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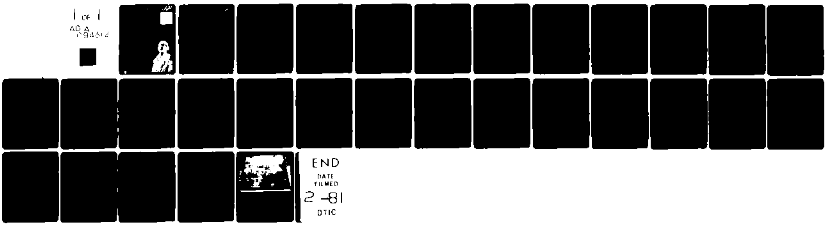

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(6) KALMAN FILTER TECHNIQUES FOR CONTROL OF REPEATED ECONOMIC SURVEYS.

by

(10) Wray Smith
Zeev Barzily

(9) Scientific rept.

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In this paper we discuss the determination of sample sizes and inter-survey intervals in repeated economic surveys. Three models are discussed. The first two maintain the intersurvey intervals at a constant length while the third allows the intersurvey intervals to vary in length. In the first model we assume that the number of obtained observations in each survey is equal to the number of designated observations. The second and third models

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20. Abstract (continued)

assume that the number of obtained observations is a random variable. The costs which are taken into account are the fixed and varying costs of surveying and a cost which is due to the use of estimates from the repeated surveys.

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KALMAN FILTER TECHNIQUES FOR CONTROL OF REPEATED ECONOMIC SURVEYS*

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1. Introduction and Background

Statistical data from household surveys and administrative records are widely used in formulas to allocate grant-in-aid funds or other governmental benefits to state and local jurisdictions. A review of U.S. programs appears in Gonzalez (1978). An explicit tradeoff between data collection costs in repeated surveys and an imputed cost of misallocation of resources due to the use of imprecise survey data in setting allocation levels was considered in Smith and Zalkind (1978) as a scalar problem in deterministic inventory theory. The present paper places such problems in a control theory framework accommodating vector linear models of nonstationary economic processes measured by noisy multi-item repeated surveys. Optimal rules are found for control of a sequence of surveys, where there is a fixed charge plus unit costs of

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surveying and an imputed quadratic loss associated with the imprecision of the resulting state estimates of the observed economic process.

In this paper we analyze one model in which the number of completed interviews in a survey is equal to the designated number of interviews and two models in which the number of completed interviews is a random variable. The models are analyzed with the aid of an equivalent sample size form of the Kalman filter that was derived in Smith (1979). A related inventory analysis of single-item repeated surveys under random-yield sampling is set forth in Smith and Zalkind (1980). Methods for time series analysis of repeated surveys are presented in Scott and Smith (1974). Problems of the choice of loss function for misallocation error are of practical importance, but they are not treated in the present paper, nor are problems of "distributive equity" in allocations across states or local jurisdictions, as discussed in Spencer (1979). In the models of this paper, we are estimating (tracking but not controlling) the state of the observed economic process, and we are controlling the survey measurement subsystem; see Meier, Peschon, and Dressler (1967). Also see Aoki and Li (1968), which discusses control problems with cost for observation, although not treating the fixed-plus-variable cost structure of the present paper.

2. Formulation of the Models

In this section we formulate the models to be discussed in the present paper. In the first two models (Sections 3 and 4) we are interested in the evolution of $\underset{\sim}{x}(j)$, a multivariate socioeconomic process (e.g., money income or proportion of a population in poverty), through discrete time points j , $j=0,1,2,\ldots$. Here $\underset{\sim}{x}(j)$ is an $m \times 1$ state vector, and we assume that it evolves according to the vector random walk

$$\underset{\sim}{x}(j+1) = \underset{\sim}{x}(j) + \underset{\sim}{w}(j+1) , \qquad (1)$$

where $\underset{\sim}{w}(j) \sim N(\underset{\sim}{0},\underset{\sim}{Q})$. That is, $\underset{\sim}{w}(j)$ has a multivariate normal distribution with a zero mean vector and known process disturbance covariance matrix $\underset{\sim}{Q}$. Surveys of the process $\underset{\sim}{x}(j)$ are taken once every T units of time at $k = T$, $2T$, $3T$, ..., rT, ... (In one of our stochastic models, T may vary.) We also assume that $\underset{\sim}{x}(0) \sim N(E[\underset{\sim}{x}(0)]\ ,\ \underset{\sim}{C}(0))$, where $E[\underset{\sim}{x}(0)]$ and $C(0)$ are known.

At each survey time $n_d(k)$ individuals are sampled, but only $N(k)$ completed interviews are obtained. We assume that $P[N(k) \geq 1] = 1$. The difference $n_d(k) - N(k)$ is due to refusals to respond or incomplete observations. We assume that $\underset{\sim}{y}(k)$, the m-dimensional survey measurement, is represented by the survey equation

$$\underset{\sim}{y}(k) \quad = \quad \underset{\sim}{x}(k) + (1/N(k)) \sum_{i=0}^{N(k)} \underset{\sim}{u}_i(k) \ , \tag{2}$$

where each $\underset{\sim}{u}_i(k)$ is an $m \times 1$ vector that is distributed normally with a zero mean and measurement noise covariance matrix $\underset{\sim}{R}$. The matrix $\underset{\sim}{R}$ is assumed to be known, time invariant, symmetric, and positive definite. Both $\underset{\sim}{w}(j)$ and the $\underset{\sim}{u}_i(k)$ are assumed to be serially uncorrelated.

We may find the best linear estimate of $\underset{\sim}{x}(rT+j)$, given $E[\underset{\sim}{x}(0)]$, $\underset{\sim}{C}(0)$, and the r measurements $y^r = (\underset{\sim}{y}(T), \underset{\sim}{y}(2T), \ldots, \underset{\sim}{y}(rT))$, as a linear combination of the observations y^r . We define the estimation error $\underset{\sim}{x}^e(k+j|k)$ as

$$\underset{\sim}{x}^e(k+j|k) \quad = \quad \underset{\sim}{x}(k+j) - \hat{\underset{\sim}{x}}(k+j|k) \ ,$$

where

$$\hat{\underset{\sim}{x}}(k+j|k) = E[\underset{\sim}{x}(k+j)|y^r] \ .$$

We also define the j-step-ahead estimation error covariance matrix $\underset{\sim}{C}(k+j|k)$ as the conditional expectation of the outer product of the j-step-ahead estimation error vector given the r measurements; namely,

$$\underset{\sim}{C}(k+j|k) = E[\underset{\sim}{x}^e(k+j|k)\ (\underset{\sim}{x}^e(k+j|k))'] \ . \tag{3}$$

We now state a general optimal filter theorem for the vector random walk models. Related proofs and derivations of Kalman-type filters for vector models may be found in Jazwinski (1970), Melsa and Cohn (1978), or Sage and White (1977).

