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Integration of the Trajectory and Transition Matrices for Powered Flight.

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20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

TRAJ or the Trajectory Reconstruction and Analysis method is prototype software under development of the Western Space and Missile Center that will ensure improvements in the range's capability for best Estimate of Trajectory Prediction. This report compares the modified Euler and Runge-Kutta methods of integrating the missile's powered flight trajectory and demonstrates that the Runge-Kutta is best. The report also compares the Runge-Kutta and Adams-Moulton integrators in the TRAJ transition matrix computation and again determines that the Runge-Kutta produces less error. Spline polynominals were also investigated as an

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interpolation technique in order to reduce integration time.

SUMMARY

The purpose of the Trajectory Reconstruction and Analysis Method (TRAM) project is to develop software which provides the Western Space and Missile Center (WSMC) with an improved Best Estimate of Trajectory (BET) analysis capability. Data from on-board and land based sensors have noise content of varying degrees. In order to derive information from this noisy data optimally, the TRAM program must be computationally quite complex and require significant computer time when run on IBM 360/65 or an equivalent machine. Because of this, algorithms which economize the computation have been developed.

The vehicle trajectory and transition matrix computation consume an appreciable portion of the filtering and smoothing process of TRAM and, consequently, efficient methods to compute these quantities were needed and developed. However, prior to implementing these efficient algorithms in the software code a 'standard' set of algorithms that do not degrade the accuracy of the data and preserve the accuracy of the estimation process must be constructed. The proposed set of efficient algorithms are then compared to the standards to insure that the efficient methods are sufficiently accurate.

This study undertakes to establish standards in two areas. The first is the computation of the powered flight trajectory used to compute the nominal trajectory, the nominal radar off-set data and the transition matrices. The second area in which standards are established is in the computation of the powered flight transition matrices.

Two integrators were used to integrate the powered flight trajectory, and subsequently, values within the interval of integration were obtained by interpolation. These methods took into account a trajectory having an appreciable acceleration component due to the thrust of powered flight and a trajectory reconstructed from quantized acceleration data that limits its precision.

The powered flight transition matrices are subdivided into several submatrices in [2]. The present study deals only with the matrices designated by ϕ_{aa} in Reference [2].

The characteristic matrices, which are used in the computation of the transition matrices, tested both the Runge-Kutta and Adams-Moulton integrators for integration accuracy. The Runge-Kutta method proved markedly better and met the accuracy criterion. The optimal step size in this study turned out to be as large as the one used in the free fall portion of TRAM. This result is demonstrated mathematically in this technical report.

By integrating the equations at very small step sizes, one may obtain an estimate of the accuracy of the foregoing algorithms. The results of the integrations at larger step sizes and the results from interpolation were compared to those resulting from integration at small step sizes. This comparison thus determines an estimate of accuracy for both integration and interpolation.

Reference [2], Brooks, R. A., Trajectory Reconstruction and Analysis Methodology, Vandenberg Air Force Base, Performance Analysis Department, Federal Electric Corporation, WTR Division, 1978.

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1.0 INTRODUCTION

This report is an extension of an earlier work, Reference [1], which was concerned with the efficient and accurate calculation of the state and transition matrices for the TRAM project, Reference [2]. The earlier work was concerned with the vacuum freefall portion of the trajectory; whereas, this work is concerned with the powered flight portion.

Because of the demands placed on the computer in the environment in which a program like TRAM must be executed, there is always a great need to make the execution as efficient as possible. In addition, the numerical algorithms must be sufficiently precise so that the accuracy of the instrumentation and the inherent capability of the estimation method are not degraded. This study was undertaken to guarantee that result.

Two important differences between powered flight and freefall are that the dynamics of the vehicle are greater in powered flight because of the thrust forces and that information about its acceleration due to thrust is supplied by PIGA counts. However, the PIGA counts are quantized so that their accuracy is limited and they are available only at discrete times. This is in contrast to freefall where the information about the trajectory supplied by the differential equations of motion is always continuous.

Because the dynamics are greater and the differential equations are discontinuous, it is clear that it will be more difficult and costly to obtain the required accuracy in powered flight. Therefore, the numerical methods developed for powered flight must be able to handle the greater dynamics and to make the optimum use of the data available.

The transition matrix composition differs in powered flight from free fall, Reference [2]. It is necessary to include terms related to the inertial

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Reference [1], Thompson, G. T., Computation of State Vector and Transition Matrix for TRAM, Vandenberg Air Force Base, Federal Electric Corporation, WTR Division, 1977.

Reference [2], Brooks, R. A., Trajectory Reconstruction and Analysis Methodology, Vandenberg Air Force Base, Performance Analysis Department, Federal Electric Corporation, WTR Division, 1978.

measuring unit in the state vector. These terms are static; nevertheless, they do affect position and velocity and the transition matrix must reflect this. This report is concerned only with the part of the transition matrix which is associated with the dynamic states of position and velocity. The part associated with the IMU static states will be treated elsewhere.

2.0 PROBLEM STATEMENT

2.1 The Trajectory Equations

In both the freefall and powered flight segments of the trajectory the estimation is mechanized around a nominal trajectory which can be obtained by integrating the equations of motion. The nominal trajectory is used to compute a nominal variation offset, the transition matrices, and it is used to compute the total state vector.

The equations of motion are expressed formally as:

$$\dot{P} = V$$

$$\dot{V} = A_G(P) + A_R(P, V) + A_T(t) \qquad (2.1)$$

where P and V are three component vectors of position and velocity respectively; A_G is the acceleration due to gravity; A_R is the apparent acceleration due to the rotation of the earth, and A_T is the acceleration due to thrust. Gravity, as is indicated, is a function of position. The acceleration due to the rotation of the earth is a function of both position and velocity. Thrust acceleration is a function of time alone. During freefall A_T is identically zero. During powered flight it is determined from the PIGA data telemetered every 0.03 seconds. The accuracy of the solution of equation set (2.1) is investigated in this report.

2.2 The Transition Matrices

Let x be a six component vector whose first three components are position and whose last three components are velocity. Then equations (2.1) can be written

$$\dot{x} = f(x, t) \quad .$$

Further, let

$$F(t) \underset{=}{\Delta} \frac{\partial f}{\partial x^T}(x, t)$$

Then the transition matrix satisfies the differential equation

$$\dot{\Phi}(t, t_o) = F(t)\Phi(t, t_o) \tag{2.2}$$

where $\Phi(t_o, t_o) = I$

and I is the identity matrix.

