| ] dx' \cdot [\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu \eta) \\
 &- \beta_\nu \eta \cos(\beta_\nu h) \sin(\beta_\nu \eta) + 2 \cos(\beta_\nu h) \cos(\beta_\nu \eta)] \} \\
 &+ \frac{m}{m-1} \frac{1}{\beta_\nu e K_\nu} \left[ \frac{m-2}{m} \cos(\beta_\nu h) \sin(\beta_\nu z) + \beta_\nu h \sin(\beta_\nu h) \right. \\
 &\left. \sin(\beta_\nu z) + \beta_\nu z \cos(\beta_\nu h) \cos(\beta_\nu z) \right] \cdot \left\{ \left[ - \frac{\beta_\nu^2 h^2}{x-\xi} K_0 [\beta_\nu |x-\xi| ] \right. \right. \\
 &\left. \left. - \beta_\nu^3 h^2 \frac{|x-\xi|}{(x-\xi)} K_1 [\beta_\nu |x-\xi| ] - \frac{2\beta_\nu h^2}{(x-\xi)|x-\xi|} K_1 (\beta_\nu |x-\xi|) \right. \right. \\
 &\left. \left. + \frac{2h^2}{(x-\xi)^3} \right] \cdot [\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu \eta) - \beta_\nu \eta \cos(\beta_\nu h) \sin(\beta_\nu \eta) \right. \\
 &\left. + 2 \frac{(m-1)}{m} \cos(\beta_\nu h) \cos(\beta_\nu \eta) \right] + \left[ \beta_\nu^3 h^2 \frac{|x-\xi|}{(x-\xi)} K_1 [\beta_\nu |x-\xi| ] \right. \\
 &\left. - \frac{\beta_\nu^2 h^2}{x-\xi} \right] [\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu \eta) - \beta_\nu \eta \cos(\beta_\nu h) \sin(\beta_\nu \eta) + 2 \cos(\beta_\nu h) \cos(\beta_\nu \eta)] \}
 \end{aligned}
 \tag{50}$$

and

$$\begin{aligned}
H_2 [ |x-\xi|; n, z ] &= 2 \frac{1}{\alpha_n^3} \\
\left\{ -\frac{\alpha_n^2}{x-\xi} K_0 [ \alpha_n |x-\xi| ] - \frac{2\alpha_n}{|x-\xi|(x-\xi)} K_1 [ \alpha_n |x-\xi| ] \right. & \\
+ \left. \frac{2}{(x-\xi)^3} - \frac{\alpha_n^2}{2(x-\xi)} \right\} \cos(\alpha_n \eta) \sin(\alpha_n z) & \quad (51)
\end{aligned}$$

Finally, integrating once with respect to  $x$  and  $z$  one finds\*

$$\begin{aligned}
& \frac{1}{\pi h} \sum_{\nu=1}^{\infty} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} H_1 [ |x-\xi|; n, z ] d\eta d\xi \\
& + \frac{1}{\pi h} \sum_{n=1}^{\infty} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} H_2 [ |x-\xi|; n, z ] d\eta d\xi \\
& - \frac{1}{\pi h} \int \int_{\text{crack faces}} \{ \pm v(\xi, 0, \eta) \} \left( \frac{m+1}{4m} \right) \left[ \frac{z}{x-\xi} \right] d\eta d\xi \\
& = - \left( \frac{\sigma_0}{2G} \right) xz ; |x| < c , |z| < h . \quad (52)
\end{aligned}$$

We have reduced, therefore, the problem to that of the solution of a two-dimensional singular integral equation for the unknown function  $v(\xi, 0, \eta)$ . This solution will be discussed in a subsequent paper.

It is interesting to note that equation (49) is also applicable to planar cracks of arbitrary shape that lie on the  $x$ - $z$ -plane and are symmetric with respect to both  $x$  and  $z$  -axes\*\*.

\*The reader should notice that the function  $v(\xi, 0, \eta)$  has a  $\mp$  sign also.  
 \*\*The same method of solution may also be used in order to derive a much more general integral equation which applies to any arbitrary crack shape or void. This matter is currently under investigation and the results will be reported in another paper.

Perhaps it is instructive to point out some of the advantages of the present formulation over that of reference [4]. These are:

- (i) we are seeking the solution of one integral equation
- (ii) the unknown function is real and has physical meaning
- (iii) the unknown function can be related directly to experimental observations
- (iv) the formulation applies to a large class of planar crack problems

6. Solution to Navier's equations.

Without going into the mathematical details, we may now write the displacement functions  $u^{(c)}$ ,  $v^{(c)}$  and  $w^{(c)}$  in terms of the unknown function  $v(\xi, 0, \eta)$ , for  $|x| < c$ :

$$\begin{aligned}
 u^{(c)} = & \pm \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \left\{ \frac{1-m}{4m} \frac{x-\xi}{(x-\xi)^2 + |y|^2} - \frac{m+1}{4m} \frac{\partial}{\partial x} \left[ \frac{|y|^2}{(x-\xi)^2 + |y|^2} \right] \right. \\
 & + \frac{1}{4m} \left( \frac{2h^2}{3} - z^2 - \eta^2 \right) \frac{\partial^2}{\partial x^2} \left[ \frac{x-\xi}{(x-\xi)^2 + |y|^2} \right] \left. \right\} d\eta d\xi \\
 & \pm \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \cdot \frac{\partial}{\partial y} \left( \frac{\partial^2 N}{\partial x \partial y} \right) d\eta d\xi \\
 & \pm \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \cdot \frac{\partial M}{\partial x} d\eta d\xi
 \end{aligned} \tag{53}$$

$$\begin{aligned}
 v^{(c)} = & \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \left\{ \frac{1}{2} \frac{|y|}{(x-\xi)^2 + |y|^2} + \frac{m+1}{4m} \frac{\partial}{\partial x} \left[ \frac{(x-\xi)|y|}{(x-\xi)^2 + |y|^2} \right] \right. \\
 & + \frac{1}{4m} \left[ \frac{2h^2}{3} - z^2 - \eta^2 \right] \frac{\partial^2}{\partial x^2} \left[ \frac{|y|}{(x-\xi)^2 + |y|^2} \right] \left. \right\} d\eta d\xi \\
 & - \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \cdot \frac{\partial}{\partial x} \left( \frac{\partial^2 N}{\partial x \partial y} \right) d\eta d\xi \\
 & + \frac{1}{\pi h} \int_{-c}^c \int_{-h}^h v(\xi, 0, \eta) \frac{\partial M}{\partial y} d\eta d\xi
 \end{aligned} \tag{54}$$



$$\begin{aligned}
w(c) &= \pm \frac{1}{\pi h} \int_{-h}^h \int_{-c}^c v(\xi, 0, \eta) \cdot \frac{1}{2m} \frac{\partial}{\partial x} \left[ \frac{(x-\xi)z}{(x-\xi)^2 + |y|^2} \right] d\xi d\eta \\
&\pm \frac{1}{\pi h} \int_{-h}^h \int_{-c}^c v(\xi, 0, \eta) \frac{\partial}{\partial z} (M-\bar{M}) d\xi d\eta,
\end{aligned} \tag{55}$$

where for simplicity we have adopted the following definitions:

$$N \equiv 2 \sum_{n=1}^{\infty} \frac{K_0[\alpha_n \sqrt{(x-\xi)^2 + |y|^2}]}{\alpha_n^2} \cos \alpha_n z \cos \alpha_n \eta \tag{56}$$

$$M \equiv \frac{m}{m-1} \sum_{\nu=1}^{\infty} \frac{1}{e^{K_\nu}} \left\{ \left( \frac{m-2}{m} + \cos^2 \beta_\nu h \right) \cos \beta_\nu h \cos \beta_\nu z \right.$$

$$\left. - \beta_\nu z \cos \beta_\nu h \sin \beta_\nu z \right\} \left\{ - \frac{\partial^2}{\partial x^2} K_0[\beta_\nu \sqrt{(x-\xi)^2 + |y|^2}] \cdot \right. \tag{57}$$

$$\left. \cdot [\beta_\nu h \sin \beta_\nu h \cos \beta_\nu \eta - \beta_\nu \eta \cos \beta_\nu h \sin \beta_\nu \eta + 2 \left( \frac{m-1}{m} \right) \cos \beta_\nu h \cos \beta_\nu \eta] \right.$$

$$\left. + \beta_\nu^2 K_0[\beta_\nu \sqrt{(x-\xi)^2 + |y|^2}] [\beta_\nu h \sin \beta_\nu h \cos \beta_\nu \eta - \beta_\nu \eta \cos \beta_\nu h \sin \beta_\nu \eta + \right.$$

$$\left. + 2 \cos \beta_\nu h \cos \beta_\nu \eta] \right\}$$

and

$$\begin{aligned}
\tilde{M} \equiv 4 & \sum_{\nu=1}^{\infty} \frac{1}{e^{k_{\nu}}} \cos\beta_{\nu} h \cos\beta_{\nu} z \left\{ -\frac{\partial^2}{\partial x^2} K_0[\beta_{\nu} \sqrt{(x-\xi)^2 + |y|^2}] \right. \\
& \cdot [ \beta_{\nu} h \sin\beta_{\nu} h \cos\beta_{\nu} \eta - \beta_{\nu} \eta \cos\beta_{\nu} h \sin\beta_{\nu} \eta + 2\left(\frac{m-1}{m}\right) \cos\beta_{\nu} h \cos\beta_{\nu} \eta ] \\
& + \beta_{\nu}^2 K_0[\beta_{\nu} \sqrt{(x-\xi)^2 + |y|^2}] \cdot [ \beta_{\nu} h \sin\beta_{\nu} h \cos\beta_{\nu} \eta - \beta_{\nu} \eta \cos\beta_{\nu} h \sin\beta_{\nu} \eta \\
& \left. + 2 \cos\beta_{\nu} h \cos\beta_{\nu} \eta ] \right\} .
\end{aligned} \tag{58}$$

In view of the above, it appears that the solution may not be separable either in cylindrical or spherical coordinates.

Finally, one may express the total strain energy stored in the system to be:

$$W = -\frac{1}{2} \int_{-h}^h \int_{-c}^c \{(v^+ - v^-) \sigma_y\}_{y=0} dx dz . \tag{59}$$

## 7. Discussion

Although we have put forth a considerable amount of effort to solve the double singular integral equation, we have not as yet been successful in recovering, explicitly, the unknown displacement function  $v(x, 0, z)$ , valid throughout the thickness of the plate. This is an extremely difficult problem where physical intuition can be misleading.

At present we have developed two methods which in principle should give us the desired solution. Unfortunately, in order to recover the corner singularity, one is forced to sum up, analytically, a double series of complex eigenfunctions. This is a task of monumental difficulty, for the algebra is tedious and long.

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APPENDIX I.

To find the complete homogeneous solution of equation (82) of reference [4], we proceed as follows.

Assume first a solution of the form

$$f^{(h)}(1+\zeta) = (1-\zeta)^{2-2/m} G(\zeta) ,$$

where  $G(\zeta)$  is an arbitrary function of  $\zeta$  . Next, substitute into homogeneous difference-differential equation to find

$$(1-\zeta)^{3-2/m} G'(\zeta) + (1+\zeta)^{3-2/m} G'(-\zeta) = 0 ,$$

from which one may now deduce that:

$$G(\zeta) = \sum_{n=0}^{\infty} a_{2n+1} \int_0^{\zeta} (1+\zeta)^{3-2/m} (\zeta)^{2n+1} d\zeta + c_0$$

or

$$G(\zeta) = \sum_{n=0}^{\infty} a_{2n+1} \frac{\zeta^{2n+2}}{(2n+2)} {}_2F_1 \left( -3+\frac{2}{m}, 2n+2; 2n+3; -\zeta \right)$$

## APPENDIX II

The roots of the equation  $\sin(2\beta_{\nu}h) = - (2\beta_{\nu}h)$ .

The equation has an infinite number of complex roots which appear in groups of four. However, as it was pointed out in the text, for this analysis only the roots with positive real parts are pertinent and furthermore, the only real root  $\beta_{\nu} = 0$  must be discarded. Thus, if we define the roots  $\beta_2, \beta_4, \beta_6, \dots$  to be the complex conjugates of the roots  $\beta_1, \beta_3, \beta_5, \dots$ , then by setting

$$2\beta_{\nu}h = x_{\nu} + iy_{\nu} \quad \nu = 1, 3, 5, \dots$$

and using a Newton-Rampson numerical method one finds

$\nu$	$x_{\nu}$	$y_{\nu}$
1	4.21239	2.25073
3	10.71254	3.10315
5	17.07337	3.55109
7	23.39836	3.85881
	etc.	

Furthermore, the asymptotic behavior of the roots for large  $\nu$ , i.e., for  $\nu = 15, 17, 19, \dots$ , is given by the following simple relations

$$x_{\nu} \approx (\nu + \frac{1}{2})\pi$$

$$y_{\nu} \approx \cos h^{-1}[(\nu + \frac{1}{2})\pi] .$$

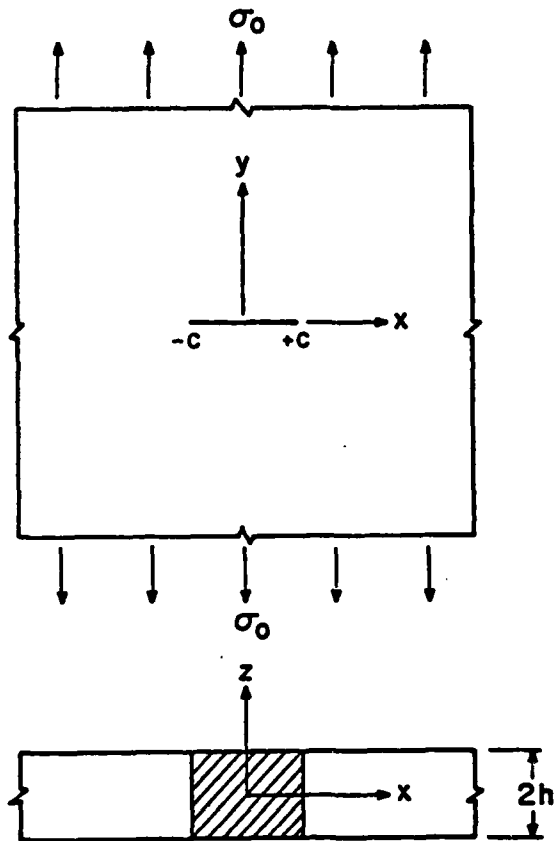


Figure 1. Geometrical representation of an infinite cracked plate with thickness  $2h$  and crack length  $2c$ .

PART II

PARTIAL THROUGH CRACK IN A PLATE  
OF FINITE THICKNESS



NOTATION

$D^2$	=	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
$\partial_1$	=	$\frac{\partial}{\partial x}$
$\partial_2$	=	$\frac{\partial}{\partial y}$
$\nabla^2$	=	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
E	=	Young's modulus
G	=	$\frac{E}{2(1+\nu)}$
h	=	thickness of the plate
m	=	$\frac{1}{\nu}$
r	=	$\sqrt{x^2 + y^2}$
s	=	Fourier transform parameter
u, v, w	=	displacement functions
$u^{(C)}, v^{(C)}, w^{(C)}$	=	displacement functions due to the complementary problem
$u^{(P)}, v^{(P)}, w^{(P)}$	=	displacement functions due to the particular problems
x, y, z	=	rectangular cartesian coordinates
$\alpha_n$	=	$n\pi/h \quad n = 1, 2, 3, \dots$
$\beta_\nu$	=	roots of the eq. $\sin(\beta_\nu h) = 0$
$\gamma_\nu$	=	roots of the eq. $\sin(\gamma_\nu h) = -1$
$\Gamma_\nu, \tilde{\Gamma}_\nu, S_n$	=	coefficients as defined in text
$\theta$	=	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$
$\nu$	=	Poisson's ratio

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$  = stress components

$\sigma_x^{(C)}, \sigma_y^{(C)}, \sigma_z^{(C)}, \sigma_{xy}^{(C)}, \sigma_{xz}^{(C)}, \sigma_{yz}^{(C)}$  = stress components due to the complementary solution

$\sigma_x^{(P)}, \sigma_y^{(P)}, \sigma_z^{(P)}, \sigma_{xy}^{(P)}, \sigma_{xz}^{(P)}, \sigma_{yz}^{(P)}$  = stress components due to the particular solution

$\bar{\sigma}_0$  = uniform applied stress

## I. INTRODUCTION

One of the problems in fracture mechanics which apparently has not received extensive theoretical treatment is that concerning the effect of a partial through crack upon the stress distribution in a plate of finite thickness. This lack of interest is primarily due to the fact that three dimensional problems present mathematical complexities which are substantially greater than those associated with plane stress or plane strain. However, it is now possible\* to study this complex phenomenon which has defied researchers for some time.

## II. FORMULATION OF THE PROBLEM

Consider the equilibrium of a homogeneous, isotropic, elastic plate which occupies the space  $|x| < \infty$ ,  $|y| < \infty$ ,  $0 < z < h$  and contains a plane crack in the  $xz$ -plane. The crack is elliptical in shape and is defined by the inequality

$$\left(\frac{x}{c}\right)^2 + \left(\frac{z}{a}\right)^2 < 1 \quad (1)$$

The plate faces  $z=0$  and  $z=h$  are free of stress and constraint. Loading is applied by the periphery of the plate  $|x|, |y| \rightarrow \infty$  and is given by

$$\sigma_x = \tau_{xy} = \tau_{yz} = 0, \quad \sigma_y = \sigma_0.$$

In the absence of body forces, the coupled differential equations governing the displacement functions  $u$ ,  $v$  and  $w$  are:

$$\frac{m}{m-2} \frac{\partial \theta}{\partial x} + \nabla^2 u = 0 \quad (2)$$

$$\frac{m}{m-2} \frac{\partial \theta}{\partial y} + \nabla^2 v = 0 \quad (3)$$

$$\frac{m}{m-2} \frac{\partial \theta}{\partial z} + \nabla^2 w = 0 \quad (4)$$

\*See references [1,2].

where

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (5)$$

and the stress-displacement relations are given by Hook's law as:

$$\sigma_x = 2G \left\{ \frac{\partial u}{\partial x} + \frac{\theta}{m-2} \right\} \quad (6)$$

$$\sigma_y = 2G \left\{ \frac{\partial v}{\partial y} + \frac{\theta}{m-2} \right\} \quad (7)$$

$$\sigma_z = 2G \left\{ \frac{\partial w}{\partial z} + \frac{\theta}{m-2} \right\} \quad (8)$$

$$\tau_{xy} = G \left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\} \quad (9)$$

$$\tau_{yz} = G \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \quad (10)$$

$$\tau_{zx} = G \left\{ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right\} \quad (11)$$

As to boundary conditions, one must require that at:

$$z = 0 : \tau_{xz} = \tau_{yz} = \sigma_z = 0 \quad (12)$$

$$z = h : \tau_{xz} = \tau_{yz} = \sigma_z = 0 \quad (13)$$

$$\left(\frac{x}{c}\right)^2 + \left(\frac{z}{a}\right)^2 < 1, y = 0^{\pm} : \tau_{xy} = \tau_{yz} = \sigma_y = 0 \quad (14)$$

$$|y| \rightarrow \infty \text{ and all } x : \tau_{xy} = \tau_{yz} = 0, \sigma_y = \bar{\sigma}_0 \quad (15)$$

$$|x| \rightarrow \infty : \sigma_x = \tau_{xy} = \tau_{zx} = 0. \quad (16)$$

It is found convenient to seek the solution to the crack plate problem in the form

$$u = u^{(P)} + u^{(C)} \text{ etc.}, \quad (17)$$

where the first component represents the usual "undisturbed" or "particular" solution of a plate without the presence of a crack. Such a particular solution can be easily constructed and for the particular problem at hand is

$$u^{(P)} = -\frac{\bar{\sigma}_0}{2G\Delta} (m-2)^2 x \quad (18)$$

$$v^{(P)} = - [1 - (m-1)^2] \frac{\bar{\sigma}_0 (m-2)}{2G\Delta} y \quad (19)$$

$$w^{(P)} = - (m-2)^2 \left( \frac{\bar{\sigma}_0}{2G\Delta} \right) z \quad (20)$$

$$\Delta = (m-1)^3 - 3(m-1) + 2 \quad (21)$$

#### MATHEMATICAL STATEMENT OF THE COMPLEMENTARY PROBLEM

In view of the particular solution, we need to find three functions  $u^{(C)}(x,y,z)$ ,  $v^{(C)}(x,y,z)$  and  $w^{(C)}(x,y,z)$ , such that they satisfy simultaneously the partial differential equations (2) - (4) and the following boundary conditions:

$$\text{at } \left(\frac{x}{c}\right) + \left(\frac{z}{a}\right)^2 < 1, |y| = 0: \tau_{xy}^{(C)} = \tau_{yz}^{(C)} = 0, \sigma_y^{(C)} = -\bar{\sigma}_0 \quad (22)$$

$$\text{at } z = 0: \tau_{xz}^{(C)} = \tau_{yz}^{(C)} = \sigma_z^{(C)} = 0 \quad (23)$$

$$\text{at } z = h: \tau_{xz}^{(C)} = \tau_{yz}^{(C)} = \sigma_z^{(C)} = 0 \quad (24)$$

$$\text{at } \sqrt{x^2 + y^2} \rightarrow \infty: u^{(C)}, v^{(C)} \text{ and } w^{(C)} \text{ are to be bounded.} \quad (25)$$

#### METHOD OF SOLUTION

In constructing a solution to the system (2) - (4) we use the method described in reference [1] to recover the following ordinary differential equation of the independent variable  $z$ :

$$\frac{d^2 u^{(C)}}{dz^2} + (D^2 + \frac{m}{m-2} \partial_1^2) u^{(C)} + (\frac{m}{m-2} \partial_1 \partial_2) v^{(C)} + (\frac{m}{m-2} \partial_1) \frac{dw^{(C)}}{dz} = 0 \quad (26)$$

$$\frac{d^2 v^{(C)}}{dz^2} + (D^2 + \frac{m}{m-2} \partial_2^2) v^{(C)} + (\frac{m}{m-2} \partial_1 \partial_2) u^{(C)} + (\frac{m}{m-2} \partial_2) \frac{dw^{(C)}}{dz} = 0 \quad (27)$$

$$\left(\frac{2m-2}{m-2}\right) \frac{d^2 w^{(C)}}{dz^2} + \left(\frac{m}{m-2} \partial_1\right) \frac{du^{(C)}}{dz} + \left(\frac{m}{m-2} \partial_2\right) \frac{dv^{(C)}}{dz} + D^2 w^{(C)} = 0 \quad (28)$$

where the symbols of differentiation  $\partial_1$ ,  $\partial_2$ ,  $D^2$  are to be interpreted as numbers.

