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DIGITAL STRAIGHTNESS AND CONVEXITY,

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ABSTRACT

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1. **Introduction**. <u>Digital image or picture processing</u> [1]
is concerned to a great extent with the extraction and de-
scription of objects or regions in pictures - individual
characters in text, components in circuit diagrams, cells
in Pap smears, tumors in chest x-rays, buildings in aerial
photographs, etc. [2]. The description often involves
geometrical properties of the regions; thus one needs to
know how to define and measure such properties when the
pictures are represented in digital form.

A <u>digital picture</u> is a rectangular array of lattice
points, with each of which a numerical "gray level" is
associated. We can "segment" a picture into regions, e.g.,
by defining subsets of the points that have characteristic
ranges of gray levels. Thus a region or object in a digital
picture is simply a finite, nonempty set of lattice points.

<u>Digital topology</u> [3], which was the subject of an earlier
paper in this Monthly [4], deals with topological properties
of digital objects, e.g., with their connectedness and adja-
cency properties, and with digital arcs and curves. Some of
the basic concepts of digital topology will be summarized in
Section 2. In this paper we discuss another important class
of properties, involving the concepts of <u>straightness</u> (of
arcs) and <u>convexity</u> (of objects). The central questions are:
When can a digital arc be the digitization of a real straight

line segment? When can a digital object be the digitization
of a real convex set? We will define "digitization" in
Section 3; these questions will be formulated more precisely
in Section 4; and their solutions will be summarized in Sec-
tions 5-6. Section 7 discusses the extension of these results
to three dimensions, and Section 8 sketches algorithms for
determining straightness and convexity based on the results.
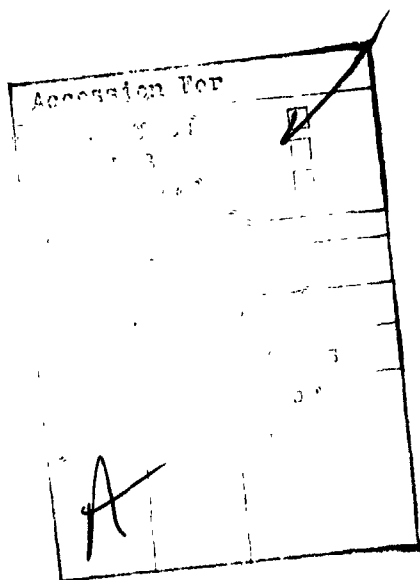
2. **Some digital topology.** Let Π be a rectangular array of lattice points having positive integer coordinates (x,y), where $1 \le x \le M$, $1 \le y \le N$. To each lattice point $P = (x,y)$ is associated a closed unit square, called a <u>cell</u>, centered at P; it will be denoted by P' or by $(x,y)'$.

The <u>4-neigbors</u> of (x,y) are its four horizontal and vertical neighbors $(x\pm 1,y)$ and $(x,y\pm 1)$, provided these are in Π. The <u>8-neighbors</u> of (x,y) consist of its y-neighbors together with its four diagonal neighbors $(x-1, y\pm 1)$ and $(x+1,y\pm 1)$ in Π. Evidently Q is a 4-neighbor of P if the cell Q' shares a side with P', and an 8-neighbor if P' and Q' share a side or a corner. We can thus also define the 4- and 8-neighbors of a cell.

A <u>path</u> from P to Q is a sequence of points $P=P_0, P_1, \ldots, P_n = Q$ such that P_i is a neighbor of P_{i-1}, $1 \le i \le n$; we speak of a 4-path or an 8-path, depending on whether "neighbor" means 4-neighbor or 8-neighbor. Let S be any subset of Π. We say that P and Q are (4- or 8-) <u>connected</u> in S if there exists a path from P to Q consisting entirely of points of S. This is readily an equivalence relation; its equivalence classes are called the (4- or 8-) <u>components</u> of S. $S \subseteq \Pi$ is called connected if any two points of S are connected in S, i.e., if there is only one component of S. The definitions for a set of cells S' are analogous.