The transition matrix will have an order which is equal in general to the size of the state vector. However, in this study it will only have an order of six, because this study is only concerned with the part of the transition matrix associated with the dynamic states of position and velocity. The accuracy with which equation set (2.2) can be solved is investigated in this report.

2.3 Integration Accuracy

The solution of the differential equation

$$\dot{x} = f(x, t)$$

$$x(0) = x_o$$

can be expressed

$$x_c(t) = x_T(t) + \varepsilon_R(t) + \varepsilon_T(t)$$

where

$x_c(t)$ is the computed value of x,

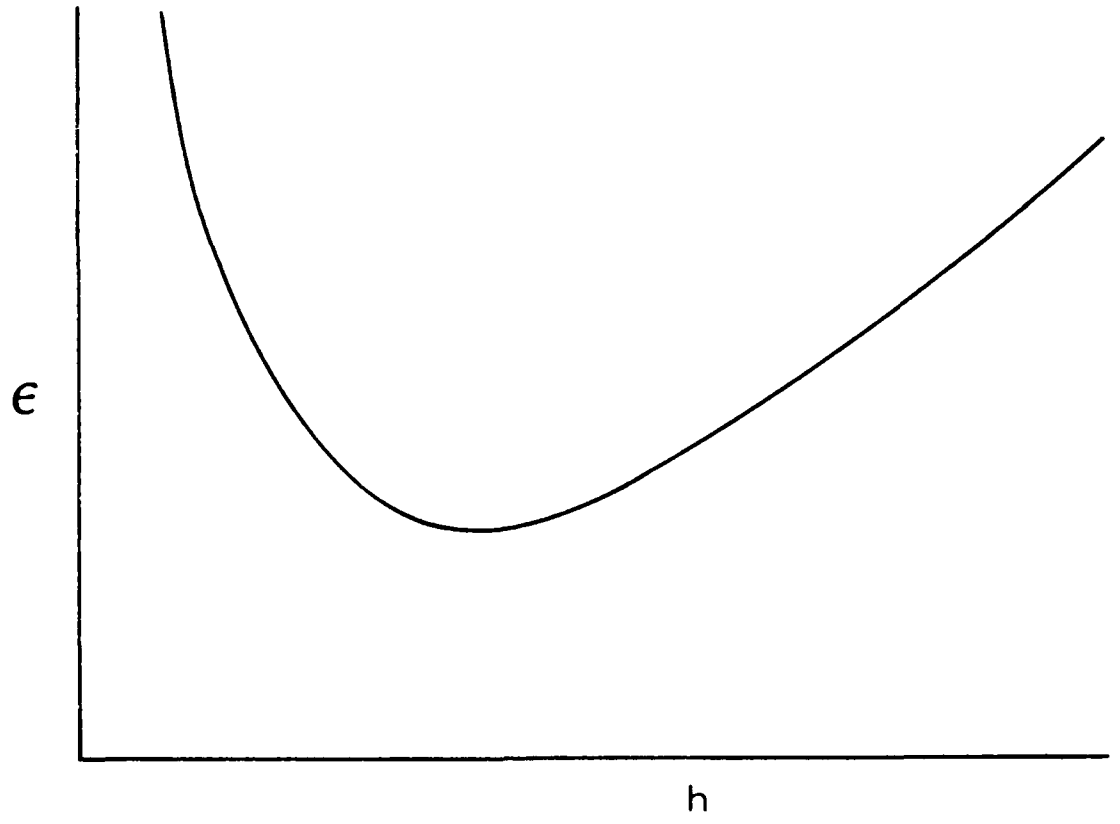
$x_T(t)$ is the true value,

$\varepsilon_R(t)$ is the round-off error,

and $\varepsilon_T(t)$ is the truncation error.

$\varepsilon_T(t)$ results from the computer's use of an approximation to the derivative. ε_R results because of the finite word length of the computer.

Let $\varepsilon = \varepsilon_R + \varepsilon_T$, then the following graph is representative of the total error ε at a fixed time as a function of the step size h.



In Figure 1 h is the integration step size. This figure indicates that, in general, there exists an optimum h at which the total error has a minimum. If the step size is increased from the optimum, then the total error increases because ε_T increases, even though ε_R decreases. If the step size is decreased from the optimum, then the round-off error causes the total error to increase even though the truncation error decreases. The determination of an optimum h is the purpose of this study. Clearly, it is desirable to use as large a value of h as possible for which the total error is within allowable bounds, because larger the values of h result in smaller execution times.

2.4 Polynominal Interpolation

Both the trajectory and transition matrices will be needed at arbitrary points. Normally, and specifically for the cases under consideration here, the integration of the differential equations is much more expensive than evaluating interpolating polynominals. It is therefore more efficient to integrate the respective differential equations over as large an interval as possible, and then interpolate for values within these intervals.

Since, after integrating the differential equations, the function and its derivative are readily available at both ends of the interval, spline polynomials are obvious choices as interpolators because the spline function and its derivatives must be continuous at the interval mesh points. However, whatever choice is made, the interpolator must contribute errors which are significantly less then the error of the integrator. The determination of the error of interpolation is one purpose of this study.

3.0 ANALYSIS

3.1 Analysis Of The Equations Of Motion

3.1.1 Formulation Of The Equations Of Motion

There were two approaches made in formulating the equations of motion. The first is designated as Set I. It is that set given in Section 2.0

$$\dot{P} = V$$

$$\dot{V} = A_G(P, V) + A_R(P, V) + A_F(t) \quad . \tag{3.0}$$

The $A_G(P, V)$ term is a simple gravity model including a J and a D term. $A_R(P,V)$ is the acceleration present because the equations are expressed in a rotating coordinate system. It includes the centrifugal and centripital terms.

$A_F(t)$ is the acceleration due to thrust. It is computed from the PIGA counts, counts from a pendulous integrating gyroscopic accelerometer. Since the PIGA counts give the change of velocity over the minor cycle, all that needs to be done to determine the average acceleration over the minor cycle is to divide the change in velocity by the length of the minor cycle, 0.03 seconds. The PIGA counts are converted from counts to feet per second by multiplying by the quantization factor of 0.12, and the results are transformed from the instrumentation coordinate system to the earth centered rotating coordinate system in which the equations of motion are integrated.