Upon integrating the above system subject to the initial conditions\*,

$$\begin{aligned}
 u^{(C)} &= u_0, \quad v^{(C)} = v_0, \quad w^{(C)} = w_0, \quad \frac{du^{(C)}}{dz} = u_0^1, \quad \frac{dv^{(C)}}{dz} = v_0^1 \\
 \frac{dw^{(C)}}{dz} &= w_0^1, \quad \text{for } z=0,
 \end{aligned}
 \tag{29}$$

one has after a few simple calculations\*\*

$$\begin{aligned}
 u^{(C)} &= -\frac{m-2}{2(m-1)} \frac{\sin(zD)}{D} \partial_1 w_0 - \frac{m}{2(m-1)} z \cos(zD) \partial_1 w_0 \\
 &+ \cos(zD) u_0 - \frac{m}{2(m-2)} \frac{z \sin(zD)}{D} \partial_1 \theta_0
 \end{aligned}
 \tag{30}$$

$$\begin{aligned}
 v^{(C)} &= -\frac{m-2}{2(m-1)} \frac{\sin(zD)}{D} \partial_2 w_0 - \frac{m}{2(m-1)} z \cos(zD) \partial_2 w_0 \\
 &+ \cos(zD) v_0 - \frac{m}{2(m-2)} \frac{z \sin(zD)}{D} \partial_2 \theta_0
 \end{aligned}
 \tag{31}$$

$$\begin{aligned}
 w^{(C)} &= \cos(zD) w_0 + \frac{m}{2(m-1)} Dz \sin(zD) w_0 \\
 &- \frac{1}{m-2} \frac{\sin(zD)}{D} \theta_0 + \frac{m}{2(m-2)} \left[ \frac{\sin(zD)}{D} - z \cos(zD) \right] \theta_0
 \end{aligned}
 \tag{32}$$

where

$$\theta_0 = \frac{m-2}{m-1} (\partial_1 u_0 + \partial_2 v_0)
 \tag{33}$$

Finally, in order to satisfy the boundary conditions (24) we require that

$$[mhD \sin(hD) \partial_1] u_0 + [mhD \partial_2 \sin(hD)] v_0 - mD [\sin(hD) - hD \cos(hD)] w_0 = 0
 \tag{34}$$

$$\left[ \frac{\sin(hD)}{D} (-\partial_1^2 - (m-1)D^2) - mh \cos(hD) \partial_1^2 \right] u_0 + \left[ -\frac{\sin(hD)}{D} \partial_1 \partial_2 \right.
 \tag{35}$$

$$\left. - mh \cos(hD) \partial_1 \partial_2 \right] v_0 + [mh \partial_1 D \sin(hD)] w_0 = 0$$

\* $u_0, v_0, w_0, u_0^1, v_0^1, w_0^1$  are arbitrary functions of  $x$  and  $y$   
 \*\*Note that in equations (30) - (32) we have let

$$\begin{aligned}
 u_0 &= -\partial_1 w_0 \\
 v_0 &= -\partial_2 w_0 \\
 w_0 &= -\frac{1}{m-2} \theta_0
 \end{aligned}$$

in order to simultaneously satisfy the boundary condition (23).

$$\begin{aligned}
 & -[mh \cos(hD) \partial_1 \partial_2 + \frac{\sin(hD)}{D} \partial_1 \partial_2] u_0 - [mh \cos(hD) \partial_2^2 + \frac{\sin(hD)}{D} \partial_2^2 \\
 & + (m-1) \sin(hD) D] v_0 + [hm \sin(hD) D \partial_2] w_0 = 0
 \end{aligned} \tag{36}$$

or

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

where the differential operators  $d_{ik}$  are defined as

$$\begin{aligned}
 d_{11} &= mh \partial_1 D \sin(hD) \\
 d_{12} &= mh \partial_2 D \sin(hD) \\
 d_{13} &= -mD [\sin(hD) - hD \cos(hD)] \\
 d_{21} &= \frac{1}{D} [\sin(hD) (\partial_2^2 - mD^2) - mhD \partial_1^2 \cos(hD)] \\
 d_{22} &= -\frac{1}{D} [\sin(hD) + mhD \cos(hD)] \partial_1 \partial_2 \\
 d_{23} &= mh \partial_1 D \sin(hD) \\
 d_{31} &= -[\sin(hD) + mhD \cos(hD)] \frac{1}{D} \partial_1 \partial_2 \\
 d_{32} &= -\frac{1}{D} [\sin(hD) (mD^2 - \partial_1^2) + mh \partial_2^2 D \cos(hD)] \\
 d_{33} &= mh \partial_2 D \sin(hD) .
 \end{aligned} \tag{38}$$

Keeping in mind that the differential operators  $\partial_1$ ,  $\partial_2$ ,  $D^2$  obey the same formal rules of addition and multiplication as numbers, the solution of system (37) is given by

$$\begin{aligned}
 u_0 &= \chi_1(x, y) \\
 v_0 &= \chi_2(x, y) \\
 w_0 &= \chi_3(x, y)
 \end{aligned} \tag{39}$$

where the unknown displacement functions  $\chi_1$ ,  $\chi_2$ ,  $\chi_3$  satisfy the differential relations

$$\nabla^2 \chi_i = 0 \quad i = 1, 2, 3 \tag{40}$$

with

$$Q = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = m^2(m-1) D^3 \sin(hD) \{h^2 D^2 - \sin^2(hD)\} \quad (41)$$

We construct next the following integral representations for  $u_0$ ,  $v_0$  and  $w_0$  which have the proper behavior at infinity

$$u_0(x, y^\pm) = \int_0^\infty \{ (P_1 + |y|Q_1) e^{-s|y|} + \sum_{\nu=1}^\infty R_\nu^{(1)} e^{-\sqrt{s^2 + \beta_\nu^2} |y|} + \sum_{\nu=1}^\infty \tilde{R}_\nu^{(1)} e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} + \sum_{n=1}^\infty S_n^{(1)} e^{-\sqrt{s^2 + \alpha_n^2} |y|} \} \sin(xs) ds \quad (42)$$

$$v_0(x, y^\pm) = \mp \int_0^\infty \{ (P_2 + |y|Q_2) e^{-s|y|} + \sum_{\nu=1}^\infty R_\nu^{(2)} e^{-\sqrt{s^2 + \beta_\nu^2} |y|} + \sum_{\nu=1}^\infty \tilde{R}_\nu^{(2)} e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} + \sum_{n=1}^\infty S_n^{(2)} e^{-\sqrt{s^2 + \alpha_n^2} |y|} \} \cos(xs) ds \quad (43)$$

$$w_0(x, y^\pm) = \int_0^\infty \{ (P_3 + |y|Q_3) e^{-s|y|} + \sum_{\nu=1}^\infty R_\nu^{(3)} e^{-\sqrt{s^2 + \beta_\nu^2} |y|} + \sum_{\nu=1}^\infty \tilde{R}_\nu^{(3)} e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} + \sum_{n=1}^\infty S_n^{(3)} e^{-\sqrt{s^2 + \alpha_n^2} |y|} \} \cos(xs) ds \quad (44)$$

The  $\pm$  signs refer to  $y > 0$  and  $y < 0$  respectively,  $\alpha_n = \frac{n\pi}{h}$  ( $n=1, 2, 3, \dots$ ) and  $\beta_\nu, \gamma_\nu$  are the roots of the equations

$$\sin(\beta_\nu h) = (\beta_\nu h) \quad (45)$$

$$\sin(\gamma_\nu h) = -(\gamma_\nu h) \quad (46)$$

The equations have an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are chosen to be the complex conjugate pairs with positive real parts. The only real roots  $\beta_\nu = \gamma_\nu = 0$  must be ignored.



Finally, an examination of the solution shows that the unknown functions  $P$ ,  $Q$ ,  $R$  etc. are not all independent. Assuming, therefore, that one can differentiate under the integral sign and inserting equations (42)-(44) into (37) one finds

$$Q_2 = -Q_1 \quad (47)$$

$$(1+m) s (P_1 + P_2) + (3m-1) Q_1 + 2smh Q_3 = 0 \quad (48)$$

$$hR_V^{(1)} = -\frac{(1 - \cos(\beta_V h))}{\beta_V^2} sR_V \quad (49)$$

$$hR_V^{(2)} = \frac{\sqrt{s^2 + \beta_V^2}}{\beta_V^2} (1 - \cos(\beta_V h)) R_V \quad (50)$$

$$R_V^{(3)} = R_V \quad (51)$$

$$h\tilde{R}_V^{(1)} = -\frac{s}{\gamma_V^2} (1 + \cos \gamma_V h) \tilde{R}_V \quad (52)$$

$$h\tilde{R}_V^{(2)} = \frac{\sqrt{s^2 + \gamma_V^2}}{\gamma_V^2} (1 + \cos \gamma_V h) \tilde{R}_V \quad (53)$$

$$\tilde{R}_V^{(3)} = \tilde{R}_V \quad (54)$$

$$S_n^{(3)} = 0 \quad (55)$$

$$S_n^{(1)} = -\frac{\sqrt{s^2 + \alpha_n^2}}{s} S_n \quad (56)$$

$$S_n^{(2)} = S_n \quad (57)$$

In order to facilitate our subsequent discussion it is found convenient at this stage to summarize our results:

(i) complementary displacements\*:

$$u^{(C)} = \int_0^\infty \left\{ [P_1 + sz P_3 + \frac{3m-2}{2(m-1)} z^2 s Q_1 + \frac{m}{2(m-1)} z^2 s^2 (P_1 + P_2) + |y| Q_1] e^{-s|y|} \right.$$

\*It can be shown, that in order to satisfy the remaining boundary conditions  $Q_3$  must vanish. This information is used when writing the complementary displacements and stresses.

$$\begin{aligned}
& - \sum_{\nu=1}^{\infty} s R_{\nu} \left\{ \left( \frac{1 - \cos \beta_{\nu} h}{\beta_{\nu}^2 h} \right) \left[ \cos \beta_{\nu} z - \frac{m}{2(m-1)} \beta_{\nu} z \sin \beta_{\nu} z \right] \right. \\
& - \frac{m-2}{2(m-1)} \frac{1}{\beta_{\nu}} \left[ \sin(\beta_{\nu} z) + \frac{m}{m-2} \beta_{\nu} z \cos(\beta_{\nu} z) \right] \left. \right\} e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|} \\
& - \sum_{\nu=1}^{\infty} s \tilde{R}_{\nu} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu}^2 h} \left[ \cos(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z) \right] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_{\nu}} \left[ \sin(\gamma_{\nu} z) + \frac{m}{m-2} \gamma_{\nu} z \cos(\gamma_{\nu} z) \right] \right\} e^{-\sqrt{s^2 + \gamma_{\nu}^2} |y|} \\
& - \sum_{n=1}^{\infty} \frac{\sqrt{s^2 + \alpha_n^2}}{s} S_n \cos(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2} |y|} \left. \right\} \sin(xs) ds
\end{aligned} \tag{58}$$

$$\begin{aligned}
v^{(C)} &= \mp \int_0^{\infty} \left\{ [P_2 - sz P_3 - \frac{3m-2}{2(m-1)} z^2 s Q_1 - \frac{m}{2(m-1)} z^2 s^2 (P_1 + P_2) - |y| Q_1] e^{-s|y|} \right. \\
& \quad \left. + \sum_{n=1}^{\infty} S_n \cos(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2} |y|} \right. \\
& + \sum_{\nu=1}^{\infty} R_{\nu} \sqrt{s^2 + \beta_{\nu}^2} \left\{ \frac{(1 - \cos(\beta_{\nu} h))}{\beta_{\nu}^2 h} \left[ \cos(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z) \right] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_{\nu}} \left[ \sin \beta_{\nu} z + \frac{m}{m-2} \beta_{\nu} z \cos \beta_{\nu} z \right] \right\} e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|} \\
& + \sum_{\nu=1}^{\infty} \tilde{R}_{\nu} \sqrt{s^2 + \gamma_{\nu}^2} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu}^2 h} \left[ \cos(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \sin \gamma_{\nu} z \right] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_{\nu}} \left[ \sin(\gamma_{\nu} z) + \frac{m}{m-2} \gamma_{\nu} z \cos(\gamma_{\nu} z) \right] \right\} e^{-\sqrt{s^2 + \gamma_{\nu}^2} |y|} \left. \right\} \cos(xs) ds .
\end{aligned} \tag{59}$$

$$\begin{aligned}
w^{(C)} &= \int_0^{\infty} \left\{ [P_3 - \frac{1}{m-1} sz (P_1 + P_2) - \frac{1}{m-1} z Q_1] e^{-s|y|} \right. \\
& + \sum_{\nu=1}^{\infty} R_{\nu} \left\{ \frac{(1 - \cos(\beta_{\nu} h))}{\beta_{\nu} h} \left[ \frac{m-2}{2(m-1)} \sin(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \cos(\beta_{\nu} z) \right] \right. \\
& \quad \left. + \cos(\beta_{\nu} z) + \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z) \right\} e^{-\sqrt{s^2 + \beta_{\nu}^2} |y|} \\
& + \sum_{\nu=1}^{\infty} \tilde{R}_{\nu} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu} h} \left[ \frac{m-2}{2(m-1)} \sin(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \cos(\gamma_{\nu} z) \right] \right. \\
& \quad \left. + \cos(\gamma_{\nu} z) + \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z) \right\} e^{-\sqrt{s^2 + \gamma_{\nu}^2} |y|} \left. \right\} \cos(xs) ds .
\end{aligned} \tag{60}$$

$$\begin{aligned}
\theta^{(c)} = & \int_0^\infty \left\{ -2 \left( \frac{m-2}{m+1} \right) Q_1 e^{-s|y|} + \frac{m-2}{m-1} \sum_{\nu=1}^\infty \frac{1}{h} R_\nu \left[ (1 - \cos(\beta_\nu h)) \cos(\beta_\nu z) \right. \right. \\
& - \left. \beta_\nu h \sin(\beta_\nu z) \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \\
& + \frac{m-2}{m-1} \sum_{\nu=1}^\infty \frac{\tilde{R}_\nu}{h} \left[ (1 + \cos(\gamma_\nu h)) \cos(\gamma_\nu z) - \gamma_\nu h \sin(\gamma_\nu z) \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} \left. \right\} \cos(xs) ds .
\end{aligned} \tag{61}$$

(ii) complementary stresses:

$$\begin{aligned}
\frac{\tau_{yz}}{G} &^{(c)} = \int_0^\infty \left\{ -\frac{m}{m-1} \sum_{\nu=1}^\infty R_\nu \sqrt{s^2 + \beta_\nu^2} \left[ \frac{(1 - \cos(\beta_\nu h))}{\beta_\nu h} (\sin(\beta_\nu z) + \beta_\nu z \cos(\beta_\nu z)) \right. \right. \\
& - \left. \beta_\nu z \sin(\beta_\nu z) \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \\
& - \frac{m}{m-1} \sum_{\nu=1}^\infty \tilde{R}_\nu \sqrt{s^2 + \gamma_\nu^2} \left[ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu h} (\sin(\gamma_\nu z) + \gamma_\nu z \cos(\gamma_\nu z)) \right. \\
& \left. \left. - \gamma_\nu z \sin(\gamma_\nu z) \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} - \sum_{n=1}^\infty S_n \alpha_n \sin(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2} |y|} \right\} \cos(xs) ds .
\end{aligned} \tag{62}$$

$$\begin{aligned}
\frac{\tau_{zx}}{2G} &^{(c)} = \int_0^\infty \left\{ \frac{m}{m-1} \sum_{\nu=1}^\infty s R_\nu \left[ \frac{(1 - \cos(\beta_\nu h))}{\beta_\nu h} (\sin(\beta_\nu z) + \beta_\nu z \cos(\beta_\nu z)) - \beta_\nu z \sin(\beta_\nu z) \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \right. \\
& + \frac{m}{m-1} \sum_{\nu=1}^\infty s \tilde{R}_\nu \left[ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu h} (\sin(\gamma_\nu z) + \gamma_\nu z \cos(\gamma_\nu z)) - \gamma_\nu z \sin(\gamma_\nu z) \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} \\
& \left. + \sum_{n=1}^\infty \frac{\sqrt{s^2 + \alpha_n^2}}{s} \alpha_n S_n \sin(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2} |y|} \right\} \sin(xs) ds .
\end{aligned} \tag{63}$$

$$\begin{aligned}
\frac{\sigma_z}{2G} &^{(c)} = \int_0^\infty \left\{ \frac{m}{2(m-1)} \sum_{\nu=1}^\infty \frac{R_\nu}{h} \left[ (1 - \cos(\beta_\nu h)) \beta_\nu z \sin(\beta_\nu z) - \beta_\nu h \sin(\beta_\nu z) \right. \right. \\
& \left. + \beta_\nu h \beta_\nu z \cos(\beta_\nu z) \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \\
& + \frac{m}{2(m-1)} \sum_{\nu=1}^\infty \frac{\tilde{R}_\nu}{h} \left[ (1 + \cos(\gamma_\nu h)) \gamma_\nu z \sin(\gamma_\nu z) - \gamma_\nu h \sin(\gamma_\nu z) \right. \\
& \left. + \gamma_\nu h \gamma_\nu z \cos(\gamma_\nu z) \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} \left. \right\} \cos(sx) ds .
\end{aligned} \tag{64}$$

$$\begin{aligned}
\frac{\sigma_y}{2G} &= \int_0^\infty \left\{ [s P_2 - s^2 z P_3 - \frac{z^2 s^2 Q_1}{m+1} - s|y|Q_1 + \frac{(m-1)}{m+1}Q_1] e^{-s|y|} \right. \\
&+ \frac{1}{m-1} \sum_{\nu=1}^{\infty} \frac{R_\nu}{h} [(1 - \cos(\beta_\nu h)) \cos(\beta_\nu z) - \beta_\nu h \sin(\beta_\nu z)] e^{-\sqrt{s^2 + \beta_\nu^2}|y|} \\
&+ \sum_{\nu=1}^{\infty} R_\nu (s^2 + \beta_\nu^2) \left[ \frac{(1 - \cos(\beta_\nu h))}{h} [\cos(\beta_\nu z) - \frac{m}{2(m-1)} \beta_\nu z \sin(\beta_\nu z)] \right. \\
&\quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_\nu} [\sin(\beta_\nu z) + \frac{m}{m-2} \beta_\nu z \cos(\beta_\nu z)] \right] e^{-\sqrt{s^2 + \beta_\nu^2}|y|} \\
&+ \sum_{n=1}^{\infty} \sqrt{s^2 + \alpha_n^2} S_n \cos(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2}|y|} \\
&+ \frac{1}{m-1} \sum_{\nu=1}^{\infty} \frac{R'_\nu}{h} [(1 + \cos(\gamma_\nu h)) \cos(\gamma_\nu z) - \gamma_\nu h \sin(\gamma_\nu z)] e^{-\sqrt{s^2 + \gamma_\nu^2}|y|} \\
&+ \sum_{\nu=1}^{\infty} R'_\nu (s^2 + \gamma_\nu^2) \left[ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu h} [\cos(\gamma_\nu z) - \frac{m}{2(m-1)} \gamma_\nu z \sin(\gamma_\nu z)] \right. \\
&\quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_\nu} [\sin(\gamma_\nu z) + \frac{m}{m-2} \gamma_\nu z \cos(\gamma_\nu z)] \right] e^{-\sqrt{s^2 + \gamma_\nu^2}|y|} \left. \right\} \cdot \cos(xs) ds . \tag{65}
\end{aligned}$$

$$\begin{aligned}
\frac{\tau_{xy}}{G} &= - \int_0^\infty \{ [s(P_1 - P_2) + 2s^2 z P_3 + \frac{2}{m+1} z^2 s^2 Q_1 + 2|y|sQ_1 - Q_1] e^{-s|y|} \\
&- \sum_{\nu=1}^{\infty} s \sqrt{s^2 + \beta_\nu^2} R_\nu \left[ \frac{(1 - \cos(\beta_\nu h))}{\beta_\nu^2 h} (2 \cos(\beta_\nu z) - \frac{m}{m-1} \beta_\nu z \sin(\beta_\nu z)) \right. \\
&\quad \left. - \frac{m-2}{m-1} \frac{1}{\beta_\nu} (\sin(\beta_\nu z) + \frac{m}{m-2} \beta_\nu z \cos(\beta_\nu z)) \right] e^{-\sqrt{s^2 + \beta_\nu^2}|y|} \\
&- \sum_{\nu=1}^{\infty} s \sqrt{s^2 + \gamma_\nu^2} R'_\nu \left[ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu h} (2 \cos(\gamma_\nu z) - \frac{m}{m-1} \gamma_\nu z \sin(\gamma_\nu z)) \right. \\
&\quad \left. - \frac{m-2}{m-1} \frac{1}{\gamma_\nu} (\sin(\gamma_\nu z) + \frac{m}{m-2} \gamma_\nu z \cos(\gamma_\nu z)) \right] e^{-\sqrt{s^2 + \gamma_\nu^2}|y|} \\
&- \sum_{n=1}^{\infty} \frac{(2s^2 + \alpha_n^2)}{s} S_n \cos(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2}|y|} \left. \right\} \sin(xs) ds . \tag{66}
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma_x}{2G} &= \int_0^\infty \{ [s P_1 + s^2 z P_3 + \frac{1}{m+1} z^2 s^2 Q_1 + y s Q_1] e^{-s|y|} \\
&- \sum_{\nu=1}^{\infty} s^2 R_\nu \left[ \frac{(1 - \cos(\beta_\nu h))}{\beta_\nu^2 h} \left[ \cos(\beta_\nu z) - \frac{m}{2(m-1)} \beta_\nu z \sin(\beta_\nu z) \right] \right. \\
&\quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_\nu} \left[ \sin(\beta_\nu z) + \frac{m}{m-2} \beta_\nu z \cos(\beta_\nu z) \right] \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \\
&+ \frac{1}{m-1} \sum_{\nu=1}^{\infty} \frac{R_\nu}{h} \left[ (1 - \cos(\beta_\nu h)) \cos(\beta_\nu z) - \beta_\nu h \sin(\beta_\nu z) \right] e^{-\sqrt{s^2 + \beta_\nu^2} |y|} \\
&- \sum_{\nu=1}^{\infty} s^2 \gamma_\nu \left[ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu^2 h} \left[ \cos(\gamma_\nu z) - \frac{m}{2(m-1)} \gamma_\nu z \sin(\gamma_\nu z) \right] \right. \\
&\quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_\nu} \left[ \sin(\gamma_\nu z) + \frac{m}{m-2} \gamma_\nu z \cos(\gamma_\nu z) \right] \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} \\
&+ \frac{1}{m-1} \sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{h} \left[ (1 + \cos(\gamma_\nu h)) \cos(\gamma_\nu z) - \gamma_\nu h \sin(\gamma_\nu z) \right] e^{-\sqrt{s^2 + \gamma_\nu^2} |y|} \\
&- \sum_{n=1}^{\infty} \sqrt{s^2 + \alpha_n^2} S_n \cos(\alpha_n z) e^{-\sqrt{s^2 + \alpha_n^2} |y|} \} \cos(xs) ds.
\end{aligned} \tag{67}$$

By direct substitution, it can easily be ascertained that the above complementary displacements satisfy Navier's equations and furthermore the corresponding stresses  $\sigma_z^{(c)}$ ,  $\tau_{zx}^{(c)}$ ,  $\tau_{yz}^{(c)}$  do vanish at the plate faces  $z = 0$  and  $z = h$ .

Moreover, to satisfy the continuity conditions, one must require that:

$$\begin{aligned}
& \int_0^{\infty} \left\{ [P_2 - sz P_3 - \frac{3m-2}{2(m-1)} z^2 s Q_1 - \frac{m}{2(m-1)} z^2 s^2 (P_1 + P_2)] \right. \\
& \quad \left. + \sum_{n=1}^{\infty} s_n \cos(\alpha_n z) \right. \\
& + \sum_{v=1}^{\infty} R_v \sqrt{s^2 + \beta_v^2} \left\{ \frac{(1 - \cos(\beta_v h))}{\beta_v^2 h} [\cos(\beta_v z) - \frac{m}{2(m-1)} \beta_v z \sin(\beta_v z)] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_v} [\sin(\beta_v z) + \frac{m}{m-2} \beta_v z \cos(\beta_v z)] \right\} \\
& + \sum_{v=1}^{\infty} R_v \sqrt{s^2 + \gamma_v^2} \left\{ \frac{(1 + \cos(\gamma_v h))}{\gamma_v^2 h} [\cos(\gamma_v z) - \frac{m}{2(m-1)} \gamma_v z \sin(\gamma_v z)] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_v} [\sin(\gamma_v z) + \frac{m}{m-2} \gamma_v z \cos(\gamma_v z)] \right\} \cos(xs) ds \\
& = \begin{cases} + v^{(c)}(x, 0, z) ; \frac{x^2}{c^2} + \frac{z^2}{a^2} < 1 \\ 0 ; \frac{x^2}{c^2} + \frac{z^2}{a^2} > 1 , \end{cases}
\end{aligned} \tag{68}$$

$$\begin{aligned}
& \int_0^{\infty} \left\{ [sP_1 + s^2z P_3 + \frac{3m-1}{2(m-1)} z^2 s^2 Q_1 + \frac{m}{2(m-1)} z^2 s^3 (P_1 + P_2)] \right. \\
& - \sum_{\nu=1}^{\infty} \sqrt{s^2 + \beta_{\nu}^2} s R_{\nu} \left\{ \left( \frac{1 - \cos \beta_{\nu} h}{\beta_{\nu}^2 h} \right) [\cos(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z)] \right. \\
& - \left. \frac{m-2}{2(m-1)} \frac{1}{\beta_{\nu}} [\sin(\beta_{\nu} z) + \frac{m}{m-2} \beta_{\nu} z \cos(\beta_{\nu} z)] \right\} \\
& \left. - \sum_{\nu=1}^{\infty} \sqrt{s^2 + \gamma_{\nu}^2} s \tilde{R}_{\nu} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu}^2 h} [\cos(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z)] \right. \right. \\
& \left. \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_{\nu}} [\sin(\gamma_{\nu} z) + \frac{m}{m-2} \gamma_{\nu} z \cos(\gamma_{\nu} z)] \right\} \right. \\
& \left. - \sum_{n=1}^{\infty} \frac{s^2 + \alpha_n^2}{s} S_n \cos(\alpha_n z) \right\} \sin(xs) ds \\
& \left\{ \begin{array}{ll} + \left( \frac{\partial u(c)}{\partial y} \right)_{y=0} & ; \frac{x^2}{c^2} + \frac{z^2}{a^2} < 1 \\ 0 & ; \frac{x^2}{c^2} + \frac{z^2}{a^2} > 1 \end{array} \right.
\end{aligned}$$

and:

$$\begin{aligned}
& \int_0^{\infty} \left\{ [sP_3 - \frac{1}{m-1} s^2 z (P_1 + P_2) - \frac{1}{m-1} szQ_1] \right. \\
& + \sum_{\nu=1}^{\infty} \sqrt{s^2 + \beta_{\nu}^2} R_{\nu} \left\{ \frac{(1-\cos(\beta_{\nu}h))}{\beta_{\nu}h} \left[ \frac{m-2}{2(m-1)} \sin(\beta_{\nu}z) - \frac{m}{2(m-1)} \beta_{\nu}z \cos(\beta_{\nu}z) \right] \right. \\
& \quad \left. \left. + \cos(\beta_{\nu}z) + \frac{m}{2(m-1)} \beta_{\nu}z \sin(\beta_{\nu}z) \right\} \right. \\
& + \sum_{\nu=1}^{\infty} \sqrt{s^2 + \gamma_{\nu}^2} R_{\nu} \left\{ \frac{(1+\cos(\gamma_{\nu}h))}{\gamma_{\nu}h} \left[ \frac{m-2}{2(m-1)} \sin(\gamma_{\nu}z) - \frac{m}{2(m-1)} \gamma_{\nu}z \cos(\gamma_{\nu}z) \right] \right. \\
& \quad \left. \left. + \cos(\gamma_{\nu}z) + \frac{m}{2(m-1)} \gamma_{\nu}z \sin(\gamma_{\nu}z) \right\} \right\} \cos(xs) ds =
\end{aligned} \tag{70}$$

$$\begin{cases} + \left( \frac{\partial w}{\partial y} \right)_{y=0}^{(c)} & ; \frac{x^2}{c^2} + \frac{z^2}{a^2} < 1 \\ 0 & ; \frac{x^2}{c^2} + \frac{z^2}{a^2} > 1 . \end{cases}$$