Let $\overline{S}$ be the complement of S in Π. Components of $\overline{S}$ that do not meet the border of Π (the points for which x=1 or M, y=1 or N) are called <u>holes</u> in S. If S has no holes, it is called <u>simply connected</u>. It turns out to be desirable to always use opposite types of connectedness for S and $\overline{S}$, i.e., if we use 4- for S, we use 8- for $\overline{S}$, and vice versa. Thus, e.g., S is simply 8-connected if it is 8-connected and has no 4-holes.

A⊆Π is called a (4- or 8-) <u>arc</u> if it is connected, and all but two of its points (the "endpoints") have exactly two neighbors in A, while those two have exactly one. A cellular arc is defined analogously. Since we use opposite types of connectedness for A and $\overline{A}$, it is easily shown that an arc is simply connected. For further results on digital arcs and closed curves, see [3-4].

3. Digitization of regions and arcs. For any $S \subseteq \Pi$, S' denotes the set of cells corresponding to the points of S. The union of these cells regarded as subsets of the real plane will be denoted by <S'>. Similarly, we will use the notation <P'> when we regard the cell P' as a subset of the plane.

Let R be any subset of <Π'>. We say that $S \subseteq \Pi$ is the digital image of R, and that S' is the cellular image of R, if

(a) $R \subseteq$ <S'>

(b) For all $P \in S$ we have $(P')^{\circ} \cap R \neq \emptyset$,

where $(P')^{\circ}$ is the interior of P'.

We will sometimes refer to the process of forming the digital or cellular image of a set as "digitization". A simple example of a set and its digital and cellular images is shown in Figure 1. A slightly different definition was used in [5-7], which required only $P' \cap R \neq \emptyset$ in (b); but that definition does not specify a unique digital image, and furthermore the definition just given turns out to yield better results when we study digital straightness and convexity. Note, however, that by this definition, a nonempty set can have an empty image (indeed, any set that meets no cell interior has an empty image); but a nonempty open set must have a non-empty image.

Let C be any curve in <Π'>, and consider the Cartesian grid lines defined by the points of Π. Whenever C crosses a

grid line, the point of Π nearest the crossing becomes a point of the digital image of C. (If the crossing is exactly midway between two lattice points, we use the one with smaller coordinate [8]). Evidently, this is not a special case of the definition of the digital image of R. Similarly, we say that S' is the cellular image of C [9] if

   (a) $C \subseteq <S'>$

   (b) For all $P \in S$ we have $(P')^{\circ} \cap C \neq \emptyset$; or else $(P')^{\circ}$
       $\cap C = \emptyset$, $P' \cap C \neq \emptyset$, and P lies to the right of C

       (with respect to a given sense defined on C)

Again, this is not a special case of the definition of the cellular image of R. However, the definitions become the same if we thicken C slightly on its right side. Examples of the digital and cellular images of an arc are shown in Figure 2.

   Note that (for historical reasons) we are not using the same definitions for the digital and cellular images of a curve; the latter definition is given in order to make it equivalent to the definition for an arbitrary set R by thickening. Note also that the digitization of a simple arc in the plane need not be a digital or cellular arc; even if the curvature of the real arc is sufficiently small, its digital and cellular images may not be arcs. However, the image of a real straight line segment is always a (digital, cellular) arc.

4. **Digital convexity and straightness.** We can now formulate the two central questions that are the subject of this paper:

(a) When is a set the digital or cellular image of a convex set? Note that any set S or S' is always the image of a non-convex set, e.g. having small concavities that are missed by the digitization process; but not every S or S' can be the image of a convex set.

Sklansky [7], using a slightly different definition of digitization, called such an S' "digitally convex". Other authors [5,10] attempted to characterize digitally convex sets in various ways, e.g. by requiring that for all P, Q ∈ S, any point of Π on the (real) line segment PQ is a point of S; but these conditions were not necessary and sufficient for digital convexity. In Section 5 we will see that when our definition of digitization is used, several characterizations of digital convexity do in fact turn out to be equivalent to each other and to the property of being the image of a convex set.