The A_F term is discontinuous, and therefore adds to the difficulty of integrating (3.0) with as much accuracy as desired, so an alternate approach was formulated. It is designated as Set II.

First the position and velocity resulting from thrust is calculated.

$$V_F(t) = \int_0^t A_F(\sigma)d\sigma \tag{3.1}$$

$$P_F(t) = \int_0^t V_F(\tau)d\tau \quad .$$

Then primed variables are introduced

$$V' \triangleq V - V_F \tag{3.2}$$

$$P' \triangleq P - P_F \quad .$$

With these variables the differential equations of motion are written as:

$$\dot{P}' = V'$$

$$\dot{V}' = A_G(P' + P_F) + A_R(P' + P_F, V' + V_F) \quad . \tag{3.3}$$

Finally, the total position and velocity are computed by

$$V = V' + V_F \tag{3.4}$$

$$P = P' + P_V \quad .$$

From (3.3) it is seen that the discontinuous term is not present in the differential equations. This formulation will aid in the determination of the solution accurately.

The terms V_F and P_F are calculated directly from the PIGA data, and so the method for doing this needs to be set forth explicitly.

Since the PIGA counts represent changes in velocity, it is natural to represent the counts for a minor cycle as ΔV_i. Therefore,

$$V_F(t_n) = \sum_{i=1}^{n} \Delta V_i$$

because $V_F(0) = 0$.

The position due to thrust is approximated by integrating velocity by the trapezoidal rule.

$$P_F(t_n) = P_F(t_{n-1}) + \tfrac{1}{2}(V_F(t_n) + V_F(t_{n-1}))(.03)$$

$$P_F(0) = 0 \quad .$$

The integration step size is the minor cycle time of 0.03 seconds.

$A_F(t_n)$ is computed by

$$A_F(t_n) = \Delta V_n/0.03 \quad .$$

By the use of the above formulas $A_F(t_n)$, $V_F(t_n)$ and $P_F(t_n)$ n = 0, 1, 2 .... can be computed. The values at any time can be found by interpolating, as follows:

$$A_F(t) = A_F(t_i) \quad , \quad t_i \leq t < t_{i+1}$$

$$V_F(t) = V_F(t_i) + (t - t_i)A_F(t_i) \quad t_i \leq t < t_{i+1}$$

$$P_F(t) = P_F(t_i) + (t - t_i)V_F(t_i) + \tfrac{1}{2}(t - t_i)^2 A_F(t_i) \quad t_i \leq t \leq t_{i+1}$$

The results of the interpolation are multiplied by the scale factor to change them from counts to feet per seconds, and then they are transformed from the instrumentation coordinate system to the earth centered rotating system.

3.1.2 Integration Of The Equations Of Motion

The methods chosen to integrate the equations of motion are well known, Reference [3]. Two methods were selected, the modified Euler and the classical fourth order Runge-Kutta method. They were chosen because of their stability and because they are one-step methods.

Reference [3], Bellman, Richard, Introduction to Matrix Analysis, New York: McGraw Hill Book Company, 1960.

One-step methods have the advantage that only information within the interval of integration is used and the function and its derivatives, up to the order of the method, need to be continuous within the interval. Multi-step methods require information from several steps and the function and its derivatives, up to the order of the method, need be continuous over all the intervals used. Unfortunately, the later condition is not met in powered flight because the accelerations are discontinuous from one minor cycle to another.

The equations of motion were formulated in two different ways and the two methods of integration were used on both formulations.

The modified Euler method is as follows:

Given the differential equations

$$\dot{y} = f(y, t) \tag{3.5}$$

$$y(t_o) = y_o$$

where y is the unknown variable, in general a vector, and specified at some initial time; then by the modified Euler method

$$p_{n+1} = y_n + hf(y_n, t_n) \tag{3.6}$$

$$y_{n+1} = y_n + \frac{h}{2}(f(p_{n+1}, t_{n+1}) + f(y_n, t_n))$$

where h is the step size of the integration. p_{n+1} is considered to be the predicted value of the function y by t_{n+1}. y_{n+1} is the corrected value. From (3.6) it is clear that the values of y at one step are completely determined by values at the preceeding step. This method is said to be a second order method because if y were a polynominal in t of no higher degree than two, then the modified Euler method would integrate the differential equations exactly if the arithmetic were done in infinite precision.

For any function y which has a third derivative, the truncation error incurred by using this method is given by

$$\varepsilon_T = \frac{-h^3}{12} y^{(3)}(\zeta) \tag{3.7}$$

where ζ lies somewhere within the interval of integration.

The second method used was the classical fourth order Runge-Kutta method. It is as follows:

Given the equations of (3.5), the solution is determined by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + k_2 + k_3 + k_4] \tag{3.8}$$

where

$$k_1 = hf(y_n, t_n)$$

$$k_2 = hf(y_n + \tfrac{1}{2}k_1, t_n + \tfrac{1}{2}h)$$

$$k_3 = hf(y_n + \tfrac{1}{2}k_2, t_n + \tfrac{1}{2}h)$$

$$k_4 = hf(y_n + k_3, t_n + h) \quad .$$

This method is fourth order because if y were a polynomial of degree no higher than four, then (3.8) would give the answer exactly at any time if the arithmetic were carried out in infinite precision.

Its truncation error is difficult to determine, which is one of the weaknesses of the method; however, for any function which has a fifth derivative, there exists a positive constant c, such that

$$\varepsilon_T \leq ch^5 \tag{3.9}$$

where h is the step size.