Optimal Filter Theorem for Vector Random Walk Models. The optimal (minimum variance) filter for the discrete system (1), (2) consists of difference equations for the conditional mean $\hat{\underset{\sim}{x}}(k+j|k)$ and the estimation error covariance matrix $\underset{\sim}{C}(k+j|k)$, for $T=1,2,\ldots,T_{max}$:

between surveys,

$$\begin{aligned} \hat{\underset{\sim}{x}}(k+j|k) &= \hat{\underset{\sim}{x}}(k|k) \ , \quad \text{for} \quad k=T,2T,\ldots \ , \text{ and } \ j=1,\ldots,T \ , \\ \underset{\sim}{C}(k+j|k) &= \underset{\sim}{C}(k|k) + j\underset{\sim}{Q} \ ; \end{aligned} \tag{4}$$

at surveys

$$\begin{aligned} \hat{\underset{\sim}{x}}(k|k) &= \hat{\underset{\sim}{x}}(k|k-T) + \underset{\sim}{K}(k)\,[\underset{\sim}{y}(k) - \hat{\underset{\sim}{x}}(k|k-T)] \ , \\ \underset{\sim}{C}(k|k) &= [\underset{\sim}{I} - \underset{\sim}{K}(k)]\,\underset{\sim}{C}(k|k-T) \ , \end{aligned} \tag{5}$$

where $\underset{\sim}{K}(k)$ is the Kalman gain

$$\underset{\sim}{K}(k) = \underset{\sim}{C}(k|k-T)\,[\underset{\sim}{C}(k|k-T) + \underset{\sim}{B}(k)]^{-1} \ , \tag{6}$$

and $\underset{\sim}{B}(k)$ is the sample noise covariance matrix obtained from $\underset{\sim}{R}$ by dividing each of its elements by $N(k)$. The second equations in (4) and (5) are called the error covariance equations.

The first two models differ in the number of observations obtained. In Model A we assume that the number of observations in each survey is a known constant $n_d(k) = n_d$. In other words,

$$P[N(k) = n_d] = 1 \ .$$

Model B assumes that a constant scalar designated sample size n_d is to be used for all survey times, that the obtained sample size is a random variable with expectation θn_d , where $0 < \theta < 1$ is known, and that the inter-survey intervals T are nonvarying.

The third model (Model C) involves a continuous-time process--a Brownian motion process. Here we assume that the sample sizes obtained are i.i.d. random variables and we allow the intersurvey periods to vary as a function of the sample sizes obtained. The remaining assumptions and the analysis parallel those of the first two models.

In all three models we are interested in determining $n_d(k)$, or n_d, the designated sample size to be used at time k. We determine the $n_d(k)$ that minimizes a cost per unit time function J. We define J by

$$J = E[(1/T)(c_0 + c_1 n_d(k) + c_2 N(k)) + L_{avg}] , \tag{7}$$

where c_0 is the fixed start-up cost of ordering a survey, c_1 is a unit cost per designated item-interview (for m items), and c_2 is the additional unit cost per obtained item-interview. Letting a_1 be a loss weighting coefficient set by the decision maker, L_{avg} is defined by

$$\begin{aligned} L_{avg} &= a_1 \text{ trace } [(1/T) \sum_{j=0}^{T-1} \underset{\sim}{C}(k+j|k)] \\ &= a_1 \text{ trace } [\underset{\sim}{C}(k|k) + ((T-1)/2)\underset{\sim}{Q}] . \end{aligned} \tag{8}$$

The L_{avg} is an imputed loss attributed to the use of survey estimates of the state of the process and is proportional to the sum of the estimation error variances. By (6), $\underset{\sim}{C}(k|k)$ is dependent on the realization of $N(k)$ through $\underset{\sim}{K}(k)$.

3. Model A

We specify for Model A that there is a sequence of m-item surveys of one homogeneous population with obtained sample sizes $N(k) = n_d(k)$. The filter gain $\underset{\sim}{K}(k)$ of (6) may be written as

$$\underset{\sim}{K}(k) = \underset{\sim}{C}(k|k-T) [\underset{\sim}{C}(k|k-T) + \underset{\sim}{R}/n_d(k)]^{-1} .$$

Let us now introduce the concept of *equivalent sample size* and show how the error covariance equations in the optimal filter theorem may be usefully analyzed using this concept. Let the matrix $\underset{\sim}{N}_0(k+j|k)$ be defined by

$$\underset{\sim}{N}_0(k+j|k) = \underset{\sim}{R}^{1/2}\, \underset{\sim}{C}^{-1}(k+j|k)\, \underset{\sim}{R}^{1/2} , \quad j=0,1,\ldots,T-1 , \tag{9}$$

where $\underset{\sim}{R}^{1/2}\underset{\sim}{R}^{1/2} = \underset{\sim}{R}$. The Kalman gain may then be written as

$$\underset{\sim}{K}(k) = \underset{\sim}{R}^{1/2}\, \underset{\sim}{N}_0^{-1}(k|k-T)\, \underset{\sim}{R}^{1/2}\, [\underset{\sim}{R}^{1/2}\, \underset{\sim}{N}_0^{-1}(k|k-T)\, \underset{\sim}{R}^{1/2} + \underset{\sim}{R}/n_d(k)]^{-1} , \tag{10}$$

and thus we obtain a recursive relation in $\underset{\sim}{N}_0(k|k)$,

$$\underset{\sim}{N}_0(k|k) = n_d(k)\underset{\sim}{I} + \underset{\sim}{N}_0(k-T|k-T)\, [\underset{\sim}{I} + T\underset{\sim}{R}^{-1/2}\, \underset{\sim}{Q}\, \underset{\sim}{R}^{-1/2}\, \underset{\sim}{N}_0(k-T|k-T)]^{-1} . \tag{11}$$

