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A NONLINEAR STABILITY THEORY FOR PLANE BOUNDARY-LAYER FLOWS(U)

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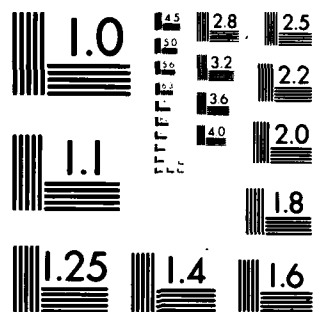
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A NONLINEAR STABILITY THEORY FOR PLANE
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JULY 1980

By: TOSHI KUBOTA, TONY CHAN, A.K. LIU
AND SUPHI OZGUR

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- ii) the rate of change of the undisturbed boundary layer with streamwise distance is small,
- iii) the rate of amplification of disturbances is small.

The computation was checked against published results for plane Poiseuille flow and the Orr-Sommerfeld solutions for Blasius flow and a numerical solution of Navier-Stokes flow along a flat plate. The present theory agrees well with published results for these flow fields.

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LIST OF SYMBOLS

A	Complex disturbance amplitude
k	Wave number
l	Reference length
L_ω	Orr-Sommerfeld operator
M	} Forcing terms, defined in Equation (2.20,a,b,c)
N	
Q	
R	Reynolds number
t	Time
U_∞	Freestream velocity
x	Distance along the wall
X	$=\mu x$, slow x-variable
y	Distance normal to the wall
∇^2	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^4	$= (\nabla^2)^2$
θ	Complex phase angle
λ	Landau constant
μ	Small parameter, measure of streamwise variation
η	Kinematic viscosity coefficient
$\phi^{(n)}$	Fourier component (in x and t) of ψ , corresponding to frequency $n\omega$
ψ	Streamfunction
ω	Radian Frequency

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1. INTRODUCTION

It is generally accepted that the transition of laminar flow to turbulent flow in channels and boundary layers is preceded by amplification of small-amplitude disturbances super-imposed on the steady laminar flow. In the early stages, nonlinear effects are negligible, and furthermore, if the slow variation of undisturbed flow with distance along the flow is neglected - quasi-one-dimensional flow - the small disturbances may be decomposed into Fourier components in time and streamwise distance. The Fourier components satisfy the homogeneous Orr-Sommerfeld equation, whose eigenvalues determine the stability of the Fourier components.

The results from linear stability theory are applied in the prediction of transition by Smith (Smith and Gamberoni, 1956; and Jaffe, Okamura and Smith, 1970) and by Mack (1975). The Smith method is based on the correlation of the amplitude ratio computed by linear theory with experimental data on the transition Reynolds number. Transition is assumed to take place when the computed amplitude for the most amplified wave reaches e^N with N ranging from 8 to 11. The Mack method, on the other hand, is based on the magnitude of the computed amplitude. The initial amplitude of a disturbance at the critical Reynolds number is estimated from the power spectrum of free-stream turbulence, and the subsequent amplitude is computed based on linear stability theory. Transition is assumed to take place when the computed amplitude reaches a certain reference value. Mack applied the method to predict the effect of freestream turbulence level on the transition Reynolds number and obtained good agreement with the measured data.

These prediction methods apply linear stability theory which one fully recognizes does not apply when the disturbance amplitudes are no longer very small, and one would hope that, if nonlinear effects are included

in the computation, one could predict transition with better precision and confidence. Inclusion of nonlinear effects, however, magnifies the complexity by a large factor. We do not have the exact analytical solution and are forced to employ some approximation methods. Because the problem is nonlinear, we cannot superimpose elementary solutions to generate a new solution. Every plausible mechanism has to be studied individually. Though, at present, the consensus among the people who are investigating boundary-layer transition is that the three-dimensional effects are an essential element in the nonlinear regime leading to transition. The nonlinear stability of two-dimensional disturbances has been investigated by many people, beginning with the pioneering work of Stuart and Watson (1960), which is based on the expansion in small amplitude and the first approximation is the linear stability theory. The nonlinear stability theory is in reasonably good shape for confined flows that are weakly unstable in the linear regime, for example, two-dimensional Poiseuille flow (Itoh, 1977). The theory, however, has not been worked out for boundary-layer flows, because the treatment of the effect on the mean flow is more subtle in boundary-layer flows than in the flow between parallel walls.

Quite apart from this analytical approach, numerical integration of the Navier-Stokes equation was undertaken to study the stability of large amplitude disturbances (Fasel, 1976; Murdock, 1977). Between the two computations, however, there was some disagreement on the relative phase between the fundamental-frequency component and the second-harmonic component.

The present investigation was undertaken to establish a nonlinear stability theory for two-dimensional disturbances in boundary-layer flows. The results of the present study serve as a check on the numerical integration of the Navier-Stokes equation. This type of approach is expected to give a good representation of the early development of the nonlinear effect for a growing disturbance and, presumably, can provide further insight into the boundary-layer transition process.

The basic assumptions made in the formulation are:

- (1) The amplitude of disturbances is small.
- (2) The rate of change of the undisturbed boundary layer with streamwise distance is small.
- (3) The rate of amplification of disturbance is small.

The first assumption allows us to expand the solution in powers of small amplitude. In the present work we retained terms to the third power of the amplitude, but we did not calculate the third harmonic in our analysis since it does not affect the form of the evolution equation to the order we considered. The third assumption together with the second assumption assures that the mean flow (disturbed as well as undisturbed) variation with the streamwise distance is small, so that the disturbed mean flow remains boundary-layer like.