With the truncation error, or the upper bound for it, expressed as it is, it is possible to formulate an approach which will determine an upper bound for the total error incurred in integrating the trajectory. Figure 1 shows that

if the step size is taken large enough, the truncation error is much larger than the round-off error. In this case $\varepsilon \sim \varepsilon_T$. The approach was to run the program at such a step size to a fixed time and then cut the size in half and rerun the program to the same time. By subtracting the two results a very good estimate was obtained of the error incurred by using the larger step size. The equations are as follows:

$$x_c(t, h) = x_T(t) + \varepsilon_T(t, h) + \varepsilon_R(t, h) \quad .$$

The computed value at time t for a given step size h is equal to the true value $x_T(t)$ plus the truncation error and round-off error for the step size. However, we have assumed that, at the step size chosen, the round-off error is much smaller than the truncation error. $\varepsilon_R(t, h) << \varepsilon_T(t, h)$. Therefore,

$$x_c(t, h) \sim x_T(t) + \varepsilon_T(t, h) \quad . \tag{3.9}$$

Now, in both the modified Euler and in the Runge-Kutta methods, the truncation error is expressed as

$$\varepsilon_T(t, h)_i = c_i h^k$$

where k = 3 for the modified Euler method, and 5 for Runge-Kutta method. The subscript i indicates that the error term applies only to making a single step. However, when the step size is halved, then

$$\varepsilon_T(t, h/2)_i = \frac{c_i}{2^k} h^k$$

assuming the kth derivatives are nearly constant in the interval. Since two steps need to be taken to integrate to the same place as in the large step, the truncation error for the smaller step at time t is approximated by

$$\varepsilon_T(t, h/2)_i \sim 2 \frac{c_i}{2^k} \cdot h^h = \frac{c_i}{2^{k-1}} h^k$$

$$= \frac{1}{2^{k-1}} \varepsilon_T(t, h)_i \quad . \tag{3.10}$$

Now the computed value at half the step size is equal to

$$x_c(t, h/2) \cong x_T(t) + \varepsilon_T(t, h/2) \quad . \tag{3.11}$$

By subtracting (3.11) from (3.9) it follows

$$x_c(t, h) - x_c(t, h/2) \cong \varepsilon_T(t, h) - \varepsilon_T(t, h/2) \quad .$$

Since from (3.10) it was shown that the truncation error at each step for the half step integration is a fraction of the error at the full step, the total truncation error up to time t is mainly that from the larger step size. Therefore

$$x_c(t, h) - x_c(t, h/2) \cong \varepsilon_T(t, h) \quad . \tag{3.12}$$

The above approach is only valid when the truncation error needs to be determined to an order of magnitude. It is not possible to make a precise determination in this manner. It is also clear that the assumptions are better met by the higher order Runge-Kutta method.

For the above approach to give meaningful results, the size of the step and the half-step must be large enough so that the truncation error is the dominant term. This fact is verified if, for a smaller value of h and half its value, closer agreement is obtain in the computed values. If the round-off is becoming the larger term, for smaller values of h the differences should be about the same or become larger.

It is clear from comparing the number of times the right hand side of the differential equation has to be evaluated in (3.6) and (3.8) that the Runge-Kutta method will take longer than the modified Euler method. The amount of work is doubled for the Runge-Kutta; however, because the Runge-Kutta permits larger step sizes, this method turns out to be the most economical in obtaining the high accuracy desired.

3.1.3 Spline Polynomials For The Trajectory

The error caused by interpolation needs to be controlled as carefully as the error from numerical integration. The manner for determining the error is rather straight forward. After the coefficients of the interpolating polynomial have been determined, it is tested by integrating over the interval of interpolation with a very short step size. The integrated values are, consequently, very accurate. The interpolated values from the polynomial are compared to them.

The algorithms for determining the coefficients of the polynomials and the programs to do this are given in Reference [1]. For the trajectory, it is possible to use both cubic and quintic splines. The cubic spline is determined by using position and velocity at each end of the interpolating interval. The quintic spline is determined by using position, velocity and acceleration at each end of the interval.

The polynomials are normalized to interpolate over in interval (0, 1). This improves the round-off error appreciably. With the position, velocity and acceleration at each end of the interval the normalized polynomial coefficients are determined in the following *manner*.

Let

$$\tilde{p}_o = p_o$$

$$\tilde{v}_o = v_o \cdot h$$

$$\tilde{a}_o = a_o \cdot h^2$$

$$\tilde{p}_1 = p_1$$

$$\tilde{v}_1 = v_1 \cdot h$$

$$\tilde{a}_1 = a_1 \cdot h^2$$

Reference [1], Thompson, G. T., Computation of State Vector and Transition Matrix for TRAM, Vandenberg Air Force Base, Federal Electric Corporation, WTR Division, 1977.

where p_o, v_o and a_o are the position, velocity and acceleration at the beginning of the interval, and p_1, v_1 and a_1 are position, velocity and acceleration at the end of the interval. Each of these quantities is a vector with three elements. h is the length of the interval, $h = t_1 - t_o$.

Then the coefficients for the cubic polynomial are determined as follows:

$$c_3 = \tilde{v}_1 + \tilde{v}_o - 2(\tilde{p}_1 - \tilde{p}_o) \tag{3.13}$$

$$c_2 = \tilde{p}_1 - \tilde{p}_o - \tilde{v}_o - c_3$$

$$c_1 = \tilde{v}_o$$

$$c_o = \tilde{p}_o \quad .$$

The polynomial is then

$$\bar{p}(t) = c_o + c_1 t + c_2 t^2 + c_3 t^3$$

and is computed for each of the three components of the vector.

The quintic polynomial coefficients are determined by the following algorithm.

$$a_1 = \tilde{p}_1 - \tilde{p}_o - \tilde{v}_o - \tilde{a}_o/2 \tag{3.14}$$

$$a_2 = \tilde{v}_1 - \tilde{v}_o - \tilde{a}_o$$

$$a_3 = (\tilde{a}_1 - \tilde{a}_o)/2$$

$$d_5 = a_3 - 3a_2 + 6a_1$$

$$d_4 = a_2 - 3a_1 - 2d_5$$

$$d_3 = a_1 - d_4 - d_5$$

$$d_2 = a_o/2$$

$$d_1 = \dot{v}_o$$

$$d_o = \dot{p}_o$$

The polynomial is

$$\bar{q}(t) = d_o + d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4 + d_5 t^5 \quad .$$

Since the polynomials are normalized, position and velocity at any time t, $t_o \leq t \leq t_1$, are obtained from the cubic by these equations

$$p(t) = \bar{p}(u(t)) \tag{3.15}$$

$$v(t) = \bar{p}'(u(t))/h$$

where

$$0 \leq u(t) = (t - t_o)/h \leq 1 \quad .$$

When the quintic spline is used, the position and velocity for any time t is given by

$$p(t) = \bar{q}(u(t)) \tag{3.16}$$

$$v(t) = \bar{q}'(u(t))/h$$

where

$$0 \leq u(t) = (t - t_o)/h \leq 1 \quad .$$

3.2.1 Mathematical Formulation Of The Transition Matrices

The mathematical treatment of the transition matrices is given in Reference [2]. The transition matrices are related to the fundamental matrices by the equation

$$\Phi(t, s) = \Psi(t)\Psi^{-1}(s) \tag{3.17}$$

and the fundamental matrices satisfy the differential equation

$$\dot{\Psi}(t) = F(t)\Psi(t) \qquad t' \leq t \leq t'' \tag{3.18}$$

$$\Psi(t') = I$$

where I is the identity matrix and F(t) is given by (2.2).