(b) When is an arc the digital or cellular image of a straight line segment? Here again, note that any arc A or A' is always the image of a non-straight real arc, e.g. having small irregularities, but not every A or A' is the image of a straight line.

Freeman [11] gave a semi-formal set of conditions for a digital arc to be the digital image of a straight line, and Rosenfeld [12] gave a necessary and sufficient condition. In

Section 6 we will see that this condition is analogous to one of the characterizations of digital convexity; a digital or cellular arc is convex iff it is the image of a straight line. We will also see that a set is digitally convex iff a digital line segment joining any two of its points is contained in it, which is analogous to a standard characterization of convexity in the real plane.

5. <u>Digital convexity</u>. In this section we summarize the main results on characterization of digitally convex objects. The proofs can be found in [13-15].

<u>Theorem 5.1</u>. The following properties of a set $S \subseteq \Pi$ are all equivalent:

(a) For any $P, Q \in S$, there exists no point of $\bar{S}$ lying on the (real) line segment PQ.

> In other words: There exists no triple of collinear points of Π such that the first and last ones lie in S and the middle one lies in $\bar{S}$.

(b) For any $P, Q \in S$, and any point (u,v) of the real line segment PQ, there exists a point (x,y) of S such that $\max \{|x-u|, |y-v|\} < 1$.

> In other words: The line segment joining any two points of S lies everywhere "near" S, in the sense that every point of it is strictly within city block distance 1 of some point of S.

(c) For any $P, Q \in S$, let R_{PQ} be the subset of $<\bar{S}'>$ bounded by the real line segment PQ and the boundary of $<S'>$; then R_{PQ} contains no point of $\bar{S}$.

(d) Let $H(S)$ be the (real) convex hull of S; then no point of $\bar{S}$ lies in $H(S)$.

> An alternative form of (d) is: For any $(u,v) \in H(S)$, there exists an $(x,y) \in S$ such that $\max \{|u-x|, |v-y|\} < 1$.

Given a set $S \subseteq \Pi$, a point P of S is called <u>semi-isolated</u> if only one of its 4-neighbors is in S. A set S is <u>regular</u> if it does not have any semi-isolated points.

<u>Theorem 5.2</u>. Let $S \subseteq \Pi$ be a regular set. Then it is the digital image of a convex subset of the real plane if and only if it has the properties in Theorem 5.1.

A set S that satisfies Theorem 5.1 will be called <u>digitally convex</u>. Therefore, a regular set S is digitally convex if and only if there exists a convex subset R of the real plane whose digital image is S. However, if S is not regular, then S may be digitally convex but have no convex preimage as shown in Figure 3.

In the real plane, a region R is convex if the midpoint of any pair of points of R also lies in R. In the digital case, however, the analogous condition is necessary for convexity, but not sufficient. Let $P=(a,b)$ and $Q=(c,d)$ be two points of Π, and let $u = \dfrac{a+c}{2}$, $v = \dfrac{b+d}{2}$. The <u>set of midpoints</u> of P and Q is the set of lattice points $M_{PQ} \equiv \{ (\lfloor u \rfloor, \lfloor v \rfloor), (\lfloor u \rfloor, \lceil v \rceil), (\lceil u \rceil, \lfloor v \rfloor), (\lceil u \rceil, \lceil v \rceil) \}$. Note that this set consists of one, two, or four points, depending on whether a and c, b and d have the same or opposite parity.

<u>Theorem 5.3</u>. If S is digitally convex, then for all $P, Q \in S$ we have $M_{PQ} \cap S \neq \emptyset$, but not conversely.

However, we can prove that another property involving midpoints

is equivalent to convexity. Let $P = (a,b)$, $Q = (c,d)$, (u,v) $= (\frac{a+c}{2}, \frac{b+d}{2})$. On the real line segment $\overline{P(u,v)}$, let P* (possibly the same as P) be the lattice point closest to (u,v), and similarly let Q* be the lattice point on $\overline{(u,v)Q}$ closest to (u,v). Then S is digitally convex if, for all $P,Q \in S$, either P* or Q* is in S.

