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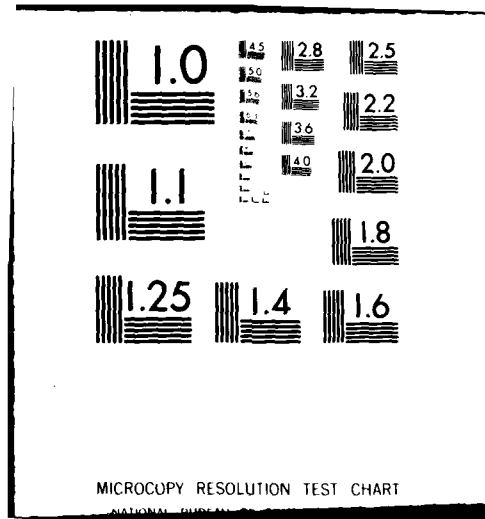
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APPROACHES FOR CONVERGENCE OF A BASIC ITERATIVE METHOD FOR THE --ETC(U)
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APPROACHES FOR CONVERGENCE OF A BASIC ITERATIVE
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Jong-Shi Pang

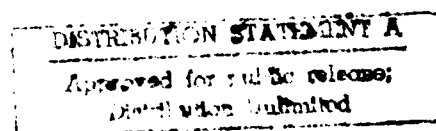


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APPROACHES FOR CONVERGENCE OF A BASIC ITERATIVE
METHOD FOR THE LINEAR COMPLEMENTARITY PROBLEM*

Jong-Shi Pang

Abstract. Iterative methods have been found very useful for solving many large linear complementarity problems arising from applications. In this paper, we formulate a basic algorithm and use it as a unifying framework for the study of such methods. Next, we apply various strategies to investigate the convergence of the basic algorithm. Finally, we discuss the possibility of extending the analysis presented here to treat other complementarity and variational problems.

Key Words. Iterative methods, linear complementarity problem, matrix splittings, convergence.

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1. INTRODUCTION

The linear complementarity problem has become a very important subject in mathematical programming. Over the years, many methods have been proposed for the numerical solution of the problem. These methods may be classified as (i) direct, (ii) iterative or (iii) hybrid. Direct methods are based on pivoting techniques and have the property of finite termination. Some of the best-known direct methods are Lemke's almost complementary pivoting algorithm [27], Cottle's principal pivoting algorithm [9] and the parametric version of Graves' principal pivoting algorithm [8, 20]. Iterative methods generally converge only in the limit. Hybrid methods are iterative in nature but use some other (in many cases, direct) methods to solve subproblems.

Both the direct and iterative methods have their own advantages and disadvantages. Typically, direct methods are most suited for small to medium sized problems. Their efficiency tends to decrease as the problem size increases. This is partly due to the fact that both the number of pivots and the amount of work required in each pivot are expected to grow rapidly with the problem size. Another disadvantage of direct methods for solving large problems is the excessive amount of computer storage. On the other hand, iterative methods usually require very little extra storage because they often operate on the given data only. As a result of this nice feature, iterative methods have become very useful for solving many large linear complementarity problems arising from applications. See [10, 12, 13, 16, 36].

Our purpose in this paper is to continue our previous work in [38] to develop a general convergence theory of iterative methods for solving the linear complementarity problem. The basic framework used throughout the analysis is built on the rather old concept of matrix splittings [41].

Specifically, given a splitting of the matrix defining the linear complementarity problem, a basic algorithm (in the sense of Zangwill [43]) is constructed which generates a sequence of iterates by solving some linear complementarity subproblems. Depending on the properties of the splitting involved, four different approaches may be used to establish the convergence (to a desired solution of the given linear complementarity problem) of the sequence generated. In what follows, we give an overview of these approaches.

The first approach, to be called the symmetry approach is based on the assumption that a certain quadratic function can be used to monitor the progress of the algorithm. Typically, this approach is used in the case when the matrix defining the original linear complementarity problem possesses some symmetry property (although the matrix itself need not be symmetric). For instance, if the given complementarity problem arises as the set of Karush-Kuhn-Tucker optimality conditions of a convex quadratic program, then the objective function of the program is a natural candidate to be used in the monitoring process. Some references on this approach are [5, 10, 11, 12, 13, 16, 30].

The second and third approaches were initially used by Ahn [3] in his doctoral dissertation to establish the convergence of the PIES algorithm [21]. Both approaches are based on some standard contraction argument. One of them involves "norm-contraction" and the other "vector-contraction". (More detailed explanation of these two terms will be given later.) In the earlier paper [38], the author has extended Ahn's work [4] and has established a rather general convergence result (based on the vector-contraction approach) for the basic algorithm mentioned previously. Later in the paper, this result will be improved. Incidentally, the norm-contraction approach has also been

used by Aganagic [1] in his convergence proof of a particular iterative method for solving the linear complementarity problem with an asymmetric matrix.

Finally, the fourth approach, to be called the monotone approach, involves showing that the sequence of iterates is both nonincreasing and bounded below (in the vector sense). Once these two properties are established, the desired convergence of the iterates is obtained easily. Typically, the monotonicity of the iterates is implied by some least-element connection of the linear complementarity subproblems. See [37].

Once the convergence of the iterates is established, an equally important question is how fast the convergence is, in other words, what the convergence rate of the algorithm is. Typically, this rate may not be easy to obtain if the symmetry or monotone approach is used. The two contraction approaches usually give rise to a geometrical rate of convergence. In this paper, we shall be concerned merely with the convergence of the basic algorithm and shall not address its rate of convergence.

The organization of the rest of this paper is as follows. In the next section, we formally introduce the basic algorithm and show how it includes a block relaxation method for a certain strictly convex quadratic program as a special case. In the four sections following Section 2, we investigate the convergence of the basic algorithm using the four approaches outlined above. Finally in the last section, we draw some concluding remarks concerning the possibility of extending our analysis given here to treat other complementarity and variational problems.

2. THE BASIC ALGORITHM

Given an n -vector q and n by n matrix M , the linear complementarity problem (q, M) is to find an n -vector x so that the conditions below are satisfied

$$u = q + Mx \geq 0, \quad x \geq 0 \quad \text{and} \quad u^T x = 0.$$

By a splitting of the matrix M , we mean the representation

$$M = B + C$$

where B and C are matrices of the same order as M . We denote such a splitting by (B, C) . In [38], we have introduced the following algorithm as a unifying framework for the study of many iterative methods for solving the linear complementarity problem (q, M) .