Mathematical formulation is presented in Section 2, the detailed discussion of numerical method in Section 3, and the discussion of results in Section 4.

2. FORMULATION OF NONLINEAR STABILITY PROBLEM

In this section we formulate a nonlinear stability theory for two-dimensional disturbances in an incompressible boundary layer along a flat plate, including the effect of non-parallelness of the mean flow. Our starting point is the Navier-Stokes equation written in terms of the streamfunction ψ :

$$(\nabla^2 \psi)_t + \psi_y (\nabla^2 \psi)_x - \psi_x (\nabla^2 \psi)_y = \frac{1}{R} \nabla^4 \psi \quad (2.1)$$

with boundary conditions

$$\psi = \psi_y = 0 \text{ at } y = 0, \quad \psi_y \rightarrow 1 \text{ as } y \rightarrow \infty.$$

Here, x, y, t and ψ are made dimensionless by dividing distances by a reference length ℓ (comparable with the boundary-layer thickness), time by ℓ/U_∞ (U_∞ freestream velocity) and streamfunction by $U_\infty \ell$. R denotes the Reynolds number $U_\infty \ell / \nu$.

The undisturbed flow changes very slowly with x , and we denote the rate of change by a small parameter μ . We introduce a slow variable

$$X = \mu x \quad (2.2)$$

and then the basic flow is given by

$$\bar{\psi} = \bar{\psi}(X, y) \quad (2.3)$$

so that

$$\bar{\psi}_y = O(1), \quad \bar{\psi}_X = \mu \bar{\psi}_X = O(\mu)$$

Furthermore, we assume that the perturbation depends on x and t (fast variables) only through the complex phase angle θ , defined by

$$\theta_x = k, \quad \theta_t = -\omega \quad (2.4)$$

The wave number k , however, depends on the slow variable X . We assume ω to be real, but k complex with a small imaginary part; namely we have a spatial stability problem with a weak amplification rate.

Then, the streamfunction assumes the following form:

$$\psi = \psi^{(0)}(X, y) + \sum_{n=1}^{\infty} \left\{ \psi^{(n)}(X, y) e^{in\theta} + \tilde{\psi}^{(n)}(X, y) e^{-in\tilde{\theta}} \right\} \quad (2.5)$$

where $(\tilde{})$ denotes the complex conjugate. Substituting the above expression into equation (2.1) and collecting the terms with equal values of n , we obtain the following infinite set of equations:

MEAN FLOW:

$$\begin{aligned} & \frac{1}{\mu R} \Delta_0^2 \psi^{(0)} - \psi_y^{(0)} \Delta_0 \psi_X^{(0)} + \psi_X^{(0)} \Delta_0 \psi_y^{(0)} \\ &= \frac{1}{\mu} \sum_{n=1}^{\infty} e^{-2n\theta} i \left\{ \psi_y^{(n)} \cdot (-in\tilde{k} + \mu \partial_X) \tilde{\Delta}_n \tilde{\psi}^{(n)} + \tilde{\psi}_y^{(n)} (ink + \mu \partial_X) \Delta_n \psi^{(n)} \right. \\ & \quad \left. - (ink + \mu \partial_X) \psi^{(n)} \cdot \tilde{\Delta}_n \tilde{\psi}_y^{(n)} - (-in\tilde{k} + \mu \partial_X) \tilde{\psi}^{(n)} \cdot \Delta_n \psi_y^{(n)} \right\} \quad (2.6) \end{aligned}$$

FUNDAMENTAL:

$$\begin{aligned}
 & \frac{1}{R} \Delta_1^2 \psi^{(1)} + i\omega \Delta_1 \psi^{(1)} - \psi_y^{(0)} (ik + \mu \partial_X) \Delta_1 \psi^{(1)} \\
 & + (\Delta_0 \psi_y^{(0)}) (ik + \mu \partial_X) \psi^{(1)} - \mu (\Delta_0 \psi_X^{(0)}) \cdot \psi_y^{(1)} + \mu \psi_X^{(0)} \Delta_1 \psi_y^{(1)} \\
 & = \sum_{m=1}^{\infty} e^{-2m\theta} i \left\{ \psi_y^{(m+1)} (-im\tilde{k} + \mu \partial_X) \tilde{\Delta}_m \tilde{\psi}^{(m)} + \tilde{\psi}_y^{(m)} [i(m+1)k + \mu \partial_X] \Delta_{m+1} \psi^{(m+1)} \right. \\
 & \left. - [i(m+1)k + \mu \partial_X] \psi^{(m+1)} \cdot \tilde{\Delta}_m \tilde{\psi}^{(m)} - (-im\tilde{k} + \mu \partial_X) \tilde{\psi}^{(m)} \cdot \Delta_{m+1} \psi^{(m+1)} \right\} \quad (2.7)
 \end{aligned}$$