6. <u>Digital and cellular straightness</u>. We next present the main results on digital straightness and its relationship to digital convexity. For further details see [9,14].

<u>Theorem 6.1</u>. The following properties of a digital arc A⊆Π are equivalent:

(a) For any P,Q∈A, and any point (u,v) of the real line segment PQ, there exists a point (x,y) of A such that max {|x-u|, |y-v|} < 1.

   Note that this is the same as (b) of Theorem 5.1.

(b) There exists a straight line segment whose digital image is A.

A digital arc satisfying Theorem 6.1 will be called a <u>digital straight line segment</u>.

<u>Corollary 6.2</u>. A digital arc is a digital straight line segment iff it is digitally convex.

<u>Theorem 6.3</u>. S⊆Π is digitally convex if any two points of S lie on a digital straight line segment contained in S.

<u>Theorem 6.4</u>. A cellular arc A'⊆Π' is the cellular image of a straight line segment iff there exists a straight line segment whose cellular image is A'.

The proof, which is quite complicated, is based on establishing that property (a) of Theorem 6.1 holds for such cellular arcs.

A cellular arc satisfying Theorem 6.4 will be called a
cellular straight line segment.

Corollary 6.5. A cellular arc is a cellular straight
line segment iff it is cellularly convex.

Theorem 6.6. $S' \subseteq \Pi'$ is cellularly convex if any two cells
of S' lie on a cellular straight line segment contained in S.

7. <u>The three-dimensional case</u>. It is straightforward to extend the basic concepts of digital topology from two to three (or more) dimensions [17]. A point $Q = (u,v,w)$ is a 6-neighbor of $P = (x,y,z)$ if $|x-u| + |y-u| + |z-w| = 1$, and a 26-neighbor if max $\{|x-u|, |y-v|, |z-w|\}= 1$. Given a finite set S of three-dimensional lattice points, a point P of S is called semi-isolated if four of its 6-neighbors are in $\bar{S}$ and they are mutually 26-neighbors. A set S is regular if it has no semi-isolated point.

The properties that characterize digitally convex sets are also easily extendable to three dimensions. However, it turns out that many of these extensions are no longer equivalent [18]. In fact, we have

<u>Theorem 7.1</u>. The following properties of a finite regular set S of three-dimensional lattice points are equivalent:

(a) For any $O,P,Q \in S$, and any point (u,v,w) of the real triangle OPQ, there exists a point (x,y,z) of S such that max $\{|u-x|, |v-y|, |w-z|\} < 1$.

(b) Let $H(S)$ be the (real) convex hull of S; then for any point (u,v,w) of $H(S)$, there exists a point (x,y,z) of S such that max $\{|u-x|, |v-y|, |w-z|\} < 1$.

(c) S is the digital image of a convex subset of real 3-space.

An S (not necessarily regular) that has properties a) and b) of Theorem 7.1 is called a <u>digital convex solid</u>.

Theorem 7.2. The following properties of S are neces-
sary, but not sufficient, for S to be a digital convex solid:

(a') For any P,Q∈S, there exists no point of $\bar{S}$ on the
(real) line segment PQ.

(b') For any P,Q∈S, and any (u,v,w) on the real line
segment PQ, there exists an (x,y,z)∈S such that
max {|x-u|, |y-v|, |z-w|} < 1.

(c') H(S) contains no point of $\bar{S}$.

The non-sufficiency of these properties is illustrated in
Figure 4, where set S is not convex but satisfies (a') and
(b'), and T is not convex but satisfies (c').

8. __Algorithms.__ We describe algorithms that determine

   i) whether or not a given set $S \subseteq \Pi$ is digitally convex.

  ii) whether or not a given set $S \subseteq \Pi$ is a digital straight
      line segment.

 iii) whether or not a given set S of three-dimensional
      lattice points is digitally convex.

   Let $\Pi = \{(x,y) \mid 1 \leq x \leq M \text{ and } 1 \leq y \leq N\}$ and $S \subseteq \Pi$.
A point P is a corner point of $S(\bar{S})$ if two of its 4-neighbors
are points of $\bar{S}(S)$ and they are mutually 8-neighbors.