Basic Algorithm. Let (B, C) be a splitting of the matrix M and let E be an n by n nonnegative diagonal matrix with $E_{ii} < 1$ for all i . Let x^0 be an arbitrary initial vector. Generate the sequence of vectors $\{x^k\}$ as follows. Given x^k , let x^{k+1} be a solution to the complementarity problem

$$u = q + Cx^k + Bx \geq 0, \quad x \geq Ex^k \quad \text{and} \quad u^T(x - Ex^k) = 0. \quad (1)$$

Alternatively, the algorithm may be described in terms of the associated algorithmic map G where for each given vector $y \in R^n$, $G(y)$ is the set of all solutions to the complementarity problem (1) with y replacing x^k . Using the map G , we may rephrase the algorithm as : $x^{k+1} \in G(x^k)$. It is easy to show that any fixed point of G , i.e., any vector y such that $y \in G(y)$, is a solution to the linear complementarity problem (q, M) .

The problem (1) is not in the exact form of a standard linear complementarity problem (q, M) . However, by means of the simple translation of

variables $z = x - Ex^k$, (1) is obviously equivalent to the linear complementarity problem (q^k, B) where $q^k = q + (C + BE)x^k$.

So far, we have not stated any condition on the splitting (B, C) to guarantee that each linear complementarity subproblem (1) is solvable. From now on, we simply declare this as a blanket assumption of the paper and will not be concerned with the exact manner in which x^{k+1} is computed. The analysis in the subsequent sessions will be devoted entirely on investigating the convergence of the sequence $\{x^k\}$ generated. (For practical purposes, the splitting (B, C) should be such that each linear complementarity subproblem (1) can be solved fairly easily.)

By appropriately choosing the splitting (B, C) and the diagonal matrix E , we [38] have known how Mangasarian's iterative scheme [30] and the point successive overrelaxation (SOR) method of Cottle, Golub and Sacher [17] can be cast as special cases of the basic algorithm described above. In what follows, we demonstrate that the relaxation method of Cea and Glowinski [5] specialized to solve strictly convex quadratic programs with separable constraints is also a special realization of our basic algorithm. Specifically, consider the quadratic program

$$\text{minimize } \frac{1}{2} v^T Q v + f^T v \quad \text{subject to } F_i v_i \leq e_i, \quad v_i \geq 0, \quad i = 1, \dots, K \quad (QP)$$

where the matrix Q is symmetric positive definite and the vector v is partitioned into K subvectors v_i ($i = 1, \dots, K$). Let the matrix $Q = (Q_{ij})$ and vector $f = (f_i)$ be partitioned accordingly. Then the relaxation method of Cea and Glowinski works as follows. Let $\omega_i < 2$ be positive scalars. Given $v^k = (v_i^k)$, compute $v^{k+1} = (v_i^{k+1})$ by successively solving the quadratic subprogram

$$\text{minimize } \frac{1}{2} v_i^T Q_{ii} v_i - [Q_{ii} v_i^k - \omega_i (\sum_{j < i} Q_{ij} v_j^{k+1} + \sum_{j \geq i} Q_{ij} v_j^k + f_i)]^T v_i \quad (QP_i)$$

$$\text{subject to } F_i v_i \leq e_i \quad \text{and} \quad v_i \geq 0$$

and letting v_i^{k+1} be its unique solution. The ω_i are the relaxation parameters.

To simplify the notations, let's fix $K = 3$. The Karush-Kuhn-Tucker conditions for the program (QP) and the subprogram (QP_i) can be formulated respectively, as the linear complementarity problems (q, M) and (q_i^k, N_{ii}) where

$$q = \begin{pmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{pmatrix} \quad M = \begin{pmatrix} Q_{11} & F_1^T & Q_{12} & 0 & Q_{13} & 0 \\ -F_1 & 0 & 0 & 0 & 0 & 0 \\ Q_{21} & 0 & Q_{22} & F_2^T & Q_{23} & 0 \\ 0 & 0 & -F_2 & 0 & 0 & 0 \\ Q_{31} & 0 & Q_{32} & 0 & Q_{33} & F_3^T \\ 0 & 0 & 0 & 0 & -F_3 & 0 \end{pmatrix} \quad (2)$$

and

$$q_i^k = \begin{pmatrix} f_i + \sum_{j < i} Q_{ij} v_j^{k+1} + \sum_{j > i} Q_{ij} v_j^k - (1 - \omega_i) Q_{ii} v_i^k / \omega_i \\ e_i \end{pmatrix}, \quad N_{ii} = \begin{pmatrix} Q_{ii} / \omega_i & F_i^T \\ -F_i & 0 \end{pmatrix}.$$

By letting λ_i^{k+1} be an optimal Lagrange multiplier of the subprogram (QP_i) , it is easy to see that the vector $x^{k+1} = (v^{k+1}, \lambda^{k+1})$ solves the linear complementarity problem (1) with $E = 0$,

$$B = \begin{pmatrix} Q_{11}/\omega_1 & F_1^T & 0 & 0 & 0 & 0 \\ -F_1 & 0 & 0 & 0 & 0 & 0 \\ Q_{21} & 0 & Q_{22}/\omega_2 & F_2^T & 0 & 0 \\ 0 & 0 & -F_2 & 0 & 0 & 0 \\ Q_{31} & 0 & Q_{32} & 0 & Q_{33}/\omega_3 & F_3^T \\ 0 & 0 & 0 & 0 & -F_3 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \tilde{\omega}_1 Q_{11} & 0 & Q_{12} & 0 & Q_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\omega}_2 Q_{22} & 0 & Q_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\omega}_3 Q_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

and where $\tilde{\omega}_i = -(1-\omega_i)/\omega_i$. Note that (B, C) is in fact a splitting of the matrix M given in (2). Consequently, we have shown that with this particular splitting, our basic algorithm reduces precisely to the relaxation method of Cea and Glowinski. A noteworthy point here is that the above matrix B is generally not such that all linear complementarity problems (r, B) are solvable. However, due to the special structure of B , each subproblem (1) will have a solution if the original quadratic program (QP) is feasible.

Basically, the above relaxation method of Cea and Glowinski is just one member of the family of block SOR methods for solving convex quadratic programs. Another version of such methods was first proposed in [12] and recently refined in [13]. We have not been able to show whether this latter version can also be cast in the form of the basic algorithm, except in the case when the relaxation parameter does not exceed one. See [38]. This is partly due to the fact that the relaxation step is performed in a slightly different way from the one in the method of Cea and Glowinski. For more discussion on the relationship between these two block SOR methods, see [11].