If only one item is surveyed (m=1), then Q and R (and K) are scalars, the matrix $\underset{\sim}{N}_0(k|k-T)$ is replaced by the scalar $N_0(k|k-T)$, and

$$K(k) = n_d(k) / [N_0(k|k-T) + n_d(k)]$$

and

$$N_0(k|k) = n_d(k) + N_0(k|k-T) .$$

This development leads to equivalent sample size relations to replace the error variance equations in the optimal filter theorem for the scalar case of Model A:

between surveys,

$$N_0(k+j|k) = N_0(k|k)\, [1 + jQR^{-1}\, N_0(k|k)]^{-1} ;$$

at surveys,

$$N_0(k|k) = n_d(k) + N_0(k-T|k-T)\, [1 + TQR^{-1}\, N_0(k-T|k-T)]^{-1} .$$

Here $N_0(k+j|k)$ is the equivalent sample size remaining at time $k+j$, j time units after ordering new stock. If the system is in steady state, $N_0(k|k-T)$ or $N_0(k+T|k)$ may be interpreted in inventory terms as the "reorder point." Since $n_d(k)$ is the sample size of the survey conducted at time k, we may interpret $N_0(k|k)$ as the size of a survey that would be required to obtain the same degree of precision of estimate at time k as that provided by an optimally combined data set of size $n_d(k) + N_0(k|k-T)$.

Figure 1 depicts the pattern of deterministic decay and replenishment of equivalent sample size for the scalar case of Model A.

Returning to the case where $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ are matrices, using (8), (5), and (9) in (7) we obtain

$$\begin{aligned} J(N_0(k|k),T) &= (1/T)[c_0 + c_1 m\, n_d(k)] + a_1 \text{ trace } [\underset{\sim}{N}_0^{-1}(k|k)\underset{\sim}{R} + ((T-1)/2)\underset{\sim}{Q}] \\ &= (1/T)\Big[c_0 + c_1 \text{ trace } [\underset{\sim}{N}_0(k|k) - N_0(k-T|k-T) \cdot \\ &\qquad (\underset{\sim}{1} + T\underset{\sim}{Q}\underset{\sim}{R}^{-1}\underset{\sim}{N}_0(k-T|k-T))^{-1}]\Big] \\ &\qquad + a_1 \text{ trace } [\underset{\sim}{N}_0^{-1}(k|k)\underset{\sim}{R} + ((T-1)/2)\underset{\sim}{Q}] \;. \end{aligned} \tag{12}$$

The optimal pair $(n_d^*(k), T^*)$ or, equivalently, $(\underset{\sim}{N}_0^*(k|k), T^*)$, can easily be determined, since for our survey systems there always exists a T, denoted T_{max}, beyond which it is not possible to lengthen the intersurvey interval T without violating $1 \le \text{trace}[\underset{\sim}{N}_0(k+T|k)]$. Thus we calculate $(n_d^*(k), T^*)$ by setting T at successive integer values, $T=1,2,\ldots,T_{max}$ and computing the value of the sample size $n_d(k)$ which minimizes J for each T. We then adopt the T corresponding to the minimum of J over T.

It can be shown that when k is large the error covariance matrices $\underset{\sim}{C}(k|k)$, $\underset{\sim}{C}(k+T|k+T)$, ... , will approach a limit, say $\underset{\sim}{C}$. Equivalently, $\underset{\sim}{N}_0(k|k)$ will approach a steady-state matrix $\underset{\sim}{N}_0$. If the system is nearing steady state, then $\underset{\sim}{C}(k|k-T)$ will also approach a limit. From the covariance equation in (4), we then have

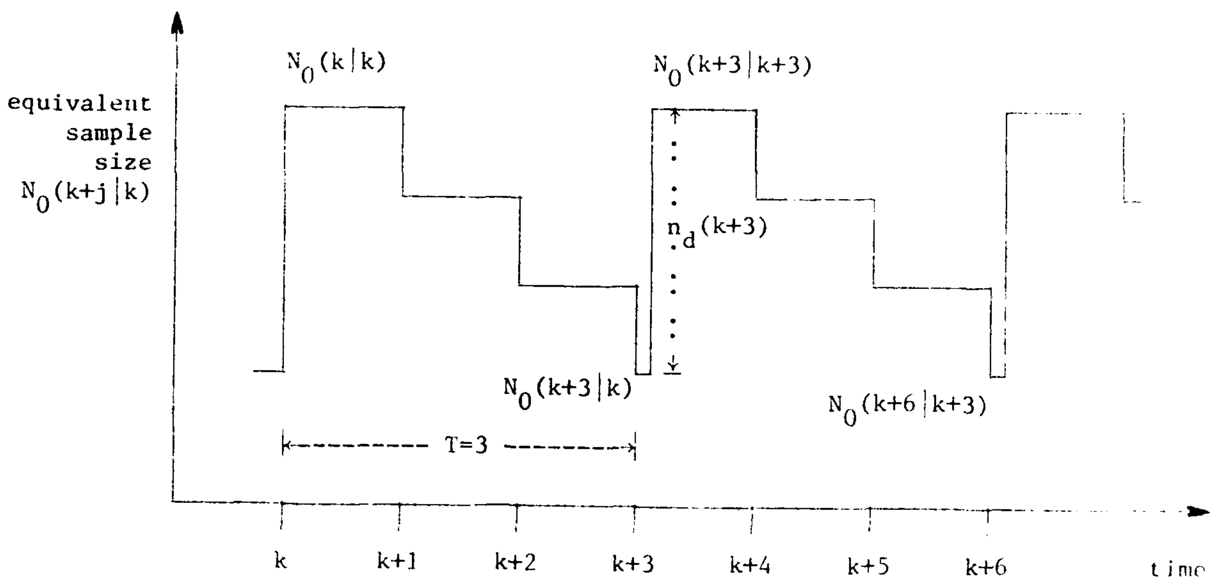

Figure 1.--Deterministic decay and replenishment of equivalent sample size for Model A with $m=1$.