HIGHER HARMONICS: $n \geq 2$

$$\begin{aligned}
 & \frac{1}{R} \Delta_n^2 \psi^{(n)} + i n \omega \Delta_n \psi^{(n)} - \psi_y^{(0)} (ink + \mu \partial_X) \Delta_n \psi^{(n)} + (\Delta_0 \psi^{(0)}) (ink + \mu \partial_X) \psi^{(n)} \\
 & - \mu (\Delta_0 \psi_X^{(0)}) \psi_y^{(n)} + \mu \psi_X^{(0)} \Delta_n \psi_y^{(n)} \\
 & = \sum_{m=1}^{\infty} \left\{ \psi_y^{(n-m)} (imk + \mu \partial_X) \Delta_m \psi^{(m)} - [i(n-m)k + \mu \partial_X] \psi^{(n-m)} \cdot \Delta_m \psi_y^{(m)} \right\} \\
 & + \sum_{m=1}^{\infty} e^{-2m\theta} i \left\{ \psi_y^{(n+m)} (-im\tilde{k} + \mu \partial_X) \tilde{\Delta}_m \tilde{\psi}^{(m)} + \tilde{\psi}_y^{(m)} [i(n+m)k + \mu \partial_X] \Delta_{n+m} \psi^{(n+m)} \right. \\
 & \left. - [i(n+m)k + \mu \partial_X] \psi^{(n+m)} \cdot \tilde{\Delta}_m \tilde{\psi}^{(m)} - (-im\tilde{k} + \mu \partial_X) \tilde{\psi}^{(m)} \cdot \Delta_{n+m} \psi^{(n+m)} \right\} \quad (2.8)
 \end{aligned}$$

In these equations, the following notations are used:

∂_x = partial differentiation with respect to the slow variable x .

∂_y = partial differentiation with respect to y .

$$\begin{aligned}\Delta_n &= \partial_y^2 + (ink + \mu \partial_x)^2 \\ &= \partial_y^2 - n^2 k^2 + \mu i n (2k \partial_x + k_x) + \mu^2 \partial_x^2\end{aligned}$$

Mean Flow Solution - Lowest Order:

To the lowest order, the mean-flow equation is

$$\frac{1}{\mu R} \partial_y^4 \bar{\psi} - \bar{\psi}_y \partial_y^2 \bar{\psi}_x + \bar{\psi}_x \partial_y^3 \bar{\psi} = 0$$

for $\psi^{(0)} = \bar{\psi} + \dots$. Integrating once with respect to y , we obtain

$$\frac{1}{\mu R} \bar{\psi}_{yyy} - \bar{\psi}_y \bar{\psi}_{xy} + \bar{\psi}_x \bar{\psi}_{yy} = P(x) \quad (2.9)$$

which we recognize as the usual boundary-layer equation, after identifying $P(x)$ to be the pressure gradient imposed by the flow external to the boundary layer.

Fundamental Component - Lowest Order

The fundamental component equation to the lowest order is the Orr-Sommerfeld equation:

$$L_\omega(\psi^{(1)}) \equiv \frac{1}{R} (\partial_y^2 - k^2)^2 \psi^{(1)} - i \left[(k \bar{\psi}_y - \omega) (\partial_y^2 - k^2) \psi^{(1)} - k \bar{\psi}_{yyy} \psi^{(1)} \right] = 0 \quad (2.10)$$

and the solution is

$$\psi^{(1)} = A(X) \phi^{(1)}(y; X, \omega) \quad (2.11)$$

where $A(X)$ ($\phi^{(1)}$) is the amplitude function, and $\phi^{(1)}$ satisfies

$L_{\omega}(\phi^{(1)}) = 0$. With the boundary conditions;

$$\phi^{(1)}(0) = \phi_y^{(1)}(0) = \phi^{(1)}(\infty) = 0.$$

Thus, we obtain the eigenvalue solution

$$\phi^{(1)} = \phi^{(1)}(y; X, \omega) \quad (2.12a)$$

$$k = k(X, \omega) \quad (2.12b)$$

Second Harmonic Component

The equation for the second-harmonic component, generated by the above fundamental component, is then

$$\begin{aligned} L_{2\omega}(\psi^{(2)}) &\equiv \frac{1}{R} (\partial_y^2 - 4k^2)^2 \psi^{(2)} - 2i \left[(k\bar{\psi}_y - \omega) (\partial_y^2 - 4k^2) \psi^{(2)} - k\bar{\psi}_{yyy} \psi^{(2)} \right] \\ &= ikA^2(X) \left[\phi_y^{(1)} \phi_{yy}^{(1)} - \phi^{(1)} \phi_{yyy}^{(1)} \right] + \dots \end{aligned}$$

Therefore, to the lowest order, the second harmonic will be of the form:

$$\psi^{(2)} = A^2(X) \phi^{(2)}(y; X, \omega) \quad (2.13a)$$

$$L_{2\omega}(\phi^{(2)}) = ik \left[\phi_y^{(1)} \phi_{yy}^{(1)} - \phi^{(1)} \phi_{yyy}^{(1)} \right] \quad (2.13b)$$

Correction to Mean Flow:

Also, from the lowest-order fundamental component, we obtain the following equation for the correction to the mean flow,

$$\begin{aligned} \frac{1}{\mu R} \psi_{yyyy}^{(0)} - \bar{\psi}_y \psi_{xyy}^{(0)} - \bar{\psi}_{xyy} \psi_y^{(0)} + \bar{\psi}_x \psi_{yyy}^{(0)} + \bar{\psi}_{yyy} \psi_x^{(0)} \\ = \frac{1}{\mu} A \tilde{A} e^{-2\theta} i \left[i(k\tilde{\phi}^{(1)} \phi_y^{(1)} - k\phi^{(1)} \tilde{\phi}_y^{(1)}) - 4k_i \phi_y^{(1)} \tilde{\phi}_y^{(1)} \right]_y + \dots \end{aligned}$$

The neglected terms are of the order of μ^2 and ϵ^4/μ .