We should say a few more words about the basic algorithm before ending this section. Matrix splittings are extremely useful in the study of iterative methods for solving systems of linear equations [31, 41]. The construction of the basic algorithm is partly based on the belief that the matrix splittings should be equally useful in the case of the linear complementarity problem. Furthermore, because of its generality, one could choose (provided that one knows how) the splitting that is most appropriate for the individual problems. Hopefully, the various results established in subsequent sections would then allow one to draw some affirmative conclusion about the convergence of the algorithm.

3. THE SYMMETRY APPROACH FOR CONVERGENCE

Starting in this section, we shall derive some convergence results for the basic algorithm presented in the last section. As mentioned in the introduction, we shall follow four different approaches. In this section, we give result based on the symmetry approach. The following is the main convergence theorem of this approach. Recall that a copositive matrix M is one such that $x^T M x \geq 0$ for all $x \geq 0$ and a copositive-plus matrix is a copositive matrix M which satisfies the implication: $[x^T M x = 0, x \geq 0] \Rightarrow (M + M^T)x = 0$.

Theorem 1. Let E be a nonnegative diagonal matrix with $E_{ii} < 1$ and let (B, C) be a splitting of M satisfying

$$(3.1) \quad B = U + V + C^T ;$$

(3.2) There exists a permutation matrix P and a nonempty index set α (with complement β) such that

$$P^T V P = \begin{pmatrix} V_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix}, \quad P^T C P = \begin{pmatrix} C_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix}, \quad P^T E P = \begin{pmatrix} E_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P^T U P = \begin{pmatrix} 0 & U_{\alpha\beta} \\ -U_{\alpha\beta}^T & 0 \end{pmatrix}$$

where $V_{\alpha\alpha}$ is symmetric positive definite;

(3.3) Each of the linear complementarity subproblems (1) is solvable.

Suppose also that the initial vector x^0 is nonnegative and satisfies

$$q_\beta - U_{\alpha\beta}^T x_\alpha^0 \geq 0.$$

Then every accumulation point of the sequence $\{x^k\}$ generated by the basic algorithm solves the linear complementarity problem (q, M) . Moreover, if in addition, the condition below is satisfied

(3.4) The matrix $A_{\alpha\alpha} = (V + C + C^T)_{\alpha\alpha}$ is copositive-plus and there exist vectors y_α^1 and y_α^2 so that

$$q_\alpha + A_{\alpha\alpha} y_\alpha^1 > 0 \quad (4i)$$

and

$$y_\alpha^2 \geq 0 \quad \text{and} \quad q_\beta - U_{\alpha\beta}^T y_\alpha^2 > 0. \quad (4ii)$$

Then the sequence $\{x^k\}$ is bounded and therefore has an accumulation point which solves the linear complementarity problem (q, M) .

We point out several remarks about the above theorem. First of all, in order to satisfy all the conditions (3.1)-(3.4), the matrix M need not be symmetric or positive semi-definite, although it must be copositive-plus (if (3.4) holds). In fact, the matrix M after a principal rearrangement is given by

$$\begin{pmatrix} A_{\alpha\alpha} & U_{\alpha\beta} \\ -U_{\alpha\beta}^T & 0 \end{pmatrix} \quad (5)$$

which is bisymmetric. Second, the matrix V must be symmetric positive semi-definite by condition (3.2), whereas B needs not be so. Third, it is easy to see that the matrix M in (2) with the splitting (B, C) defined in (3) satisfies conditions (3.1) and (3.2) of the theorem. In fact, it suffices to choose

$$U = \begin{pmatrix} 0 & F_1^T & 0 & 0 & 0 & 0 \\ -F_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_2^T & 0 & 0 \\ 0 & 0 & -F_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_3^T \\ 0 & 0 & 0 & 0 & -F_3 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \frac{2-\omega_1}{\omega_1} Q_{11} & & & & & \\ & 0 & & & & \\ & & \frac{2-\omega_2}{\omega_2} Q_{22} & & & \\ & & & 0 & & \\ & & & & \frac{2-\omega_3}{\omega_3} Q_{33} & \\ & & & & & 0 \end{pmatrix}.$$

The requirement that each relaxation parameter $\omega_i \in (0, 2)$ is needed in order for the submatrix

$$V_{\alpha\alpha} = \begin{pmatrix} \frac{2-\omega_1}{\omega_1} Q_{11} & & \\ & \frac{2-\omega_2}{\omega_2} Q_{22} & \\ & & \frac{2-\omega_3}{\omega_3} Q_{33} \end{pmatrix}$$

to be positive definite. As noted in the last section, condition (3.3) holds if the quadratic program (QP) is feasible. As for condition (3.4), the matrix $A_{\alpha\alpha} = Q$ is positive definite (and thus copositive-plus) by assumption. The system (4i) is consistent for the same reason. The consistency of the system (4ii) is implied by the existence of a Slater point in the program (QP). In essence, this last condition (consistency of (4ii)) is not required in the proof of convergence of the sequence of primal vectors v^k in the relaxation method of Cea and Glowinski. It is needed in order for the sequence of optimal multipliers to converge as well.

The main convergence results obtained by Mangasarian [30] for his iterative scheme are special cases of Theorem 1. In fact, as pointed out in [38], Mangasarian's scheme is a special case of our basic algorithm with $E = (1-\lambda)I$ and the splitting (B, C) given by

$$B = (\lambda\omega^*)^{-1}D + K \quad \text{and} \quad C = (M - K) - (\lambda\omega^*)^{-1}D$$

where D is a positive diagonal matrix, K is a strictly triangular matrix, $0 < \lambda < 1$ and $\omega^* > 0$. (In the original statement of the algorithm, Mangasarian allows the matrices D and K to vary from one iteration to the next. In essence, we could do the same thing in our basic algorithm. That is, we could let the splitting (B, C) depend on the iteration. However for the sake of simplicity, we choose not to include this generality in the treatment.) In stating his algorithm, Mangasarian requires that there

exists a $\gamma > 0$ such that

$$y^T [(\lambda \omega^*)^{-1} D + K - \frac{M}{2}] y \geq \gamma \|y\|^2 \quad \text{for all } y.$$

obviously, this last condition is equivalent to the positive definiteness of the matrix

$$V = 2(\lambda \omega^*)^{-1} D + K^T + K - M^T.$$

If M is symmetric (this is a crucial assumption in Mangasarian's proofs), then so is V . It is now easy to see that conditions (3.1) ~ (3.3) are satisfied (α is the entire index set $\{1, \dots, n\}$ and P is the identity matrix). Condition (3.4) reduces to the one in Theorem 2 of [30].