Thus, by Fourier inversion, one has:



$$\begin{aligned}
& [P_2 - szP_3 - \frac{3m-2}{2(m-1)} z^2 sQ_1 - \frac{m}{2(m-1)} z^2 s^2 (P_1 + P_2)] \\
& + \sum_{n=1}^{\infty} S_n \cos(\alpha_n a) \\
& + \sum_{\nu=1}^{\infty} R_{\nu} \sqrt{s^2 + \beta_{\nu}^2} \left\{ \frac{(1 - \cos(\beta_{\nu} h))}{\beta_{\nu}^2 h} [\cos(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z)] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_{\nu}} [\sin(\beta_{\nu} z) + \frac{m}{m-2} \beta_{\nu} z \cos(\beta_{\nu} z)] \right\} \\
& + \sum_{\nu=1}^{\infty} R_{\nu} \sqrt{s^2 + \gamma_{\nu}^2} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu}^2 h} [\cos(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z)] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_{\nu}} [\sin(\gamma_{\nu} z) + \frac{m}{m-2} \gamma_{\nu} z \cos(\gamma_{\nu} z)] \right\} = \\
& = + \frac{2}{\pi} \int_0^S c \nu^{(c)} (\xi, 0, z) \cos(s\xi) d\xi
\end{aligned} \tag{71}$$

$$\begin{aligned}
& [sP_1 + s^2 z P_3 + \frac{3m-2}{2(m-1)} z^2 s^2 \overset{\varphi_1}{P_1} + \frac{m}{2(m-1)} z^2 s^2 (P_1 + P_2)] + \\
& - \sum_{\nu=1}^{\infty} \sqrt{s^2 + \beta_{\nu}^2} s R_{\nu} \left\{ \left( \frac{1 - \cos \beta_{\nu} h}{\beta_{\nu}^2 h} \right) [\cos(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z)] \right. \\
& - \left. \frac{m-2}{2(m-1)} \frac{1}{\beta_{\nu}} [\sin(\beta_{\nu} z) + \frac{m}{m-2} \beta_{\nu} z \cos(\beta_{\nu} z)] \right\} \\
& - \sum_{\nu=1}^{\infty} \sqrt{s^2 + \gamma_{\nu}^2} s R_{\nu} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu}^2 h} [\cos(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z)] \right. \\
& \quad \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_{\nu}} [\sin(\gamma_{\nu} z) + \frac{m}{m-2} \gamma_{\nu} z \cos(\gamma_{\nu} z)] \right\} \quad (72)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{n=1}^{\infty} \frac{s^2 + \alpha_n^2}{s} S_n \cos(\alpha_n z) = \\
& = + \frac{2}{\pi} \int_0^c \left( \frac{\partial u(c)}{\partial y} \right)_{y=0} \sin(s\xi) d\xi \\
& = + \frac{2}{\pi} \int_0^c \left( \frac{\partial v(c)}{\partial \xi} \right)_{y=0} \sin(s\xi) d\xi \\
& = + \frac{2s}{\pi} \int_0^c v(c) (\xi, 0, z) \cos(s\xi) d\xi
\end{aligned}$$

$$\begin{aligned}
& [sP_3 - \frac{1}{m-1} s^2 z (P_1 + P_2) - \frac{1}{m-1} szQ_1] \\
& + \sum_{\nu=1}^{\infty} \sqrt{s^2 + \beta_{\nu}^2} R_{\nu} \left\{ \frac{(1 - \cos(\beta_{\nu} h))}{\beta_{\nu} h} \left[ \frac{m-2}{2(m-1)} \sin(\beta_{\nu} z) - \frac{m}{2(m-1)} \beta_{\nu} z \cos(\beta_{\nu} z) \right] \right. \\
& \quad \left. + \cos(\beta_{\nu} z) + \frac{m}{2(m-1)} \beta_{\nu} z \sin(\beta_{\nu} z) \right\} \\
& + \sum_{\nu=1}^{\infty} \sqrt{s^2 + \gamma_{\nu}^2} \tilde{R}_{\nu} \left\{ \frac{(1 + \cos(\gamma_{\nu} h))}{\gamma_{\nu} h} \left[ \frac{m-2}{2(m-1)} \sin(\gamma_{\nu} z) - \frac{m}{2(m-1)} \gamma_{\nu} z \cos(\gamma_{\nu} z) \right] \right. \\
& \quad \left. + \cos(\gamma_{\nu} z) + \frac{m}{2(m-1)} \gamma_{\nu} z \sin(\gamma_{\nu} z) \right\} = \\
& = + \frac{2}{\pi} \int_0^S c \left( \frac{\partial w^{(c)}}{\partial y} \right)_{y=0} \cos(s\xi) d\xi \\
& = + \frac{2}{\pi} \int_0^S c \left( \frac{\partial v^{(c)}}{\partial z} \right)_{y=0} \cos(s\xi) d\xi
\end{aligned} \tag{73}$$

where for simplicity we have defined  $S_c = c\sqrt{1 - \frac{z^2}{a^2}}$  and in equations (72) and (73) we have made use of the remaining two boundary conditions:

$$\tau_{xy}^{(c)} = \tau_{zy}^{(c)} = 0 \quad \text{at } y = 0$$

The reader should notice that by adding eqs. (71) and (72) one concludes that

$$\begin{aligned}
\sum_{n=1}^{\infty} s S_n \cos(\alpha_n z) & = + \frac{4s}{\pi} \int_0^S c v^{(c)}(\xi, 0, z) \cos(s\xi) d\xi \\
& - s(P_1 + P_2)
\end{aligned} \tag{74}$$

from which\*,

$$s S_n = \bar{\alpha} \frac{8s}{\pi h} \int_0^a \int_0^{S_c} v(\xi, 0, \eta) \cos(s\xi) \alpha s (\alpha \eta) d\xi d\eta \quad (75a)$$

and

$$s S_0 = -2s (P_1 + P_2) \bar{\alpha} \frac{8s}{\pi h} \int_0^a \int_0^{S_c} v^{(c)}(\xi, 0, \eta) \cdot \cos(s\xi) d\xi d\eta. \quad (75b)$$

Finally, we would like to solve for the unknown coefficients  $R_\nu$  and  $\bar{R}_\nu$ . To accomplish this, we proceed as follows. Using eq. (74) into (71) and, upon differentiating w.r.t.  $z$  once, one has

$$\begin{aligned} & -sP_3 - \frac{3m-2}{m-1} zsQ_1 - \frac{m}{m-1} zs^2 (P_1 + P_2) \\ & + \sum_{\nu=1}^{\infty} R_\nu \sqrt{s^2 + \beta_\nu^2} \left\{ \frac{(1 - \cos(\beta_\nu h))}{\beta_\nu h} \left[ -\frac{3m-2}{2(m-1)} \sin(\beta_\nu z) - \frac{m}{2(m-1)} \beta_\nu z \cos(\beta_\nu z) \right] \right. \\ & \quad \left. + \left[ -\cos(\beta_\nu z) + \frac{m}{2(m-1)} \beta_\nu z \sin(\beta_\nu z) \right] \right\} \\ & + \sum_{\nu=1}^{\infty} \bar{R}_\nu \sqrt{s^2 + \gamma_\nu^2} \left\{ \frac{(1 + \cos(\gamma_\nu h))}{\gamma_\nu h} \left[ -\frac{3m-2}{2(m-1)} \sin(\gamma_\nu z) - \frac{m}{2(m-1)} \gamma_\nu z \cos(\gamma_\nu z) \right] \right. \\ & \quad \left. + \left[ -\cos(\gamma_\nu z) + \frac{m}{2(m-1)} \gamma_\nu z \sin(\gamma_\nu z) \right] \right\} \\ & = \pm \left( \frac{2}{\pi} \right) \int_0^{S_c} \left( \frac{\partial v^{(c)}}{\partial z} \right)_{y=0} \cos(s\xi) d\xi \end{aligned} \quad (76)$$

\* Notice that  $S_c$  is a function of  $\eta$  now.

Utilizing, next, the orthogonality condition\*, it is possible to determine the coefficients  $R_{\nu}$  and  $\tilde{R}_{\nu}$  in terms of the function  $(\frac{\partial v^{(c)}}{\partial z})_{y=0}$ .

Thus

$$R_{\nu} = \int_0^a \int_0^c (\frac{\partial v^{(c)}}{\partial \eta})_{y=0} B(\beta_{\nu} h, \beta_{\nu} \eta) \cos(s\xi) d\xi \quad (77)$$

and

$$\tilde{R}_{\nu} = \int_0^a \int_0^c (\frac{\partial v^{(c)}}{\partial \eta})_{y=0} \tilde{B}(\gamma_{\nu} h, \gamma_{\nu} \eta) \cos(s\xi) d\xi \quad (78)$$

Finally, inserting into equation (65) one reduces the problem to that of the solution of a double singular integral equation\*\*, i.e.

$$\iint_{\substack{\text{crack} \\ \text{faces}}} (\frac{\partial v^{(c)}}{\partial \eta})_{y=0} H[x - \xi; \beta_{\nu}, \gamma_{\nu}; z, \eta] d\xi d\eta \quad (79)$$

$$= -\frac{\sigma_0}{2g}; \quad \frac{x^2}{c^2} + \frac{z^2}{a^2} < 1,$$

where the kernel  $H$  consists of the sum of three infinite series of the type found in the through-the-thickness crack.

The explicit solution of this double singular integral equation, will determine the displacement and stress fields. Unfortunately, we have not been successful in extracting the solution to the equation explicitly. It appears, however, that the solution is not separable either in spherical or cylindrical coordinates.

\* See reference [2].

\*\* Eq. (79) may be integrated to give another singular integral eq. with  $v^{(c)}(\xi, 0, \eta)$  as the unknown.

## REFERENCES

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PART III

UNIQUENESS THEOREMS FOR DISPLACEMENT FIELDS WITH LOCALLY  
FINITE ENERGY IN LINEAR ELASTOSTATICS

## INTRODUCTION

The Classical Theory of Linear Elastostatics. The fundamental problem of linear elastostatics is to determine the equilibrium displacement field that is produced in an elastic body of known shape and composition by the action of known body forces and surface tractions or displacements. In the classical formulation of the theory the displacements and stresses are required to be differentiable and satisfy the differential equations of equilibrium in the interior of the body and to be continuous and satisfy the prescribed surface traction or displacement conditions on the boundary. This boundary value problem has a history that begins with A. L. Cauchy's discovery of the equilibrium equations in 1822; see reference [18, p. 8]. The uniqueness of classical solutions for bounded bodies with smooth surfaces was proved by G. Kirchhoff in 1859 [12]. General existence theorems for classical solutions were first proved during the period 1906-1908 by integral equation methods. The principal contributors were I. Fredholm [6], G. Lauricella [17], R. Marcolongo [19], A. Korn [15, 16] and T. Boggio [2, 3]. More recently G. Fichera has proved the existence of classical solutions in bounded bodies with smooth boundaries by the methods of modern functional analysis [4, 5]. Thus the theory of the classical boundary value problems of linear elastostatics is essentially complete.

The Need for a More General Theory. Unfortunately the classical theory described above provides an inadequate foundation for the analysis of most of the problems studied by applied scientists in their applications of linear elastostatics. Examination of any of the numerous books



on theoretical elasticity, beginning with the classical treatise of A. E. H. Love [18], reveals that most of the problems treated in them involve unbounded bodies, such as infinite plates or bars, and/or bodies having sharp edges or corners. Moreover, the stress fields are known to have singularities at re-entrant edges and corners. Examples of these difficulties can be found in the theory of cracks; see I. N. Sneddon and M. Lowengrub [22]. It is sometimes argued that the classical theory is a sufficient foundation for applications because real bodies are always bounded and boundaries with sharp edges and corners can be approximated by smooth ones. However, although this procedure simplifies the problems from the viewpoint of the classical theory, it makes them inaccessible to techniques such as separation of variables and integral transform methods that are used by applied scientists. Thus the real issue is whether a mathematical theory can be devised that is sufficiently general to provide a foundation for the analysis of the singular problems that are actually studied by applied scientists. The purpose of this paper is to provide the beginnings of such a theory comprising a formulation of the elastostatic boundary value problems that is applicable to bodies of arbitrary shape and corresponding uniqueness theorems.

Remarks on the Formulation of Boundary Value Problems. A

"formulation" of a boundary value problem is a definition of the class of functions in which solutions are to be sought. The classical formulation of the elastostatic boundary value problem was described above. Many other formulations are possible. For example, the continuity conditions may be replaced at some or all boundary points by boundedness or integrability conditions, the equilibrium equations may be required to hold in a weak sense, etc. In principle, any formulation is acceptable if

there is an existence theorem, stating that there is at least one solution in the class, and a uniqueness theorem, stating that there is at most one solution in the class. In practice the choice of a solution class turns on technical considerations. The proof of an existence theorem is facilitated by choosing a large solution class but uniqueness is lost if the class is too large. The proof of a uniqueness theorem is facilitated by choosing a small solution class but existence is lost if the class is too small. For example, Kirchhoff's theorem on the uniqueness of classical solutions of the elastostatic boundary value problem can be proved for bodies having re-entrant sharp edges but in this case no classical solution exists.

The Role of Existence and Uniqueness Theorems. A pure existence theorem for a boundary value problem demonstrates that the properties chosen to define the solution class are not contradictory; i.e., there are functions with these properties. In the presence of an existence theorem a uniqueness theorem shows that the defining properties of the solution class characterize the solution completely. However, a uniqueness theorem can be even more valuable when no general existence theorem is known. In such cases it may still be possible in certain instances, corresponding to special choices of the boundary or data, to construct a solution in the chosen solution class. A uniqueness theorem then shows that the solution is the correct one. An interesting example of this occurred in the theory of the diffraction of electromagnetic waves by a perfectly conducting circular disk. In 1948 J. Meixner [20] proved a uniqueness theorem for this problem and used it to show that a solution that had been published in 1927 was incorrect. Of course, in the absence of a general existence theorem it is desirable to prove uniqueness in as

large a solution class as possible since this facilitates application of the uniqueness theorem in specific instances.

The Boundedness Question for the Displacement Fields. Linear elastostatics is an approximation that is valid for small displacements. If the displacements are bounded then by suitable scaling they may be made arbitrarily small. Hence it is natural to make boundedness of the displacements a defining property of the solution class. Indeed, this property has often been employed in constructing solutions of particular problems. It has also been used by J. K. Knowles and T. A. Pucik [14] in the formulation and proof of a general uniqueness theorem for plane crack problems. However, it is shown in this paper that uniqueness holds in the larger class of solutions with locally finite energy, without boundedness conditions. This result shows that the boundedness hypothesis is redundant and the boundedness property, in instances where it holds, must be derivable from the other hypotheses.

Displacement Fields with Locally Finite Energy. In this paper it is taken as a fundamental principle that equilibrium displacement fields in elastic bodies must have finite strain energy in bounded portions of the bodies. Such displacement fields will be called displacement fields with locally finite energy (or, for brevity, fields wLFE). The equilibrium displacement field corresponding to prescribed body forces will be characterized among all fields wLFE, by the principle of virtual work. The class of displacement fields that obey these two principles will be called the solutions with locally finite energy (for brevity, solutions wLFE) of the elastostatic boundary value problems. The principal results of this paper are uniqueness theorems for this class of solutions. In particular, the uniqueness of solutions wLFE in

bounded bodies is proved without additional hypotheses concerning the boundary or the displacement field. The uniqueness of solutions wLFE in unbounded bodies is proved under a growth restriction on the behavior of the stress or displacement fields at infinity. Moreover, it is shown by examples that a growth restriction is necessary for uniqueness.

The remainder of the paper is organized as follows. The class of displacement fields wLFE is defined in §1. §2 contains the definition of the class of solutions wLFE in homogeneous elastic bodies of arbitrary shape, subject to prescribed surface tractions, prescribed body forces and prescribed displacements or stresses at infinity. The regularity properties of solutions wLFE are also discussed in this section. §3 presents the uniqueness theorems for solutions wLFE of problems with prescribed surface tractions. In §4 the methods and results of §3 are extended to the other classical boundary value problems of linear elastostatics including problems with prescribed surface displacements, problems with mixed boundary conditions, problems for inhomogeneous elastic bodies and n-dimensional generalizations. §5 contains a discussion of related literature.

## 1. DISPLACEMENT FIELDS WITH LOCALLY FINITE ENERGY

A fixed system of Cartesian coordinates is used throughout the paper and points of Euclidean space are identified with their coordinate triples  $(x_1, x_2, x_3) = x \in R^3$ . With this convention each elastic body in space is associated with a domain (open connected set)  $\Omega \subset R^3$  that describes the set of interior points of the body. The closure and boundary of  $\Omega$  are denoted by  $\bar{\Omega}$  and  $\partial\Omega = \bar{\Omega} - \Omega$ , respectively. The notation of Cartesian tensor analysis [11] is used to describe the physical variables associated with elastic bodies. In particular, tensors of various orders are denoted by subscripts and the summation convention is used.

The fundamental unknown of elastostatic boundary value problems is the displacement field. It is denoted below by  $u_i = u_i(x)$ . The notation  $u_{i,j} = \partial u_i / \partial x_j$  is used for the covariant derivative of  $u_i$ . The strain tensor field  $e_{ij}(u)$  associated with  $u_i$  is defined by the differential operator

$$(1.1) \quad e_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

It is assumed, following G. Green [7 and 18, pp. 11-12 and 95-99], that for quasi-static isothermal small deformations of an elastic body there is a positive definite quadratic function of  $e_{ij}$ ,

$$(1.2) \quad w = \frac{1}{2} c_{ijkl} e_{ij} e_{kl}$$

such that for all  $K \subset \Omega$

$$(1.3) \quad W_K = \frac{1}{2} \int_K c_{ijkl} e_{ij}(u) e_{kl}(u) dx$$

is the strain energy of the displacement field  $u_i$  in the set  $K$ . The positivity assumption means that

$$(1.4) \quad c_{ijkl} e_{ij} e_{kl} > 0 \text{ for all } e_{ij} = e_{ji} \neq 0$$

The stress-strain tensor  $c_{ijkl}$  is uniquely determined by  $w$  if the natural symmetries

$$(1.5) \quad c_{ijkl} = c_{jikl} = c_{klij}$$

are assumed. The stress tensor field  $\sigma_{ij}(u)$  associated with  $u_i$  is given by the differential operator

$$(1.6) \quad \sigma_{ij}(u) = c_{ijkl} e_{kl}(u)$$

The positive definiteness of  $w$  implies that  $\sigma_{ij} = c_{ijkl} e_{kl}$  has a unique solution  $e_{ij} = \gamma_{ijkl} \sigma_{kl}$  and  $w = \frac{1}{2} \sigma_{ij} e_{ij} = \frac{1}{2} \gamma_{ijkl} \sigma_{ij} \sigma_{kl}$ . In particular,

$$(1.7) \quad W_K = \frac{1}{2} \int_K \sigma_{ij}(u) e_{ij}(u) dx = \frac{1}{2} \int_K \gamma_{ijkl} \sigma_{ij}(u) \sigma_{kl}(u) dx$$

is a functional of  $\sigma_{ij}(u)$  alone. A body is homogeneous if and only if  $c_{ijkl}$  is constant in  $\Omega$ . It is isotropic if and only if [11, 18]

$$(1.8) \quad c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

where  $\lambda$  and  $\mu$  are scalars such that  $\mu > 0$ ,  $3\lambda + 2\mu > 0$ . The results in §2 and §3 are formulated for the case of homogeneous anisotropic bodies. In §4 it is shown that the uniqueness theorems hold for the more general case of inhomogeneous anisotropic media with bounded uniformly positive

definite stress-strain tensor. This means that the components  $c_{ijkl}(x)$  are Lebesgue measurable and there exist positive constants  $c_0$  and  $c_1 \geq c_0$  such that

$$(1.9) \quad c_0 e_{ij} e_{ij} \leq c_{ijkl}(x) e_{ij} e_{kl} \leq c_1 e_{ij} e_{ij} \text{ for all } x \in \Omega$$

and all  $e_{ij} = e_{ji}$ .

The most general uniqueness theorems for solutions wLFE will be obtained by making the class of displacement fields wLFE as large as possible subject to the LFE condition. Hence it is natural to define the energy integrals  $W_K(u)$  to be Lebesgue integrals and to interpret the differential operators  $e_{ij}$  in the distribution-theoretic sense. It can be shown that this choice has the additional advantage that the set of displacement fields wLFE is a complete space in the sense of convergence in energy on bounded sets. It was by using such complete function spaces that Fichera proved the existence of solutions of the elastostatic boundary value problems in bounded domains.

In the remainder of this section several function spaces are defined that are needed for the formulation and proof of the uniqueness theorems. In the definitions  $\Omega \subset \mathbb{R}^3$  denotes an arbitrary domain.

The definitions are based on the Lebesgue space

$$(1.10) \quad L_2(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \mid u(x) \text{ is } L\text{-measurable, } \int_{\Omega} u(x)^2 dx < \infty\}$$

and the associated spaces

$$(1.11) \quad L_2^{loc}(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \mid u \in L_2(K) \text{ for every bounded measurable } K \subset \Omega\}$$

$$(1.12) \quad L_2^{\text{int}}(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \mid u \in L_2(C) \text{ for every compact } C \subset \Omega\}$$

and

$$(1.13) \quad L_2^{\text{com}}(\Omega) = L_2(\Omega) \cap \{u \mid u(x) \text{ is equivalent to } 0 \text{ outside a bounded set}\}$$

It is clear that  $L_2^{\text{com}}(\Omega) \subset L_2(\Omega) \subset L_2^{\text{loc}}(\Omega) \subset L_2^{\text{int}}(\Omega)$ . Moreover,  $L_2^{\text{com}}(\Omega) = L_2(\Omega) = L_2^{\text{loc}}(\Omega)$  if and only if  $\Omega$  is bounded. Note that the condition  $u \in L_2^{\text{loc}}(\Omega)$  restricts the behavior of  $u$  near  $\partial\Omega$  because the sets  $K$  in (1.11) can be any bounded open subsets of  $\Omega$ . The condition  $u \in L_2^{\text{int}}(\Omega)$  is weaker because it does not restrict the behavior of  $u$  near  $\partial\Omega$ . All of the function spaces used below are spaces of tensor fields on  $\Omega$  whose components lie in certain linear subspaces of  $L_2^{\text{int}}(\Omega)$ .

The space  $L_2^{\text{int}}(\Omega)$  may be interpreted as a linear subspace of L. Schwartz's space  $\mathcal{D}'(\Omega)$  of all distributions on  $\Omega$  [21]. Thus functions  $u \in L_2^{\text{int}}(\Omega)$  have derivatives of all orders in  $\mathcal{D}'(\Omega)$  and if

$$(1.14) \quad A = \sum_{0 \leq |\alpha| \leq m} A_\alpha \partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$$

(where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ ) is a partial differential operator with constant coefficients then  $Au \in \mathcal{D}'(\Omega)$ . The notation  $Au \in L_2^{\text{int}}(\Omega)$  (resp.  $L_2^{\text{loc}}(\Omega)$ ,  $L_2(\Omega)$ ,  $L_2^{\text{com}}(\Omega)$ , etc.) will be interpreted to mean that the distribution  $Au$  is in the subspace  $L_2^{\text{int}}(\Omega)$  (resp.  $L_2^{\text{loc}}(\Omega)$ ,  $L_2(\Omega)$ ,  $L_2^{\text{com}}(\Omega)$ , etc.). If  $A_1, A_2, \dots, A_n$  is a set of partial differential operators with constant coefficients the following notation will be used.

$$(1.15) \quad L_2(A_1, A_2, \dots, A_n; \Omega) = L_2(\Omega) \cap \{u \mid A_j u \in L_2(\Omega), j = 1, 2, \dots, n\}$$



$$(1.16) \quad L_2^{\text{loc}}(A_1, A_2, \dots, A_n; \Omega) = L_2^{\text{loc}}(\Omega) \cap \{u \mid A_j u \in L_2^{\text{loc}}(\Omega), j = 1, 2, \dots, n\}$$

$$(1.17) \quad L_2^{\text{int}}(A_1, A_2, \dots, A_n; \Omega) = L_2^{\text{int}}(\Omega) \cap \{u \mid A_j u \in L_2^{\text{int}}(\Omega), j = 1, 2, \dots, n\}$$

$$(1.18) \quad L_2^{\text{com}}(A_1, A_2, \dots, A_n; \Omega) = L_2^{\text{com}}(\Omega) \cap L_2(A_1, A_2, \dots, A_n; \Omega)$$

In particular, if  $\{A_1, A_2, \dots, A_n\} = \{\partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \mid 0 \leq |\alpha| \leq m\}$

the following notation will be used.

$$(1.19) \quad L_2^m(\Omega) = L_2(A_1, A_2, \dots, A_n; \Omega)$$

$$(1.20) \quad L_2^{m, \text{loc}}(\Omega) = L_2^{\text{loc}}(A_1, A_2, \dots, A_n; \Omega)$$

$$(1.21) \quad L_2^{m, \text{int}}(\Omega) = L_2^{\text{int}}(A_1, A_2, \dots, A_n; \Omega)$$

$$(1.22) \quad L_2^{m, \text{com}}(\Omega) = L_2^{\text{com}}(A_1, A_2, \dots, A_n; \Omega)$$

Notations such as  $u_i \in L_2^{\text{loc}}(\Omega)$ ,  $e_{ij} \in L_2(\Omega)$ , etc. will be interpreted to mean that each component of the tensor field is in the indicated space. With this convention the classes of displacement fields wFE (with finite energy) and wLFE may be defined as follows.

Definition. A vector field  $u_i$  on  $\Omega$  is said to be a displacement field wFE if and only if it is in the function space

$$(1.23) \quad E(\Omega) = \{u \mid u_i \in L_2^{\text{loc}}(\Omega), e_{ij}(u) \in L_2(\Omega)\}$$

Similarly,  $u_i$  is said to be a displacement field wLFE if and only if it is in the function space

$$(1.24) \quad E^{\text{loc}}(\Omega) = \{u \mid u_i \in L_2^{\text{loc}}(\Omega), e_{ij}(u) \in L_2^{\text{loc}}(\Omega)\}$$

Note that  $E^{\text{loc}}(\Omega) = E(\Omega)$  if and only if  $\Omega$  is bounded.

The terminology used in the definition is justified by the observation that if the stress-strain tensor satisfies (1.9) then  $e_{ij}(u) \in L_2(\Omega)$  implies  $\sigma_{ij}(u) \in L_2(\Omega)$  and hence  $u \in E(\Omega)$  implies

$$(1.25) \quad W_\Omega = \frac{1}{2} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx < \infty$$

Similarly, if (1.9) holds then  $e_{ij}(u) \in L_2^{loc}(\Omega)$  implies  $\sigma_{ij}(u) \in L_2^{loc}(\Omega)$  and hence  $u \in E^{loc}(\Omega)$  implies

$$(1.26) \quad W_K = \frac{1}{2} \int_K \sigma_{ij}(u) e_{ij}(u) dx < \infty$$

for all bounded measurable sets  $K \subset \Omega$ .

Each of the function spaces defined above is a complete space with respect to a suitable topology. Several examples of this will be indicated. It is well known that  $L_2(\Omega)$  is a Hilbert space with scalar product

$$(1.27) \quad (u, v) = \int_\Omega u(x) v(x) dx$$

Similarly,  $E(\Omega)$  and  $E^{loc}(\Omega)$  are Fréchet spaces [28] with respect to the families of semi-norms defined by

$$(1.28) \quad \rho_{K,E}(u) = \left( \int_K u_i(x) u_i(x) dx + \int_\Omega \sigma_{ij}(u) \sigma_{ij}(u) dx \right)^{1/2}$$

and

$$(1.29) \quad \rho_{K,E}^{loc}(u) = \left( \int_K \{u_i(x) u_i(x) + \sigma_{ij}(u) \sigma_{ij}(u)\} dx \right)^{1/2}$$

respectively, where  $K$  is any bounded measurable subset of  $\Omega$ . In particular, if  $\Omega$  is bounded then  $E^{loc}(\Omega) = E(\Omega)$  is a Hilbert space. These

completeness results play no role in the uniqueness theorems given below. However, they are essential for the validity of existence theorems for solutions wLFE. This is evident from the proofs of Fichera's existence theorems for bounded bodies.