The inverse of $\Psi(t)$, $\Psi^{-1}(t)$ also satisfies a differential equation

$$\dot{\Psi}^{-1}(t) = -\Psi^{-1}(t)F(t)$$

$$\Psi^{-1}(0) = I \ .$$

$\Psi^{-1}(t)$ may be found by integrating the differential equations or by finding $\Psi(t)$ and inverting it.

In this study only those terms of the transition matrix which are associated with position and velocity are considered. This means the F(t) above is equal to $F_{aa}(t)$ and $\Psi(t)$ is equal to Ψ_{aa} of Section 6.3. The subscripts are dropped to simplify notation.

It should be pointed out that this is apparently the simplest part of the transition matrix to compute. Ψ_{ab} of Section 6.3 evidently presents some significant problems. How these problems are handled will appear in another report.

Reference [2], Brooks, R. A., Trajectory Reconstruction and Analysis Methodology, Vandenberg Air Force Base, Performance Analysis Department, Federal Electric Corporation, WTR Division, 1978.

3.2.2 Integration Of The Transtion Matrices

The problem is how to integrate (3.18) accurately and efficiently and then to explore what accuracies can be realized with the chosen methods. It is important to note that most of the integration time will be spent in the integration of fundamental matrices simply because of their size. For whatever methods are chosen, a standard with which to compare them is necessary in order to maintain the desired accuracy. The inverse of $\Psi(t)$ will be integrated the same way $\Psi(t)$ is.

The integration of the fundamental matrices for free fall was examined, Reference [1]. The elements of the matrices are the same. The only difference is the more dynamic and discontinuous powered flight trajectory. From the mathematics, it appears these differences would make only second order changes. The equations to show this follow

$$\dot{\Psi}(t) = F(t)\Psi(t)$$

$$\Psi(0) = I$$

Where I is a 6×6 identity matrix. Whatever integrator is chosen, how well it performs depends on how well the higher derivatives of Ψ with respect to t behave.

$$\ddot{\Psi}(t) = \dot{F}(t)\Psi(t) + F(t)\dot{\Psi}(t)$$

$$= \dot{F}(t)\Psi(t) + F(t)F(t)\Psi(t) \quad .$$

The $\dot{F}(t)\psi(t)$ term gives a clue as to what might be expected from the integrators.

F(t) is a mat··· ···e ijth term can be denoted as $q_{ij}(t)$; therefore

$$F(t) \quad (q_{ij}(t)) \quad i, j = 1, \ldots, 6 \quad .$$

From the definit··n of F(t) it is clear that it is an explicit function of the position and velocity, which are functions of time. As a matter of fact t does not appear explicitly in F(t); therefore, it may be written as

---

Reference [1], Thompson, G. T., Computation of State Vector and Transition Matrix for TRAM, Vandenberg Air Force Base, Federal Electric Corporation, WTR Division, 1977.

$$F(t) = (g_{ij}(x(t))) \quad i, j = 1, \ldots, 6 \quad .$$

Then

$$\dot{F}(t) = (\dot{g}_{ij}(x(t))) \quad .$$

But

$$\dot{g}_{ij}(x(t)) = \left(\frac{\partial g_{ij}}{\partial x^T}\right) \cdot \dot{x}(t) \quad .$$

The term $\dot{x}(t)$ was discussed in Section 3.1.2, where it was noted that the powered flight trajectory had greater dynamics and that the $A_F(t)$ term in these differential equations was discontinuous. In this expression, however, each term in $\dot{x}(t)$ is multipled by coefficients that are either zero or very small. For $i = 1, 2, 3$ the coefficients are identically zero, and for $i = 4, 5, 6$ the terms are bounded by 10^{-12} in absolute value. As a result the greater dynamics and the discontinuities in $\dot{x}(t)$ will have a second order effect on the integration of the characteristic equation.

For the term $\dot{F}(t)\Psi(t)$, with the use of the Schwartz inequality

$$\| \dot{F}(t)\Psi(t) \| \leq \| \frac{\partial g_{ij}}{\partial x^T} \| \quad \| \dot{x}(t) \| \, \| \Psi(t) \|$$

where $\| \cdot \|$ indicates the norm, Reference [3], but since

$$\| \frac{\partial g_{ij}}{\partial x^T} \| \leq 10^{-12}$$

$$\| \dot{x}(t) \| < 10^{5}$$

$$\| \Psi(t) \| \sim 1$$

Reference [3], Bellman, Richard, Introduction to Matrix Analysis, pages 161 and 162 , New York: McGraw Hill Book Company, 1960.

then

$$\| \dot{F}(t)\Psi(t) \| \leq 10^{-12} \cdot 10^{5} = 10^{-7} .$$

Similarly, we can see by inspection that higher partials of $g_{ij}(t)$ will be smaller still. The higher derivatives of $\psi(t)$ are discontinuous but the discontinuities are multiplied by very small partials, and as a result, it is expected that rather large step sizes might be used in the integration of $\dot{\Psi}(t)$.

The second term in $\ddot{\Psi}(t)$ is $F(t)F(t)\Psi(t)$.

Since $\| F \| \leq 10^{-5}$, $\| \Psi(t) \| \leq 1$, and this term is continuous, it presents no difficulty in the integration of $\dot{\Psi}(t)$.