Proof of Theorem 1. First note that by the choice of x^0 , the sequence $\{x^k\}$ is nonnegative and satisfies

$$q_\beta - U_{\alpha\beta}^T x_\alpha^k \geq 0 \quad \text{for all } k \geq 0.$$

We shall let the quadratic function

$$f(x) = q_\alpha^T x_\alpha + \frac{1}{2} x_\alpha^T M x_\alpha$$

monitor the progress of the algorithm. By the structure (5) of M , it follows that

$$f(x) = q_\alpha^T x_\alpha + \frac{1}{2} x_\alpha^T A_{\alpha\alpha} x_\alpha.$$

We now show that the sequence $\{f(x^k)\}$ is nonincreasing. In fact, we have

$$\begin{aligned} f(x^{k+1}) - f(x^k) &= (q_\alpha + A_{\alpha\alpha} x_\alpha^k)^T (x^{k+1} - x^k)_\alpha + \frac{1}{2} (x^{k+1} - x^k)_\alpha^T A_{\alpha\alpha} (x^{k+1} - x^k)_\alpha \\ &= (q_\alpha + C_{\alpha\alpha} x_\alpha^k + (V + C^T)_{\alpha\alpha} x_\alpha^{k+1})^T (x^{k+1} - x^k)_\alpha - \frac{1}{2} (x^{k+1} - x^k)_\alpha^T V_{\alpha\alpha} (x^{k+1} - x^k)_\alpha \\ &= (q_\alpha + C_{\alpha\alpha} x_\alpha^k + (V + C^T)_{\alpha\alpha} x_\alpha^{k+1} + U_{\alpha\beta} x_\beta^{k+1})^T (x^{k+1} - x^k)_\alpha - (x_\beta^{k+1})^T (U_{\alpha\beta}^T x_\alpha^{k+1} - U_{\alpha\beta}^T x_\alpha^k) \\ &\quad - \frac{1}{2} (x^{k+1} - x^k)_\alpha^T V_{\alpha\alpha} (x^{k+1} - x^k)_\alpha \end{aligned}$$

$$\begin{aligned}
 &= (q_\alpha + C_{\alpha\alpha} x_\alpha^k + (V + C^T)_{\alpha\alpha} x_\alpha^{k+1} + U_{\alpha\beta} x_\beta^{k+1})^T [(x^{k+1} - Ex^k) + (E - I)x^k]_\alpha \\
 &\quad + (x_\beta^{k+1})^T [(q_\beta - U_{\alpha\beta}^T x_\alpha^{k+1}) - (q_\beta - U_{\alpha\beta}^T x_\alpha^k)] - \frac{1}{2}(x^{k+1} - x^k)_\alpha^T V_{\alpha\alpha} (x^{k+1} - x^k)_\alpha \\
 &\leq -\frac{1}{2}(x^{k+1} - x^k)_\alpha^T V_{\alpha\alpha} (x^{k+1} - x^k)_\alpha \leq 0,
 \end{aligned}$$

where the next to last inequality follows from the fact that x^{k+1} solves the linear complementarity problem (1) and from the observation made at the start of the proof concerning the sequence $\{x^k\}$.

Now let x^* be an accumulation point of the sequence $\{x^k\}$ and let $\{x^{k_j}\}$ be a subsequence converging to x^* . As noted in [30], the sequence $\{f(x^k)\}$ must then converge. It therefore follows that

$$\lim_{j \rightarrow \infty} x_\alpha^{k_j-1} = \lim_{j \rightarrow \infty} x_\alpha^{k_j} = x_\alpha^*. \quad (6)$$

For each k_j , we have

$$u_\alpha^{k_j} = q_\alpha + C_{\alpha\alpha} x_\alpha^{k_j-1} + B_{\alpha\alpha} x_\alpha^{k_j} + B_{\alpha\beta} x_\beta^{k_j} \geq 0, \quad x_\alpha^{k_j} \geq E_{\alpha\alpha} x_\alpha^{k_j-1} \quad (7i)$$

$$u_\beta^{k_j} = q_\beta + B_{\beta\alpha} x_\alpha^{k_j} \geq 0, \quad x_\beta^{k_j} \geq 0 \quad (7ii)$$

and

$$(u_\alpha^{k_j})^T (x_\alpha^{k_j} - E_{\alpha\alpha} x_\alpha^{k_j-1}) = (u_\beta^{k_j})^T x_\beta^{k_j} = 0. \quad (7iii)$$

By passing the limit $k_j \rightarrow \infty$ and using (6), we conclude immediately that x^* solves the linear complementarity problem (q, M) .

Suppose now that the condition (3.4) also holds. It suffices to show that the sequence $\{x^k\}$ is bounded. We first show that $\{x_\alpha^k\}$ is. Suppose not. Then by Lemma 3 in [30], we may deduce that there exists a vector z_α^* satisfying

$$0 \neq z_\alpha^* \geq 0, \quad q_\alpha^T z_\alpha^* \leq 0 \quad \text{and} \quad (z_\alpha^*)^T A_{\alpha\alpha} z_\alpha^* = 0.$$

By the assumed property of $A_{\alpha\alpha}$, we obtain $A_{\alpha\alpha} z_\alpha^* = 0$. But this would contradict the consistency of the system (4i). Hence $\{x_\alpha^k\}$ must be bounded. Finally, we

show that $\{x_\beta^k\}$ is bounded. Suppose not; then there is a subsequence $\{x_\beta^{k_j}\}$ such that $\|x_\beta^{k_j}\| \rightarrow \infty$. The normalized subsequence $\{x_\beta^{k_j}/\|x_\beta^{k_j}\|\}$ has an accumulation point z_β^* . With no loss of generality, we may assume that $z_\beta^{k_j}/\|z_\beta^{k_j}\| \rightarrow z_\beta^*$. Since the sequence x_α^k is bounded, it has an accumulation point z_α^* . By considering a suitable subsequence if necessary, we may assume that $x_\alpha^k \rightarrow z_\alpha^*$. Dividing (7i) by $\|z_\beta^{k_j}\|$ and passing the limit $k_j \rightarrow \infty$, we obtain

$$B_{\alpha\beta} z_\beta^* \geq 0 \quad (8)$$

Moreover, we have from (7iii)

$$[(I - E_{\alpha\alpha})z_\alpha^*]^T B_{\alpha\beta} z_\beta^* = 0$$

which implies by the fact that $I - E_{\alpha\alpha}$ is a positive diagonal matrix

$$(z_\alpha^*)^T B_{\alpha\beta} z_\beta^* = 0.$$

Similarly, we obtain from (7ii) and (7iii)

$$(z_\beta^*)^T [q_\beta + B_{\beta\alpha} z_\alpha^*] = 0.$$

Since $B_{\alpha\beta} = U_{\alpha\beta} = -B_{\beta\alpha}^T$, it follows that $(z_\beta^*)^T q_\beta = 0$. This together with (8) and the fact that $z_\beta^* \geq 0$, would contradict the consistency of the system (4ii). The contradiction completes the proof of the theorem.