Integrating once with respect to y , we get

$$\begin{aligned} \frac{1}{\mu R} \psi_{yyy}^{(0)} - \bar{\psi}_y \psi_{xy}^{(0)} + \bar{\psi}_x \psi_{yy}^{(0)} - \bar{\psi}_{xy} \psi_y^{(0)} + \bar{\psi}_{yy} \psi_x^{(0)} \\ = \frac{1}{\mu} A \tilde{A} e^{-2\theta} i \left[i(k\tilde{\phi}^{(1)} \phi_y^{(1)} - k\phi^{(1)} \tilde{\phi}_y^{(1)}) - 4k_i \phi_y^{(1)} \tilde{\phi}_y^{(1)} \right] + P_1(x) + \dots \end{aligned}$$

where $P_1(x)$ again is a term due to streamwise pressure gradient and will be $O(\mu^2)$ for a flat-plate boundary layer. We assume $|A|^2/\mu \gg \mu^2$, and let

$$\psi_1^{(0)} = |A|^2 e^{-2\theta} i \phi_1^{(0)}(x, y) \quad (2.15a)$$

Thus, the equation for $\phi_1^{(0)}$ becomes

$$\begin{aligned} \frac{1}{R} \phi_{yyy}^{(0)} - \bar{\psi}_y \phi_{xy}^{(0)} + \bar{\psi}_x \phi_{yy}^{(0)} - \bar{\psi}_{xy} \phi_y^{(0)} + \bar{\psi}_y \phi_x^{(0)} + 2k_i (\bar{\psi}_y \phi_y^{(0)} - \bar{\psi}_{yy} \phi_x^{(0)}) \\ = i(k\tilde{\phi}^{(1)} \phi_y^{(1)} - k\phi^{(1)} \tilde{\phi}_y^{(1)}) - 4k_i \phi_y^{(1)} \tilde{\phi}_y^{(1)} \end{aligned} \quad (2.15b)$$

In this equation, we have neglected $d/dx \ln|A|$ compared with $k_i = \theta_{ix}$ since the former is of the order of μ . Also, we have gone back to the original x by multiplying through by μ .

Corrections to Fundamental Component:

The modification to the fundamental component consists of two parts: one part due to the variation of $\bar{\psi}, \phi^{(1)}$ and k with X , and the other part due to nonlinear feedback. By putting

$$\frac{dA}{dX} = k_1 A + \frac{1}{\mu} \lambda A^2 \bar{A} \quad (2.16)$$

and

$$\psi^{(1)} = A\phi^{(1)} + \mu A\phi_1^{(1)} + |A|^2 A\phi_2^{(1)} \quad (2.17)$$

we obtain the following equations for the corrections $\phi_1^{(1)}$ and $\phi_2^{(1)}$

$$L_\omega(\phi_1^{(1)}) = k_1 M(\phi^{(1)}) + Q(\bar{\psi}, \phi^{(1)}, k) \quad (2.18)$$

$$L_\omega(\phi_2^{(1)}) = \lambda M(\phi^{(1)}) + iN(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}) \quad (2.19)$$

where,

$$M(\phi^{(1)}) = 2\omega k \phi^{(1)} + \bar{\psi}_y (\phi_{yy}^{(1)} - 3k^2 \phi^{(1)}) - \frac{4i}{R} k (\phi_{yy}^{(1)} - k^2 \phi^{(1)}) \quad (2.20a)$$

$$\begin{aligned} Q(\bar{\psi}, \phi^{(1)}, k) = & \bar{\psi}_{yyx} \phi^{(1)} - \bar{\psi}_x (\phi_{yyy}^{(1)} - k^2 \phi_y^{(1)}) - \bar{\psi}_{yyx} \phi_y^{(1)} \\ & + 2\omega k \phi_x^{(1)} + \bar{\psi}_y (\phi_{yyx}^{(1)} - 3k^2 \phi_x^{(1)}) - \frac{4i}{R} k (\phi_{yyx}^{(1)} - k^2 \phi_x^{(1)}) \\ & + \left[\omega \phi^{(1)} - 3k \bar{\psi}_y \phi^{(1)} - \frac{2i}{R} (\phi_{yy}^{(1)} - 3k^2 \phi^{(1)}) \right] k_x \end{aligned} \quad (2.20b)$$

and

$$\begin{aligned}
 N(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}) = & e^{-2\theta} \left\{ 2k\tilde{\phi}_y^{(1)} (\phi_{yy}^{(2)} - 4k^2\phi^{(2)}) - \tilde{k}\phi_y^{(2)} (\tilde{\phi}_{yy}^{(1)} - \tilde{k}^2\phi^{(1)}) \right. \\
 & + \tilde{k}\phi^{(1)} (\phi_{yyy}^{(2)} - 4k^2\phi_y^{(2)}) - 2k\phi^{(2)} (\tilde{\phi}_{yyy}^{(1)} - \tilde{k}^2\tilde{\phi}_y^{(1)}) \\
 & \left. + k\phi_y^{(0)} (\phi_{yy}^{(1)} - k^2\phi^{(1)}) - k\phi_{yyy}^{(0)} \phi^{(1)} \right\} \quad (2.20c)
 \end{aligned}$$