The methods of integration chosen were the fourth order Adams-Moulton and Runge-Kutta methods. The Runge-Kutta does not differ conceptionally from the way in which it was discussed in Section 3.1.2, although the differential equations are different and the application is somewhat different. The explicit algorithm is given by (3.8).

The Adams-Moulton method is discussed in detail in Reference [4]. By it the solution of (3.5) is as follows:

$$\bar{y}_{n+1} = y_n + \frac{h}{24} (55f(y_n, t_n) - 59f(y_{n-1}, t_{n-1}) + 37f(y_{n-2}, t_{n-2}) - 9f(y_{n-3}, t_{n-3}))$$

$$y_{n+1} = y_n + \frac{h}{24} (9f(\bar{y}_{n+1}, t_{n+1}) + 19f(y_n, t_n) - 5f(y_{n-1}, t_{n-1}) + f(y_{n-2}, t_{n-2}))$$

where h is the integration step size and $f(y_n, t_n)$ is the value of the derivative of y at t_n.

Reference [4], Lapidas, Leon and Seinfiled, John H., Numerical Solution of Ordinary Differential Equations, New York: Academic Press, Inc., 1971.

Several past values of the function are used, and if the method is to be effective, it must have high order derivates that are continuous in the interval from t_{n-3} to t_{n+1}. In integrating from t_n to t_{n+1} the method first computes a predicted value $\bar{y}_{n+1}$ at t_{n+1}, and then the derivative, evaluated at t_{n+1} with $\bar{y}_{n+1}$, is used to compute the corrected value y_{n+1} at t_{n+1}.

Since to step from t_n to t_{n+1} only two evaluations of the differential equations are required, this method is nearly twice as fast as the Runge-Kutta method which requires four.

This method is known to be very stable and the truncation error for the predictor and corrector is given by

$$T_p = \frac{251}{720} h^5 y[5]$$

$$T_c = \frac{-19}{720} h^5 y[5] \quad .$$

The corrector, as is always the case, is more accurate. Both error terms depend upon the fifth power of the step size and the fifth derivative of the function being integrated.

The first three values of the variable after the initial conditions have to be obtained by other means before this method can be used for the first time. These initial three values were obtained by the Runge-Kutta method.

3.2.3 Interpolators For The Transition Matrices

Two interpolators were used. One was the cubic spline described in Section 3.1.3. The other was a interpolator which uses the function and the derivative at three points. This is not a spline interpolator in the correct sense but it is similar to one. The quintic spline described in Section 3.1.3 was not used because the second derivative of the fundamental matrices are needed for it and these are difficult and very costly to compute.

The cubic spline has already been discussed, but the quintic interpolator needs to be expressed explicitly. What is given are three values of a function and its derivative at equally spaced points. So then these values are given

$$\Psi(t_1),\ \Psi'(t_1),\ \Psi(t_2),\ \Psi'(t_2),\ \Psi(t_3),\ \Psi'(t_3),\ h, \quad h = t_3 - t_1 .$$

Compute first

$$r_1 = -\frac{h}{2}\Psi'(t_1) - \Psi(t_1) + \Psi(t_2)$$

$$r_2 = -h\Psi'(t_1) + h\Psi(t_2)$$

$$r_3 = -h\Psi'(t_1) - \Psi(t_2) + \Psi(t_3)$$

$$r_4 = -h\Psi'(t_1) + h\Psi(t_3) .$$

Then compute

$$a_0 = \Psi(t_1)$$

$$a_1 = h\Psi'(t_1)$$

$$a_2 = 16r_1 - 8r_2 + 7r_3 - r_4$$

$$a_3 = 32r_1 - 6r_3 + r_4 - 4a_2$$

$$a_4 = 32r_1 - r_3 - 3a_3 - 7a_2$$

$$a_5 = r_3 - a_4 - a_3 - a_2 .$$

Then the polynomial

$$p(u) = a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5, \quad 0 \le u \le 1$$

$$u = (t - t_1)/h$$

is a polynomial, normalized over the interval t_1 to t_3 which matches the given functional values and derivatives at each end of the interval and at the mid-point. That is,

$$p(0) = \Psi(t_1)$$

$$p'(0)/h = \Psi'(t_1)$$

$$p(\tfrac{1}{2}) = \Psi(t_2)$$

$$p'(\tfrac{1}{2})/h = \Psi'(t_2)$$

$$p(1) = \Psi(t_3)$$

$$p'(1)/h = \Psi'(t_3) \quad .$$

To find the interpolated value of any element of the fundamental matrix or its derivative at any value t, $t_1 \leq t \leq t_3$, u is computed first

$$u = (t - t_1)/h$$

and then p(u) and p'(u)/h.

$$\Psi(t) = p(u)$$

$$\Psi'(t) = p'(u)/h \quad .$$

4.0 NUMERICAL RESULTS

4.1 Computation Of The Trajectory

4.1.1 Integration Of The Trajectory

With the equations formulated as in Set I and as in Set II discussed in Section 3.1.1, both sets were integrated using the modified Euler and the Runge-Kutta methods described in Section 3.1.2 to 200 seconds.

The only parameter varied as both sets of equations were integrated by the two methods was the integration step size. The step size was changed in such a way that either a minimum total error was reached (as described in Section 2.0) or the error in position became less than 10^{-4} feet and simultaneously the error in velocity became less than 10^{-6} feet/second. In the numerical experimentation, the errors fell below the bounds before the minimum error was reached.

Table 1 gives the results from the various integrations. The equations are identified as Set I or Set II; the step size and the method of integration are indicated. The error incurred is given. The error was determined in the manner described in Section 3.1.2. Also given is the time taken to do the integration. h is the step size. ε_p is the error in position, and ε_v is the error in velocity.