4. THE NORM-CONTRACTION APPROACH FOR CONVERGENCE

An implication of the assumptions of Theorem 1 is that the matrix M must be bisymmetric with a zero diagonal block. In this and the next two sections, we derive some convergence results by replacing the symmetry requirement with some other assumptions. We first review some matrix concepts [31].

If A is a symmetric positive definite matrix, then it has a unique Cholesky factorization $A = (A^{\frac{1}{2}}) (A^{\frac{1}{2}})^T$ where $A^{\frac{1}{2}}$ denotes the Cholesky factor (or square root) of A . The ℓ_p -norm ($1 \leq p \leq \infty$) of the matrix A is defined by

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$$

where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ is the ℓ_p -norm of an n -vector x . (Note: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.) The Hölder inequality [31] states that

$$|x^T y| \leq \|x\|_p \|y\|_q \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

with equality holding if and only if x is a scalar multiple of y . We denote by $\rho(A)$ the spectral radius of the matrix A . It is known that $\rho(A) \leq \|A\|_p$ for all $p \in [1, \infty]$.

Theorem 2. Let (B, C) be a splitting of the n by n matrix M such that

(4.1) $B = U + V$ where U is skew-symmetric;

(4.2) There exists a permutation matrix P and a nonempty index set α (with complement β) such that

$$P^T V P = \begin{pmatrix} v_{\alpha\alpha} & 0 \\ 0 & v_{\beta\beta} \end{pmatrix}, \quad P^T C P = \begin{pmatrix} c_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix}$$

with $V_{\alpha\alpha}$ being symmetric positive definite and $V_{\beta\beta}$ symmetric positive semi-definite;

(4.3) Each of the linear complementarity subproblems (1) with $E = 0$ has a solution;

$$(4.4) \quad \|V_{\alpha\alpha}^{-\frac{1}{2}} C_{\alpha\alpha} (V_{\alpha\alpha}^{-\frac{1}{2}})^T\|_p < 1 \quad \text{for some } p \in [1, \infty].$$

Then any accumulation point of the sequence $\{x^k\}$ generated by the basic algorithm with $E = 0$ is a solution to the linear complementarity problem (q, M) . Moreover, if in addition, the condition below is satisfied

(4.5) There exists a vector y_α^2 such that

$$y_\alpha^2 \geq 0 \quad \text{and} \quad q_\beta - U_{\alpha\beta}^T y_\alpha^2 > 0,$$

then the sequence $\{x^k\}$ is bounded and therefore has an accumulation point which solves the problem (q, M) .

Before proving the theorem, we discuss some implications of its assumptions. First of all, the matrix M , after a principal rearrangement, is of the form

$$\begin{pmatrix} V_{\alpha\alpha} + C_{\alpha\alpha} + U_{\alpha\alpha} & U_{\alpha\beta} \\ -U_{\alpha\beta}^T & V_{\beta\beta} + U_{\beta\beta} \end{pmatrix} \quad (9)$$

which is similar to (5). However, the properties of the two diagonal blocks in the two forms are quite different. In (5), the two diagonal blocks are symmetric (in fact, one of them is zero). They are not in (9). On the other hand, it can be shown easily that if $p = 2$ in condition (4.4), the matrix $V_{\alpha\alpha} + C_{\alpha\alpha}$ and thus $A_{\alpha\alpha} = V_{\alpha\alpha} + C_{\alpha\alpha} + U_{\alpha\alpha}$ is positive definite (see [3] e.g.,) The corresponding diagonal block in (5) may not even be semi-definite. The whole matrix in (9) is positive semi-definite. Summarizing, we could say

that in the present case, we are trading symmetry of the splitting (B, C) with a stronger form of positive definiteness (condition (4.4)).

In [3], Ahn noted that the PIES algorithm can be applied to a linear complementarity problem with a bisymmetric matrix. In general, such a linear complementarity problem is of the form

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} M & A^T \\ -A & N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \quad w_1^T x_1 = w_2^T x_2 = 0.$$

The PIES algorithm operates by solving a sequence of linear complementarity problems

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} (M-\hat{M})x_1^k \\ (N-\hat{N})x_2^k \end{pmatrix} + \begin{pmatrix} \hat{M} & A^T \\ -A & \hat{N} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \quad w_1^T x_1 = w_2^T x_2 = 0$$

where \hat{M} and \hat{N} are the diagonal parts (assumed positive) of M and N respectively.

Note that each such subproblem is a strictly convex quadratic program. It is easy to see that the PIES algorithm is a special case of our basic algorithm with $E = 0$,

$$B = \begin{pmatrix} \hat{M} & A^T \\ -A & \hat{N} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} M-\hat{M} & 0 \\ 0 & N-\hat{N} \end{pmatrix}.$$

Obviously with α chosen as the entire index set, P the identity matrix

$$U = \begin{pmatrix} 0 & A^T \\ -A & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \hat{M} & 0 \\ 0 & \hat{N} \end{pmatrix}$$

conditions (4.1) and (4.2) are satisfied. Condition (4.3) holds because the matrix B is positive definite. Finally, with $p = 2$, (4.4) is precisely the norm condition derived for the convergence of the PIES algorithm [3].

(Condition (4.5) is void in this instance because β is empty.)

The matrix in (9) is also bisymmetric. This does not imply, however, the PIES algorithm applied to the linear complementarity problem (\tilde{q}, \tilde{M}) where \tilde{M} is the matrix in (9) will converge. One reason is that the norm assumption required in the convergence proof of the PIES algorithm would imply that \tilde{M} should be positive definite. In general, \tilde{M} is only semi-definite.

Another special case of Theorem 2 is Aganagic's convergence result for his iterative scheme [1]. As noted in [38], Aganagic's scheme corresponds to our basic algorithm with $E = 0$,

$$B = \frac{1}{\omega^*} I \quad \text{and} \quad C = M - \frac{1}{\omega^*} I$$

where ω^* is a positive parameter. Specialized to this splitting, condition (4.4) becomes $\|I - \omega^* M\|_p < 1$ which is precisely the one required by Aganagic.