$$C(k|k-T) \rightarrow C + TQ \; .$$

The steady-state Kalman gain K will be

$$K = (C + TQ)\,(C + TQ + R/n_d)^{-1} \; ,$$

so that

$$C = (I - K)\,(C + TQ) \; .$$

That is,

$$\begin{aligned} TQ &= K(C + TQ) \\ &= (C + TQ)\,(C + TQ + R/n_d)^{-1}\,(C + TQ) \; , \end{aligned}$$

or

$$C\,Q^{-1}\,C + T\,C - (T/n_d)\,R = 0 \; . \qquad (13)$$

We solve (13) for C, obtaining

$$C = -(T/2)Q + Q^{1/2}\,[(T/n)Q^{-1/2}\,R\,Q^{-1/2} + (T^2/4)I]^{1/2}\,Q^{1/2} \; . \qquad (14)$$

<u>A numerical example of Model A</u>

Suppose for $m=2$ the cost coefficients are

$$c_0 = \$10^6 \; , \quad c_1 = \$67.50 \; , \quad c_2 = 0 \; , \quad \text{and} \quad a_1 = 2\times10^9 \; ,$$

and the Q and R noise covariance matrices have the following numerical entries:

$$Q = \begin{bmatrix} .00005 & .00004 \\ .00004 & .00007 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0.10 & 0.05 \\ 0.05 & 0.25 \end{bmatrix} .$$

Let D be the matrix of eigenvalues of $Q^{-1/2}\,R\,Q^{-1/2}$,

$$Q^{-1/2}\,R\,Q^{-1/2} = \begin{bmatrix} 3140 & -1978 \\ -1978 & 5018 \end{bmatrix} ; \quad \text{then} \quad D = \begin{bmatrix} 1889 & 0 \\ 0 & 6269 \end{bmatrix} .$$

We can construct an orthogonal matrix V whose columns are normalized eigenvectors of $Q^{-1/2}\,R\,Q^{-1/2}$, so that $Q^{-1/2}\,R\,Q^{-1/2} = V\,D\,V'$. We will also have need of the matrix $\underset{\sim}{S} = \underset{\sim}{V}'\, \underset{\sim}{Q}\, \underset{\sim}{V}$. For the numerical values above,

$$\underset{\sim}{V} = \begin{bmatrix} .8453 & -.5344 \\ .5344 & .8453 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{S} = \begin{bmatrix} .00009184 & .00002619 \\ .00002619 & .00002816 \end{bmatrix} .$$

We will find it convenient to make the change of variable $z = T/n_d$, where $0 < n_d$ is the sample size at each survey time. We assume the loss function appropriate to the end-use allocation formula is a quadratic loss

$$L_{avg} = a_1 \text{ trace } [(1/T) \sum_{j=0}^{T-1} (\underset{\sim}{C}+j\underset{\sim}{Q})] .$$

Using (14) we obtain

$$L_{avg} = a_1 \text{ trace } [-\underset{\sim}{Q}/2 + (z\underset{\sim}{D} + (T^2/4)\underset{\sim}{I})^{1/2}\, \underset{\sim}{V}'\underset{\sim}{Q}\underset{\sim}{V}] ,$$

where we have used the property that $\text{trace}[\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C}] = \text{trace}[\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{A}]$, that if $\underset{\sim}{U}$ is an orthogonal matrix that diagonalizes the matrix $\underset{\sim}{A}$, then $\underset{\sim}{U}$ also diagonalizes $\underset{\sim}{A}^{1/2}$, or if $\underset{\sim}{A} = \underset{\sim}{U}\underset{\sim}{B}\underset{\sim}{U}'$, then $\underset{\sim}{A}^{1/2} = \underset{\sim}{U}\underset{\sim}{B}^{1/2}\underset{\sim}{U}'$, and that $\text{trace}[\underset{\sim}{A}] = \text{trace}[\underset{\sim}{U}'\underset{\sim}{A}\underset{\sim}{U}] = \text{trace}[\underset{\sim}{B}]$; see, for example, Bellman (1970).

We may now write our cost function J for the two-item repeated survey problem with scalar sample size n_d and sampling interval T as

$$\begin{aligned} J &= (1/T)\,[c_0 + c_1 m n_d] + a_1 \text{ trace } [-\underset{\sim}{Q}/2 + ((T/n_d)\underset{\sim}{D} + (T^2/4)\underset{\sim}{I})^{1/2}\, \underset{\sim}{S}] \\ &= c_0/T + c_1\, m/z + a_1\, [s_{11}(z\, d_{11} + T^2/4)^{1/2} + s_{22}(z\, d_{22} + T^2/4)^{1/2} \\ &\qquad - (q_{11} + q_{22})/2] . \end{aligned}$$

It can be shown that J is convex in n_d given T and convex in T given n_d . In practice, (n_d^*, T^*) will be found in a region in which J is convex in (n_d, T) . Fixing T successively at $T=1,2,\ldots,10$

years, we solve $\partial J/\partial z = 0$ for z , and hence for $n_d = 1/z$, by a numerical search procedure. We then compute J for each T using the minimizing n_d and pick the T and n_d combination with lowest average annual cost J . For the given cost coefficients, numerical results are displayed in Figure 2. The T and n_d combination for which the average cost is minimized is $T=3$ years and n_d = 5,26[illegible] units, with J = \$875,473 per year. As Figure 2 makes clear, the average cost curve is rather flat in the neighborhood of the optimal T^* and typically the survey administrator will not incur major additional costs by choosing $T^* + 1$ or $T^* - 1$ instead of T^* . It may also be seen that sampling too frequently is relatively more costly than sampling too seldom, assuming the correctness of the underlying random walk process model for the socioeconomic variables. An administrator who is concerned that the underlying process parameters may take unexpected jumps or exhibit turning points, which are not modeled by the simple time-invariant random walk models, would presumably opt for sampling more frequently than the optimal interval found by this method.