TABLE 1

TRAJECTORY INTEGRATION

	MODIFIED EULER		RUNGE-KUTTA	
h	SET 1	SET 2	SET 1	SET 2
.015	T = 411 $\varepsilon_p = 1.5\times10^{-4}$ $\varepsilon_v = 10^{-6}$	T = 419 $\varepsilon_p = 8\times10^{-5}$ $\varepsilon_v = 9\times10^{-7}$		
.03	T = 218 $\varepsilon_p = 6\times10^{-4}$ $\varepsilon_v = 4\times10^{-6}$	T = 231 $\varepsilon_p = 5\times10^{-4}$ $\varepsilon_v = 9\times10^{-6}$	T = 346 $\varepsilon_p = 6\times10^{-6}$ $\varepsilon_v = 1\times10^{-5}$	T = 351 $\varepsilon_p = 7\times10^{-6}$ $\varepsilon_v = 3\times10^{-8}$
.06	T = 97 $\varepsilon_p = 10^{-1}$ $\varepsilon_v = 5\times10^{-5}$	T = 119 $\varepsilon_p = 10^{-3}$ $\varepsilon_v = 2\times10^{-5}$	T = 168 $\varepsilon_p = 4\times10^{-2}$ $\varepsilon_v = 1.5\times10^{-5}$	T = 170 $\varepsilon_p = 9\times10^{-4}$ $\varepsilon_v = 5\times10^{-6}$
.09				T = 94 $\varepsilon_p = 3\times10^{-3}$ $\varepsilon_v = 2\times10^{-5}$

* In Table 1 T has units of seconds, ε_p has units of feet, ε_v has units of feet/second, and h has units of second.

The times given must be divided by three to give the time of the integration at the specified step size. This is because the step size was divided by 2 to integrate a value that was used as a standard for comparison. The integration time at half the step size plus the integration time at the full step size is about three times the value at the specified step size.

The cases where no data is presented in the table were not run because either the error had become too large or too small.

4.1.2 Trajectory Interpolation

The algorithms for the cubic and quintic splines used to interpolate the trajectory are given in Section 3.1.1. The equations to compute the coefficients of the polynomials are given by (3.13) and (3.14).

Both the cubic and quintic splines were used, but the results were so similar that it was not meaningful to treat them separately. There was, however, a difference in the results when the equations of motion were formulated differently, and, of course, the results changed markedly when the step size was changed.

Table 2 gives the results of the numerical experimentation on spline interpolation. It gives the error in interpolation obtained by subtracting the very accurately integrated values from spline interpolation. The parameters are the same as in Table 1.

TABLE 2

SPLINE INTERPOLATION

h	SET 1	SET 2
.03	T = 946 $\varepsilon_p = 10^{-8}$ $\varepsilon_v = 10^{-7}$	T = 1023 $\varepsilon_p = 10^{-8}$ $\varepsilon_v = 10^{-7}$
.06	T = 506 $\varepsilon_p = 10^{-2}$ $\varepsilon_v = 10^{-7}$	T = 519 $\varepsilon_p = 10^{-8}$ $\varepsilon_v = 10^{-7}$
.12		T = 249 $\varepsilon_p = 10^{-7}$ $\varepsilon_v = 10^{-6}$
.24		T = 136 $\varepsilon_p = 3\times10^{-7}$ $\varepsilon_v = 2.5\times10^{-6}$
.50		T = 73 $\varepsilon_p = 3\times10^{-6}$ $\varepsilon_v = 10^{-5}$
1.0		T = 39 $\varepsilon_p = 10^{-4}$ $\varepsilon_v = 10^{-5}$

4.2 Computation Of The Characteristic Matrices

4.2.1 Integration Of The Characteristic Matrices

The characteristic matrices and their inverses were integrated in two ways. One was by the Runge-Kutta method and the other was by the Adams-Moulton method which were described in Sections 3.1.2 and 3.2.2. Only the characteristic matrices associated with the dynamic terms (Ψ_{aa}) were integrated, Reference [2]. The inverses were calculated both by integration and by inversion.

The Adams-Moulton integration was allowed to run over the entire span of integration (100 seconds) without being reinitialized, whereas, the Runge-Kutta integration was reinitialized to the identity at every step.

The standard with which the integration of these matrices and their inverses by both methods were compared to was the result of Runge-Kutta integrations with much smaller step sizes.

Since the point of view is different in the two different methods, the results of the integrations are presented in two different tables.

4.2.2 Interpolation Of The Characteristic Matrices

The interpolation of the characteristic matrices was accomplished by cubic and quintic polynomials described in Section 3.2.2. When the Adams-Moulton method was used as an integrator, both the cubic spline polynomial and quintic polynomial was used. The standard used for comparison was the result of Runge-Kutta integration with small step sizes. The polynomials were evaluated at the step points of the Runge-Kutta integration.

When the Runge-Kutta method was used as an integrator, only the cubic spline polynomial was used as an interpolator. The standard for comparison was the output of the Runge-Kutta method small with step sizes.

Reference [2], Brooks, R. A., Trajectory Reconstruction and Analysis Methodology, Vandenberg Air Force Base, Performance Analysis Department, Federal Electric Corporation, WTR Division, 1978.

4.2.3 Error In The Characteristic Matrix Computation

Tables 3 and 4 give the results of the computation of characteristic matrices for the Adams-Moulton and Runge-Kutta methods respectively. The error of interpolation is given for each of the methods. Both the cubic and quintic polynomials were use as interpolators. Since the results were nearly the same, only the values from the cubic were considered.

The standard matrices were subtracted from the matrices determined by integration and by interpolation to obtain error matrices. The norms of the error matrice were taken to be the maximum absolute value of any element of the matrix. The norms of the error matrices are given in the tables.

In the same manner, Reference [1], the error matrices are broken up into four parts. The upper left nine elements are those errors which propagate errors in position into position. The upper right nine elements submatrix is the matrix which propagates errors in velocity into position. The lower left nine element submatrix propagates errors in position into velocity, and the lower right nine element submatrix propagates errors in velocity into velocity. The thirty-six element characteristic error matrix is represented by four numbers, a norm for each of the submatrices described above.