In the definition of  $E^{\text{loc}}(\Omega)$  the operators  $e_{ij}(u)$  defined by (1.1) are interpreted in the distribution-theoretic sense. Hence the condition  $u \in E^{\text{loc}}(\Omega)$  does not necessarily imply that the individual derivatives  $u_{i,j} \in L_2^{\text{loc}}(\Omega)$ . However, it is known that if  $u \in E^{\text{loc}}(\Omega)$  then  $u_{i,j} \in L_2(C)$  for every compact set  $C \subset \Omega$ . This is a consequence of Korn's inequality in the form

$$(1.30) \quad \|u_{i,j}\|_{L_2(C)}^2 \leq \gamma \left[ \sum_{i=1}^3 \|u_i\|_{L_2(K)}^2 + \sum_{i,j=1}^3 \|e_{ij}(u)\|_{L_2(K)}^2 \right]$$

which is valid for all  $u \in E^{\text{loc}}(\Omega)$ , all bounded open sets  $K \subset \Omega$  and all compact sets  $C \subset K$  with a constant  $\gamma = \gamma(C,K)$ . This result can be derived from the version of Korn's inequality due to J. Gobert [8]. Moreover if  $\Omega$  has the cone property [1, 9] then one may take  $C = K$  in (1.30). Hence in this case

$$(1.31) \quad u \in E^{\text{loc}}(\Omega) \Rightarrow u_i \in L_2^{1,\text{loc}}(\Omega)$$

In particular, for domains that are bounded and have the cone property

$$(1.32) \quad u \in E(\Omega) \Rightarrow u_i \in L_2^1(\Omega)$$

## 2. EQUILIBRIUM PROBLEMS WITH PRESCRIBED SURFACE TRACTIONS

In this section elastostatic equilibrium problems are formulated, and regularity properties of the solutions are discussed, for homogeneous anisotropic elastic bodies of arbitrary shape that are subject to prescribed body forces, prescribed surface tractions and, in the case of unbounded bodies, prescribed displacements or stresses at infinity. The cases of prescribed body forces  $F_i$ , zero surface tractions and zero displacements or stresses at infinity are discussed first.

The Principle of Virtual Work. Let  $\Omega \subset \mathbb{R}^3$  be an arbitrary domain and let  $u \in E^{loc}(\Omega)$  be the equilibrium displacement field wLFE corresponding to body forces  $F_i \in L_2^{com}(\Omega)$  and zero surface tractions. Imagine that the equilibrium is disturbed slightly by changing  $u_i$  to  $u_i + v_i$  where  $v_i$  is a field wFE from the set

$$(2.1) \quad E^{com}(\Omega) = E(\Omega) \cap \{v \mid e_{ij}(v) \in L_2^{com}(\Omega)\}$$

Let  $K \subset \Omega$  be a bounded measurable set such that  $e_{ij}(v)$  is equivalent to zero in  $\Omega - K$ . Then  $W_K(\sigma(u))$  and  $W_K(\sigma(u + v))$  are the strain energies in  $K$  before and after the disturbance. Hence the work done against internal forces during the disturbance is  $W_K(\sigma(u + v)) - W_K(\sigma(u))$ . The energy norm of  $v$  can be made arbitrarily small. If this is done and terms quadratic in  $v$  are dropped, in keeping with the linear theory, the difference becomes

$$(2.2) \quad \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) dx = \text{Work done against internal forces}$$

Moreover, if the body forces are constant during the displacement then

$$(2.3) \quad -\int_{\Omega} F_i v_i \, dx = \text{Work done against body forces}$$

No further work is done during the disturbance if the surface tractions are zero. The principle of virtual work states that the true equilibrium field  $u_i(x)$  is characterized by the property that the total work done against the internal and external forces in any (small) disturbance of  $u_i$  consistent with the constraints is zero [23]. Thus in the present case

$$(2.4) \quad \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) \, dx - \int_{\Omega} F_i v_i \, dx = 0$$

for all  $v \in E^{\text{com}}(\Omega)$ . This motivates the following

Definition. A displacement field  $u_i$  is said to be a solution wLFE of the equilibrium problem for the domain  $\Omega$  with body forces  $F_i \in L_2^{\text{com}}(\Omega)$  and zero surface tractions if and only if  $u \in E^{\text{loc}}(\Omega)$  and (2.4) holds for all  $v \in E^{\text{com}}(\Omega)$ .

Necessary Conditions for the Solvability of Problems with Zero Surface Traction. The fields

$$(2.5) \quad v_i(x) = a_i + \epsilon_{ijk} b_j x_k, \quad x \in R^3$$

where  $a_i$  and  $b_i$  are constant vectors and  $\epsilon_{ijk}$  is the alternating tensor [11] satisfy  $e_{ij}(v) = 0$  in  $R^3$  and hence  $v \in E^{\text{com}}(R^3)$ . In particular,  $v \in E^{\text{com}}(\Omega)$  for every domain  $\Omega$ . It follows from (2.4) with this choice of  $v$  that necessary conditions for the existence of a solution wLFE are

$$(2.6) \quad \int_{\Omega} F_i \, dx = 0$$

$$(2.7) \quad \int_{\Omega} (F_i x_j - F_j x_i) \, dx = 0$$

Physically, these conditions mean that the body forces  $F_i$  exert no net resultant or moment on the body. They are assumed to be satisfied in the remainder of the discussion of problems with zero surface tractions.

Non-uniqueness of the Displacements for Problems with Zero Surface Traction. Equations (2.5) define a displacement field that describes a rigid body displacement [11]. Moreover, since  $e_{ij}(v) = 0$  in  $R^3$  the fields (2.5) may be added to any solution  $u$  of (2.4). Physically, this means that the equilibrium displacement fields are determined only up to rigid body displacements. Hence, the natural uniqueness theorem for problems with zero surface tractions asserts that the stress and strain fields are unique while the displacement fields are unique modulo fields of the form (2.5).

Bounded Bodies and Displacement Fields wFE. If  $\Omega$  is bounded then  $E^{loc}(\Omega) = E(\Omega)$  and every solution wLFE actually has finite total strain energy in  $\Omega$ . More generally, if  $u$  is a solution wLFE for an arbitrary domain  $\Omega$  and if  $u \in E(\Omega)$  then  $u$  is said to be a solution wFE. The uniqueness of solutions wFE is proved in §3 without additional hypotheses concerning  $\Omega$  or the displacement field.

Unbounded Bodies and Equilibrium States with Prescribed Stresses or Displacements at Infinity. If  $\Omega$  is unbounded then, in general, solutions wLFE in  $\Omega$  are not unique. Simple examples of non-uniqueness are available for the case  $\Omega = R^3$ . The field  $u_i(x) = b_{ij}x_j$  with constant  $b_{ij} = b_{ji} \neq 0$  is a solution wLFE in  $R^3$  with  $F_i(x) \equiv 0$  and  $\sigma_{ij}(u) = c_{ijkl} b_{kl} \neq 0$  since  $e_{ij}(u) = b_{ij}$  and  $\sigma_{ij}(u) e_{ij}(u) = c_{ijkl} b_{ij} b_{kl} > 0$ . A second example is provided by the homogeneous isotropic plate with domain  $\Omega = \{x \mid x_1, x_2 \in R, |x_3| < h\}$  and stress-strain tensor (1.8). In this case  $u_1 = (\lambda + 2\mu)x_1$ ,  $u_2 = (\lambda + 2\mu)x_2$ ,  $u_3 = -2\lambda x_3$  defines a

displacement field in  $\Omega$  with  $F_1(x) \equiv 0$ , zero surface tractions and constant non-zero stress field  $\sigma_{11} = \sigma_{22} = 6\lambda\mu + 4\mu^2$ , all other  $\sigma_{ij} = 0$ . These examples show that uniqueness theorems for solutions wLFE in unbounded domains cannot hold without some growth restrictions at infinity on  $u_i$  or  $\sigma_{ij}$ .

The problem of finding suitable growth restrictions on  $u_i$  or  $\sigma_{ij}$  that guarantee the uniqueness of solutions wLFE is a special case of the classical problem of elastostatics of finding equilibrium displacement fields that have prescribed stresses or displacements at infinity. Many problems of this type are discussed in the treatise of Love [18]. To formulate the problem with prescribed stresses at infinity let

$$(2.8) \quad \Omega_{R,\infty} = \Omega \cap \{x \mid |x| > R\}$$

and let  $\sigma_{ij}^\infty(x)$  be a stress field that is defined in  $\Omega_{R,\infty}$  for some  $R$ , and has the desired behavior at infinity. A solution wLFE in  $\Omega$  is sought such that  $\sigma_{ij}(u)(x)$  is close to  $\sigma_{ij}^\infty(x)$  at infinity, in a suitable sense. One possibility is to require that  $\sigma_{ij}(u) - \sigma_{ij}^\infty \in L_2(\Omega_{R,\infty})$  or, equivalently,

$$(2.9) \quad W_{\Omega_{R,\infty}}(\sigma(u) - \sigma^\infty) < \infty$$

This suggests the

Definition. A solution wLFE of the equilibrium problem for an unbounded domain  $\Omega$  is said to have prescribed stresses  $\sigma_{ij}^\infty$  at infinity if and only if (2.9) holds for some  $R > 0$ .

Solutions wLFE with stresses  $\sigma_{ij}^\infty = 0$  at infinity are just the solutions wFE defined above. Condition (2.9) is correct in this case, at least for exterior domains where the stresses generated by body forces  $F_1 \in L_2^{\text{com}}(\Omega)$  are known to satisfy  $\sigma_{ij}(u)(x) = O(|x|^{-2})$ ,  $|x| \rightarrow \infty$  [13].

To formulate the problem with prescribed displacements at infinity let  $u_1^\infty(x)$  be a displacement field that is defined in  $\Omega_{R,\infty}$  for some  $R$ , and has the desired behavior at infinity. A solution wLFE in  $\Omega$  is sought such that  $u_1(x)$  is close to  $u_1^\infty(x)$  at infinity, in a suitable sense. One might try the condition  $u_1 - u_1^\infty \in L_2(\Omega_{R,\infty})$ , in analogy with (2.9). However, this condition is too strong. In fact, it is known that if  $u_1^\infty = 0$  and  $\Omega$  is an exterior domain then the displacements generated by body forces  $F_1 \in L_2^{\text{com}}(\Omega)$  have the exact order  $u_1(x) = O(|x|^{-1})$ ,  $|x| \rightarrow \infty$  [10]. Thus a weaker condition consistent with this estimate is needed. In what follows the condition

$$(2.10) \quad \|u - u^\infty\|_{r,\delta}^2 = O(r), \quad r \rightarrow \infty$$

is used where

$$(2.11) \quad \|u\|_{r,\delta}^2 = \int_{\Omega_{r,\delta}} u_1(x) u_1(x) dx,$$

$$(2.12) \quad \Omega_{r,\delta} = \Omega \cap \{x \mid r \leq |x| \leq r + \delta\}$$

and  $\delta > 0$  is a constant.

Definition. A solution wLFE of the equilibrium problem for an unbounded domain  $\Omega$  is said to have prescribed displacements  $u_1^\infty$  at infinity if and only if (2.10) holds for some  $\delta > 0$ .

A sufficient condition for (2.10) to hold with  $u_1^\infty = 0$  is

$$(2.13) \quad u_1(x) = O(|x|^{-1/2}), \quad |x| \rightarrow \infty$$

Of course, the precise order condition on  $u_1$  that is sufficient to guarantee (2.10) in particular cases will depend on the geometry of  $\Omega$  near infinity. For example, if  $\Omega = \{x \mid |x_3| < h\}$  then  $\int_{\Omega_{r,\delta}} dx = O(r)$ ,



$r \rightarrow \infty$ , and  $u_i(x) = O(1)$  is a sufficient condition for (2.10) with  $u_i^\infty = 0$ . If  $\Omega = \{x \mid (x_1, x_2) \in G, x_3 \in R\}$  where  $G \subset R^2$  is bounded then  $\int_{\Omega_{r,\delta}} dx = O(1)$  and  $u_i(x) = (|x|^{1/2})$ ,  $|x| \rightarrow \infty$ , is sufficient.

Ellipticity of the Cauchy-Green Operator. The principle of virtual work (2.4) with  $v_i \in C_0^\infty(\Omega) \subset E^{\text{com}}(\Omega)$  implies that the equilibrium fields  $u_i$  are weak solutions of the system of partial differential equations  $\sigma_{ij,j}(u) + F_i = 0$  in  $\Omega$ . If the body is homogeneous, as is assumed in this section, then the system may be written

$$(2.14) \quad A_{ik} u_k + F_i = 0$$

where

$$(2.15) \quad A_{ik} = c_{ijkl} \partial^2 / \partial x_j \partial x_l$$

The matrix differential operator  $(A_{ik})$ , with coefficients that satisfy the positivity and symmetry conditions (1.4), (1.5), will be called the Cauchy-Green operator. Conditions (1.4), (1.5) imply that  $(A_{ik})$  is strongly elliptic ( $c_{ijkl} \eta_i \eta_k \xi_j \xi_l \neq 0$  for all non-zero  $\eta_i, \xi_j$ ) and hence elliptic ( $\det(c_{ijkl} \xi_j \xi_l) \neq 0$  for all non-zero  $\xi_j$ ) [12, p. 20]. G. Fichera [5] has used the theory of elliptic boundary value problems to prove both interior and boundary regularity theorems for weak solutions of (2.14). The interior and boundary regularity properties of solutions wLFE that are implied by Fichera's results and methods are described here briefly.

Interior Regularity of Solutions wLFE. Fichera's interior regularity theorem [5, p. 36] implies the following results.

Theorem 2.1. Let  $\Omega \subset R^3$  be an arbitrary domain. Let  $u_i \in L_2^{\text{int}}(\Omega)$ ,  $e_{ij}(u) \in L_2^{\text{int}}(\Omega)$  and  $F_i \in L_2^{m,\text{int}}(\Omega)$  where  $m \geq 0$  is an

integer. Assume that (2.4) holds for all  $v_i \in C_0^\infty(\Omega)$ . Then  $u_i \in L_2^{m+2, \text{int}}(\Omega)$ .

Corollary 2.2. Let  $\Omega \subset R^3$  be an arbitrary domain and let  $u$  be a solution wLFE of the equilibrium problem for  $\Omega$  with  $F_i \in L_2^{m, \text{com}}(\Omega)$ . Then  $u_i \in L_2^{m+2, \text{int}}(\Omega)$ .

Corollary 2.3. If the hypotheses of Theorem 2.1 or Corollary 2.2 hold then  $u_i \in C^m(\Omega)$ .

Corollary 2.4. Let  $\Omega \subset R^3$  be an arbitrary domain and let  $u \in E^{\text{loc}}(\Omega)$  satisfy  $e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) = 0$  in  $L_2^{\text{loc}}(\Omega)$ . Then there exist constants  $a_i, b_i$  such that  $u_i(x) = a_i + \epsilon_{ijk} b_j x_k$  in  $\Omega$ .

Fichera proved Theorem 2.1 in [5] under the hypotheses  $f \in L_2^m(\Omega)$ ,  $u \in L_2(\Omega)$ . However, the theorem as stated above is an immediate consequence of his theorem. Corollary 2.2 is a special case of Theorem 2.1. Corollary 2.3 follows from Theorem 2.1 and Sobolev's imbedding theorem [5, p. 26]. Corollary 2.4 may be verified by noting that  $u$  is a solution wLFE in  $\Omega$  with body forces  $F_i = 0$  in  $\Omega$ . Thus  $u_i \in C^\infty(\Omega)$ , by Corollary 2.3, and  $u_{i,j} + u_{j,i} = 0$  in  $\Omega$ . The proof that every such  $u_i$  has the form  $u_i = a_i + \epsilon_{ijk} b_j x_k$  is classical [11, p. 71].

Boundary Regularity of Solutions wLFE. Fichera's theorems on regularity at the boundary imply the following results (see [5, Chapters 10 and 12]).

Theorem 2.5. Let  $\Omega \subset R^3$  be a domain with boundary  $\partial\Omega \in C^\infty$ . Let  $u$  be a solution wLFE of the equilibrium problem for  $\Omega$  with  $F_i \in L_2^{\text{com}}(\Omega) \cap C^\infty(\bar{\Omega})$ . Then  $u_i \in C^\infty(\bar{\Omega})$  and

$$(2.16) \quad \sigma_{ij}(u) n_j = 0 \text{ on } \partial\Omega$$

where  $n_i$  is the unit exterior normal field on  $\partial\Omega$ .

Corollary 2.6. Let  $x_0 \in \partial\Omega$  and assume that there is a neighborhood  $N_\delta(x_0) = \{x \mid |x - x_0| < \delta\}$  such that  $\partial\Omega \cap N_\delta(x_0) \in C^\infty$ . Moreover, let  $F_i \in L_2^{\text{com}}(\Omega) \cap C^\infty(\bar{\Omega} \cap N_\delta(x_0))$ . Then  $u_i \in C^\infty(\bar{\Omega} \cap N_\delta(x_0))$  and  $\sigma_{ij}(u)n_j = 0$  on  $\partial\Omega \cap N_\delta(x_0)$ .

Corollary 2.6 is an immediate consequence of Theorem 2.5 since boundary regularity is a local property. Boundary regularity results can also be proved when  $\partial\Omega$  and  $F_i$  have a finite number of derivatives. The following results can be proved by the methods of [5]; see also [1].

Theorem 2.7. Let  $\Omega \subset \mathbb{R}^3$  have a boundary point  $x_0$  such that  $\partial\Omega \cap N_\delta(x_0) \in C^{k+2}$  for some  $\delta > 0$  where  $k \geq 0$  is an integer. Let  $u$  be a solution wLFE of the equilibrium problem for  $\Omega$  with  $F_i \in L_2^{\text{com}}(\Omega) \cap L_2^k(\Omega \cap N_\delta(x_0))$ . Then  $u_i \in L_2^{k+2}(\Omega \cap N_\delta(x_0))$ .

Corollary 2.8. Under the hypotheses of Theorem 2.7,  $u_i \in C^k(\bar{\Omega} \cap N_\delta(x_0))$ . Moreover, if  $k \geq 1$  then  $\sigma_{ij}(u)n_j = 0$  on  $\partial\Omega \cap N_\delta(x_0)$ .

Corollary 2.9. Let  $\Omega \subset \mathbb{R}^3$  be a domain with boundary  $\partial\Omega \in C^{k+2}$ ,  $k \geq 0$ . Let  $u$  be a solution wLFE of the equilibrium problem for  $\Omega$  with  $F_i \in L_2^{k,\text{com}}(\Omega)$ . Then  $u_i \in C^k(\bar{\Omega})$ . Moreover, if  $k \geq 2$  then  $u_i$  is a classical solution of the equilibrium boundary value problem with body forces  $F_i \in C_0^{k-2}(\bar{\Omega}) \subset L_2^{k,\text{com}}(\Omega)$  and zero surface tractions; i.e.,  $u_i$  satisfies (2.10) and

$$(2.17) \quad c_{ijkl} u_{k,jl} + F_i = 0 \text{ in } \Omega$$

Bodies whose boundary  $\partial\Omega$  is a piece-wise smooth surface with piece-wise smooth edges with corners are of great interest for applications. A class of bodies of this type are the C-domains, defined and studied by N. Weck [24]. Solutions wLFE in such domains are regular and

satisfy the boundary condition (2.16) near smooth points of  $\partial\Omega$ , by Corollary 2.8. At edge and corner points of  $\partial\Omega$  condition (2.16) is meaningless, because  $n_i$  is undefined, and the only regularity property that remains is the LFE condition. For this reason the LFE condition is sometimes called the "edge condition" [20].

Equilibrium Problems with Non-Zero Surface Traction. The formulation (2.4) of the principle of virtual work is appropriate for the case of zero surface tractions. The surface traction at a point  $x_0 \in \partial\Omega$  is by definition the vector  $\sigma_{ij}(u(x_0)) n_j(x_0)$  and hence is defined only at boundary points where the boundary values  $\sigma_{ij}(u(x_0))$  and the normal vector  $n_j(x_0)$  exist. If a portion  $S \subset \partial\Omega$  is sufficiently smooth for  $n_j$  and boundary values of  $\sigma_{ij}(u)$  to exist on it then the principle of virtual work can be extended to include the boundary condition

$$(2.18) \quad \sigma_{ij}(u) n_j = \begin{cases} t_i & \text{on } S \\ 0 & \text{on } \partial\Omega - S \end{cases}$$

To do this the term

$$(2.19) \quad - \int_S t_i v_i dS = \text{Work done against surface tractions}$$

must be added to (2.4), so that the extended principle becomes

$$(2.20) \quad \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) dx - \int_{\Omega} F_i v_i dx - \int_S t_i v_i dS = 0$$

for all  $v \in E^{\text{com}}(\Omega)$ . Moreover, it is known from Sobolev's imbedding theorem that every  $v \in E^{\text{com}}(\Omega)$  has boundary values  $v \in L_2(S)$  on smooth portions  $S \subset \partial\Omega$  [1, p. 38]. In the important special case where  $\partial\Omega$  is piece-wise smooth then  $\sigma_{ij}(u) n_j$  exists almost everywhere on  $\partial\Omega$  and  $S$  may be replaced by  $\partial\Omega$  in (2.18), (2.19) and (2.20).

### 3. UNIQUENESS THEOREMS FOR PROBLEMS WITH PRESCRIBED SURFACE TRACTIONS

The strain energy theorem for classical solutions of the elastostatic equilibrium problem with body forces  $F_i$  and zero surface tractions states that [18, p. 173]

$$(3.1) \quad W_\Omega = \frac{1}{2} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx = \frac{1}{2} \int_\Omega F_i u_i dx$$

The uniqueness of classical solutions is a corollary. In this section the strain energy theorem is extended to arbitrary domains  $\Omega$  and all solutions wFE (= solutions wLFE and zero stresses at infinity if  $\Omega$  is unbounded) and solutions wLFE and zero displacements at infinity. The uniqueness of solutions wLFE with prescribed stresses or displacements at infinity follow as corollaries. The simple case of solutions wFE is treated first.

Theorem 3.1. Let  $u$  be a solution wFE of the equilibrium problem with body forces  $F_i \in L_2^{\text{com}}(\Omega)$  and zero surface tractions in a domain  $\Omega \subset \mathbb{R}^3$ . Then the strain energy equation (3.1) holds.

The proof is immediate from the representation (1.7) for  $W_\Omega$  and the definition of solution wFE, since one may take  $v_i = u_i \in E(\Omega)$  in (2.4).

Corollary 3.2. Uniqueness of Solutions wFE. Let  $u_i^{(1)}, u_i^{(2)}$  be two solutions wFE of the equilibrium problem with the same body forces  $F_i \in E^{\text{com}}(\Omega)$  and zero surface tractions. Then

$$(3.2) \quad \sigma_{ij}(u^{(1)}) = \sigma_{ij}(u^{(2)}) \text{ in } \Omega$$

and there exist constant vectors  $a_i, b_i$  such that

$$(3.3) \quad u_i^{(1)}(x) - u_i^{(2)}(x) = a_i + \epsilon_{ijk} b_j x_k \text{ in } \Omega$$

Proof.  $u_i = u_i^{(1)} - u_i^{(2)}$  is a solution wFE with body forces  $F_i \equiv 0$  in  $\Omega$  and zero surface tractions. Thus (3.1) holds with  $F_i = 0$  and  $\sigma_{ij}(u) = 0$  in  $L_2(\Omega)$  by the positive-definiteness of the energy. Moreover,  $\sigma_{ij}(u) \in C^\infty(\Omega)$  by Corollary 2.3 and hence  $\sigma_{ij}(u)(x) \equiv 0$  in  $\Omega$  which implies (3.2). Finally, Corollary 3.4 implies  $u_i(x) = a_i + \epsilon_{ijk} b_j x_k$  which implies (3.3).

Corollary 3.3. Uniqueness of Solutions wFE with Prescribed Stresses at Infinity. Let  $\Omega \subset R^3$  be unbounded and let  $u_i^{(1)}, u_i^{(2)}$  be two solutions wFE of the equilibrium problem with the same body forces  $F_i$ , zero surface tractions and the same stresses  $\sigma_{ij}^\infty$  at infinity. Then (3.2) and (3.3) hold.

Proof. By hypothesis, both  $\sigma_{ij}(u^{(1)}) - \sigma_{ij}^\infty$  and  $\sigma_{ij}(u^{(2)}) - \sigma_{ij}^\infty$  are in  $L_2(\Omega_{R,\infty})$  for some  $R > 0$ . It follows that the difference field  $u_i = u_i^{(1)} - u_i^{(2)}$  satisfies  $\sigma_{ij}(u) \in L_2(\Omega_{R,\infty})$ . Hence  $u_i$  is a solution wFE with body forces  $F_i \equiv 0$  in  $\Omega$  and zero surface tractions. Equations (3.2), (3.3) follow as in the proof of Corollary 3.2.

The uniqueness theorem for solutions wFE with prescribed displacements at infinity will be based on the following generalization of Theorem 3.1.

Theorem 3.4. Let  $u$  be a solution wFE of the equilibrium problem with body forces  $F_i \in L_2^{\text{com}}(\Omega)$  and zero surface tractions in an unbounded domain  $\Omega \subset R^3$ . Moreover, let  $u$  satisfy

$$(3.4) \quad \int_R^\infty |u|_{r,\delta}^{-2} dr = +\infty$$

for some  $R > 0$  and  $\delta > 0$ . Then  $u$  is a solution wLFE in  $\Omega$  and the strain energy equation (3.1) holds.

A proof of Theorem 3.4 is given at the end of the section, following the statement and discussion of the remaining uniqueness theorems.

Corollary 3.5. Uniqueness of Solutions wLFE with Prescribed Displacements at Infinity. Let  $\Omega \subset \mathbb{R}^3$  be unbounded and let  $u_i^{(1)}, u_i^{(2)}$  be two solutions wLFE of the equilibrium problem with the same body forces  $F_i$ , zero surface tractions and the same displacements  $u_i^\infty$  at infinity. Then (3.2) and (3.3) hold.