4. Model B

In this section we present a steady-state analysis of a vector random yield model in which a fixed scalar designated sample size n_d is used for all survey times, and T remains constant. In the use of a fixed order size, the development resembles the treatment of random supply in the inventory literature, but in our present analysis we have level-dependent deterministic decay rather than the independent stochastic demands of Karlin (1958). The stochastic nature of $N(k)$ enters the steady-state analysis in the case of a fixed n_d only through the expected obtained sample size,

$$E[N(k)] = \theta\, n_d \, .$$

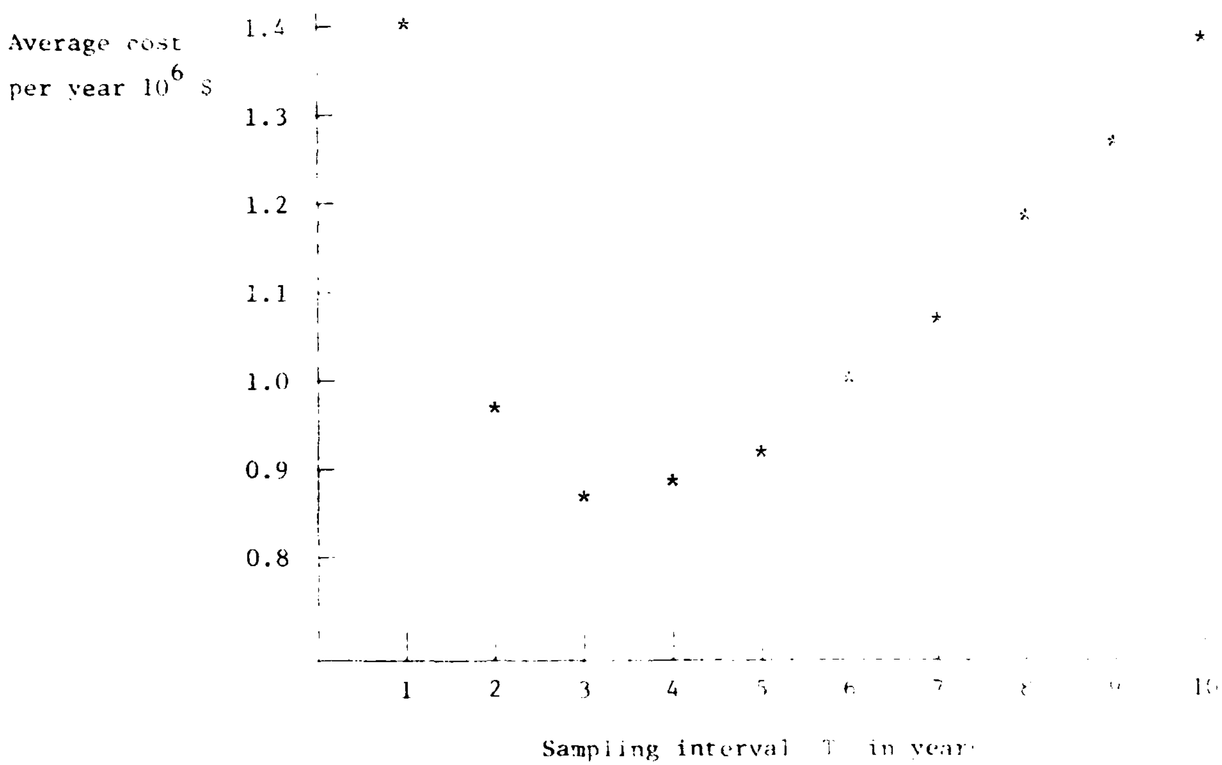

Figure 2.--Average cost versus sampling interval: Model A with m

In a random yield model, an ordering rule that requires us to order a fixed amount at every survey time is an "open loop" approach in the sense that not all of the information available or to become available from the evolving history of the survey system is used in choosing the sampling interval T and the designated sample size n_d. In the special case treated here, we do not accommodate the general case under random yield sampling where T may vary stochastically. Under the assumptions of Model B, the cost function [illegible] becomes, using (8),

$$\begin{aligned} L &= E[L'] = E[c_0 + c_1 n_d + c_2 N(k)] + a_1 \operatorname{trace} [C(k|k) + ((T-1)/2) Q] \\ &= c_0 + (c_1 + c_2) n_d + a_1 \operatorname{trace} [E[C(k|k)] + ((T-1)/2) Q] . \end{aligned} \qquad (15)$$

Note that $C(k|k)$ is now stochastically dependent on $N(k)$.

Since we assume that the stochastic system is in steady state, we define [illegible], $\hat{C} = C(k|k) = C(k|k-T)$, as the conditional expected error covariance

$$C = E[C(k|k) | [illegible], n_d, T, Q, R] ,$$

where [illegible] is a known constant and $n_d = n_d(k)$ for all k. Since Q and R are positive definite, so are C and $\hat{C}$. For scalar sample size n_d, the C we seek is found by methods similar to those employed in Section 3 for Model A, which is a fixed-yield vector model with scalar sample size. The Kalman relations must now be expressed in the form of relations among expected error covariances, expected Kalman gain, and expected obtained sample size. We now have

$$C = [I - K] (\hat{C} + TQ) ,$$

where

$$K = (\hat{C} + TQ) [\hat{C} + TQ + E[R/N(k)]]^{-1} .$$

This leads to the matrix equation

$$\hat{C} Q^{-1} \hat{C} + T C - T \underset{\sim}{R} E[1/N(k)] = 0 . \qquad (16)$$

This equation is of the form of Equation (13) except that it involves expectations rather than deterministic quantities.

Solving (16) for $\hat{C}$ we obtain, strictly paralleling (14),

$$\underset{\sim}{C} = -(T/2)\,\underset{\sim}{Q} + \underset{\sim}{Q}^{1/2}\,[T\,E[1/N(k)]\,\underset{\sim}{Q}^{-1/2}\,\underset{\sim}{R}\,\underset{\sim}{Q}^{-1/2} + (T^2/4)\underset{\sim}{I}]^{1/2}\,\underset{\sim}{Q}^{1/2} ,$$

so that, since $E[\underset{\sim}{C} + ((T-1)/2)\underset{\sim}{Q}] = \hat{\underset{\sim}{C}} + ((T-1)/2)\underset{\sim}{Q}$, we may substitute the value of $\hat{\underset{\sim}{C}}$ found above into the cost function J given by (15). The solution procedures for Model A may then be applied to find (n_d^*, T^*) .