The boundary conditions for $\phi_n^{(1)}$ ($n = 1, 2$) are

$$\phi_n^{(1)}(0) = \phi_{ny}^{(1)}(0) = \phi_n^{(1)}(\infty) = 0$$

The equation $L_\omega(\phi_n^{(1)}) = 0$ with above boundary conditions has nontrivial solutions (eigensolutions). Therefore, if the nonhomogeneous equations (2.18) and (2.19) are to have solutions, the terms on the right hand side must be orthogonal to the eigensolutions of the homogeneous equation (hereinafter this is referred to as the solvability condition), which yields:

$$k_1 = - \frac{\int_0^\infty \phi^{(1)*} Q \, dy}{\int_0^\infty \phi^{(1)*} M \, dy} \quad (2.21)$$

$$\lambda = -1 \frac{\int_0^\infty \phi^{(1)*} N \, dy}{\int_0^\infty \phi^{(1)*} M \, dy} \quad (2.22)$$

where $\phi^{(1)*}$ is the solution of the adjoint problem

$$\frac{1}{R} (\partial_y^2 - k^2)^2 \phi^{(1)*} - i \left[(k\bar{\psi}_y - \omega) (\partial_y^2 - k^2)^2 \phi^{(1)*} + 2\bar{\psi}_{yy} \phi_y^{(1)*} \right] = 0 \quad (2.23)$$

with

$$\phi^{(1)*}(0) = \phi_y^{(1)*}(0) = \phi^{(1)*}(\infty) = 0.$$

SUMMARY:

We summarize the mathematical problems that need to be solved:

(1) Fundamental-Mode Eigenvalue and Eigensolution

$$L_{\omega}(\phi^{(1)}, k_0) \equiv \frac{1}{R} \frac{\partial^2}{\partial y^2} - k_0^2 \phi^{(1)} \\ - i \left[(k_0 \bar{\psi}_y - \omega) \phi_{yy}^{(1)} - k_0^2 \phi^{(1)} - k_0 \bar{\psi}_{yyy} \phi^{(1)} \right] \\ = 0$$

$$\text{with } \phi^{(1)}(0) = \phi_y^{(1)}(0) = 0 \quad ; \quad \phi^{(1)}(y) \rightarrow 0 \text{ as } y \rightarrow \infty$$

Here L_{ω} is the Orr-Sommerfeld linear differential operator.

(2) Adjoint Orr-Sommerfeld Problem

$$L_{\omega}^*(\phi^{(1)*}, k_0) \equiv \frac{1}{R} \left(\frac{\partial^2}{\partial y^2} - k_0^2 \right)^2 \phi^{(1)*} - i \left[(k_0 \bar{\psi}_y - \omega) \left(\phi_{yy}^{(1)*} - k_0^2 \phi^{(1)*} \right) \right. \\ \left. + 2k_0 \frac{\partial^2 \bar{\psi}}{\partial y^2} \frac{\partial \phi^{(1)*}}{\partial y} \right] = 0$$

subject to the boundary conditions, $\phi^{(1)*}(0) = \phi_y^{(1)*}(0) = 0$,
 $\phi^{(1)*}(y) \rightarrow 0$ as $y \rightarrow \infty$ and with k_0 obtained from (1).

(3) Correction to Fundamental Mode and Determination of Amplification Factor k_1 Due to Non-Parallel Effects

$$L_{\omega}(\phi_1^{(1)}, k_0) = k_1 M(\phi^{(1)}) + Q(\phi^{(1)})$$

with boundary conditions identical to those for $\phi^{(1)}$.
 k_1 is determined from the solvability condition:

$$k_1 \equiv k_1(\phi^{(1)}, \phi^{(1)*}, k_0) = - \frac{\int_0^{\infty} Q \phi^{(1)*} dy}{\int_0^{\infty} M \phi^{(1)*} dy}$$

where

$$M(\phi^{(1)}) = i \left[2\omega k_0 \phi^{(1)} + \bar{\psi}_y (\phi_{yy}^{(1)} - 3k_0^2 \phi^{(1)}) - \bar{\psi}_{yyy} \phi^{(1)} \right] \\ + \frac{4}{R} k_0 (\phi_{yy}^{(1)} - k_0^2 \phi^{(1)})$$

and

$$Q(\phi^{(1)}) = i \left\{ 2\omega k_0 \phi_X^{(1)} + \omega k_{0X} \phi^{(1)} - \bar{\psi}_{yyy} \phi_X^{(1)} \right\} \\ + i \bar{\psi}_y \left\{ \phi_{yyX}^{(1)} - 3k_0^2 \phi_X^{(1)} - 3k_0 k_{0X} \phi^{(1)} \right\} \\ - \frac{1}{R} \left\{ 4k_0 \phi_{yyX}^{(1)} + 2k_{0X} \phi_{yy}^{(1)} - 4k_0^3 \phi_X^{(1)} - 6k_0^2 k_{0X} \phi^{(1)} \right\} \\ + i \bar{\psi}_{yyX} \phi_y^{(1)} - i \bar{\psi}_X (\phi_{yyy}^{(1)} - k_0^2 \phi_y^{(1)})$$

(4) Equations for $\phi_x^{(1)}$ and k_{0x}

$\phi_x^{(1)}$ and k_{0x} appearing in Q are obtained by solving the equation

$$L_\omega(\phi_x^{(1)}, k_0) = \{M k_{0x} + P\}$$

subject to the boundary conditions identical to those for $\psi^{(1)}$.