Proof. By hypothesis  $\|u^{(k)} - u_i^\infty\|_{r,\delta} = O(r^{1/2})$ ,  $r \rightarrow \infty$ ,  $k = 1, 2$ . It follows by the triangle inequality that the difference field  $u_i = u_i^{(1)} - u_i^{(2)}$  satisfies  $\|u\|_{r,\delta} = O(r^{1/2})$ ,  $r \rightarrow \infty$ , or equivalently

$$(3.5) \quad \|u\|_{r,\delta}^2 = O(r), \quad r \rightarrow \infty$$

which implies condition (3.4). Moreover,  $u$  is a solution wLFE with  $F_i = 0$  and zero surface tractions. Hence (3.1) holds with  $F_i = 0$ , by Theorem 3.4, and the conclusions (3.2), (3.3) follow as before.

Uniqueness Theorems for Problems with Non-Zero Surface Tractions.

The uniqueness theorems proved above are valid for arbitrary bounded and unbounded domains  $\Omega \subset \mathbb{R}^3$ . No local or global restrictions are imposed on  $\Omega$  or  $\partial\Omega$ . If a portion  $S \subset \partial\Omega$  is smooth enough for the surface tractions  $\sigma_{ij}(u) n_j$  and surface integrals (2.19) to be defined then solutions wLFE with non-zero surface tractions  $t_i$  on  $S$  are defined by the principle of virtual work. The uniqueness theorems for solutions with zero surface tractions extend immediately to this case because the

difference of two solutions with the same surface tractions  $t_i$  is a solution with zero surface tractions.

Other Growth Conditions at Infinity. It is clear from condition (3.4) of Theorem 3.4 that condition (3.5) is only one sufficient condition for uniqueness. Generalizations are obtained by replacing (3.5) by

$$(3.6) \quad |u|_{r,\delta}^2 = O(p(r)), \quad r \rightarrow \infty$$

where  $p(r)$  is a function such that

$$(3.7) \quad \int_R^\infty p(r)^{-1} dr = +\infty$$

If  $\Omega$  is an exterior domain ( $\{x \mid |x| > R\} \subset \Omega$  for  $R \geq R_0$ ) and if the body is isotropic as well as homogeneous; i.e., (1.8) holds, then the uniqueness theorem can be proved under weaker growth restrictions than (3.4). Indeed, under these conditions Fichera [4] has proved that

$$(3.8) \quad u_i(x) = o(1) \Rightarrow u_i(x) = O(|x|^{-1}) \text{ and } \sigma_{ij}(x) = O(|x|^{-2})$$

M. E. Gurtin and E. Sternberg [10] have rederived this result and proved the complementary result that

$$(3.9) \quad \sigma_{ij}(x) = o(1) \Rightarrow u_i(x) = O(|x|^{-1}) \text{ and } \sigma_{ij}(x) = O(|x|^{-2})$$

Moreover, these results are based on an expansion theorem for biharmonic functions in a neighborhood of infinity and are independent of  $\partial\Omega$ . Thus the uniqueness theorems for solutions wLFE with prescribed displacements or stresses at infinity in homogeneous isotropic solids are valid for arbitrary exterior domains  $\Omega$  under the conditions

$$(3.10) \quad u_i(x) - u_i^\infty(x) = o(1), \quad |x| \rightarrow \infty$$



and

$$(3.11) \quad \sigma_{ij}(u)(x) - \sigma_{ij}^{\infty}(x) = o(1), \quad |x| \rightarrow \infty$$

respectively.

Proof of Theorem 3.4. The idea of the proof is to put  $v_i = u_i$  in the principle of virtual work identity (2.4), as in the proof of Theorem 3.1. However, this cannot be done directly when  $u$  is a solution wLFE because  $v \in E^{\text{com}}(\Omega)$  must have compact support. Instead, let  $v_i(x) = \phi(x) u_i(x)$  where

$$(3.12) \quad \phi(x) = \psi((|x| - R)/\delta), \quad R > 0, \quad \delta > 0, \quad x \in \mathbb{R}^3$$

and  $\psi \in C^{\infty}(\mathbb{R})$  is a function such that  $\psi'(\tau) \leq 0$ ,  $0 \leq \psi(\tau) \leq 1$  and

$$(3.13) \quad \psi(\tau) = \begin{cases} 1, & \tau \leq 0 \\ 0, & \tau \geq 1 \end{cases}$$

These properties imply that  $\phi \in C_0^{\infty}(\mathbb{R}^3)$ ,  $0 \leq \phi(x) \leq 1$  and

$$(3.14) \quad \phi(x) = \begin{cases} 1, & |x| \leq R \\ 0, & |x| \geq R + \delta \end{cases}$$

It follows that for all  $u \in E^{\text{loc}}(\Omega)$ ,  $v = \phi u \in E^{\text{com}}(\Omega)$  and

$$(3.15) \quad v_{i,j} = \phi u_{i,j} + \phi_{,j} u_i$$

Moreover,

$$(3.16) \quad \phi_{,j}(x) = \psi'((|x| - R)/\delta) x_j / \delta |x|$$

and

$$(3.17) \quad \text{supp } \phi_{,j} \subset \Omega_{R,\infty}$$

With this choice of  $v_i$

$$(3.18) \quad e_{ij}(v) = \phi e_{ij}(u) + \frac{1}{2} (\phi_{,i} u_j + \phi_{,j} u_i)$$

and hence

$$(3.19) \quad \begin{aligned} \sigma_{ij}(u) e_{ij}(v) &= \phi \sigma_{ij}(u) e_{ij}(u) + \sigma_{ij}(u) \phi_{,i} u_j \\ &= \phi \sigma_{ij}(u) e_{ij}(u) + \delta^{-1} \psi'(|x| - R/\delta) \sigma_{ij}(u) \hat{x}_i u_j \end{aligned}$$

where  $\hat{x}_j = x_j/|x|$ . By assumption  $F_1 \in L_2^{\text{com}}(\Omega)$ . Choose  $R_0$  so large that  $\text{supp } F_1 \subset \{x \mid |x| \leq R_0\}$  and substitute  $v_i = \phi u_i$  and (3.19) in (2.4) with  $R \geq R_0$ . The result can be written

$$(3.20) \quad \begin{aligned} \int_{\Omega} \phi \sigma_{ij}(u) v_{ij}(u) dx + \delta^{-1} \int_{\Omega_{R,\delta}} \psi' \sigma_{ij}(u) \hat{x}_i u_j dx \\ - \int_{\Omega} F_1 u_i dx = 0 \end{aligned}$$

The goal of the remainder of the proof is to calculate the limit of equation (3.20) for  $R \rightarrow \infty$  and to show that the limiting form is the energy equation (3.1). To this end define

$$(3.21) \quad f(R) = \int_{\Omega} \psi(\delta^{-1}(|x| - R)) \sigma_{ij}(u) e_{ij}(u) dx - \int_{\Omega} F_1 u_i dx, \quad R \geq R_0$$

By equation (3.20) an alternative representation is

$$(3.22) \quad f(R) = -\delta^{-1} \int_{\Omega_{R,\delta}} \psi'(\delta^{-1}(|x| - R)) \sigma_{ij}(u) \hat{x}_i u_j dx$$

The properties of  $f(R)$  that are needed to complete the proof of Theorem 3.4 are described by

Lemma 3.6.  $f \in C^1[R_0, \infty)$  and has derivative

$$(3.23) \quad f'(R) = -\delta^{-1} \int_{\Omega_{R,\delta}} \psi'(\delta^{-1}(|x| - R)) \sigma_{ij}(u) e_{ij}(u) dx \geq 0$$

In particular,  $f(R)$  is monotone non-decreasing on  $[R_0, \infty)$ . Moreover,

$$(3.24) \quad f^2(R) \leq M^2 \|u\|_{R, \delta}^2 f'(R), \quad R \geq R_0$$

where  $M^2 = (\delta^{-1} c_1) \max_{0 \leq \tau \leq 1} |\psi'(\tau)|$ .

Proof of Lemma 3.6. Form the difference quotient

$$(3.25) \quad h^{-1}\{f(R+h) - f(R)\} = \int_{\Omega_{R, R+h+\delta}} h^{-1}\{\psi(\delta^{-1}(|x|-R-h)) - \psi(\delta^{-1}(|x|-R))\} \\ \times \sigma_{ij}(u) e_{ij}(u) dx$$

The quotient

$$(3.26) \quad h^{-1}\{\psi(\delta^{-1}(|x|-R-h)) - \psi(\delta^{-1}(|x|-R))\} \rightarrow -\delta^{-1} \psi'(\delta^{-1}(|x|-R)), \quad h \rightarrow 0$$

uniformly for  $x$  in bounded sets in  $R^3$ . Moreover,  $\sigma_{ij}(u) e_{ij}(u)$  is Lebesgue integrable on bounded subsets of  $\Omega$ . Thus passage to the limit  $h \rightarrow 0$  in (3.25) is permissible by Lebesgue's dominated convergence theorem. Hence  $f'(R)$  exists for all  $R \geq R_0$  and is given by (3.23). It is easy to show that the integral in (3.23) defines a continuous function of  $R$  which is non-negative. The monotonicity of  $f(R)$  follows.

To prove the inequality (3.24) note that (3.22) implies the estimate

$$(3.27) \quad |f(R)| \leq \delta^{-1} \int_{\Omega_{R, \delta}} |\psi'(\delta^{-1}(|x|-R))| |\sigma_{ij}(u) \hat{x}_i u_j| dx, \quad R \geq R_0$$

Moreover, by repeated application of Schwarz's inequality

$$(3.28) \quad |\sigma_{ij}(u) \hat{x}_i u_j| \leq (\sigma_{ij}(u) \hat{x}_i \sigma_{kj}(u) \hat{x}_k)^{1/2} (u_j u_j)^{1/2}$$

$$(3.29) \quad |\sigma_{ij}(u) \hat{x}_j| \leq \left( \sum_{i=1}^3 \sigma_{ij}^2(u) \right)^{1/2}$$

$$\begin{aligned}
 (3.30) \quad \sigma_{ij}(u) \hat{x}_i \sigma_{kj}(u) \hat{x}_k &\leq \left\{ \sum_{j=1}^3 (\sigma_{ij}(u) \hat{x}_i)^2 \right\}^{1/2} \left\{ \sum_{k=1}^3 (\sigma_{kj}(u) \hat{x}_k)^2 \right\}^{1/2} \\
 &= \sum_{j=1}^3 (\sigma_{ij}(u) \hat{x}_j)^2 \leq \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij}^2(u) = \sigma_{ij}(u) \sigma_{ij}(u)
 \end{aligned}$$

Now  $e_{ij} = \gamma_{ijkl} \sigma_{kl}$  together with (1.9) imply

$$(3.31) \quad c_1^{-1} \sigma_{ij} \sigma_{ij} \leq \sigma_{ij} e_{ij} = \gamma_{ijkl} \sigma_{ij} \sigma_{kl} \leq c_0^{-1} \sigma_{ij} \sigma_{ij}$$

for all  $\sigma_{ij} = \sigma_{ji}$ . Combining these inequalities gives

$$(3.32) \quad |\sigma_{ij}(u) \hat{x}_i u_j| \leq c_1^{1/2} (\sigma_{ij}(u) e_{ij}(u))^{1/2} (u_j u_j)^{1/2}$$

Substituting in (3.27) and using Schwarz's inequality again and equation (3.23) gives

$$\begin{aligned}
 (3.33) \quad |f(R)| &\leq \delta^{-1} c_1^{1/2} \int_{\Omega_{R,\delta}} |\psi'(\delta^{-1}(|x|-R))| (\sigma_{ij}(u) e_{ij}(u))^{1/2} (u_j u_j)^{1/2} dx \\
 &\leq \delta^{-1} c_1^{1/2} \left( \int_{\Omega_{R,\delta}} |\psi'| \sigma_{ij}(u) e_{ij}(u) dx \right)^{1/2} \left( \int_{\Omega_{R,\delta}} |\psi'| u_j u_j dx \right)^{1/2} \\
 &\leq \delta^{-1} c_1^{1/2} \mu^{1/2} (\delta f'(R))^{1/2} \|u\|_{R,\delta}
 \end{aligned}$$

where  $\mu = \text{Max } |\psi'(x)|$ . Squaring (3.33) gives (3.24).

Proof of Theorem 3.4 Concluded. Lemma 3.6 implies that  $f(+\infty)$  exists as a finite number or  $+\infty$ . It will be shown that  $f(+\infty) = 0$ . There are three cases to consider.

Case 1.  $0 < f(+\infty) \leq +\infty$ . In this case there exists  $R_1 \geq R_0$  such that  $f(R) \geq f(R_1) > 0$  for  $R \geq R_1$ . Hence (3.24) can be written

$$(3.34) \quad -\frac{d}{dR} \left( \frac{1}{f(R)} \right) = \frac{f'(R)}{f^2(R)} \geq M^{-2} \|u\|_{R,\delta}^{-2}, \quad R \geq R_1$$

and integration gives

$$(3.35) \quad \frac{1}{f(R_1)} - \frac{1}{f(R)} \geq M^{-2} \int_{R_1}^R |u|_{r,\delta}^{-2} dr, \quad R \geq R_1$$

In particular, since  $f(R) > 0$  for  $R \geq R_1$ ,

$$(3.36) \quad \frac{M^2}{f(R_1)} \geq \int_{R_1}^R |u|_{r,\delta}^{-2} dr \quad \text{for } R \geq R_1$$

But this contradicts hypothesis (3.4) of the theorem. Hence Case 1 cannot occur.

Case 2.  $f(+\infty) \leq 0$  and  $f(R_1) = 0$  for some  $R_1 \geq R_0$ . In this case  $0 \leq f(R_1) \leq f(+\infty) \leq 0$ ; i.e.  $f(+\infty) = 0$ .

Case 3.  $f(+\infty) \leq 0$  and  $f(R) < 0$  for all  $R \geq R_0$ . In this case (3.34) and (3.35) hold and the latter can be written, since  $|f(R)| = -f(R)$ ,

$$(3.37) \quad \frac{1}{|f(R)|} \geq \frac{1}{|f(R_1)|} + M^{-2} \int_{R_1}^R |u|_{r,\delta}^{-2} dr, \quad R \geq R_1$$

Hence condition (3.4) implies that  $f(+\infty) = 0$ .

It has been shown that (3.4) implies  $f(+\infty) = 0$ ; that is,

$$(3.38) \quad \lim_{R \rightarrow +\infty} \int_{\Omega} \psi(\delta^{-1}(|x|-R)) \sigma_{ij}(u) e_{ij}(u) dx = \int_{\Omega} F_i u_i dx$$

Since  $\psi(\delta^{-1}(|x|-R))$  is a monotone increasing function of  $R$  for each fixed  $x \in \mathbb{R}^3$  and tends to 1 everywhere when  $R \rightarrow \infty$ , (3.38) implies equation (3.1). In particular  $W_{\Omega} < \infty$  because  $\int_{\Omega} F_i u_i dx$  is finite. This completes the proof.

#### 4. UNIQUENESS THEOREMS FOR OTHER EQUILIBRIUM PROBLEMS

The purpose of this section is to show how the methods and results developed above can be extended to the most general equilibrium problems of linear elastostatics. Equilibria subject to other boundary conditions, equilibria in inhomogeneous anisotropic bodies and n-dimensional generalizations are discussed. In each case the boundary conditions for displacement fields wFE and wLFE are defined by appropriate subspaces of  $E(\Omega)$  and  $E^{\text{loc}}(\Omega)$ , respectively, and a corresponding form of the principle of virtual work is given. Regularity and uniqueness results for the new problems are indicated without proofs. In fact, the proofs of sections 2 and 3 are valid with minor modifications.

Equilibrium Problems with Prescribed Surface Displacements. The case of zero surface displacements is discussed first. Suitable subspaces of displacements fields are

$$(4.1) \quad E_0(\Omega) = \text{Closure in } E(\Omega) \text{ of } E^{\text{com}}(\Omega) \cap \{u \mid \text{supp } u \subset \Omega\}$$

$$(4.2) \quad E_0^{\text{loc}}(\Omega) = \text{Closure in } E^{\text{loc}}(\Omega) \text{ of } E^{\text{com}}(\Omega) \cap \{u \mid \text{supp } u \subset \Omega\}$$

The topologies in  $E(\Omega)$  and  $E^{\text{loc}}(\Omega)$  are those defined by (1.28) and (1.29), respectively. The notation

$$(4.3) \quad E_0^{\text{com}}(\Omega) = E^{\text{com}}(\Omega) \cap E_0(\Omega)$$

is also used. A solution wFE of the equilibrium problem with body forces  $F_1 \in L_2^{\text{com}}(\Omega)$  and zero surface displacements in a field  $u \in E_0(\Omega)$  that satisfies (2.4) for all  $v \in E_0(\Omega)$ . Similarly, a solution wLFE of the same problem is a field  $u \in E_0^{\text{loc}}(\Omega)$  that satisfies (2.4) for all

$v \in E_0^{\text{com}}(\Omega)$ . Problems with non-zero surface displacements

$$(4.4) \quad u_i(x) = f_i(x), \quad x \in \partial\Omega$$

may be reduced to the preceding problem if there exists a field  $u_i^0 \in E^{\text{loc}}(\Omega) \cap \{u \mid \sigma_{ij}(u^0) \in L_2^{\text{com}}(\Omega)\}$ . Then  $u_i^1 = u_i - u_i^0$  is a solution wLFE with zero boundary displacements.

The remaining boundary conditions can be formulated only when  $\partial\Omega$  is piecewise smooth. It will be assumed that  $\partial\Omega$  is a C-domain in the sense of [24]. For such domains the unit exterior normal field  $n_i(x)$  is defined and continuous at all points of  $\partial\Omega$  except edges and corners and one can define the normal and tangential components of vector field on  $\partial\Omega$  by

$$(4.5) \quad u_i = u_i^v + u_i^t, \quad u_i^v = (u_j v_j) v_i$$

Moreover,  $u_i^v v_i^t = 0$  for all  $u_i, v_i$  and hence

$$(4.6) \quad u_i v_i = u_i^v v_i^v + u_i^t v_i^t$$

Equilibrium Problems with Prescribed Tangential Surface Traction  
and Normal Surface Displacements. Suitable subspaces of displacement fields are defined by

$$(4.7) \quad E_v(\Omega) = E(\Omega) \cap \{u \mid u^v = 0 \text{ on } \partial\Omega\}$$

$$(4.8) \quad E_v^{\text{loc}}(\Omega) = E^{\text{loc}}(\Omega) \cap \{u \mid u^v = 0 \text{ on } \partial\Omega\}$$

The existence of  $u^v$  and  $u^t$  on  $\partial\Omega$  for all  $u \in E^{\text{loc}}(\Omega)$  follows from Korn's inequality and Sobolev's imbedding theorem. A solution wFE of the equilibrium problem with body forces  $F_i \in L_2^{\text{com}}(\Omega)$ , zero tangential surface tractions and zero normal surface displacements is a field

$u \in E_V(\Omega)$  that satisfies (2.4) for all  $v \in E_V(\Omega)$ . Similarly, a solution wLFE of the same problem is a field  $u \in E_V^{\text{loc}}(\Omega)$  such that (2.4) holds for all  $v \in E_V^{\text{com}}(\Omega) = E_V(\Omega) \cap E^{\text{com}}(\Omega)$ . Problems with non-zero surface tractions and displacements are treated by reducing them to the preceding case through subtraction of a suitable field.

Equilibrium Problems with Prescribed Normal Surface Tractions and Tangential Surface Displacements. This problem is dual to the preceding one. Appropriate classes of displacements are

$$(4.9) \quad E_T(\Omega) = E(\Omega) \cap \{u \mid u^T = 0 \text{ on } \partial\Omega\}$$

$$(4.10) \quad E_T^{\text{loc}}(\Omega) = E^{\text{loc}}(\Omega) \cap \{u \mid u^T = 0 \text{ on } \partial\Omega\}$$

Equilibrium Problems with Elastically Supported Surface. Physically, this corresponds to the case where surface displacements produce surface tractions that satisfy Hooke's law:

$$(4.11) \quad \sigma_{ij}(u) n_j + \beta u_i = 0 \text{ on } \partial\Omega$$

where  $\beta > 0$  is defined on  $\partial\Omega$ . A solution wLFE is a field  $u \in E^{\text{loc}}(\Omega)$  such that

$$(4.12) \quad \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) dx - \int_{\Omega} F_i v_i dx + \int_{\partial\Omega} \beta u_i v_i dS = 0$$

for all  $v \in E^{\text{com}}(\Omega)$ . Identity (4.12) is the principle of virtual work for this problem, the last term being the virtual work done against the induced surface tractions by the virtual displacement  $v$ . It follows from (4.12) that (4.11) holds at smooth points of  $\partial\Omega$ .

Equilibrium Problems with Mixed Boundary Conditions. A mixed problem that includes the preceding problems as special cases can be formulated by decomposing  $\partial\Omega$  into five portions and imposing one of the



boundary conditions defined above on each portion. Thus, if

$$(4.13) \quad \partial\Omega = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \text{ (disjoint union)}$$

and

$$(4.14) \quad E_m^{\text{loc}}(\Omega) = E^{\text{loc}}(\Omega) \cap \{u \mid u = 0 \text{ on } S_1, u^{\nu} = 0 \text{ on } S_2, u^{\tau} = 0 \text{ on } S_3\}$$

then the principle of virtual work

$$(4.15) \quad \int_{\Omega} \sigma_{ij}(u) e_{ij}(v) dx - \int_{\Omega} F_i v_i dx + \int_{S_5} \beta u_i v_i dS = 0$$

for all  $v \in E_m^{\text{com}}(\Omega) = E_m^{\text{loc}}(\Omega) \cap E^{\text{com}}(\Omega)$  characterizes the solutions of the equilibrium problem that satisfy  $u = 0$  on  $S_1$ ,  $u^{\nu} = 0$  and  $(\sigma_{ij}(u) n_j)^{\tau} = 0$  on  $S_2$ ,  $u^{\tau} = 0$  and  $(\sigma_{ij}(u) n_j)^{\nu} = 0$  on  $S_3$ ,  $\sigma_{ij}(u) n_j = 0$  on  $S_4$  and  $\sigma_{ij}(u) n_j + \beta u_i = 0$  on  $S_5$ .

Regularity and uniqueness theorems will be discussed for this mixed problem since it includes the others as special cases.

Regularity Theorems. The interior regularity properties of solutions wLFE of the mixed problem follow from Theorem 2.1 and are exactly the same as for the case discussed in section 2. Concerning boundary regularity, it can be shown by the methods of Fichera's monograph [5] that if  $\Omega$  is a C-domain of class  $C^{\infty}$  such that  $S_k^0 =$  interior of  $S_k$  in  $\partial\Omega$  is a  $C^{\infty}$  manifold for  $k = 1, \dots, 5$ , and if  $F_i \in C^{\infty}(\bar{\Omega}) \cap L_2^{\text{com}}(\Omega)$  then solutions wLFE of the mixed problem satisfy

$u_i \in C^{\infty}(\Omega \cup S_1^0 \cup S_2^0 \cup S_3^0 \cup S_4^0 \cup S_5^0) \cap L_2^{1, \text{loc}}(\Omega)$ . The condition  $u_i \in L_2^{1, \text{loc}}(\Omega)$ , which follows from Korn's inequality and Sobolev's theorem, is the "edge condition" that is needed for uniqueness. The boundary conditions on  $S_2$ ,  $S_3$  and  $S_5$  are not discussed by Fichera in [5] but can be treated by his methods.

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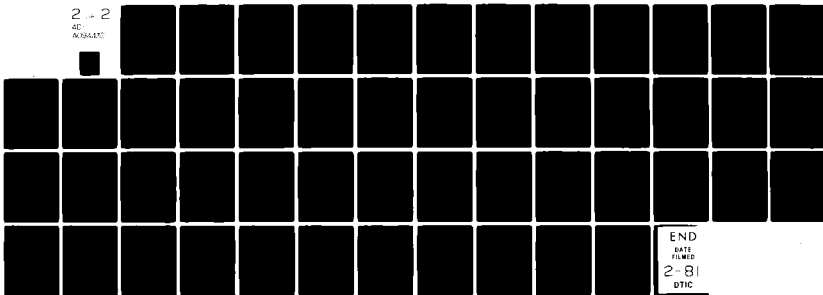
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Uniqueness Theorems. Solutions wFE of the mixed problem lie in

$$(4.16) \quad E_m(\Omega) = E(\Omega) \cap \{u \mid u = 0 \text{ on } S_1, u^{\nu} = 0 \text{ on } S_2, u^{\tau} = 0 \text{ on } S_3\}$$

and satisfy (4.15) for all  $v \in E_m(\Omega)$ . The strain energy theorem for the problem is

$$(4.17) \quad W_{\Omega} = \frac{1}{2} \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx + \frac{1}{2} \int_{S_3} \beta u_i u_i dS = \frac{1}{2} \int_{\Omega} F_i u_i dx$$

where the first equation defines the strain energy for the mixed problem.

The uniqueness of solutions wFE is an immediate corollary. Solutions with prescribed stresses or displacements at infinity will be defined by (2.9) and (2.10), respectively, as in the surface tractions problem.

Moreover, the strain energy theorem, Theorem 3.4, extends to solutions wLFE of the mixed problem. In fact, the same proof is valid because if  $u \in E_m^{loc}(\Omega)$  and  $\phi \in C_0^{\infty}(R^3)$  then  $v = \phi u \in E_m^{com}(\Omega) = E_m(\Omega) \cap E^{com}(\Omega)$ . The uniqueness of solutions wLFE of the mixed problem with prescribed displacements at infinity is an immediate corollary. It can also be shown that the displacement fields for the mixed problem are unique except in the special case of the pure surface tractions boundary condition ( $S_3 = \partial\Omega$ ).

Inhomogeneous Bodies. The uniqueness and energy theorems given above remain valid if the constant stress strain tensor  $c_{ijkl}$  is replaced by a field  $c_{ijkl}(x)$  that is Lebesgue measurable in  $\Omega$  and satisfies (1.9). The interior and boundary regularity theorems of section 2 are valid when  $c_{ijkl}(x)$  has sufficient differentiability in  $\Omega$  and  $\bar{\Omega}$ , respectively; cf. [1, p. 132].

n-Dimensional Problems. Fichera [5] has developed his theory for an n-dimensional generalization of the equations of elastostatics. All of the theorems given above extend to this n-dimensional problem with only

notational changes. The cases  $n = 1$  and  $n = 2$  are applicable to elastostatic fields that are functions of only one or two of the Cartesian coordinates.