As a final remark on the conditions of Theorem 2, we point out that since the matrix B is positive semi-definite, condition (4.3) is equivalent to the feasibility of each of the subproblems (1) [7].

Proof of Theorem 2. For $k \geq 1$, we have

$$\begin{aligned} u^k &= q + Cx^{k-1} + Bx^k \geq 0, \quad x^k \geq 0, \quad (u^k)^T x^k = 0 \\ u^{k+1} &= q + Cx^k + Bx^{k+1} \geq 0, \quad x^{k+1} \geq 0, \quad (u^{k+1})^T x^{k+1} = 0. \end{aligned}$$

Hence it follows that

$$0 \geq (x^{k+1} - x^k)^T (u^{k+1} - u^k) = (x^{k+1} - x^k)^T [B(x^{k+1} - x^k) + C(x^k - x^{k-1})]$$

By (4.1) and (4.2), we obtain

$$(x^{k+1} - x^k)^T_{\alpha} V_{\alpha\alpha} (x^{k+1} - x^k)_{\alpha} \leq (x^{k+1} - x^k)^T V (x^{k+1} - x^k) \leq - (x^{k+1} - x^k)^T_{\alpha} C_{\alpha\alpha} (x^k - x^{k-1})_{\alpha}.$$

Hence by Hölder's inequality, it follows that

$$\|V_{\alpha\alpha}^{-\frac{1}{2}}(x^{k+1} - x^k)_{\alpha}\|_p \leq \gamma \|V_{\alpha\alpha}^{-\frac{1}{2}}(x^k - x^{k-1})_{\alpha}\|_p$$

where $\gamma = \|V_{\alpha\alpha}^{-\frac{1}{2}} C_{\alpha\alpha} (V_{\alpha\alpha}^{-\frac{1}{2}})^T\|_p$. Since $\gamma < 1$, a standard contraction argument gives immediately

$$\lim_{k \rightarrow \infty} (x^{k+1} - x^k)_{\alpha} = 0.$$

Moreover, the sequence $\{x_{\alpha}^k\}$ is bounded for the same reason. The rest of the proof resembles that of Theorem 1 and is not repeated here.

Remark. It is easy to see from the above proof that Theorem 2 remains valid if condition (4.4) is replaced by the following one:

$$(4.4)' \quad \|C_{\alpha\alpha}\|_2 < \lambda$$

where λ is the least eigenvalue of the matrix $V_{\alpha\alpha}$. In fact, one can even drop the symmetry assumption on $V_{\alpha\alpha}$ if (4.4)' holds with

$$\lambda = \min_{\|x_{\alpha}\|_2=1} (x_{\alpha}^T V_{\alpha\alpha} x_{\alpha})^{\frac{1}{2}}.$$

5. THE VECTOR-CONTRACTION APPROACH FOR CONVERGENCE

In this section, we establish a convergence result of the basic algorithm by using a vector-contraction argument. To start, we review some more matrix concepts.

Let M be a real square matrix. Its comparison matrix $\hat{M} = (\hat{M}_{ij})$ is defined by

$$\hat{M}_{ij} = \begin{cases} |M_{ii}| & \text{for } i = j \\ -|M_{ij}| & \text{for } i \neq j. \end{cases}$$

Obviously, \hat{M} is a Z-matrix, i.e., it has nonpositive off-diagonal entries. If \hat{M} is a P-matrix as well, i.e., if \hat{M} has all principal minors positive, then M is said to be an H-matrix [42]. It has been proved in [34] that H-matrices with positive diagonal entries are themselves P-matrices. In particular, if M is an H-matrix with positive diagonals, then the linear complementarity problem (q, M) has a unique solution for all vectors q [39]. It is obvious that principal submatrices of H-matrices are themselves H-matrices.

A Z-matrix which is also a P-matrix is known as a K-matrix [19]. An equivalent way to define an H-matrix is to say that its comparison matrix is a K-matrix. If M is a K-matrix, then M^{-1} exists and is nonnegative. If M and N are both Z-matrices such that $M \leq N$ and M is a K-matrix, then so is N and $M^{-1} \geq N^{-1}$. See [19] for more details on these and other properties of K-matrices.

If M and N are two nonnegative matrices, then $\rho(M) \leq \rho(N)$. If M is any matrix, by $|M|$ we denote the matrix whose entries are the absolute values of those of M . Similarly, if x is a vector, by $|x|$ we denote the vector whose components are the absolute values of the ones in x .

The following convergence result was proved in [38].

Theorem 3. Let E be a nonnegative diagonal matrix with $E_{ii} < 1$ for all i .

Let (B, C) be a splitting of the matrix M such that

(5.1) B is an H-matrix with positive diagonal entries;

(5.2) $\rho(\hat{B}^{-1} \max(DE, |C|)) < 1$ where \hat{B} is the comparison matrix of B and D is the diagonal part of B .

Then each of the linear complementarity subproblems (1) has a unique solution.

Moreover, the sequence $\{x^k\}$ generated converges to the unique solution of the linear complementarity problem (q, M) .

The theorem below shows that condition (5.2) can be replaced by a weaker one.

Theorem 3'. Theorem 3 remains valid if condition (5.2) is weakened to

(5.2)' There exists a matrix G with $\rho(G) < 1$ such that for any index set α (with complement β)

$$G \geq \begin{pmatrix} I & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix}^{-1} \begin{pmatrix} E_{\alpha\alpha} & 0 \\ |C_{\beta\alpha}| & |C_{\beta\beta}| \end{pmatrix} \quad (10)$$

Before proving Theorem 3', we show that if condition (5.1) holds, then (5.2) implies (5.2)' but not conversely. In fact, we claim that (10) holds with $G = \hat{B}^{-1} \max(DE, |C|)$. To prove this, note that the product matrix in (10) is equal to

$$\begin{pmatrix} E_{\alpha\alpha} & 0 \\ \hat{B}_{\beta\beta}^{-1} (-\hat{B}_{\beta\alpha} E_{\alpha\alpha} + |C_{\beta\alpha}|) & \hat{B}_{\beta\beta}^{-1} |C_{\beta\beta}| \end{pmatrix}. \quad (11)$$

Since

$$\hat{B} \leq \begin{pmatrix} D_{\alpha\alpha} & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix} \quad \text{and} \quad \max(DE, |C|) \geq \begin{pmatrix} D_{\alpha\alpha} E_{\alpha\alpha} & 0 \\ |C_{\beta\alpha}| & |C_{\beta\beta}| \end{pmatrix},$$

we obtain

$$\hat{B}^{-1} \max(DE, |C|) \geq \begin{pmatrix} D_{\alpha\alpha} & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix}^{-1} \begin{pmatrix} D_{\alpha\alpha} E_{\alpha\alpha} & 0 \\ |C_{\beta\alpha}| & |C_{\beta\beta}| \end{pmatrix}$$