Of the terms on the right hand side, P is defined by

$$P = ik_0 \left[\bar{\psi}_{xy} (\phi_{yy}^{(1)} - k_0^2 \phi^{(1)}) - \bar{\psi}_{xyyy} \phi^{(1)} \right]$$

M is as defined in (3).

(5) Second Harmonic Mode

$$L_{2\omega}(\phi^{(2)}, 2k_0) = F_2(\phi^{(1)}, k_0)$$

where $L_{2\omega}$ is the Orr-Sommerfeld operator with k_0 obtained from (1)

$$\text{and } F_2(\phi^{(1)}, k_0) = -ik_0 \left[\phi^{(1)} \phi_{yyy}^{(1)} - \phi_y^{(1)} \phi_{yy}^{(1)} \right].$$

with

$$\phi^{(2)}(0) = \phi_y^{(2)}(0) = 0, \quad \phi^{(2)}(y) \rightarrow 0 \text{ as } y \rightarrow \infty$$

(6) Correction to Mean Flow Due to Fundamental Mode and Non-Parallel Effects

$$G(\phi^{(0)}) = F_1(\phi^{(1)}, k_0)$$

where

$$G(\phi^{(0)}) \equiv \bar{\psi}_{yx} \phi_y^{(0)} - \bar{\psi}_{yy} \phi_x^{(0)} + \bar{\psi}_y \phi_{yx}^{(0)} - \bar{\psi}_x \phi_{yy}^{(0)}$$

$$- \frac{1}{R} \phi_{yyy}^{(0)} + 2k_{0i} \left(\bar{\psi}_{yy} \phi^{(0)} - \bar{\psi}_y \phi_y^{(0)} \right)$$

$$F_1(\phi^{(1)}, k_0) = i \left[k_0 \phi^{(1)} \tilde{\phi}_y^{(1)} - \tilde{k}_0 \tilde{\phi}^{(1)} \phi_y^{(1)} \right]_y$$

$$+ 4k_{0i} |\phi_y^{(1)}|^2$$

subject to the boundary conditions,

$$\left. \begin{aligned} \phi^{(0)} = \phi_y^{(0)} = 0 \quad \text{at } y = 0 \\ \phi_y^{(0)} \rightarrow 0 \quad (\text{as } y \rightarrow \infty) \end{aligned} \right\} \quad \text{for all } x = x_1$$

For the initial condition we arbitrarily choose;

$$\phi^{(0)} = 0 \quad \text{at } x = x_1, \quad \text{for all } y.$$

(7) Correction of Fundamental Mode and Determination of Landau Constant λ , Due to Nonlinear Effects

$$L_{\omega}(\phi_2^{(1)}, k_0) = \lambda M(\phi^{(1)}) + i N(\phi^{(0)}, \phi^{(1)}, \phi^{(2)})$$

where M and N are defined below. The boundary conditions are identical to those for $\phi^{(1)}$.

The Landau Constant λ is determined from the solvability condition that the right-hand-side of the above equation has to be orthogonal to $\phi^{(1)}$. Thus, we have

$$\lambda \equiv \lambda(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \phi^{(1)*}, k_0) = - \frac{\int_0^{\infty} N \phi^{(1)*} dy}{\int_0^{\infty} M \phi^{(1)*} dy}$$

where

$$\begin{aligned} N(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}) = & i e^{-2\theta_1} \left\{ - \tilde{k}_0 \phi_y^{(2)} (\tilde{\phi}_{yy}^{(1)} - \tilde{k}_0^2 \phi^{(1)}) \right. \\ & + 2k_0 \tilde{\phi}_y^{(1)} (\phi_{yy}^{(2)} - 4k_0^2 \phi^{(2)}) - 2k_0 \phi^{(2)} (\tilde{\phi}_{yyy}^{(1)} - \tilde{k}_0^2 \tilde{\phi}_y^{(1)}) \\ & + \tilde{k}_0 \tilde{\phi}^{(1)} (\phi_{yyy}^{(2)} - 4k_0^2 \phi_y^{(2)}) + k_0 \phi_y^{(0)} (\phi_{yy}^{(1)} - k_0^2 \phi^{(1)}) \\ & \left. - k_0 \phi_{yyy}^{(0)} \phi^{(1)} \right\} \end{aligned}$$

and M is as defined in Problem (3) and k_{0x} is determined by the solvability condition to be

$$k_{0x} = \frac{- \int_0^{\infty} P \phi^{(1)*} dy}{\int_0^{\infty} M \phi^{(1)*} dy}.$$

(8) The Amplitude Equation

$$A_x = k_{0i} A + k_{1r} A + \lambda A^2 \tilde{A}$$

subject to the appropriate initial condition for A. The quantities k_0 , k_1 and λ are determined in Problems (1), (3) and (7), respectively. By expressing A in the polar form

$$A = |A| e^{i\phi}$$

and substituting into equation (2.16), we obtain the following equations for the amplitude |A| and phase ϕ :

$$|A|_x = (-k_{0i} + k_{1r}) |A| + \lambda_r |A|^3$$

$$\phi_x = k_{1i} + \lambda_i |A|^2$$

Here the subscript "r" and "i" refers to the real and imaginary components of the complex quantities k and λ .