## 5. A DISCUSSION OF RELATED LITERATURE

Fichera's paper [4] of 1950 provided the first significant extension of Kirchhoff's uniqueness theorem to unbounded domains. His result (3.8) implies that equilibrium fields in homogeneous isotropic bodies in exterior domains have finite energy if the displacements vanish at infinity. The uniqueness of equilibrium fields in such bodies is an immediate corollary. Corresponding results for fields whose stresses vanish at infinity follow from the 1961 result (3.9) of Gurtin and Sternberg [10]. The author knows of no general uniqueness results for anisotropic bodies in exterior domains or for bodies whose boundary is unbounded.

In Fichera's monograph [5] of 1965 the existence and uniqueness of classical solutions to elastostatic equilibrium problems in bounded domains with smooth boundaries is proved by the methods of functional analysis. This provides an alternative to the classical integral equation methods cited in the introduction. However, the formulation and techniques employed by Fichera can provide more general results. Fichera's semi-weak solutions (Lecture 7) are essentially the solutions wFE of this paper. Hence, Fichera's results (Lectures 7 and 12) imply the uniqueness of solutions wFE for bounded domains and boundary conditions for which Korn's inequality is valid. For the zero surface displacements problem the inequality holds for every bounded domain. For the zero surface tractions problem it holds for domains with the cone property.

The literature on uniqueness theorems in linear elastostatics up to 1970 was surveyed in a monograph by R. J. Knops and L. E. Payne [13] published in 1971. This work also contains uniqueness theorems for a class of weak solutions. However, the hypothesis that the displacement fields are continuous in  $\bar{\Omega}$  restricts the scope of these results.

Uniqueness theorems for plane crack problems were proved by J. K. Knowles and T. A. Pucik in 1973 [14] under the assumption that the displacements are bounded, but not necessarily continuous, at the crack tips. The elegant differential inequality method used in this work provided the inspiration for the proof of Theorem 3.4.

The methods employed in this paper to prove uniqueness theorems for solutions wLFE in arbitrary domains were introduced by the author during the period 1962-64 in a series of papers on boundary value problems of the theory of wave propagation [25, 26, 27]. The article [27] contains as a special case uniqueness theorems for elastodynamic problems in arbitrary domains.

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PART IV  
COMPLETENESS OF THE EIGEN FUNCTIONS FOR  
GRIFFITH CRACKS IN PLATES OF FINITE THICKNESS

### Introduction.

E.S. Folias [1] has constructed the displacement and stress fields near a Griffith crack as an expansion in eigenfunctions. The eigenfunctions were derived by an operational method due to Luré [2] and the question of completeness arises. The purpose of this report is to prove the completeness by a constructive method. The method employed is to solve the boundary value problem by Fourier analysis and to evaluate the resulting integrals as residue series. The terms in these series are precisely the eigenfunctions used by Folias.

Notation.

A system of Cartesian coordinates  $(x,y,z)$  is used. The plate occupies the region defined by

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad -h < z < h$$

The crack is defined by

$$-c < x < c, \quad y = 0, \quad -h < z < h$$

The components of the displacement field in the stressed plate are

$$u(x,y,z) , v(x,y,z) , w(x,y,z)$$

The corresponding stress tensor components are

$$\sigma_x = \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial u}{\partial x}$$

$$\tau_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau_{yx}$$

$$\tau_{xz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \tau_{zx}$$

$$\sigma_y = \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial v}{\partial y}$$

$$\tau_{yz} = G \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \tau_{zy}$$

$$\sigma_z = \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial w}{\partial z}$$

The displacement field satisfies the field equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + a^2 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + a^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + a^2 \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

where  $a^2 = \frac{\lambda+G}{G}$ . The boundary conditions are

$$\tau_{xz} = 0, \tau_{yz} = 0, \sigma_z = 0 \text{ for } -\infty < x, y < \infty, z = \pm h$$

$$\tau_{xy} = 0, \tau_{zy} = 0, \sigma_y = -\sigma_0 \text{ for } -c < x < c, y = 0 \pm, |z| < h$$

$$u, v, w = 0(1) \text{ for } x^2 + y^2 \rightarrow \infty, |z| < h$$

Symmetries.

	x	y	z
u(x,y,z)	odd	even	even
v(x,y,z)	even	odd	even
w(x,y,z)	even	even	odd

It follows from the symmetries that

$$v = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial z} = 0 \text{ for } |x| > c, y = 0, |z| < h$$

$$\frac{\partial u}{\partial y} = 0, \frac{\partial w}{\partial y} = 0 \text{ for } |x| > c, y = 0, |z| < h$$

whence

$$\tau_{xy} = 0, \tau_{zy} = 0 \text{ for } |x| > c, y = 0, |z| < h$$

Moreover, if

$$\left. \begin{aligned} u^\pm(x,0,z) &= \lim_{\epsilon \rightarrow 0} u(x, \pm\epsilon, z) \\ [u](x,0,z) &= u^+(x,0,z) - u^-(x,0,z) \end{aligned} \right\} \text{etc.}$$

then

$$\left. \begin{aligned} [u] = \left[ \frac{\partial u}{\partial x} \right] = \left[ \frac{\partial u}{\partial z} \right] &= 0 \quad \text{for } -\infty < x < \infty, \quad |z| \leq h \\ [w] = \left[ \frac{\partial w}{\partial x} \right] = \left[ \frac{\partial w}{\partial z} \right] &= 0 \quad \text{for } -\infty < x < \infty, \quad |z| \leq h \\ \left[ \frac{\partial v}{\partial y} \right] &= 0 \quad \text{for } -\infty < x < \infty, \quad |z| \leq h \end{aligned} \right\} \text{even in } y$$

while

$$\left[ \frac{\partial u}{\partial y} \right] = 2 \left( \frac{\partial u}{\partial y} \right)^+, \quad [v] = 2v^+, \quad \text{etc.} \} \text{odd in } y$$

Note: these vanish at points of continuity.

Application of the Fourier Transform in  $x$  and  $y$ .

Define

$$\hat{u}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ipx} u(x,y,z) dx$$

etc.

$$\begin{aligned} \tilde{u}(p,q,z) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-iqy} \hat{u}(p,y,z) dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(px+qy)} u(x,y,z) dx dy \end{aligned}$$

etc.

Then

$$\left(\frac{\partial u}{\partial x}\right)^{\wedge} = ip\hat{u}, \quad \left(\frac{\partial^2 u}{\partial x^2}\right)^{\wedge} = -p^2\hat{u}$$

$$\left(\frac{\partial u}{\partial y}\right)^{\wedge} = \frac{\partial \hat{u}}{\partial y}, \quad \text{etc.}$$

However,  $u, v, w$  and their derivatives may have discontinuities across the crack. Note that, if  $f^{(k)}(y) \in L_1(\mathbb{R})$  for  $k = 0, 1, 2$  and  $f \in C^2(\mathbb{R}_+) \cap C^2(\mathbb{R}_-)$  and  $f(0\pm), f'(0\pm)$  are finite

$$\begin{aligned} (f')^{\wedge}(q) &= \frac{1}{(2\pi)^{1/2}} \left( \int_0^{\infty} + \int_{-\infty}^0 \right) e^{-iqy} f'(y) dy \\ &= \frac{1}{(2\pi)^{1/2}} \left( e^{-iqy} f(y) \Big|_{0^+}^{\infty} + e^{-iqy} f(y) \Big|_{-\infty}^{0^-} \right) + iq \int_{-\infty}^{\infty} e^{-iqy} f(y) dy \\ &= - \frac{[f]}{(2\pi)^{1/2}} + iq \hat{f}(q) \end{aligned}$$

and hence

$$\begin{aligned} (f')^{\wedge}(q) &= - \frac{[f']}{(2\pi)^{1/2}} + iq (f')^{\wedge}(q) \\ &= - \frac{[f']}{(2\pi)^{1/2}} - iq \frac{[f]}{(2\pi)^{1/2}} - q^2 \hat{f}(q) \end{aligned}$$

These results and the symmetries (p. 3) imply

$$\left(\frac{\partial u}{\partial y}\right)^{\sim} = iq \tilde{u}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)^{\sim} = - \left(\frac{z}{\pi}\right)^{1/2} \left(\frac{\partial u}{\partial y}\right)_0^{\wedge} - q^2 \tilde{u} \quad \left(\left(\frac{\partial u}{\partial y}\right)_0^{\wedge} = \left(\frac{\partial u}{\partial y}\right)^{\wedge}(p, 0+, z)\right) \\ \text{etc.}$$

$$\left(\frac{\partial v}{\partial y}\right)^{\sim} = - \left(\frac{z}{\pi}\right)^{1/2} \hat{v}_0 + iq \tilde{v}$$

$$\left(\frac{\partial^2 v}{\partial y^2}\right)^{\sim} = - iq \left(\frac{z}{\pi}\right)^{1/2} \hat{v}_0 - q^2 \tilde{v}$$

$$\left(\frac{\partial w}{\partial y}\right)^{\sim} = iq \tilde{w}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)^{\sim} = - \left(\frac{z}{\pi}\right)^{1/2} \left(\frac{\partial w}{\partial y}\right)_0^{\wedge} - q^2 \tilde{w}$$

Taking the Fourier transform of the field equations (p. 2) and using the above results gives

$$\frac{d^2 \tilde{u}}{dz^2} - p^2 \tilde{u} - q^2 \tilde{u} - \left(\frac{z}{\pi}\right)^{1/2} \left(\frac{\partial u}{\partial y}\right)_0^{\wedge} + a^2 (-p^2 \tilde{u} - pq \tilde{v} - ip \left(\frac{z}{\pi}\right)^{1/2} \hat{v}_0 + ip \frac{d\tilde{w}}{dz}) = 0$$

$$\frac{d^2 \tilde{v}}{dz^2} - p^2 \tilde{v} - q^2 \tilde{v} - iq \left(\frac{z}{\pi}\right)^{1/2} \hat{v}_0 + a^2 (-pq \tilde{u} - q^2 \tilde{v} - iq \left(\frac{z}{\pi}\right)^{1/2} \tilde{v}_0 + iq \frac{d\tilde{w}}{dz}) = 0$$

$$\frac{d^2 \tilde{w}}{dz^2} - p^2 \tilde{w} - q^2 \tilde{w} - \left(\frac{z}{\pi}\right)^{1/2} \left(\frac{\partial w}{\partial y}\right)_0^{\wedge} + a^2 (ip \frac{d\tilde{u}}{dz} + iq \frac{d\tilde{v}}{dz} - \left(\frac{z}{\pi}\right)^{1/2} \left(\frac{\partial v}{\partial z}\right)_0^1 + \frac{d^2 \tilde{w}}{dz^2}) = 0$$

or



$$\frac{d^2 \tilde{u}}{dz^2} - (p^2 + q^2) \tilde{u} - a^2 p (\tilde{u} + q \tilde{v} - i \frac{d \tilde{w}}{dz}) = \left(\frac{2}{\pi}\right)^{1/2} \left[ \left(\frac{\partial u}{\partial y}\right)_0^{\wedge} + i p a^2 \tilde{v}_0 \right]$$

$$\frac{d^2 \tilde{v}}{dz^2} - (p^2 + q^2) \tilde{v} - a^2 q (\tilde{u} + q \tilde{v} - i \frac{d \tilde{w}}{dz}) = \left(\frac{2}{\pi}\right)^{1/2} [i q \tilde{v}_0 + i q a^2 \tilde{v}_0]$$

$$(1+a^2) \frac{d^2 \tilde{w}}{dz^2} - (p^2 + q^2) \tilde{w} + i a^2 \left( p \frac{d \tilde{u}}{dz} + q \frac{d \tilde{v}}{dz} \right) = \left(\frac{2}{\pi}\right)^{1/2} \left[ \left(\frac{\partial w}{\partial y}\right)_0^{\wedge} + a^2 \left(\frac{\partial v}{\partial z}\right)_0^{\wedge} \right]$$

Note that

$f(x)$  is even  $\leftrightarrow \hat{f}(p)$  is even and

$$\hat{f}(p) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \cos px f(x) dz = F_c f(p)$$

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \cos px \hat{f}(p) dp$$

while

$f(x)$  is odd  $\leftrightarrow \hat{f}(p)$  is odd and

$$\hat{f}(p) = -i \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \sin px f(x) dz = -i F_s f(p)$$

$$f(x) = i \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \sin px \hat{f}(p) dp = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \sin px F_s f(p) dp$$

The analogous formulas are valid for functions of  $y$ . Hence the symmetries, p. 3, imply

$$\tilde{u} = -i U, \quad \tilde{v} = -i V, \quad \tilde{w} = W$$

where  $U, V, W$  are real-valued. In fact,

$$U = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \sin px \cos qy u(x,y,z) dx dy$$

$$V = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \cos px \sin qy v(x,y,z) dx dy$$

$$W = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \cos px \cos qy w(x,y,z) dx dy$$

Hence the differential equations for  $\tilde{u}, \tilde{v}, \tilde{w}$  on p. 7 are equivalent to

$$\frac{d^2 U}{dz^2} - a^2 p \frac{dW}{dz} - a^2 p(p U + q V) - (p^2 + q^2)U = \left(\frac{2}{\pi}\right)^{1/2} [F_s \left(\frac{\partial u}{\partial y}\right)_0 - pa^2 F_c v_0]$$

$$\frac{d^2 V}{dz^2} - a^2 q \frac{dW}{dz} - a^2 q(p U + q V) - (p^2 + q^2)V = \left(\frac{2}{\pi}\right)^{1/2} [ - (1+a^2)q F_c v_0]$$

$$(1+a^2) \frac{d^2 W}{dz^2} + a^2 (p \frac{dU}{dz} + q \frac{dV}{dz}) - (p^2 + q^2)W = \left(\frac{2}{\pi}\right)^{1/2} [F_c \left(\frac{\partial w}{\partial y}\right)_0 + a^2 F_c \left(\frac{\partial v}{\partial z}\right)_0]$$

This can be written as a 2nd order  $3 \times 3$  matrix system of ODE's, namely

$$L \begin{pmatrix} U \\ V \\ W \end{pmatrix} - (p^2 + q^2) \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

where, if  $U = (U, V, W)^T$  ( $T = \text{transpose}$ )

$$LU = A \frac{d^2U}{dz^2} + B \frac{dU}{dz} + CU$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & -a^2 p \\ 0 & 0 & -a^2 q \\ a^2 p & a^2 q & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -a^2 p^2 & -a^2 pq & 0 \\ -a^2 pq & -a^2 q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that

$$A^T = A, \quad B^T = -B, \quad C^T = C$$

It follows that  $L$  is formally selfadjoint with respect to the scalar product

$$(U, V) = \int_{z_1}^{z_2} U^T V \, dz$$

In fact, integration by parts gives

$$\begin{aligned}
(\mathbf{U}, \mathbf{L}\mathbf{V}) &= \int_{z_1}^{z_2} (\mathbf{U}^T \mathbf{A} \mathbf{V}' + \mathbf{U}^T \mathbf{B} \mathbf{V}'' + \mathbf{U}^T \mathbf{C} \mathbf{V}''') dz \\
&= \mathbf{U}^T \mathbf{A} \mathbf{V}' + \mathbf{U}^T \mathbf{B} \mathbf{V}'' \Big|_{z_1}^{z_2} - \int_{z_1}^{z_2} (\mathbf{U}'^T \mathbf{A} \mathbf{V}' + \mathbf{U}'^T \mathbf{B} \mathbf{V}'' - \mathbf{U}^T \mathbf{C} \mathbf{V}''') dz \\
&= \mathbf{U}^T \mathbf{A} \mathbf{V}' - \mathbf{U}'^T \mathbf{A} \mathbf{V} + \mathbf{U}^T \mathbf{B} \mathbf{V}'' \Big|_{z_1}^{z_2} \\
&\quad + \int_{z_1}^{z_2} (\mathbf{U}'^T \mathbf{A} - \mathbf{U}'^T \mathbf{B} + \mathbf{U}^T \mathbf{C}) \mathbf{V} dz \\
&= [\mathbf{U}, \mathbf{V}] \Big|_{z_1}^{z_2} + \int_{z_1}^{z_2} (\mathbf{A}\mathbf{U}' + \mathbf{B} \mathbf{U}'' + \mathbf{C} \mathbf{U}''')^T \mathbf{V} dz \\
&= [\mathbf{U}, \mathbf{V}] \Big|_{z_1}^{z_2} + (\mathbf{L} \mathbf{U}, \mathbf{V})
\end{aligned}$$

where

$$[\mathbf{U}, \mathbf{V}] = \mathbf{U}^T \mathbf{A} \mathbf{V}' - \mathbf{U}'^T \mathbf{A} \mathbf{V} + \mathbf{U}^T \mathbf{B} \mathbf{V}''$$

If the index notation

$$\mathbf{U} = (u_1, u_2, u_3)^T, \quad \mathbf{V} = (v_1, v_2, v_3)^T$$

is used the bilinear form  $[\mathbf{U}, \mathbf{V}]$  can be written

$$\begin{aligned}
[U, V] &= U_1 V_1' + U_2 V_2' + (1+a^2) U_3 V_3' \\
&- U_1' V_1 - U_2' V_2 - (1+a^2) U_3' V_3 \\
&- a^2 p U_1 V_3 - a^2 q U_2 V_3 + a^2 p U_3 V_1 + a^2 q U_3 V_2
\end{aligned}$$

Boundary Conditions Associated with L .

The symmetry properties of the displacement field wrt  $z$  (p. 3) imply that

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0, \quad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0, \quad (w)_{z=0} = 0$$

It follows that

$$\frac{dU(0)}{dz} = 0, \quad \frac{dV(0)}{dz} = 0, \quad W(0) = 0$$

Note that

$$\text{B.C.1} \quad U_1'(0) = 0, \quad U_2'(0) = 0, \quad U_3(0) = 0$$

is selfadjoint for  $L$ ; i.e.

$$U \text{ and } V \text{ satisfy B.C.1} \Rightarrow [U, V]_{z=0} = 0$$

The B.C.'s at  $z = \pm h$  imply corresponding B.C.'s for  $\bar{U}$ . To write then note that (p. 6)

$$\tilde{\tau}_{xz} = G\left(\frac{d\tilde{u}}{dz} + ip \tilde{w}\right) = -i G\left(\frac{dU}{dz} - p W\right)$$

$$\tilde{\tau}_{yz} = G\left(\frac{d\tilde{v}}{dz} + iq \tilde{w}\right) = -i G\left(\frac{dV}{dz} - q W\right)$$

$$\begin{aligned}\tilde{\sigma}_z &= \lambda(ip \tilde{u} + iq \tilde{v} - \left(\frac{2}{\pi}\right)^{1/2} \hat{v}_0 + \frac{d\tilde{w}}{dz}) + 2G \frac{d\tilde{w}}{dz} \\ &= \lambda(p U + q V) + (\lambda + 2G) \frac{dW}{dz} - \left(\frac{2}{\pi}\right)^{1/2} \lambda \hat{v}_0 \\ &= \lambda\left[p U + q V + \frac{a^2+1}{a^2-1} \frac{dW}{dz}\right] - \left(\frac{2}{\pi}\right)^{1/2} \lambda \hat{v}_0\end{aligned}$$

since

$$\lambda = \frac{2G}{m-2}, \quad \frac{2G}{\lambda} = m-2, \quad 1 + \frac{2G}{\lambda} = m-1 = \frac{a^2+1}{a^2-1}$$

$$a^2 = \frac{m}{m-2}, \quad ma^2 - 2a^2 = m, \quad m(a^2 - 1) = 2a^2$$

$$m = \frac{2a^2}{a^2-1}, \quad \frac{\lambda}{2G} = \frac{a^2-1}{2}, \quad \frac{m}{m-1} = \frac{2a^2}{a^2+1}$$

It follows that

$$\frac{dU(h)}{dz} - p W(h) = 0, \quad \frac{dV(h)}{dz} - q W(h) = 0$$

$$p U(h) + q V(h) + \frac{a^2+1}{a^2-1} \frac{dW(h)}{dz} = \left(\frac{2}{\pi}\right)^{1/2} \hat{v}_0(h)$$

Note that

$$\underline{\text{B.C.2}} \quad U_1'(h) - p U_3(h) = 0, \quad U_2'(h) - q U_3(h) = 0$$

$$p U_1(h) + q U_2(h) + \frac{a^2+1}{a^2-1} U_3'(h) = 0$$

is also selfadjoint for  $L$ . In fact, if  $U$  and  $V$  satisfy B.C.2 then

$$\begin{aligned} [\bar{U}, \bar{V}]_{z=h} &= p U_1 V_3 + q U_2 V_3 - (a^2-1) U_3 (p V_1 + q V_2) \\ &\quad - p U_3 V_1 - q U_3 V_2 + (a^2-1) (p U_1 + q U_2) V_3 \\ &\quad - a^2 p U_1 V_3 - a^2 q U_2 V_3 + a^2 p U_3 V_1 + a^2 q U_3 V_2 = 0 \end{aligned}$$

$$\underline{\text{BV Problem for } U = (U_1, U_2, U_3)^T = (U, V, W)^T .}$$

$$L U - (p^2 + q^2) \bar{U} = \bar{F}, \quad 0 < z < h$$

$$M_0 \bar{U}(0) + N_0 \bar{U}'(0) = 0$$

$$M_h \bar{U}(h) + N_h \bar{U}'(h) = \bar{G}(h)$$

where

$$F(z) = \left(\frac{z}{\pi}\right)^{1/2} \begin{bmatrix} F_s \left(\frac{\partial u}{\partial y}\right)_0 - p a^2 F_c v_0 \\ - (1+a^2) q F_c v_0 \\ F_c \left(\frac{\partial w}{\partial y}\right)_0 + a^2 F_c \left(\frac{\partial v}{\partial z}\right)_0 \end{bmatrix}$$

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_h = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -q \\ p & q & 0 \end{pmatrix}, \quad N_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{a^2+1}{a^2-1} \end{pmatrix}$$

and

$$\bar{G}(h) = \begin{pmatrix} 0 \\ 0 \\ \left(\frac{2}{\pi}\right)^{1/2} F_c v_0 \end{pmatrix}$$

Method of Solving the BV Problem for U,V,W .

To solve the BV problem the general solution of  $L\bar{U} - (p^2 + q^2)\bar{U} = \bar{F}$  and BC at  $z = 0$  will be constructed as a function of the parameters

$$u_0 = U(0), \quad v_0 = V(0), \quad w'_0 = \frac{dW(0)}{dz}$$

The B.C.'s at  $z = h$  will then be used to calculate  $u_0, v_0, w'_0$ .

Solutions of the Equations  $L\bar{U} - (p^2 + q^2)\bar{U} = \bar{0}$ .

This equation, written in terms of components  $(U_1, U_2, U_3) = (U, V, W)$ , is obtained from the system on p.8 by setting the right-hand side equal to 0. Note that this system coincides with Lur e, p. 150 (3.2.12) under the correspondence

$$-i U \leftrightarrow u, \quad -i V \leftrightarrow v, \quad W \leftrightarrow w$$

$$ip \leftrightarrow \partial_1, \quad iq \leftrightarrow \partial_2, \quad -(p^2 + q^2) \leftrightarrow D^2$$

$$ip(-i U) + iq(-i V) + \frac{dW}{dz} = p U + q V + \frac{dW}{dz} \leftrightarrow \theta$$



Luré has given a complete solution of the system  $LU = (p^2 + q^2)U$  in equation (3.2.15), (3.2.17). To adapt them to the present notation write

$$s^2 = p^2 + q^2, \quad s = \sqrt{p^2 + q^2} > 0, \quad D = is$$

$$\cos zD = \cos isz = \cosh sz, \quad \sin zD = \sin isz = i \sinh sz$$

Then the solution (3.2.15) becomes

$$U = (\cosh sz)u_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (p^2 u_0 + pqv_0 + pw'_0)$$

$$V = (\cosh sz)v_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (pqu_0 + q^2 v_0 + qw'_0)$$

$$W = \frac{\sinh sz}{s} w'_0 + \frac{a^2}{2} \left( \frac{\sinh sz}{s} - z \cosh sz \right) (pu_0 + qv_0 + w'_0)$$

This solution satisfies B.C.1 at  $z = 0$ . The solution (3.2.17) becomes

$$U = \frac{\sinh sz}{s} u'_0 + \frac{a^2}{2(a^2+1)} \left( \frac{i \sinh sz}{-i s^3} + \frac{z \cosh sz}{s^2} \right) (p^2 u'_0 + pqv'_0 + ps^2 w_0)$$

$$V = \frac{\sinh sz}{s} v'_0 + \frac{a^2}{2(a^2+1)} \left( \frac{\sinh sz}{-s^3} + \frac{z \cosh sz}{s^2} \right) (pqu'_0 + q^2 v'_0 + qs^2 w_0)$$

$$W = (\cosh sz)w_0 - \frac{a^2}{2(a^2+1)} \frac{z \sinh sz}{s} (pu'_0 + qv'_0 + s^2 w_0)$$

This solution satisfies

$$\text{B.C.1'} \quad U(0) = 0, \quad V(0) = 0, \quad \frac{dW(0)}{dz} = 0$$

and

$$u'_0 = \frac{dU(0)}{dz}, \quad v'_0 = \frac{dV(0)}{dz}, \quad w_0 = W(0)$$

Solution Basis for  $L\bar{U} = (p^2 + q^2)\bar{U}$ .