It is easy to see that the matrix in the right side of the above inequality is precisely the one given in (11). This establishes our claim that (10) holds with $G = \hat{B}^{-1} \max(DE, |C|)$. To show that condition (5.2)' does not necessarily imply (5.2), consider the data

$$B = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

It is easy to see that condition (5.2)' holds with

$$G = \begin{pmatrix} \frac{1}{2} & 1/4 \\ 1/4 & \frac{1}{2} \end{pmatrix}$$

On the other hand, we have

$$\hat{B}^{-1} \max(DE, |C|) = \frac{2}{3} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

and $\rho(\hat{B}^{-1} \max(DE, |C|)) = 1$.

It has been proved in [38] that the assumptions of Theorem 3 imply that the original matrix M must be an H -matrix with positive diagonals. (This is why the linear complementarity problem (q, M) has a unique solution.) In what follows, we show that the same conclusion about the matrix M remains valid if condition (5.2) is replaced by (5.2)'. Indeed, if (5.2)' holds,

then by taking α to be the empty set, we obtain $\rho(\hat{B}^{-1} |C|) \leq \rho(G) < 1$. This implies that $\hat{B} - |C|$ is a K-matrix. By the same argument used in Corollary 3.5 of [38], we may easily deduce that M is an H-matrix with positive diagonals.

Proof of Theorem 3'. Given $k \geq 1$, let $\alpha = \alpha_1 \cup \alpha_2$ where

$$\begin{aligned}\alpha_1 &= \{i : (x^{k+1} - x^k)_i \geq 0 \text{ and } x_i^{k+1} = E_{ii} x_i^k\} \\ \alpha_2 &= \{i : (x^{k+1} - x^k)_i < 0 \text{ and } x_i^k = E_{ii} x_i^{k-1}\}.\end{aligned}$$

Then the complement of α is $\beta = \beta_1 \cup \beta_2$ with

$$\begin{aligned}\beta_1 &= \{i : (x^{k+1} - x^k)_i \geq 0 \text{ and } (Bx^{k+1} + q + Cx^k)_i = 0\} \\ \beta_2 &= \{i : (x^{k+1} - x^k)_i < 0 \text{ and } (Bx^k + q + Cx^{k-1})_i = 0\}.\end{aligned}$$

For an index $i \in \alpha_1$, we have

$$|x^{k+1} - x^k|_i = x_i^{k+1} - x_i^k \leq E_{ii}(x_i^k - x_i^{k-1}) \leq E_{ii} |x^k - x^{k-1}|_i. \quad (124)$$

Similarly, we may deduce that for $i \in \alpha_2$,

$$|x^{k+1} - x^k|_i \leq E_{ii} |x^k - x^{k-1}|_i \quad (124i)$$

Moreover for an index $i \in \beta_1$, we have

$$|x^{k+1} - x^k|_i = x_i^{k+1} - x_i^k \leq \{[-C(x^k - x^{k-1})]_i - \sum_{j \neq i} B_{ij}(x_j^{k+1} - x_j^k)\} / B_{ii}$$

which implies

$$B_{ii} |x^{k+1} - x^k|_i \leq |C_{i.}| |x^k - x^{k-1}| + \sum_{j \neq i} |B_{ij}| |x_j^{k+1} - x_j^k|$$

or equivalently

$$\hat{B}_{i.} |x^{k+1} - x^k| \leq |C_{i.}| |x^k - x^{k-1}|. \quad (13)$$

Here $\hat{B}_{i.}$ and $C_{i.}$ denote respectively, the i -th row of the matrix \hat{B} and C . By a similar argument, we may show that (13) holds for $i \in \beta_2$.

Hence, combining (12) and (13), we obtain

$$\begin{pmatrix} I & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix} \begin{pmatrix} |x^{k+1} - x^k|_{\alpha} \\ |x^{k+1} - x^k|_{\beta} \end{pmatrix} \leq \begin{pmatrix} E_{\alpha\alpha} & 0 \\ |C_{\beta\alpha}| & |C_{\beta\beta}| \end{pmatrix} \begin{pmatrix} |x^k - x^{k-1}|_{\alpha} \\ |x^k - x^{k-1}|_{\beta} \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} I & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix}$$

has a nonnegative inverse, we obtain by (10)

$$|x^{k+1} - x^k| \leq \begin{pmatrix} I & 0 \\ \hat{B}_{\beta\alpha} & \hat{B}_{\beta\beta} \end{pmatrix}^{-1} \begin{pmatrix} E_{\alpha\alpha} & 0 \\ |C_{\beta\alpha}| & |C_{\beta\beta}| \end{pmatrix} |x^k - x^{k-1}| \leq G |x^k - x^{k-1}|.$$

The rest of the proof resembles that of Theorem 3.2 in [38] and is not repeated here.

6. THE MONOTONE APPROACH FOR CONVERGENCE

In this section, we shall derive a convergence result for the basic algorithm by exploiting a certain monotonicity property of the iterates. To start, we state a known result of this kind and refer to [37] for its proof.

Theorem 4. Let E be a nonnegative diagonal matrix with $E_{ii} < 1$ and let (B, C) be a splitting of M with B a Z -matrix and C nonpositive. Let x^0 be a feasible vector to the linear complementarity problem (q, M) . For $k \geq 0$, let x^{k+1} be the least solution of the linear complementarity subproblem (1). Then the sequence $\{x^k\}$ converges to a solution of the linear complementarity problem (q, M) .

We should explain the terminology used in the above theorem. First of all, the feasible set of the linear complementarity problem (q, M) is

$$S = \{x : q + Mx \geq 0, x \geq 0\}.$$

Vectors in S are called feasible vectors to (q, M) . It is known [40] that if M is a Z -matrix and if the problem (q, M) is feasible, i.e., if the set S is nonempty, then (q, M) has a least solution x^* satisfying $x^* \leq x$ for all vectors $x \in S$. Implicit in the statement of Theorem 4 is the assertion that each x^{k+1} exists if x^0 is chosen to be feasible.

An implication of the assumed property of the splitting (B, C) in Theorem 4 is that the matrix $M = B + C$ must itself be a Z -matrix. In what follows, we extend the theorem so that this Z -property of M need not be necessary. To provide the framework for this extension, we review some pertinent background results.