We shall briefly discuss the types of mathematical problems with which we are dealing. In the fundamental mode equation, there are three parameters appearing in the equations: R, ω , k_0 . Since we are interested in spatially amplified solutions of the Orr-Sommerfeld equation, two of the parameters, R and ω , are given and k_0 is to be determined. In this form, the fundamental mode equation is a nonlinear problem, with k_0 being the eigenvalue which is treated as an additional dependent variable. The type of system is a two-point boundary value problem (2PBVP), since the boundary conditions on $\phi^{(1)}$ (and its derivatives) are imposed on both ends of the computational interval.

The adjoint problem (2) has the same form as the fundamental mode problem, except that the eigenvalue k_0 is known in advance from the solution of the fundamental mode problem, since the operators L_ω and L_ω^* can be shown to have the same eigenvalue. On the other hand, the second harmonic problem (5) is an inhomogeneous 2PBVP, not an eigenvalue problem. Finally, the correction to the mean flow (3), in contrast to the other three above-mentioned problems, is a linear partial differential equation (PDE), similar in form to the boundary layer equation. Thus, in addition to the boundary conditions, initial conditions have to be specified for $\phi^{(0)}$.

The remaining two problems of determining λ and k_1 are pure quadratures. Since the integrands are going to be approximated on some computational grid, the integrals can be approximated by applying some simple quadrature rules, like the trapezoidal rule, to the integrands and integrating the exponential tails analytically from the end of the computational interval to infinity.

The functions $\phi_1^{(1)}$ and $\phi_2^{(1)}$ can be computed as follows. Note first that the differential operator $L_\omega(k_0)$ appearing in problems (3) and (7) is singular and the solvability conditions simply force the right-hand sides to be orthogonal to the eigensolution of $L_\omega(k_0)$ so that nontrivial solutions exist. However, $\phi_1^{(1)}$ and $\phi_2^{(1)}$ are only determined up to the addition of an arbitrary multiple of $\phi^{(1)}$, since $\phi^{(1)}$ is the eigensolution of $L_\omega(k_0)$. Hence, we need to specify normalization conditions for them in order to remove this degree of freedom. To be consistent with the assumed form of the expansion for the total stream function ψ , such a condition could be that $\phi_1^{(1)}$ and $\phi_2^{(1)}$ should be orthogonal to the eigensolution of $L_\omega(k_0)$:

$$\int_0^\infty \phi^{(1)*} B \phi_1^{(1)} dy = 0, \quad \int_0^\infty \phi^{(1)*} B \phi_2^{(1)} dy = 0$$

where B is the operator defined by (see Appendix for the orthogonality condition)

$$B = 4k_0(D^2 - k_0^2) + iR[U(D^2 - 3k_0^2) + 2uk_0 - U'''] , \quad D \equiv \frac{d}{dy}$$

Therefore, the desired solution can be recovered by

$$\phi_1^{(1)} = \phi_1^{(1)}(\text{computed}) - C_1^{(1)} \phi_1^{(1)}$$

$$\phi_2^{(1)} = \phi_2^{(1)}(\text{computed}) - C_2^{(1)} \phi_1^{(1)}$$

where

$$C_1^{(1)} = \frac{\int_0^\infty \phi_1^{(1)*} B \phi_1^{(1)}(\text{computed}) dy}{\int_0^\infty \phi_1^{(1)*} B \phi_1^{(1)} dy}$$

and $C_2^{(1)}$ is obtained by placing $\phi_2^{(1)}(\text{computed})$ instead of $\phi_1^{(1)}(\text{computed})$.

From the above brief description, we see that at least three of the first four problems are 2PBVP. It turns out that if we attempt to solve the distortion of the mean flow equation by some marching technique (e.g., the methods of lines, to be discussed later), a sequence of 2PBVP has to be solved. Since these four problems constitute the major portion of the overall computational problem, it becomes immediately obvious that we need an accurate and efficient 2PBVP solver that is able to handle the different types of 2PBVP that we have discussed above. In the next section, we shall discuss in some detail the choice of such a solver.

3. NUMERICAL METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEM

A classical technique for solving the Orr-Sommerfeld equation is to use a shooting scheme together with the Runge-Kutta method on a uniform mesh (e.g., Reynolds and Potter (1967), Ling and Reynolds (1973)). The presence of linearly independent solutions with vastly different growth rates, is usually handled by some sort of filtering to maintain the independence of the computed solution (e.g., Kaplan (1964), Scott and Watts (1975)). There are also other methods that do not take the initial-value problem approach but rather solve the boundary value problem directly using finite difference methods (e.g., Thomas (1953), Osborne (1967), Orszag (1971)). Gersting and Jankowski (1972) discussed a comparison of the performance of these methods on the Orr-Sommerfeld problem.

The development of modern numerical methods for general nonlinear 2PBVP has greatly advanced in the past decade (see, for example, the surveys by Keller (1975) and Keller (1976)). We now have available powerful packages of computer codes designed for a large class of 2PBVP, of which the Orr-Sommerfeld and related problems are particular cases. Many of these packages incorporate adaptive mesh selection techniques and can handle rather sharp boundary layers efficiently. A very high order of accuracy can be achieved through the use of extrapolation techniques. Reliable error estimates are sometimes available and nonlinearity is routinely handled. We choose to use one of these packages for the 2PBVP with which we are concerned.

The solver we use is called PASVA3, the original version of which is described in Lentini and Pereyra (1977). We shall briefly describe its underlying methods and capabilities here.