$u_0 = 1, v_0 = w_0' = 0$  gives

$$\begin{pmatrix} U^1 \\ V^1 \\ W^1 \end{pmatrix} = \begin{pmatrix} \cosh sz + \frac{a^2}{2} \frac{z \sinh sz}{s} p^2 \\ \frac{a^2}{2} \frac{z \sinh sz}{s} pq \\ \frac{a^2}{2} \left( \frac{\sinh sz}{s} - z \cosh sz \right) p \end{pmatrix}$$

$u_0 = 0, v_0 = 1, w_0' = 0$  gives

$$\begin{pmatrix} U^2 \\ V^2 \\ W^2 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2} \frac{z \sinh sz}{s} pq \\ \cosh sz + \frac{a^2}{2} \frac{z \sinh sz}{s} q^2 \\ \frac{a^2}{2} \left( \frac{\sinh sz}{s} - z \cosh sz \right) q \end{pmatrix}$$

$u_0 = v_0 = 0, w_0' = 1$  gives

$$\begin{pmatrix} U^3 \\ V^3 \\ W^3 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2} \frac{z \sinh sz}{s} p \\ \frac{a^2}{2} \frac{z \sinh sz}{s} q \\ \frac{\sinh sz}{s} + \frac{a^2}{2} \left( \frac{\sinh sz}{s} - z \cosh sz \right) \end{pmatrix}$$

Similarly

$u_0' = 1, v_0' = 0, w_0 = 0$  gives

$$\begin{pmatrix} U^4 \\ U^4 \\ W^4 \end{pmatrix} = \begin{pmatrix} \frac{\sinh sz}{s} + \frac{a^2}{2(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) \frac{p^2}{s^2} \\ \frac{a^2}{2(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) \frac{pq}{s^2} \\ - \frac{a^2}{2(a^2+1)} \frac{z \sinh sz}{s} p \end{pmatrix}$$

$u'_0 = 0, v'_0 = 1, w_0 = 0$  gives

$$\begin{pmatrix} U^5 \\ V^5 \\ W^5 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{2(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) \frac{pq}{s^2} \\ \frac{\sinh sz}{s} + \frac{a^2}{2(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) \frac{q^2}{s^2} \\ - \frac{a^2}{a(a^2+1)} \frac{z \sinh sz}{s} q \end{pmatrix}$$

$u'_0 = v'_0 = 0, w_0 = 1$  gives

$$\begin{pmatrix} U^6 \\ V^6 \\ W^6 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{a(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) p \\ \frac{a^2}{2(a^2+1)} (z \cosh sz - \frac{\sinh sz}{s}) q \\ \cosh sz - \frac{a^2}{2(a^2+1)} sz \sinh sz \end{pmatrix}$$

It is evident from the B.C. at  $z = 0$  that these six solutions are linearly independent and hence span the solution space of  $L \bar{U} = (p^2 + q^2) \bar{U}$ .

Solutions of  $L \bar{U} - (p^2 + q^2) \bar{U} = \bar{F}(z)$ .

The variation of constants formula will be used. For this purpose

it is convenient to write the equation as a 1st order system. The equation has the form

$$A \bar{U}' + B \bar{U}' - C_s \bar{U} = \bar{F}$$

where

$$C_s = s^2 I - C = \begin{pmatrix} s^2 + a^2 p^2 & a^2 pq & 0 \\ a^2 pq & s^2 + a^2 q^2 & 0 \\ 0 & 0 & s^2 \end{pmatrix}$$

Now  $A^T = A > 1$ , whence

$$A = A^{1/2} A^{1/2}, \quad A^{1/2} = (A^{1/2})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{1+a^2} \end{pmatrix}$$

Thus

$$A^{1/2} \bar{U}' + A^{-1/2} B \bar{U}' - A^{-1/2} C_s \bar{U} = A^{-1/2} \bar{F}$$

Put

$$\bar{V} = A^{1/2} \bar{U} = \begin{pmatrix} U_1 \\ U_2 \\ \sqrt{1+a^2} U_3 \end{pmatrix}$$

$$\bar{U} = A^{-1/2} \bar{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3/\sqrt{1+a^2} \end{pmatrix}$$

Then

$$\bar{V}'' + (A^{-1/2} B A^{-1/2})\bar{V}' - (A^{-1/2} C_3 A^{-1/2})\bar{V} = A^{-1/2} \bar{F}$$

or

$$\bar{V}'' + B_A \bar{V}' - C_A \bar{V} = \bar{G}$$

where

$$B_A = A^{-1/2} B A^{-1/2} = -B_A^T$$

$$C_A = A^{-1/2} C_3 A^{-1/2} = C_A^T$$

$$\bar{G} = A^{-1/2} \bar{F}$$

Explicitly,

$$B_A = \frac{a^2}{(1+a^2)^{1/2}} \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -q \\ p & q & 0 \end{pmatrix}$$

$$C_A = \begin{pmatrix} s^2 + a^2 p^2 & a^2 pq & 0 \\ a^2 pq & s^2 + a^2 q^2 & 0 \\ 0 & 0 & \frac{s^2}{1+a^2} \end{pmatrix}$$

A 1st order system equivalent to the above 2nd order system may be obtained by setting

$$Y = \bar{V}, \quad Z = \bar{V}'$$

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \bar{V} \\ \bar{V}' \end{pmatrix}$$

Then

$$Y' = \bar{V}' = Z, \quad Z' = \bar{V}'' = -B_A Z + C_A Y + \bar{G}$$

and

$$X' = \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix} X + \begin{pmatrix} 0 \\ \bar{G} \end{pmatrix}$$

or

$$X' = M X + H(z), \quad H = \begin{pmatrix} 0 \\ \bar{G} \end{pmatrix}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix}$$

A fundamental matrix for  $X' = M X$  is a  $2 \times 2$  matrix solution  $\Phi(z)$  of

$$\begin{cases} \Phi'(z) = M \Phi(z) \\ \Phi(0) = I \end{cases}$$

An explicit representation of  $\Phi(z)$  can be derived from the solution basis for  $L \bar{U} = s^2 \bar{U}$ . Indeed, each solution  $\bar{U}^j = (U^j \ V^j \ W^j)^T$  of

$L \bar{U}^j = s^2 \bar{U}^j$  gives a solution  $X^j$  of  $X^j{}' = M X^j$ , namely

$$X^j = \begin{pmatrix} \bar{V}^j \\ \bar{V}^j{}' \end{pmatrix} = \begin{pmatrix} A^{1/2} \bar{U}^j \\ A^{1/2} \bar{U}^j{}' \end{pmatrix}$$

Thus, in view of the B.C.'s at  $z = 0$ ,

$$\phi(z) = (X^1 X^2 (1+a^2)^{-1/2} X^6 X^4 X^5 (1+a^2)^{-1/2} X^3)$$

$$= \begin{pmatrix} & M_2 & & & M_1 \\ & U^1 & U^2 & (1+a^2)^{-1/2} U^6 & U^4 & U^5 & (1+a^2)^{-1/2} U^3 \\ & V^1 & V^2 & ( )^{-1/2} V^6 & V^4 & V^5 & ( )^{-1/2} V^3 \\ (1+a^2)^{1/2} W^1 & ( )^{1/2} W^2 & & W^6 & (1+a^2)^{1/2} W^4 & (1+a^2)^{1/2} W^5 & W^3 \\ U^1, & U^2, & ( )^{-1/2} U^6, & U^4, & U^5, & ( )^{-1/2} U^3, \\ V^1, & V^2, & ( )^{-1/2} V^6, & V^4, & V^5, & ( )^{-1/2} V^3, \\ (1+a^2)^{1/2} W^1, & ( )^{1/2} W^2, & W^6, & ( )^{1/2} W^4, & ( )^{1/2} W^5, & W^3, \end{pmatrix}$$

where  $U^j$ ,  $V^j$ ,  $W^j$  are defined on pp. 16-17.

The fundamental matrix makes it possible to calculate a solution of

$X' = M X + H$ , namely

$$X(z) = \phi(z) \int_0^z \phi^{-1}(\zeta) H(\zeta) d\zeta$$

Indeed,

$$\begin{aligned} X' &= \phi(z) \int_0^z \phi^{-1}(\zeta) H(\zeta) d\zeta + \phi(z) \phi^{-1}(z) H(z) \\ &= M X + H \end{aligned}$$

Moreover,

$$X(0) = 0$$

Thus the general solution of  $X' = M X + H$  is given by

$$X(z) = \phi(z) X_0 + \phi(z) \int_0^z \phi(\zeta)^{-1} H(\zeta) d\zeta$$

The direct calculation of  $\phi(\zeta)^{-1}$  is difficult, but note that if

$$\underline{P} = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} = \underline{P}^T$$

and

$$E(z) = \frac{1}{2} X(z)^T P X(z)$$

then  $X' = M X \Rightarrow$

$$E'(z) = X(z)^T P X'(z) = X(z)^T P M X(z) \equiv 0$$

because

$$PM = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ C_A & -B_A \end{pmatrix} = \begin{pmatrix} 0 & C_A \\ -C_A & B_A \end{pmatrix}$$

and hence

$$(PM)^T = -PM$$

Thus  $E(z) = \text{const.}$   $\forall$  solutions of  $X' = M X$ . Take  $X(z) = \phi(z) X_0$  ( $X_0 \in \mathbb{R}^6$  arbitrary). Then

$$\begin{aligned} 2E(z) &= (\phi(z) X_0)^T P \phi(z) X_0 = X_0^T \phi(z)^T P \phi(z) X_0 \\ &= X_0^T P X_0 = 2E(0) \quad \forall X_0 \in \mathbb{R}^6 \end{aligned}$$



It follows that

$$\phi(z)^T P \phi(z) = P \quad \forall z \in \mathbb{R}$$

Since  $P$  is non-singular

$$(P^{-1} \phi(z)^T P) \phi(z) = 1$$

whence

$$\phi(z)^{-1} = \underline{P}^{-1} \phi(z)^T \underline{P}$$

Thus

$$(*) \quad X(z) = \phi(z) X_0 + \phi(z) \underline{P}^{-1} \int_0^z \phi(\zeta)^T \underline{P} H(\zeta) d\zeta$$

Solution of the B.V. Problem of p. 13.

Recall that

$$X = \begin{pmatrix} V \\ V' \end{pmatrix} = (U_1 \ U_2 \ (1+a^2)^{1/2} \ U_3 \ U_1' \ U_2' \ (1+a^2)^{1/2} \ U_3')^T$$

$$X_0 = (u_0 \ v_0 \ (1+a^2)^{1/2} \ w_0 \ u_1' \ v_0' \ (1+a^2)^{1/2} \ w_0')$$

Thus the solution (\*) satisfies B.C.1 (at  $z = 0$ )  $\iff u_0' = v_0' = w_0' = 0$ .

Thus if we write

$$X^{\pi}(z) = \phi(z) \underline{P}^{-1} \int_0^z \phi(\zeta)^T \underline{P} H(\zeta) d\zeta$$

then substituting in (\*) gives

$$U_1(z) = \phi_{11}(z) u_0 + \phi_{12}(z) v_0 + (1+a^2)^{1/2} \phi_{16}(z) w'_0 + X_1^\pi(z)$$

$$U_2(z) = \phi_{21}(z) u_0 + \phi_{22}(z) v_0 + (1+a^2)^{1/2} \phi_{26}(z) w'_0 + X_2^\pi(z)$$

$$(1+a^2)^{1/2} U_3(z) = \phi_{31}(z) u_0 + \phi_{32}(z) v_0 + (1+a^2)^{1/2} \phi_{36}(z) w'_0 + X_3^\pi(z)$$

or (see p. 21) if  $U^\pi(z) = X_1^\pi(z)$ ,  $V^\pi(z) = X_2^\pi(z)$ ,  $W^\pi(z) = (1+a^2)^{-1/2} X_3^\pi(z)$ ,

$$U(z) = U^1(z) u_0 + U^2(z) v_0 + U^3(z) w'_0 + U^\pi(z)$$

$$V(z) = V^1(z) u_0 + V^2(z) v_0 + V^3(z) w'_0 + V^\pi(z)$$

$$W(z) = W^1(z) u_0 + W^2(z) v_0 + W^3(z) w'_0 + W^\pi(z)$$

Thus (p. 15)

$$U(z) = (\cosh sz) u_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (p^2 u_0 + pqv_0 + pw'_0) + U^\pi(z)$$

$$V(z) = (\cosh sz) v_0 + \frac{a^2}{2} \frac{z \sinh sz}{s} (pqu_0 + q^2 v_0 + qw'_0) + V^\pi(z)$$

$$W(z) = \frac{\sinh sz}{s} w'_0 + \frac{a^2}{2} \left( \frac{\sinh sz}{s} - z \cosh sz \right) (pu_0 + qv_0 + w'_0) + W^\pi(z)$$

To complete the solution of the B.V. problem of p. 13 the initial values  $u_0, v_0, w'_0$  must be chosen so that the B.C.2 at  $z = h$  is satisfied. The derivatives  $U', V', W'$  are needed. They are given by

$$U'(z) = s(\sinh sz)u_0 + \frac{a^2}{2} \left( \frac{\sinh sz + sz \cosh sz}{s} \right) (p^2 u_0 + pqv_0 + pw'_0) + U^{\pi'}(z)$$

$$V'(z) = s(\sinh sz)v_0 + \frac{a^2}{2} \left( \frac{\sinh sz + sz \cosh sz}{s} \right) (pqu_0 + q^2 v_0 + qw'_0) + V^{\pi'}(z)$$

$$W'(z) = (\cosh sz)w'_0 + \frac{a^2}{2} (-sz \sinh sz) (pu_0 + qv_0 + w'_0) + W^{\pi'}(z)$$

Thus the B.C.2 (p. 13) gives

$$\begin{aligned}
U'(h) - p W(h) &= s(\sinh sh)u_0 + \frac{a^2}{2} \left( \frac{\sinh sh + sh \cosh sh}{s} \right) (p^2 u_0 + pqv_0 + pw'_0) \\
&- p \frac{\sinh sh}{s} w'_0 - \frac{pa^2}{2} \left( \frac{\sinh sh}{s} - h \cosh sh \right) (pu_0 + qv_0 + w'_0) \\
+ U''(h) - p W''(h) &= 0
\end{aligned}$$

$$\begin{aligned}
V'(h) - q W(h) &= s(\sinh sh)v_0 + \frac{a^2}{2} \left( \frac{\sinh sh + sh \cosh sh}{s} \right) (pq u_0 + q^2 v_0 + qw'_0) \\
&- q \frac{\sinh sh}{s} w'_0 - \frac{qa^2}{2} \left( \frac{\sinh sh}{s} - h \cosh sh \right) (pu_0 + qv_0 + w'_0) \\
+ V''(h) - q W''(h) &= 0
\end{aligned}$$

$$\begin{aligned}
p U(h) + q V(h) + \frac{a^2+1}{a^2-1} W'(h) &= p(\cosh sh)u_0 + \frac{pa^2}{2} h \frac{\sinh sh}{s} (p^2 u_0 + pqv_0 + pw'_0) \\
&+ q(\cosh sh)v_0 + \frac{qa^2}{2} h \frac{\sinh sh}{s} (pq u_0 + q^2 v_0 + qw'_0) \\
&+ \left( \frac{a^2+1}{a^2-1} \right) (\cosh sh)w'_0 + \left( \frac{a^2+1}{a^2-1} \right) \frac{a^2}{2} (-sh \sinh sh) (pu_0 + qv_0 + w'_0) \\
+ p U''(h) + q V''(h) + \left( \frac{a^2+1}{a^2-1} \right) W''(h) &= \left( \frac{2}{\pi} \right)^{1/2} \hat{v}_0(h)
\end{aligned}$$

This is a system of linear equations for  $u_0, v_0, w'_0$  of the form

$$\begin{aligned}
d_{11}u_0 + d_{12}v_0 + d_{13}w'_0 &= f_1(p, q) (= -U''(h) + p W''(h)) \\
d_{21}u_0 + d_{22}v_0 + d_{23}w'_0 &= f_2(p, q) (= -V''(h) + q W''(h)) \\
d_{31}u_0 + d_{32}v_0 + d_{33}w'_0 &= f_3(p, q) (= -p U''(h) - q V''(h) - \left( \frac{a^2+1}{a^2-1} \right) W''(h) \\
&+ \left( \frac{2}{\pi} \right)^{1/2} \hat{v}_0(h))
\end{aligned}$$

where

$$\begin{aligned} d_{11} &= s(\sinh sh) + \frac{a^2 p^2}{2s} (\sinh sh + sh \cosh sh) - \frac{a^2 p^2}{2s} (\sinh sh - sh \cosh sh) \\ &= s(\sinh sh) + a^2 p^2 h \cosh sh \end{aligned}$$

$$\begin{aligned} d_{12} &= \frac{a^2}{2} pq \left( \frac{\sinh sh}{s} + h \cosh sh \right) - \frac{a^2}{2} pq \left( \frac{\sinh sh}{s} - h \cosh sh \right) \\ &= a^2 pq h \cosh sh \end{aligned}$$

$$\begin{aligned} d_{13} &= \frac{a^2}{2} p \left( \frac{\sinh sh}{s} + h \cosh sh \right) - \frac{p}{s} \sinh sh - \frac{a^2}{2} p \left( \frac{\sinh sh}{s} - h \cosh sh \right) \\ &= a^2 ph \cosh sh - p \frac{\sinh sh}{s} \end{aligned}$$

$$\begin{aligned} d_{21} &= \frac{a^2}{2} pq \left( \frac{\sinh sh}{s} + h \cosh sh \right) - \frac{a^2}{2} pq \left( \frac{\sinh sh}{s} - h \cosh sh \right) \\ &= a^2 pqh \cosh sh = d_{12} \end{aligned}$$

$$\begin{aligned} d_{22} &= s(\sinh sh) + \frac{a^2}{2} q^2 \left( \frac{\sinh sh}{s} + h \cosh sh \right) - \frac{a^2}{2} q^2 \left( \frac{\sinh sh}{s} - h \cosh sh \right) \\ &= s(\sinh sh) + a^2 q^2 h \cosh sh \end{aligned}$$

$$\begin{aligned} d_{23} &= \frac{a^2}{2} q \left( \frac{\sinh sh}{s} + h \cosh sh \right) - q \frac{\sinh sh}{s} - \frac{a^2}{2} q \left( \frac{\sinh sh}{s} - h \cosh sh \right) \\ &= -\frac{q}{s} \sinh sh + a^2 qh \cosh sh \end{aligned}$$

$$\begin{aligned} d_{31} &= p \cosh sh + \frac{a^2}{2} p^3 \frac{h}{s} \sinh sh + \frac{a^2}{2} pq^2 \frac{h}{s} \sinh sh - \left( \frac{a^2+1}{a^2-1} \right) \frac{a^2}{2} sh p \sinh sh \\ &= p \cosh sh + \left( \frac{a^2}{2} psh - \left( \frac{a^2+1}{a^2-1} \right) \frac{a^2}{2} psh \right) \sinh sh \\ &= p \cosh sh + \frac{a^2}{2} psh \left( -\frac{2}{a^2-1} \right) \sinh sh = p \cosh sh - \frac{a^2}{a^2-1} p sh \sinh sh \end{aligned}$$

$$\begin{aligned}
d_{32} &= \frac{a^2}{2} p^2 q \frac{h \sinh sh}{s} + q \cosh sh + \frac{a^2}{2} q^3 \frac{h \sinh sh}{s} - \left(\frac{a^2+1}{a^2-1}\right) \frac{a^2}{2} sh q \sinh sh \\
&= q \cosh sh + \frac{a^2}{2} qsh \sinh sh - \frac{a^2}{2} \left(\frac{a^2+1}{a^2-1}\right) qsh \sinh sh \\
&= q \cosh sh + \frac{a^2}{2} qsh \sinh sh \left(\frac{-2}{a^2-1}\right) \\
&= q \cosh sh - \frac{a^2}{a^2-1} qsh \sinh sh
\end{aligned}$$

$$\begin{aligned}
d_{33} &= \frac{a^2}{2} p^2 \frac{h \sinh sh}{s} + \frac{a^2}{2} q^2 \frac{h \sinh sh}{s} + \frac{a^2+1}{a^2-1} \cosh sh - \left(\frac{a^2+1}{a^2-1}\right) \frac{a^2}{2} hs \sinh sh \\
&= \frac{a^2}{2} sh \sinh sh - \frac{a^2}{2} \left(\frac{a^2+1}{a^2-1}\right) sh \sinh sh + \frac{a^2+1}{a^2-1} \cosh sh \\
&= \left(\frac{-2}{a^2-1}\right) \frac{a^2}{2} sh \sinh sh + \frac{a^2+1}{a^2-1} \cosh sh \\
&= \frac{a^2+1}{a^2-1} \cosh sh - \frac{a^2}{a^2-1} sh \sinh sh
\end{aligned}$$

The Cofactors of Q = (d<sub>jk</sub>) .

$$\text{Let } Q_{jk} = (\text{cof}Q)_{jk}$$

$$Q_{11} = d_{22}d_{33} - d_{32}d_{23}$$

$$\begin{aligned}
&= (s \sinh sh + a^2 q^2 h \cosh sh) \left(\frac{a^2+1}{a^2-1} \cosh sh - \frac{a^2}{a^2-1} sh \sinh s\right) \\
&+ (q \cosh sh - \frac{a^2}{a^2-1} q sh \sinh s) \left(\frac{a^2+1}{a^2-1} \cosh sh - \frac{a^2}{a^2-1} sh \sinh s\right) \\
&= \frac{1}{a^2-1} [(a^2+1)s \sinh \cosh - a^2 s^2 h \sinh^2 + (a^2+1)a^2 q^2 h \cosh^2 - a^4 q^2 h^2 s \sinh \cosh] \\
&+ q^2 \left[\frac{1}{s} \sinh \cosh - a^2 h \cosh^2 - \frac{a^2}{a^2-1} h \sinh^2 + \frac{a^4}{a^2-1} h^2 s \sinh \cosh\right] \\
&= \left(\frac{(a^2+1)a^2 q^2 h}{a^2-1} - a^2 q^2 h\right) \cosh^2 + \left(\frac{-a^2 s^2 h}{a^2-1} - \frac{a^2 q^2 h}{a^2-1}\right) \sinh^2 \\
&+ \left(\frac{a^2+1}{a^2-1} s - \frac{a^4 q^2 h s}{a^2-1} + \frac{q^2}{s} + \frac{a^4 q^2 h^2 s}{a^2-1}\right) \sinh \cosh
\end{aligned}$$

$$\begin{aligned}
&= a^2 q^2 h \frac{2}{a^2-1} \cosh^2 - \frac{a^2(s^2+q^2)h}{a^2-1} \sinh^2 + \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh \cosh \\
&= \frac{a^2(s^2+q^2)h}{a^2-1} + \frac{2a^2 q^2 h - a^2(s^2+q^2)h}{a^2-1} \cosh^2 + \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh \cosh \\
&= \frac{a^2(s^2+q^2)h}{a^2-1} - \frac{a^2 p^2 h}{a^2-1} \cosh^2 + 2 \frac{a^2(s^2+q^2) + p^2}{s(a^2-1)} \sinh 2sh
\end{aligned}$$

$$\begin{aligned}
Q_{12} &= - (d_{21}d_{33} - d_{31}d_{23}) = d_{31}d_{23} - d_{21}d_{33} \\
&= (p \cosh - \frac{a^2}{a^2-1} psh \sinh)(-\frac{q}{s} \sinh + a^2 qh \cosh) \\
&\quad - (a^2 pqh \cosh)(\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh) \\
&= -\frac{pq}{s} \sinh \cosh + a^2 pqh \cosh^2 + \frac{a^2 qph}{(a^2-1)} \sinh^2 - \frac{a^4 pqsh^2}{a^2-1} \sinh \cosh \\
&\quad - \frac{a^2(a^2+1)}{a^2-1} pqh \cosh^2 + \frac{a^4 pqsh^2}{a^2-1} \sinh \cosh \\
&= -\frac{a^2 pqh}{a^2-1} + (a^2 pqh + \frac{a^2 pqh}{a^2-1} - \frac{(a^4+a^2) pqh}{a^2-1}) \cosh^2 \\
&\quad + (-\frac{pq}{s}) \sinh \cosh \\
&= -\frac{a^2 pqh}{a^2-1} + a^2 pqh (1 + \frac{1}{a^2-1} - \frac{a^2+1}{a^2-1}) \cosh^2 - \frac{pq}{s} \sinh \cosh \\
&= -\frac{a^2 pqh}{a^2-1} - \frac{a^2 pqh}{a^2-1} \cosh^2 sh - \frac{pq}{2s} \sinh 2sh
\end{aligned}$$

$$\begin{aligned}
Q_{13} &= d_{21}d_{32} - d_{31}d_{22} \\
&= (a^2 pqh \cosh)(q \cosh - \frac{a^2}{a^2-1} q sh \sinh) \\
&\quad - (p \cosh - \frac{a^2}{a^2-1} psh \sinh)(s \sinh + a^2 q^2 h \cosh) \\
&= (\cosh - \frac{a^2}{a^2-1} sh \sinh)(-ps \sinh) \\
&= \frac{a^2}{a^2-1} phs^2 \sinh^2 sh - ps \sinh sh \cosh sh
\end{aligned}$$

$$= -\frac{a^2}{a^2-1} phs^2 + \frac{a^2}{a^2-1} phs^2 \cosh^2 sh - \frac{ps}{2} \sinh 2sh$$

$$\begin{aligned} Q_{21} &= - (d_{12}d_{33} - d_{32}d_{13}) = d_{32}d_{13} - d_{12}d_{33} \\ &= (q \cosh - \frac{a^2}{a^2-1} qsh \sinh) (a^2 ph \cosh - p \frac{\sinh}{s}) \\ &\quad - (a^2 pqh \cosh) (\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh) \\ &= a^2 pqh \cosh^2 - \frac{pq}{s} \sinh \cosh - \frac{a^4 pqsh^2}{a^2-1} \sinh \cosh + \frac{a^2 pqh}{a^2-1} \sinh^2 \\ &\quad - \frac{a^2(a^2+1)pqh}{a^2-1} \cosh^2 + \frac{a^4 pqh^2 s}{a^2-1} \sinh \cosh \\ &= -\frac{a^2 pqh}{a^2-1} + (a^2 pqh + \frac{a^2 pqh}{a^2-1} - \frac{a^2(a^2+1)pqh}{a^2-1}) \cosh^2 \\ &\quad + (-\frac{pq}{s}) \sinh \cosh \\ &= -\frac{a^2 pqh}{a^2-1} - \frac{a^2 pqh}{a^2-1} \cosh^2 sh - \frac{pq}{2s} \sinh 2sh = Q_{12} \end{aligned}$$

$$\begin{aligned} Q_{22} &= d_{11}d_{33} - d_{31}d_{13} \\ &= (s \sinh + a^2 p^2 h \cosh) (\frac{a^2+1}{a^2-1} \cosh - \frac{a^2}{a^2-1} sh \sinh) \\ &\quad - (p \cosh - \frac{a^2}{a^2-1} psh \sinh) (a^2 ph \cosh - \frac{p}{s} \sinh) \\ &= \frac{s(a^2+1)}{a^2-1} \sinh \cosh - \frac{a^2 s^2 h}{a^2-1} \sinh^2 + \frac{a^2(a^2+1)p^2 h}{a^2-1} \cosh^2 - \frac{a^4 p^2 sh^2}{a^2-1} \sinh \cosh \\ &\quad - a^2 p^2 h \cosh^2 + \frac{p^2}{s} \sinh \cosh + \frac{a^4 p^2 sh^2}{a^2-1} \sinh \cosh - \frac{a^2 p^2 h}{a^2-1} \sinh^2 \\ &= -\frac{a^2(s^2+p^2)h}{a^2-1} (\cosh^2 - 1) + (\frac{a^2(a^2+1)p^2 h}{a^2-1} - a^2 p^2 h) \cosh^2 \\ &\quad + (\frac{s(a^2+1)}{a^2-1} - \frac{p^2}{s}) \sinh \cosh \\ &= \frac{a^2(p^2+s^2)h}{a^2-1} + a^2 h (\frac{(a^2+1)p^2}{a^2-1} - p^2 - \frac{s^2+p^2}{a^2-1}) \cosh^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{s^2(a^2+1) + p^2(a^2-1)}{s(a^2-1)} \sinh \cosh \\
& = \frac{a^2(p^2+s^2)h}{a^2-1} - \frac{a^2q^2h}{a^2-1} \cosh^2 sh + \frac{a^2(s^2+p^2)+q^2}{2s(a^2-1)} \sinh 2sh
\end{aligned}$$

$$\begin{aligned}
Q_{23} & = - (d_{11}d_{32} - d_{31}d_{12}) = d_{31}d_{12} - d_{11}d_{32} \\
& = (\cosh - \frac{a^2}{a^2-1} sh \sinh) (a^2p^2qh \cosh) \\
& \quad - q (s \sinh + a^2p^2h \cosh) (\cosh - \frac{a^2}{a^2-1} sh \sinh) \\
& = (\cosh - \frac{a^2}{a^2-1} sh \sinh) (-qs \sinh) \\
& = -qs \sinh \cosh + \frac{a^2s^2qh}{a^2-1} \sinh^2 \\
& = -\frac{a^2s^2qh}{a^2-1} + \frac{a^2s^2qh}{a^2-1} \cosh^2 sh - \frac{qs}{2} \sinh 2sh
\end{aligned}$$

$$\begin{aligned}
Q_{31} & = d_{12}d_{23} - d_{22}d_{13} \\
& = (a^2pqh \cosh) (-\frac{q}{s} \sinh + a^2qh \cosh) \\
& \quad - (s \sinh + a^2q^2h \cosh) (a^2ph \cosh - \frac{p}{s} \sinh) \\
& \quad - a^2phs \sinh \cosh + p \sinh^2 \\
& = -p + p \cosh^2 sh - \frac{a^2phs}{2} \sinh 2sh
\end{aligned}$$

$$\begin{aligned}
Q_{32} & = - (d_{12}d_{23} - d_{21}d_{13}) = d_{21}d_{13} - d_{12}d_{23} \\
& = (a^2pqh \cosh) (a^2ph \cosh - \frac{p}{s} \sinh) \\
& \quad - (a^2pqh \cosh) (\frac{q}{s} \sinh + a^2qh \cosh)
\end{aligned}$$



$$\begin{aligned}
&= (a^2 p q h \cosh) (a^2 h (p-q) \cosh + \frac{q-p}{s} \sinh) \\
&= (a^2 p q h \cosh \operatorname{sh}) (p-q) (a^2 h \cosh - \frac{1}{s} \sinh)
\end{aligned}$$

$$\begin{aligned}
Q_{33} &= d_{11} d_{22} - d_{21} d_{12} \\
&= (s \sinh + a^2 p^2 h \cosh) (s \sinh + a^2 q^2 h \cosh) - a^4 p^2 q^2 h^2 \cosh^2 \\
&= s^2 \sinh^2 + (a^2 p^2 \operatorname{sh} + a^2 q^2 \operatorname{sh}) \sinh \cosh \\
&= s^2 \sinh^2 \operatorname{sh} + a^2 s^3 h \sinh \operatorname{sh} \cosh \operatorname{sh}
\end{aligned}$$