Reference [1], Thompson, G. T., Computation of State Vector and Transition Matrix for TRAM, Vandenberg Air Force Base, Federal Electric Corporation, WTR Division, 1977

TABLE 3

ADAMS-MOULTON INTEGRATION

ERROR MATRICES								
STEP SIZE	.99		1.98		3.96		5.04	
INTEGRATION	10^{-9}	10^{-8}	10^{-8}	10^{-7}	10^{-8}	10^{-6}	10^{-7}	10^{-6}
	10^{-11}	10^{-9}	10^{-10}	10^{-8}	10^{-10}	10^{-8}	10^{-9}	10^{-7}
INTERPOLATION	10^{-9}	10^{-8}	10^{-9}	10^{-7}	10^{-8}	10^{-6}	10^{-7}	10^{-6}
	10^{-11}	10^{-9}	10^{-10}	10^{-8}	10^{-10}	10^{-8}	10^{-9}	10^{-7}

TABLE 4

RUNGE-KUTTA INTEGRATION

ERROR MATRICES								
STEP SIZE	.99		1.98		3.96		5.04	
INTEGRATION	10^{-14}	10^{-13}	10^{-14}	10^{-11}	10^{-12}	10^{-9}	10^{-12}	[illegible]
	10^{-18}	10^{-15}	10^{-16}	10^{-14}	10^{-15}	10^{-12}	10^{-14}	[illegible]
INTERPOLATION	10^{-13}	10^{-11}	10^{-12}	10^{-9}	10^{-10}	10^{-8}	[illegible]	10^{-8}
	10^{-15}	10^{-13}	10^{-14}	10^{-11}	10^{-13}	10^{-10}	[illegible]	10^{-10}

5.0 CONCLUSIONS

5.1 Conclusions About the Trajectory Computation

The trajectory was integrated over two hundred seconds. From Table 1 it is seen that the only times that an error in position of less than 10^{-4} feet and simultaneously an error in velocity of less than 10^{-6} feet/second was reached were when Equation Set 1 and Set 2 were integrated at a step size of 0.03 seconds with the Runge-Kutta method and when both of these equation sets were integrated at a step size of 0.015 seconds with modified Euler method. These were the only satisfactory cases of all the cases run.

The cases that were satisfactory in the spline interpolation were when Set 1 was integrated and interpolated over an interval of 0.03 seconds and when Set 2 was integrated and then interpolated over intervals of 0.03, 0.06 and 0.12 seconds. When Set 2 was interpolated over an interval of 0.24 seconds, the position accuracy fell well within the requirement but the velocity error was slightly larger; therefore, this interval might be considered satisfactory.

5.2 Conclusions About the Characteristic Matrix Computation

Errors in the computation of the transition vector relate errors in the state vector from one step to the next. If we bound the errors in position and velocity, then the total error can be bounded by multiplying the error bound for one step by the total number of steps taken.

$$E_T = E_i N_{step} .$$

The total error is less than or equal to the bound for any step times the total number of steps. The bound for step i is given by

$$E_i = \left[E_\Phi \begin{bmatrix} p_\varepsilon \\ v_\varepsilon \end{bmatrix} \right]_i$$

where ξ_ϕ is the bound for the computation error in the transition matrix for any step and p_ε and v_ε are the bounds for the errors in position and velocity for any step.

With the same approach to bounding the error for the characersitic matrix computation, Reference [1], the following may be stated. The errors in position at each step must be less than 100 feet and in velocity less than one foot per second and since there are less than 1500 steps, errors in the characteristic matrix computation for the two methods of integration can be bounded.

For the Adams-Moulton method the results are presented in Table 5.

TABLE 5

ADAMS-MOULTON INTEGRATION ERROR FOR TRANSITION MATRICES

STEP SIZE	.99	1.98	3.96	5.04	Secs.
ε_p	.003	.02	.03	.2	Ft.
ε_v	.00003	.0003	.0003	.003	Ft./Sec.

ε_p and ε_v is the bound for position and velocity error due to numerical errors in computing the transition matrix.

Since the interpolation does not contribute any further appreciable error, the values in Table 5 can be considered the total error for this method.

From the above tabled results the following conclusions can be made. The Runge-Kutta method at 5.04 second step size is an adequate method for computing the transition matrix. With this method of integration the cubic spline can be used satisfactorly to interpolate at this step size.

Reference [1], Thompson, G. T., Computation of State Vector and Transition Matrix for TRAM, Vandenberg Air Force Base, Federal Electric Corporation, WTR Division, 1977.

This step size is in agreement with the analysis which anticipated that the computation of the transition matrix for powered flight could be accomplished by a procedure similar to that used in free fall.

From the tables it is apparent that the interpolation is the factor which limits the step size for the Runge-Kutta method.

As for the Adams-Moulton method, it is clear that the accuracies obtained by this method are not sufficiently good to warrant its use. It must be remembered that the integration was made over 100 seconds; whereas, in the Runge-Kutta the integration was reinitialized every 5 seconds. Nevertheless, even when the step size was taken to be 99 seconds the accuracy was marginal. This method should not be abandoned, however, because it has the potential for integrating as well as the Runge-Kutta and is twice as fast.

The disadvantage of using the Adams method is that it is necessary to use a different starting algorithm for the first three steps that either is the Runge-Kutta method or is as costly as the Runge-Kutta method. This means that the integration must run at least 20 seconds before any time savings are made at all, when a five second step size is used.

For the Runge-Kutta method of integration we have a similar table

TABLE 6

RUNGE-KUTTA INTEGRATION ERROR FOR TRANSITION MATRICES

STEP SIZE	.99	1.98	3.96	5.04	Secs.
ϵ_p	$?\times10^{-9}$	2×10^{-8}	2×10^{-5}	2×10^{-5}	Ft.
ϵ_v	1.5×10^{-12}	3×10^{-10}	2×10^{-8}	3×10^{-8}	Ft./Sec.

The interpolation error appreciably degraded the computation in this case. The total error, which is the sum of the integration and interpolation error, is given in Table 7.

TABLE 7

TOTAL COMPUTATION ERROR FOR RUNGE-KUTTA

STEP SIZE	.99	1.98	3.96	5.04	Secs.
ε_p	3×10^{-7}	2×10^{-5}	3×10^{-4}	3×10^{-4}	Ft.
ε_v	3×10^{-9}	2×10^{-7}	2×10^{-6}	3×10^{-6}	Ft./Sec.

6.0 RECOMMENDATIONS

Since the Runge-Kutta error is less and is faster than the modified Euler, the recommended algorithms for the trajectory are the Runge-Kutta method of integration on equation Set 2 with a step size of 0.03 seconds and then interpolation with a cubic spline for no more than 0.24 second intervals. Because the Runge-Kutta error is less and is faster, the recommended algorithms to compute the characteristic matrices are the Runge-Kutta method for integration with a step size of five seconds and then the cubic spline interpolation over the same interval.