A real square matrix B is hidden Z if there exist Z-matrices X and Y such that the following two conditions hold

$$(6.1) \quad BX = Y$$

$$(6.2) \quad r^T X + s^T Y > 0 \quad \text{for some vectors } r, s \geq 0.$$

Hidden Z-matrices were introduced by Mangasarian [28, 29] and later studied intensively in [14, 32, 33, 34, 35]. The following are some useful properties of such matrices. Their proofs can be found in the cited references.

- (i) The matrix X in (6.1) is nonsingular.
- (ii) There exist complementary index sets α and β such that the matrix

$$\begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ Y_{\beta\alpha} & Y_{\beta\beta} \end{pmatrix}$$

has a nonnegative inverse.

- (iii) For every vector p for which the linear complementarity problem (p, B) is feasible, the problem (p, B) has a least solution x^* , least with respect to the cone ordering induced by the polyhedral cone generated by the matrix X ; that is, x^* is characterized by the property that

$$X^{-1} x^* \leq X^{-1} x \quad (\text{componentwise}) \quad \text{for every } x \in S.$$

We are now ready to state the main convergence result of this section.

Theorem 4'. Let E be a nonnegative diagonal matrix with $E_{ii} < 1$ and let (B, C) be a splitting of the matrix M such that

$$(6.3) \quad B \text{ is a hidden Z-matrix;}$$

(6.4) There exists a vector b such that $Cx \leq b$ for every x feasible to the problem (q, M) ;

(6.5) $CX \leq 0$ where X is the matrix in condition (6.1).

Let x^0 be a feasible vector to the linear complementarity problem (q, M) . For $k \geq 0$, let x^{k+1} be the least solution of the linear complementarity subproblem (1), least with respect to the ordering specified in property (iii) above. Then the sequence $\{x^k\}$ converges to a solution of the problem (q, M) .

Remark. Condition (6.4) is satisfied if for instance, the matrix C is a nonpositive combination of the rows of the matrix $\begin{pmatrix} M \\ I \end{pmatrix}$.

Proof of Theorem 4'. We show that x^{k+1} exists and is feasible to the linear complementarity problem (q, M) . By means of an inductive argument, we may assume that this is true for x^k . Since x^k is feasible to the problem (q, M) , it is certainly feasible to the subproblem (1). Hence, according to property (iii) mentioned above, x^{k+1} exists and

$$X^{-1} x^{k+1} \leq X^{-1} x^k. \quad (14)$$

To show that x^{k+1} is feasible to (q, M) , observe that

$$x^{k+1} \geq Ex^k \geq 0$$

and

$$\begin{aligned} q + Mx^{k+1} &= q + CX(X^{-1} x^{k+1}) + Bx^{k+1} \geq q + CX(X^{-1} x^k) + Bx^{k+1} \\ &= q + Cx^k + Bx^{k+1} \geq 0 \end{aligned}$$

where the first inequality follows from (14) and condition (6.5). Consequently, x^{k+1} is feasible to (q, M) . Now, the sequence $\{X^{-1} x^k\}$ is

nonincreasing. We show that it is bounded below as well. Letting

$w^k = X^{-1} x^k$, we have

$$Xw^k = x^k \geq 0$$

and by letting $p = q + b$,

$$Yw^k = Bx^k \geq -(q + Cx^{k-1}) \geq -p$$

where the last inequality holds because each x^{k-1} is feasible to the problem (q, M) . By property (ii), it follows that

$$w^k \geq \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ Y_{\beta\alpha} & Y_{\beta\beta} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -p_\beta \end{pmatrix}.$$

Consequently, the sequence $\{w^k\}$ and thus $\{x^k\}$, converges. It is obvious that the limit of $\{x^k\}$ solves the linear complementarity problem (q, M) .

This completes the proof of the theorem.

7. SOME CONCLUDING REMARKS

In this paper, we have described four different approaches to establish the convergence of a basic iterative method for solving the linear complementarity problem. Here, we would like to discuss the possibility of extending the analysis to treat other complementarity and variational problems.

Let f be a mapping from R^n into itself and let K be a subset in R^n . The variational inequality problem (f, K) is to find a vector $x \in K$ such that

$$(y - x)^T f(x) \geq 0 \quad \text{for all } y \in K .$$

It is known [26] that if K is a (convex) cone, then the above variational problem is equivalent to the generalized complementarity problem: find $x \in K$ such that

$$f(x) \in K^* \quad \text{and} \quad f(x)^T x = 0 .$$

where K^* is the dual cone of K , i.e.,

$$K^* = \{y \in R^n : y^T x \geq 0 \quad \text{for all } x \in K\} .$$

Obviously, the linear complementarity problem (q, M) is a special case of the generalized complementarity problem with $f(x) = q + Mx$ and K being the nonnegative orthant of R^n .

There are at least two families of iterative methods for solving the variational inequality problem (f, K) . One of them involves the idea of function splitting. More precisely, let $f(x) = g(x) + h(x)$. Then given x^k , let x^{k+1} solve the problem:

$$(y - x)^T (g(x) + h(x^k)) \geq 0 \quad \text{for all } x \in K . \quad (15)$$

Presumably, the function g should possess certain desirable structure which would allow this latter subproblem to be solved more easily than the given

problem (f, K) . An example of such splitting is where $g(x) = x$ and $h(x) = f(x) - x$. As pointed out in [6], under this latter splitting, the subproblem (15) is equivalent to finding the projection of the point $x^k - f(x^k)$ on the set K . Convergence of this projection scheme can be established (by a contraction argument) under some strong monotonicity and Lipschitz continuity properties on f . See [2, 6, 18].

Another approach of iterative methods for solving the variational inequality problem (f, K) consists of solving a sequence of subproblems of the form : find $x \in K$ so that

$$(y - x)^T f^k(x) \geq 0 \quad \text{for all } y \in K \quad (16)$$

where $f^k(x)$ is an approximation of the given function at the current iterate. Examples of such methods include the family of Quasi-Newton methods (which include the Newton method) studied recently in [17, 24, 25] and the PIES-type algorithms [21, 22, 23]. In these examples, $f^k(x)$ is either affine or such that the subproblem (16) is equivalent to (and therefore can be solved as) a convex programming problem.

As a future research project, we intend to develop a general unifying framework for the study of various iterative methods for solving the variational inequality problem (f, K) . Such a study would include the unification and possible strengthening of known convergence results. That this effort is potentially fruitful is strongly suggested by the work of the present paper and the many special instances which currently exist in the literature.

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