$|Q| = \det(d_{jk})$  can be calculated from Luré, p. 153 and the correspondence (see p. 14)

$$ip \leftrightarrow \alpha_1, \quad iq \leftrightarrow \alpha_2, \quad is \leftrightarrow D$$

This gives

$$\begin{aligned}
|Q| &= 2a^2 h (is)^3 \sin(ish) \left(1 + \frac{\sin 2ish}{2ish}\right) \\
&= 2a^2 h s^3 \sinh \operatorname{sh} \left(1 + \frac{\sinh 2sh}{2sh}\right)
\end{aligned}$$

Solution of the System on p. 25. We can solve by means of the relations.

$$\sum_{k=1}^3 Q_{kj} d_{k\ell} = |Q| \delta_{j\ell}$$

Thus

$$(*) \quad \begin{cases} |Q| u_0 = Q_{11}f_1 + Q_{21}f_2 + Q_{31}f_3 \\ |Q| v_0 = Q_{12}f_1 + Q_{22}f_2 + Q_{32}f_3 \\ |Q| w'_0 = Q_{13}f_1 + Q_{23}f_2 + Q_{33}f_3 \end{cases}$$

The only real zero of  $|Q(s)|$  is at  $s = \sqrt{p^2+q^2} = 0$ . Thus

$\forall$  real  $(p,q) \neq (0,0)$

$$u_0(p,q) = \sum_{j=1}^3 \frac{Q_{j1}(p,q)f_j(p,q)}{|Q(s)|}$$

$$v_0(p,q) = \sum_{j=1}^3 \frac{Q_{j2}(p,q)f_j(p,q)}{|Q(s)|}$$

$$w'_0(p,q) = \sum_{j=1}^3 \frac{Q_{j3}(p,q)f_j(p,q)}{|Q(s)|}$$

Substituting for  $u_0, v_0, w'_0$  in the equations on p. 24 gives

$$U(p,q,z), \quad V(p,q,z), \quad W(p,q,z)$$

Residue Series Representation for  $\hat{u}(p,y,z)$ ,  $\hat{v}(p,y,z)$ ,  $\hat{w}(p,y,z)$ .

The equations on pp. 4-8 give

$$\hat{u}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} \tilde{u}(p,q,z) dq = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} U(p,q,z) dq$$

$$\hat{v}(p,y,z) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} V(p,q,z) dq$$

$$\hat{w}(p,y,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} W(p,q,z) dq$$

In particular,

$$\hat{u}(p, q, 0) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} u_0(p, q) dq$$

$$\hat{v}(p, y, 0) = \frac{-i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} v_0(p, q) dq$$

$$\frac{\partial \hat{w}(p, y, 0)}{\partial z} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{iyq} w'_0(p, q) dq$$

Now the equations on p. 32 and p. 24 give  $U, V, W, u_0, v_0, w'_0$  as meromorphic functions of  $q$  for each fixed  $p$ . Thus residue series for the above functions can be obtained by deforming the contour in the upper half of the  $q$ -plane for  $y > 0$  (lower half for  $y < 0$ ). The poles of the integrals  $U, \dots, w'_0$  are the zeros of  $|Q(s)|$ . The cofactors  $Q_{jk}(p, q)$  are holomorphic in the  $q$ -plane. Examinations of the formulas for  $U^\pi, V^\pi, W^\pi$  and  $f_j(p, q)$  shows that these functions are analytic everywhere except at  $s = 0$ , because  $P^{-1} = O(s^{-2})$ . Thus special care is necessary in calculating the residue at  $s = 0$  ( $q = \pm i|p|$ ).

#### Zeros of $|Q(s)|$ .

There are two families of zeros

$$1) \sinh sh = -i \sin(ish) = 0 \iff ish = is_n h = n\pi, \quad n = 0, 1, 2,$$

Thus

$$s_n = \sqrt{p^2 + q_n^2} = \frac{n\pi}{ih} = -i \frac{n\pi}{h}, \quad p^2 + q_n^2 = -\left(\frac{n\pi}{h}\right)^2$$

$$q_n^2 = -\left(p^2 + \left(\frac{n\pi}{h}\right)^2\right), \quad q_n = i \sqrt{p^2 + (n\pi/h)^2} = i \sqrt{p^2 + \alpha_n^2}$$

$$q_0 = i |p|$$

These are simple zeros of  $|Q(s)|$  for  $n > 1$ . However

$$|Q(s)| = O(s^4), \quad s \rightarrow 0$$

$$(p^2 + q^2)^2 = (q - i|p|)^2(q + i|p|)^2 - (2i|p|)^2(q - i|p|)^2$$

Thus  $q_0$  is, in general, a higher-order pole.

$$2) \quad 1 + \frac{\sin 2sh}{2sh} = 1 - i \frac{\sin 2ish}{2sh} = -i \left( \frac{2ish + \sin 2ish}{2sh} \right) = 0$$

$$\rightarrow 2ish = 2is_{\nu}h = 2\beta_{\nu}h \leftrightarrow s_{\nu} = \sqrt{p^2 + q_{\nu}^2} = -i\beta_{\nu}$$

$$p^2 + q_{\nu}^2 = -\beta_{\nu}^2, \quad q_{\nu}^2 = -(p^2 + \beta_{\nu}^2), \quad q_{\nu} = i\sqrt{p^2 + \beta_{\nu}^2}$$

Calculation of  $\bar{F}(z)$ .

$\bar{F}(z)$  is defined on p. 13. Now on  $y = 0 \pm$

$$\tau_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0$$

$$\tau_{zy} = G \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0$$

Thus

$$\left( \frac{\partial u}{\partial y} \right)_0 = \frac{\partial u(x, 0^+, z)}{\partial y} = - \left( \frac{\partial v}{\partial x} \right)_0$$

$$F_s \left( \frac{\partial u}{\partial y} \right)_0 = - F_s \left( \frac{\partial v}{\partial x} \right)_0 = + p F_c v_0$$

$$F_c \left( \frac{\partial w}{\partial y} \right)_0 = - F_c \left( \frac{\partial v}{\partial z} \right)_0$$

Thus

$$\left(\frac{\pi}{2}\right)^{1/2} F_1(z) = p (1-a^2) F_C v_0$$

$$\left(\frac{\pi}{2}\right)^{1/2} F_2(z) = -q (1+a^2) F_C v_0$$

$$\left(\frac{\pi}{2}\right)^{1/2} F_3(z) = - (1-a^2) F_C \left(\frac{\partial v}{\partial z}\right)_0$$

Calculation of  $U^\pi, V^\pi, W^\pi, U^{\pi'}, V^{\pi'}, W^{\pi}'$ .

From pp. 18-19.

$$H(z) = \begin{pmatrix} 0 \\ \overline{G(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1/2} F(z) \end{pmatrix}$$

$$PH(\zeta) = \begin{pmatrix} C_A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \overline{G(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{G(z)} \end{pmatrix}$$

Write

$$\phi(z) = \begin{pmatrix} \phi^{11}(z) & \phi^{12}(z) \\ \phi^{21}(z) & \phi^{22}(z) \end{pmatrix}$$

Then

$$\phi^T(\zeta) = \begin{pmatrix} \phi^{11T}(\zeta) & \phi^{21T}(\zeta) \\ \phi^{12T}(\zeta) & \phi^{22T}(\zeta) \end{pmatrix}$$

Hence

$$\phi^T(\zeta) PH(\zeta) = \begin{pmatrix} \phi^{11T} & \phi^{21T} \\ \phi^{12T} & \phi^{22T} \end{pmatrix} \begin{pmatrix} 0 \\ -\overline{G} \end{pmatrix} = - \begin{pmatrix} \phi^{21T} \overline{G} \\ \phi^{22T} \overline{G} \end{pmatrix}$$

$$P^{-1} \phi^T PH = - \begin{pmatrix} C_A^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi^{21T} \overline{G} \\ \phi^{22T} \overline{G} \end{pmatrix} = - \begin{pmatrix} C_A^{-1} \phi^{21T} \overline{G} \\ - \phi^{22T} \overline{G} \end{pmatrix}$$

$$\begin{aligned}
& \phi(z) P^{-1} \phi^T(\zeta) P H(\zeta) \\
&= \begin{pmatrix} \phi^{11}(z) & \phi^{12}(z) \\ \phi^{21}(z) & \phi^{22}(z) \end{pmatrix} \begin{pmatrix} -C_A^{-1} \phi^{21T}(\zeta) \bar{G}(\zeta) \\ \phi^{22T}(\zeta) \bar{G}(\zeta) \end{pmatrix} \\
&= \begin{pmatrix} -\phi^{11}(z) C_A^{-1} \phi^{21T}(\zeta) \bar{G}(\zeta) + \phi^{12}(z) \phi^{22T}(\zeta) \bar{G}(\zeta) \\ -\phi^{21}(z) C_A^{-1} \phi^{21T}(\zeta) \bar{G}(\zeta) + \phi^{22}(z) \phi^{22T}(\zeta) \bar{G}(\zeta) \end{pmatrix}
\end{aligned}$$

Thus (pp. 23-24)

$$\begin{aligned}
\bar{U}^\pi(z) &= (U^\pi(z), V^\pi(z), W^\pi(z))^T = A^{-1/2} (X_1^\pi, X_2^\pi, X_3^\pi)^T \\
&= \int_0^z A^{-1/2} (\phi^{12}(z) \phi^{22T}(\zeta) - \phi^{11}(z) C_A^{-1} \phi^{21T}(\zeta)) A^{-1/2} \bar{F}(\zeta) d\zeta
\end{aligned}$$

$$A^{-1/2} \phi^{11}(z) = (\bar{U}^1 \bar{U}^2 \bar{U}^6) A^{-1/2}$$

$$A^{-1/2} \phi^{12}(z) = (\bar{U}^4 \bar{U}^5 \bar{U}^3) A^{-1/2}$$

$$\phi^{21T}(\zeta) = (A^{1/2} (\bar{U}^1, \bar{U}^2, \bar{U}^6) A^{-1/2})^T = A^{-1/2} (\bar{U}^1, \bar{U}^2, \bar{U}^6) A^{1/2}$$

$$\phi^{22T}(\zeta) = (A^{1/2} (\bar{U}^4, \bar{U}^5, \bar{U}^3) A^{-1/2})^T = A^{-1/2} (\bar{U}^4, \bar{U}^5, \bar{U}^3) A^{1/2}$$

$$A^{-1/2} \phi^{12}(z) \phi^{22T}(\zeta) A^{-1/2} = (\bar{U}^4(z) \bar{U}^5(z) \bar{U}^3(z)) A^{-1} (\bar{U}^4(\zeta) \bar{U}^5(\zeta) \bar{U}^3(\zeta))^T$$

$$A^{-1/2} C_A^{-1} A^{-1/2} = \frac{1}{s^4(1+a^2)} \begin{pmatrix} s^2+a^2q^2 & -a^2pq & 0 \\ -a^2pq & s^2+a^2p^2 & 0 \\ 0 & 0 & s^2(1+a^2) \end{pmatrix} = C_s^{-1}$$

$$A^{-1/2} \phi^{11}(z) C_A^{-1} \phi^{21T}(\zeta) A^{-1/2} = (\bar{U}^1(z) \bar{U}^2(z) \bar{U}^6(z)) C_s^{-1} (\bar{U}^1(\zeta) \bar{U}^2(\zeta) \bar{U}^6(\zeta))^T$$

Put

$$M_1(z) = (\bar{U}^4(z)\bar{U}^5(z)\bar{U}^3(z)) , \quad M_2(z) = (\bar{U}^1(z)\bar{U}^2(z)\bar{U}^6(z))$$

Then

$$\bar{U}^{\pi}(z) = \int_0^z \{M_1(z)A^{-1}M_1'(\zeta)^T - M_2(z)C_S^{-1}M_2'(\zeta)^T\} \bar{F}(\zeta) d\zeta$$

$$\begin{aligned} \bar{U}^{\pi'}(z) &= \{M_1(z)A^{-1}M_1'(z)^T - M_2(z)C_S^{-1}M_2'(z)^T\} \bar{F}(z) \\ &+ \int_0^z \{M_1'(\zeta)A^{-1}M_1(\zeta)^T - M_2'(\zeta)C_S^{-1}M_2(\zeta)^T\} \bar{F}(\zeta) d\zeta \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{U}^{\pi'}(z) &= A^{-1/2} (X_4^{\pi} X_5^{\pi} X_6^{\pi})^T \\ &= \int_0^z A^{-1/2} (\phi^{22}(z)\phi^{22T}(\zeta) - \phi^{21}(z)C_A^{-1}\phi^{21T}(\zeta)) A^{-1/2} \bar{F}(\zeta) d\zeta \end{aligned}$$

$$A^{-1/2}\phi^{22}(z)\phi^{22T}(\zeta)A^{-1/2} = (\bar{U}^4(z)\bar{U}^5(z)\bar{U}^3(z))A^{-1}(\bar{U}^4(\zeta)\bar{U}^5(\zeta)\bar{U}^3(\zeta))^T$$

$$A^{-1/2}\phi^{21}(\zeta)C_A^{-1}\phi^{21T}(\zeta)A^{-1/2} = (\bar{U}^1(z)\bar{U}^2(\zeta)\bar{U}^6(z))A^{-1/2}C_A^{-1}A^{-1/2}(\bar{U}^1(\zeta)\bar{U}^2(\zeta)\bar{U}^6(\zeta))^T$$

$$\bar{U}^{\pi'}(z) = \int_0^z \{M_1'(z)A^{-1}M_1^T(\zeta) - M_2'(z)C_S^{-1}M_2^T(\zeta)\} \bar{F}(\zeta) d\zeta$$

$$\bar{U}^{\pi''}(z) = \{M_1'(z)A^{-1}M_1'(z)^T - M_2'(z)C_S^{-1}M_2'(z)^T\} \bar{F}(z)$$

$$+ \int_0^z \{M_1''(\zeta)A^{-1}M_1(\zeta)^T - M_2''(\zeta)C_S^{-1}M_2(\zeta)^T\} \bar{F}(\zeta) d\zeta$$

$$A \bar{U}^{\pi''}(z) + B \bar{U}^{\pi'}(z) - C_S \bar{U}^{\pi}(z) = A \{M_1'(z)A^{-1}M_1'(z)^T - M_2'(z)C_S^{-1}M_2'(z)^T\} \bar{F}(z)$$

An alternative derivation is as follows.

$$\text{Try } \bar{U}^{\pi}(z) = M_1(z)\bar{C}_2(z) + M_2(z)\bar{C}_1(z)$$

$$\begin{aligned} \bar{U}^{\pi'}(z) &= M_1'(z)\bar{C}_2(z) + M_2'(z)\bar{C}_1(z) \\ &\quad + M_1(z)\bar{C}_2'(z) + M_2(z)\bar{C}_1'(z) + \text{set} = 0 \end{aligned}$$

$$\begin{aligned} \bar{U}^{\pi''}(z) &= M_1''(z)\bar{C}_2(z) + M_2''(z)\bar{C}_1(z) \\ &\quad + M_1'(z)\bar{C}_2'(z) + M_2'(z)\bar{C}_1'(z) \end{aligned}$$

$$A \bar{U}^{\pi''} + B \bar{U}^{\pi'} - C_S \bar{U}^{\pi} = A M_1''(z)\bar{C}_2(z) + A M_2''(z)\bar{C}_1(z) = F(z)$$

Thus

$$M_1(z)\bar{C}_2'(z) + M_2(z)\bar{C}_1'(z) = \bar{0}$$

$$M_1'(z)\bar{C}_2(z) + M_2'(z)\bar{C}_1(z) = A^{-1} \bar{F}(z)$$

or

$$\begin{pmatrix} U^1 & U^2 & U^6 & U^4 & U^5 & U^3 \\ U^1 & U^2 & U^6 & U^4 & U^5 & U^3 \end{pmatrix} \begin{pmatrix} \bar{C}_1' \\ \bar{C}_2' \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}\bar{F} \end{pmatrix}$$

or

$$\phi^0(z) \begin{pmatrix} \bar{C}_1' \\ \bar{C}_2' \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1}\bar{F} \end{pmatrix}$$

Note that

$$\phi(z) = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix}.$$



Thus (p. 22)

$$P = \begin{pmatrix} A^{-1/2} C_s A^{-1/2} & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \phi^0(z)^T \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} A^{-1/2} C_s A^{-1/2} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \phi^0(z) \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix}$$

and

$$\begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} P \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} = \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

$$\phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \phi^0(z) = \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

or

$$\phi^0(z)^{-1} = \begin{pmatrix} C_s^{-1} & 0 \\ 0 & -A^{-1} \end{pmatrix} \phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

Applying this to the system on p. 38 gives

$$\phi^0(z) \begin{pmatrix} \bar{C}'_1 \\ \bar{C}'_2 \end{pmatrix} = \begin{pmatrix} M_2(z) & M_1(z) \\ M'_2(z) & M'_1(z) \end{pmatrix} \begin{pmatrix} \bar{C}'_1 \\ \bar{C}'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ A^{-1} \bar{F} \end{pmatrix}$$

$$\begin{pmatrix} \bar{C}'_1 \\ \bar{C}'_2 \end{pmatrix} = \begin{pmatrix} C_s^{-1} & 0 \\ 0 & -A^{-1} \end{pmatrix} \begin{pmatrix} M_2^T(z) & M_1^T(z) \\ M_2'^T(z) & M_1'^T(z) \end{pmatrix} \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} 0 \\ A^{-1} \bar{F} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \bar{C}_1^T(z) \\ \bar{C}_2^T(z) \end{pmatrix} &= \begin{pmatrix} C_s^{-1} & 0 \\ 0 & -A^{-1} \end{pmatrix} \begin{pmatrix} -M_2^T(z) \bar{F}(z) \\ -M_1^T(z) \bar{F}(z) \end{pmatrix} \\ &= \begin{pmatrix} -C_s^{-1} M_2^T(z) \bar{F}(z) \\ A^{-1} M_1^T(z) \bar{F}(z) \end{pmatrix} \end{aligned}$$

$$\bar{U}^T(z) = \int_0^z \{M_1(z) A^{-1} M_1^T(\zeta) - M_2(z) C_s^{-1} M_2^T(\zeta)\} \bar{F}(\zeta) d\zeta$$

$$\phi^0(z)^T \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \phi^0(z)$$

$$= \begin{pmatrix} M_2^T(z) & M_2^T(z) \\ M_1^T(z) & M_1^T(z) \end{pmatrix} \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} M_2(z) & M_1(z) \\ M_2^T(z) & M_1^T(z) \end{pmatrix} = \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

$$= \begin{pmatrix} M_2^T & M_2^T \\ M_1^T & M_1^T \end{pmatrix} \begin{pmatrix} C_s M_2 & C_s M_1 \\ -A M_2^T & -A M_1^T \end{pmatrix}$$

$$= \begin{pmatrix} M_2^T C_s M_2 - M_2^T A M_2^T & M_2^T C_s M_1 - M_2^T A M_1^T \\ M_1^T C_s M_2 - M_1^T A M_2^T & M_1^T C_s M_1 - M_1^T A M_1^T \end{pmatrix}$$

$$= \begin{pmatrix} C_s & 0 \\ 0 & -A \end{pmatrix}$$

Calculation of Coefficients in the Residue Series.

$$M_2(p, q, z) = \begin{pmatrix} U^1 & U^2 & U^6 \\ V^1 & V^2 & V^6 \\ W^1 & W^3 & W^6 \end{pmatrix}, \quad M_1(p, q, z) = \begin{pmatrix} U^4 & U^5 & U^3 \\ V^4 & V^5 & V^3 \\ W^4 & W^5 & W^3 \end{pmatrix}$$

The Poles  $q_n = i\sqrt{p^2 + \alpha_n^2}$ ,  $s_n = -i\alpha_n$ ,  $n = 1, 2, 3, \dots$  ( $\alpha_n = \frac{n\pi}{h}$ )

$$\cosh s_n h = \cos i s_n h = \cosh \alpha_n h = \cos n\pi = (-1)^n$$

$$\sinh s_n h = -i \sin i s_n h = -i \sin n\pi = 0$$

$$M_2(p, q_n, h) = \begin{pmatrix} (-1)^n & 0 & \frac{a^2 h p}{2(a^2+1)} (-1)^n \\ 0 & (-1)^n & \frac{a^2 h q_n}{2(a^2+1)} (-1)^n \\ -\frac{a^2 h}{2} (-1)^n p & -\frac{a^2 h}{2} (-1)^n q_n & (-1)^n \end{pmatrix}$$

$$M_1(p, q_n, h) = \begin{pmatrix} \frac{a^2 h p^2}{2(a^2+1)(-\alpha_n^2)} (-1)^n & \frac{a^2 h p q_n}{2(a^2+1)(-\alpha_n^2)} (-1)^n & 0 \\ \frac{a^2 h p q_n}{2(a^2+1)(-\alpha_n^2)} (-1)^n & \frac{a^2 h q_n^2}{2(a^2+1)(-\alpha_n^2)} (-1)^n & 0 \\ 0 & 0 & -\frac{a^2 h}{2} (-1)^n \end{pmatrix}$$

$$M_2'(p, q_n, h) = \begin{pmatrix} \frac{a^2 h p^2}{2} (-1)^n & \frac{a^2 h p q_n}{2} (-1)^n & 0 \\ \frac{a^2 h p q_n}{2} (-1)^n & \frac{a^2 h p^2}{2} (-1)^n & 0 \\ 0 & 0 & -\frac{a^2 (-\alpha_n^2)}{2(a^2+1)} (-1)^n \end{pmatrix}$$

$$M_1'(p, q_n, h) = \begin{pmatrix} (-1)^n & 0 & \frac{a^2}{2} h p (-1)^n \\ 0 & (-1)^n & \frac{a^2}{2} h q_n (-1)^n \\ -\frac{a^2}{2(a^2+1)} h p (-1)^n & -\frac{a^2}{2(a^2+1)} h q_n (-1)^n & (-1)^n \end{pmatrix}$$

For a simple pole at  $q = q_0 = |p|$

$$\begin{aligned} \text{Res}_{q_0} \{e^{iyq} u_0(p, q)\} &= \lim_{q \rightarrow q_0} \{(q - q_0) e^{iyq} u_0(p, q)\} \\ &= e^{-y|p|} \lim_{q \rightarrow q_0} \{(q - q_0) u_0(p, q)\} \end{aligned}$$

For a double pole

$$\begin{aligned} \text{Res}_{q_0} \{e^{iyq} u_0(p, q)\} &= \lim_{q \rightarrow q_0} \frac{\partial}{\partial q} \{(q - q_0)^2 e^{iyq} u_0(p, q)\} \\ &= \lim_{q \rightarrow q_0} [e^{iyq} \frac{\partial}{\partial q} \{(q - q_0)^2 u_0\} + iy e^{iyq} \{(q - q_0)^2 u_0\}] \\ &= e^{iy|p|} \text{Res}_{q_0} u_0(p, q) + iy e^{-y|p|} a_{-2}(u_0) \end{aligned}$$

For higher order poles, correspondingly higher order powers of  $y$  appear.

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