   __Algorithm__ 2D-CONVEX (S)

   1. Construct the convex hull H(S) of the set of corner
      points of S.

   2. Check if H(S) contains a corner point of $\bar{S}$. If it
      does, then S is not digitally convex. Otherwise,
      S is digitally convex.

   __Algorithm__ STRAIGHT-LINE(S)

   1. Check if S is a digital arc. If not, S is not a
      digital straight line segment.

   2. Using 2D-CONVEX, determine whether or not S is convex.
      If it is then S is a digital straight line segment.
      Otherwise, it is not.

   If we represent S by a method called run length code [1],
then the convex hull H(S) may be constructed in time O(M) by
the method in [16]. Thus, both algorithms 2D-CONVEX and
STRAIGHT-LINE run in time O(M).

Given a set S of three dimensional lattice points, a
point of S is a corner point of S if three of its 6-neighbors
are in $\bar{S}$ and they are mutually 26-neighbors. Let H be a
three dimensional polyhedron. Then a point P on the surfaces
of H is a semi-lattice point if two of its coordinates are
integers. A point P on the edges of H is a semi-lattice
point if one of its coordinates is an integer. A semi-lattice
point P=(x,y,z) is said to be near S if there is a point
Q=(u,v,w) of S such that max $\{|u-x|,|u-y|,|w-z|\} < 1$.

Algorithm 3D-CONVEX(S)

1. Construct the convex hull H(S) of the set of corner
   points of S.

2. Check if H(S) contains a point of $\bar{S}$. If so, S is not
   digitally convex.

3. Check if H(S) has a semi-lattice point which is not
   near S. If so, then S is not digitally convex.

4. Otherwise, S is digitally convex.

Suppose that S is a subset of the set of three dimensional
lattice points, $\{(x,y,z)|1 \le x,y,z \le M\}$. Again, if S is repre-
sented by a run length code, then the convex hull H(S) may be
constructed in time $O(M^2 \log M)$ by the method in [19]. Thus,
the algorithm 3D-CONVEX runs in time $O(M^2 \log M)$.

9. Concluding remarks

We have seen that concepts such as straightness and convexity can be defined for sets of lattice points, but properties of those concepts that are all equivalent in the real domain may become inequivalent in the discrete domain. Moreover, properties that are equivalent in two dimensions may become inequivalent in three. Thus in order to measure geometrical properties of digital objects, careful thought is needed to determine which of the standard concepts in the real domain can be safely used in the digital case.

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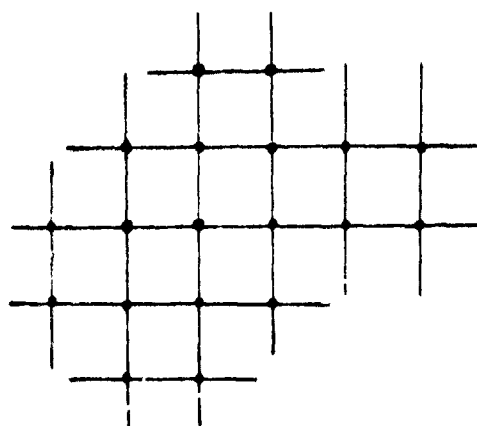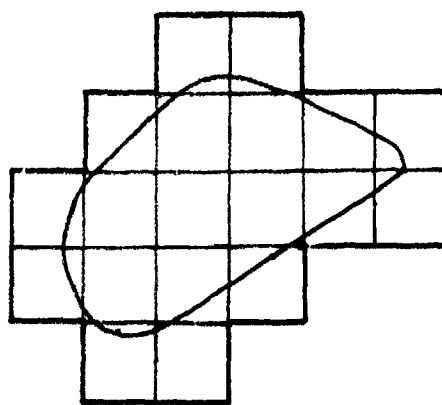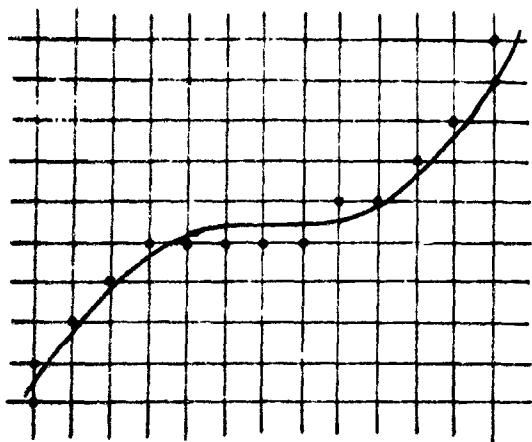
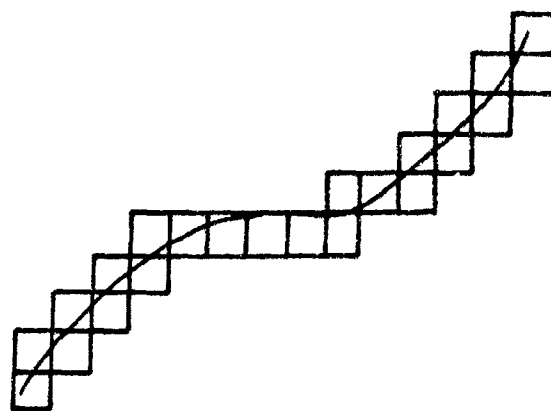(a) S   (b) S' and R