5. Model C

Model C is an extension of random-yield Model B. We consider here a scalar process which evolves in continuous time t with survey measurements taken at time points t_k , $k=1,2,\ldots$. Here we allow the intersurvey periods T_k , $k=1,\ldots$ ($T_k = t_k - t_{k-1}$) , to vary as a function of the obtained sample size $N(t_{k-1})$. We assume that the scalar process is represented by the linear stochastic differential equation

$$dx(t) = x(t)dt + d\beta(t) , \quad t_0 < t , \tag{17}$$

where $x(t)$ is a unidimensional Brownian motion process with

$$E[(d\beta(t))^2] = Q\,dt .$$

The survey equation (2) now becomes

$$y(t_k) = x(t_k) + (1/N(t_k)) \sum_{i=1}^{N(t_k)} u_i(t_k) ,$$
$$k=1,2,\ldots \quad \text{and} \quad t_0 \leq t_k < t_{k+1} , \tag{18}$$

where $y(t_k)$ and $u_i(t_k)$ are defined as before. As set forth in Jazwinski (1970, Theorem 7.1), the optimal filter for the system (17) and (18) becomes the continuous-discrete filter:

between surveys,

$$d\hat{x}(t|t)/dt = \hat{x}(t|t) ,$$
$$dC(t|t)/dt = 2C(t|t) + Q , \quad t_k < t < t_{k+1} ; \tag{19}$$

at surveys,

$$\hat{x}(t_k|t_k) = \hat{x}(t_k|t_k^-) + K(t_k)\,[y(t_k) - \hat{x}(t_k|t_k^-)] \;,$$
$$C(t_k|t_k) = [I - K(t_k)]\,C(t_k|t_k^-) \;, \tag{20}$$

where the Kalman gain $K(t_k)$ is given by

$$K(t_k) = C(t_k|t_k^-)\,[C(t_k|t_k^-) + R/N(t_k)]^{-1} \;, \tag{21}$$

and t_k^- represents the time instant just before the survey conducted (and instantaneously processed) at t_k . As shown in Jazwinski (1970), if we integrate the process equation in (19) over intervals $[t_k, t_{k+1}]$, we may write

$$x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} d\beta(\tau) \;,$$

$$= x(t_k) + w(k+1,k) \;,$$

where

$$w(k+1,k) = \int_{t_k}^{t_{k+1}} d\beta(\tau) \;.$$

By Jazwinski (1970, Theorem 4.1), $\{w(k+1,k)\}$ is a zero-mean, white Gaussian sequence with

$$E[(w(k+1,k))^2] = (t_{k+1} - t_k)Q \;,$$

and the continuous-discrete filter for Model C may be imbedded in a discrete filter paralleling Models A and B. The integrated form of (19) is

$$\hat{x}(t|t) = \hat{x}(t_k|t_k) \;,$$
$$C(t|t) = C(t_k|t_k) + (t-t_k)Q \;, \quad t_k < t < t_{k+1} \;. \tag{22}$$

We now analyze Model C with response rate $\phi(k)$. Suppose that the equivalent sample size on hand at time t_0 is $N_0(t_0)$. The equivalent sample size remaining at time t, $N_0(t|t_0)$, is

$$N_0(t|t_0) = N_0(t_0|t_0)\,[1 + (t-t_0)QR^{-1}N_0(t_0|t_0)]^{-1} . \tag{23}$$

When $N_0(t|t_0)$ just falls to a preselected value n_r (the reorder point), $1 < n_r$, a new survey of designated sample size n_d is ordered [illegible] resulting in [illegible] $N(t_j) \le n_d$. We assume that $N(t_j)$ is independent of $N(t_i)$, $i \neq j$, and hence $N_0(t_j|t_0) = n_r + N(t_j)$ is independent of $N_0(t_i|t_0) = n_r + N(t_i)$, $i \neq j$. In other words, [illegible] we have a renewal process with renewals occurring at the time points where $N_0(t|t_0)$ has just decayed to the preselected reorder point n_r, $1 < n_r$.

We now denote by $\phi(t_k)$ the response rate for the survey conducted at t_k. We assume that the $\phi(t_k)$ are i.i.d. random variables with a known distribution and that $0 < \phi(t_k) < 1$. We thus obtain

$$N_0(t_k|t_k) = n_r + \phi(t_k)n_d ,$$

and

$$E[N_0(t_k|t_k)] = n_r + \bar{\phi} n_d ,$$

where $\bar{\phi}$ is $E[\phi(t_k)]$.

In inventory terms, we thus have a problem of stochastic replenishment, with designated order size n_d, delivered order size $N(t_k)$, and updated equivalent sample size $N_0(t_k|t_k)$. Given the realization of $N_0(t_k|t_k)$, we then have level-dependent deterministic decay from $N_0(t_k|t_k)$ down to the reorder point n_r. The time required for this decay to take place, T_{k+1}, is found in terms of n_r, n_d, and $\phi(t_k)$ to be

$$T_{k+1} = (R/Q) \left[1/n_r - (n_r + \phi(t_k)n_d)^{-1}\right] . \tag{24}$$

We may also write

$$T_{k+1}^{-1} = QR^{-1} n_r \left[n_r + \phi(t_k)n_d\right] / \phi(t_k)n_d . \tag{25}$$

A typical pattern of level-dependent deterministic decay and stochastic replenishment is depicted in Figure 3.

The average ordering cost, $G_{avg}(\phi(t_k))$, over an interval of length $T_{k+1} = t_{k+1} - t_k$, is

$$\begin{aligned} G_{avg}(\phi(t_k)) &= T_{k+1}^{-1} \left[c_0 + (c_1 + c_2\phi(t_k))n_d\right] \\ &= QR^{-1} n_r (n_r + \phi(t_k)n_d) \left[c_0 + (c_1 + c_2\phi(t_k))n_d\right] / \phi(t_k)n_d . \end{aligned} \tag{26}$$

Since $C(t|t_k) = R / N_0(t|t_k)$, we may write the average quadratic loss $L_{avg}(\phi(t_k))$ over the interval T_{k+1} as

$$L_{avg}(\phi(t_k)) = (a_1/T_{k+1}) \int_{t_k}^{t_{k+1}} C(\tau|t_k)d\tau ,$$

which yields

$$L_{avg}(\phi(t_k)) = (a_1R/2) \left[1/n_r + 1/(n_r + \phi(t_k)n_d)\right] . \tag{27}$$

Using (8) and (15), we obtain the cost function J as

$$J = E_\phi \left[G_{avg}(\phi(t_k)) + L_{avg}(\phi(t_k))\right] . \tag{28}$$

It can be verified by taking second derivatives that J is convex in n_r , $1 < n_r$, for fixed n_d , $0 < n_d$, and in n_d , $0 < n_d$, for fixed n_r , $1 < n_r$. If we compute the determinant of the Hessian of $G_{avg} + L_{avg}$, we find that for some (but not all) combinations of values of n_r , n_d , and the cost coefficients the determinant is negative. Therefore, J is not convex in general. Nonetheless, in practice there

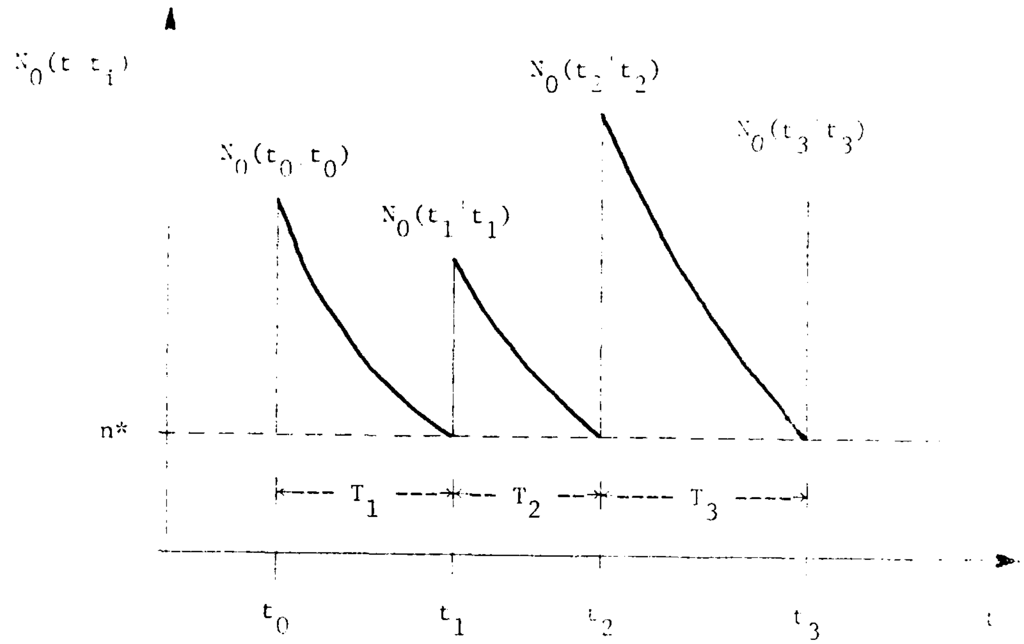

Figure 3.--Level-dependent decay and stochastic replenishment in continuous-time random yield Model C.

is an optimal (n_r^*, n_d^*) pair on the interior of a region in which J is convex and it may be found as an iterative solution to the pair of equations $\partial J/\partial n_r = 0$, $\partial J/\partial n_d = 0$.

Case of Uniform ϕ

In the case of a uniform ϕ on (a,b) ,

$$E[\phi] = \mu = (a+b)/2 \ ,$$

$$E[1/\phi] = \Lambda = (b-a)^{-1} \log(b/a) \ ,$$

and

$$E[(n_r + \phi n_d)^{-1}] = (1/n_d)(b-a)^{-1} \log[(n_r + b n_d)/(n_r + a n_d)] \ ,$$

so that

$$J = E[C_{avg}(\phi)] + E[L_{avg}(\phi)]$$

$$= QR^{-1}[c_0 n_r^2 \Lambda/n_d + c_1 n_r^2 \Lambda + c_2 n_r^2 + c_0 n_r + c_1 n_r \mu n_d + c_2 n_r \mu n_d] \qquad (29)$$

$$+ (a_1 R/2)\left[1/n_r + (1/n_d)(b-a)^{-1} \log[(n_r + b n_d)/(n_r + a n_d)]\right] .$$

Now J may be minimized over (n_r, n_d) by standard iterative methods closely paralleling those used for (s,q) inventory problems. A numerical example of Model C with a uniform ϕ is portrayed in Figure 4. As is shown in Smith (1980), a Markov chain set-up may also be used to find an optimal rule.