Figure 1. R and its digital and cellular images, S and S'.

(a)   (b)
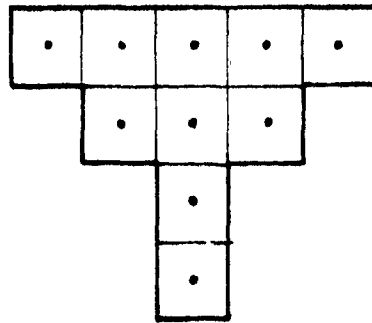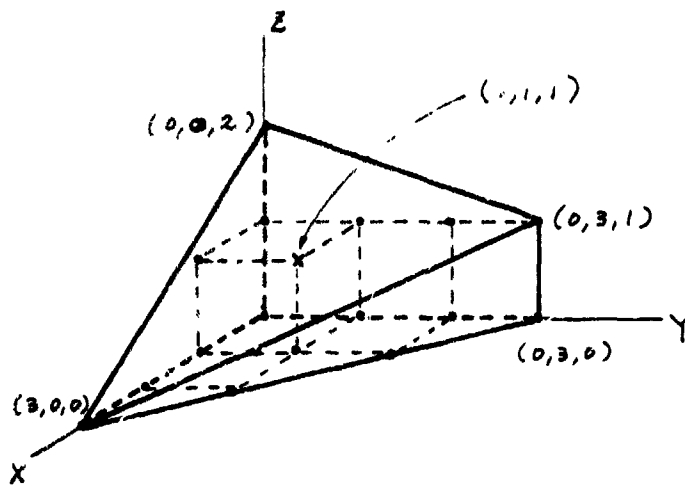
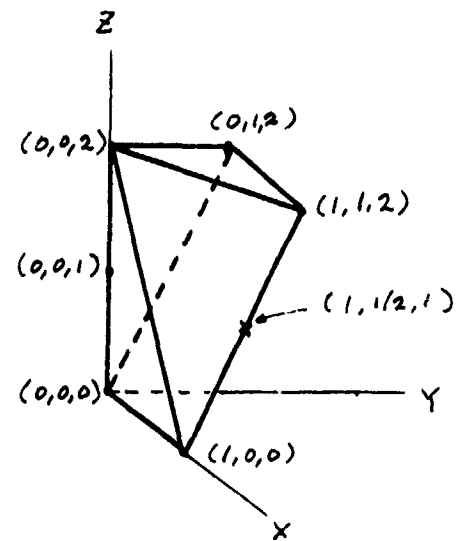Figure 2. Digital and cellular images of an arc.

Figure 3. A set which is digitally convex but has no convex preimage.

(a) S

(b) T

Figure 4. Examples illustrating the non-sufficiency of various convexity properties in three dimensions.