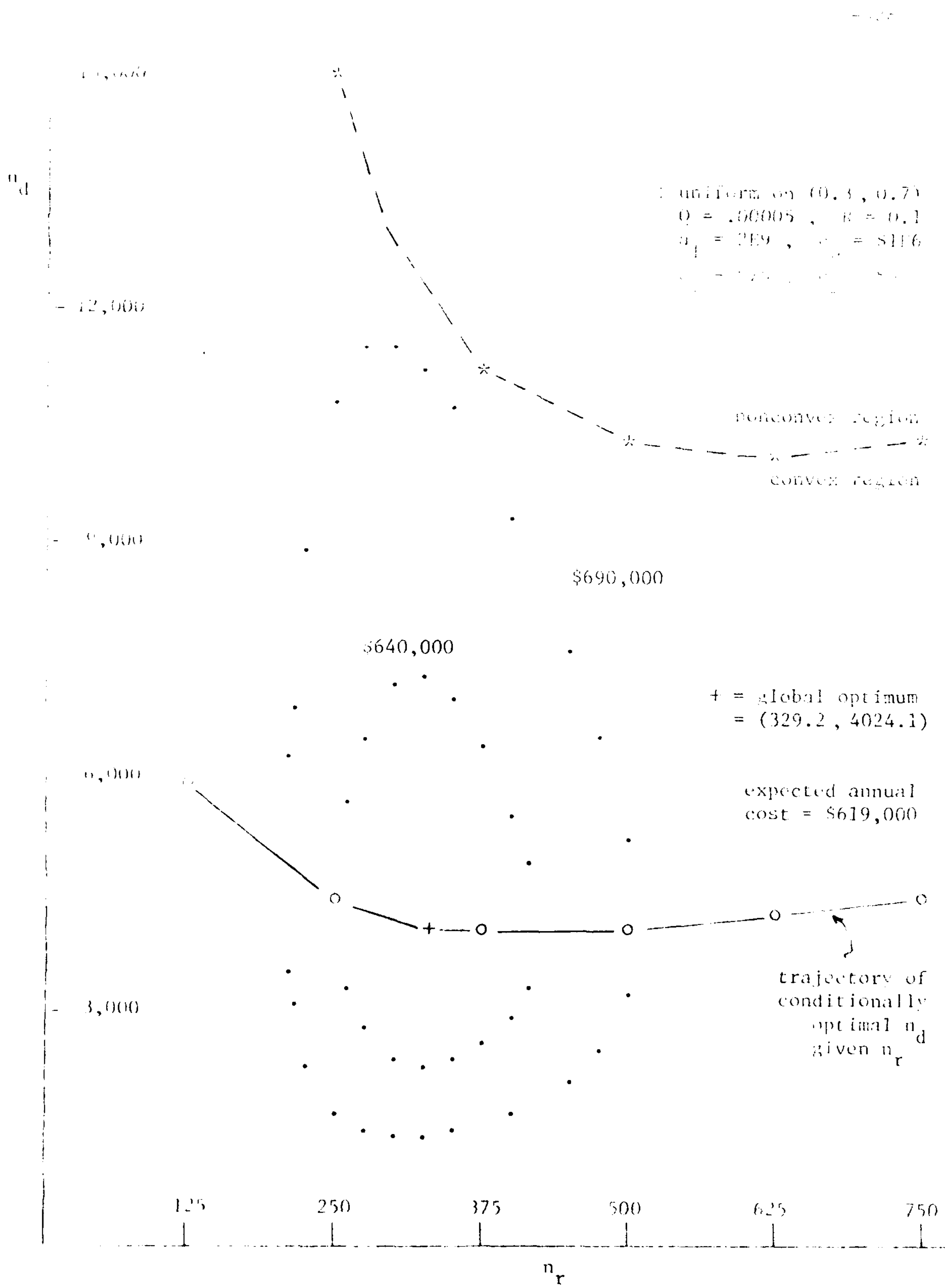

Figure 4.--Designated sample size n_d versus reorder point n_r: isocost contours for example of Model C.

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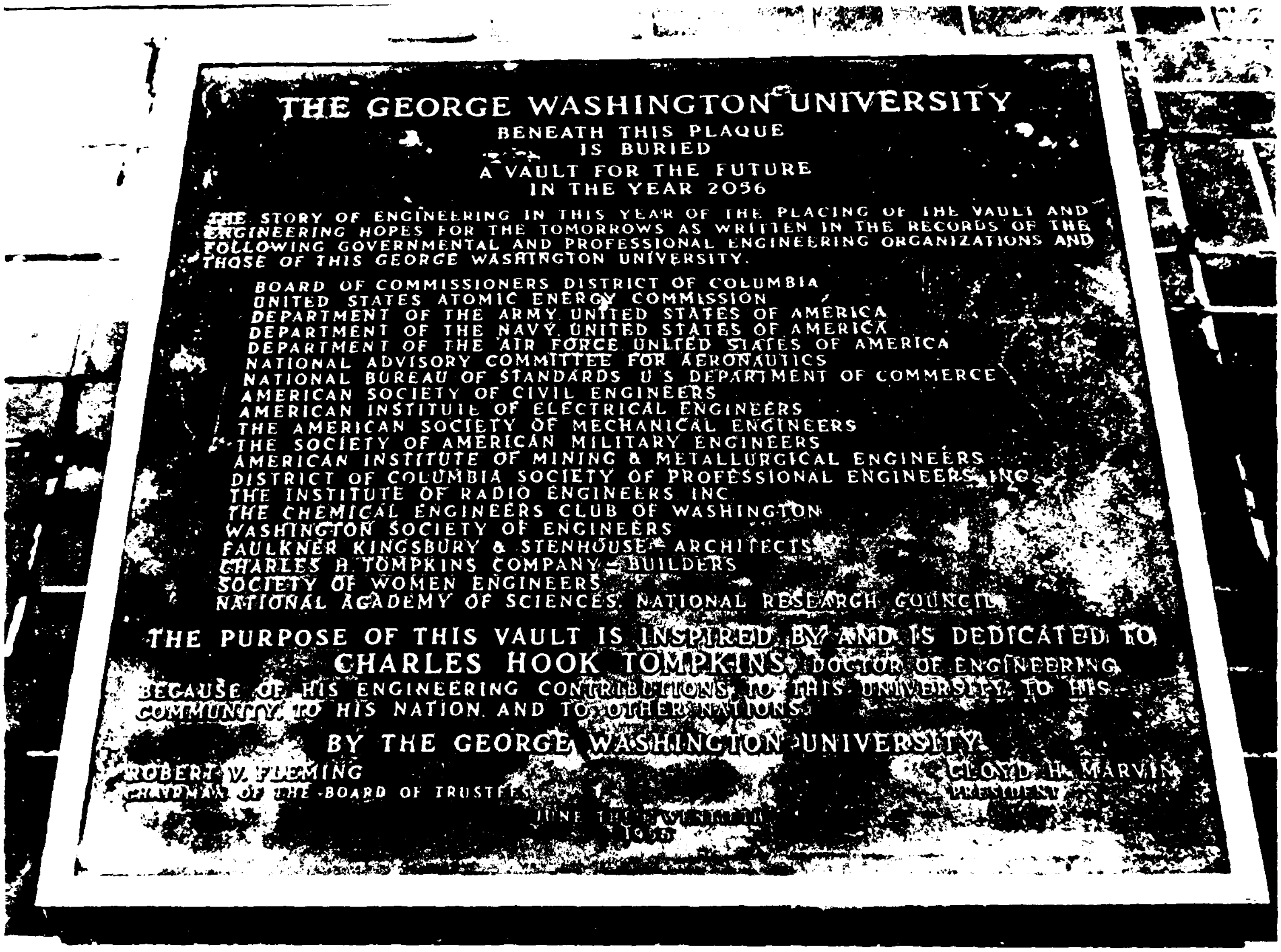

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