PASVA3 is based on the finite difference method and is applicable to first order systems of the form

$$\frac{dy}{dt} = f(t,y) \quad , \quad t \in (a,b) \quad ,$$

subject to the boundary conditions of the form:

$$g(y(a),y(b)) = 0 \quad .$$

where f and g can be quite general nonlinear functions, but sufficiently smooth so that the above problem has an isolated solution. The method of solution utilizes a global finite difference approximation rather than a shooting method. Through the use of the trapezoidal rule approximation, the differential equation is approximated by a system of (generally nonlinear) algebraic equations to be solved for the unknown function values on the grid. Newton's method and variants of it are used to find the solution to this algebraic system. The Jacobian matrix that results has a special block structure and an efficient elimination scheme is devised to exploit this structure. The Jacobian matrix is also used to obtain a higher order accuracy (higher than the second order accuracy of the basic trapezoidal rule approximation) through the use of a procedure called deferred correction. Since the factorization of the Jacobian matrix has to be computed only infrequently, high order accuracies can be achieved with very minimal costs. Even more important than that is the availability of accurate error estimates for the computed solution through the use of the same Jacobian matrix. Accurate error estimates are extremely valuable because they allow us to assign a confidence level to the computed results. Such error estimates are noticeably absent in previously published computations related to the Orr-Sommerfeld equation. The last significant feature of the code that we shall mention is the ability to automatically and adaptively select a good mesh for the problem. This feature allows the code to resolve mild boundary layers (region of relatively sharp changes in the solution)

efficiently and at the same time reduce the computer storage required for a given accuracy. A crude adaptive mesh selection procedure was first used by Hughes (1972) on the Orr-Sommerfeld problem.

Computational Domain

We used a general rectangular finite difference grid (Figure 3-1) on which the problems listed in Section 1 are to be solved. This grid is non-uniform in general.

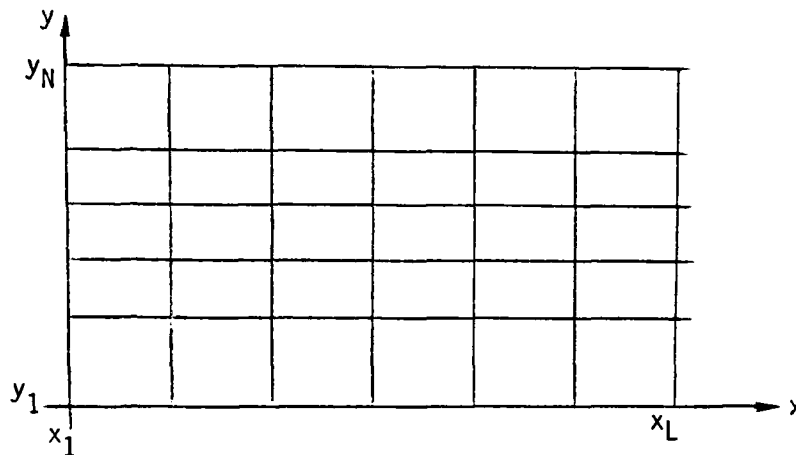


Figure 3-1. The Computational Grid

On this grid, the dimensionless mean flow stream function $\bar{\psi}$ (and its derivatives) are to be given, depending on the value of the Reynolds number and the kind of problem itself. Also, appropriate initial conditions are to be given at $x = x_1$ and appropriate boundary conditions to be given on $y = y_1$ and $y = y_N$. This formulation of the computational domain is quite general, and can handle boundary layer flows, Poiseuille flows and Couette flows. For example, for plane Poiseuille flow with walls at $y_1 = -1$, $y_N = 1$, $\bar{\psi}(y) = y - \frac{y^3}{3}$ is known exactly. For Blasius flow with Reynolds number R , the dimensionless mean-flow stream function $\bar{\psi}$ at a point (x,y) is given by

$$\left. \begin{aligned} \bar{\psi}(x,y) &= \sqrt{\frac{x}{R}} f(\eta) \\ \eta &= y\sqrt{\frac{R}{x}} \end{aligned} \right\} \quad (3.1)$$

and f satisfies the following 2PBVP:

$$\left. \begin{aligned} 2 f''' + f f'' &= 0 \\ f(0) = f'(0) &= 0, \quad f'(\infty) = 1. \end{aligned} \right\} \quad (3.2)$$

The function $f(\eta)$ is calculated using PASVA3. The mean-flow $\bar{\psi}$ on the grid can then be calculated from (3.1) by interpolation from a table of f vs. η . The value of y_N is chosen so that the corresponding values of η at all x -stations are large enough to contain most of the boundary layer. Usually, a value of η about 10 is sufficient.

The placement of the y mesh points is determined by the adaptive mesh selection procedure in PASVA3 when we use it to solve the fundamental-mode problem at $x = x_1$ to a given accuracy. If the mean flow does not vary with x too strongly, this mesh should be good enough for all subsequent x -station fundamental modes. This mesh should also be fairly good for the second harmonic and the distortion of mean flow equations because the fundamental mode constitutes the forcing functions for those equations. Our experience seems to indicate that a mesh selected this way is more than adequate for all the three other problems (including the adjoint problem). Notice that, in principle, it is not necessary to use the same mesh in y for all x -stations; its choice is for convenience, especially in connection with the marching scheme for solving the distortion of mean flow problem (6).

The placement of the x-mesh is to be specified by the user. It can be non-uniform, but for solving the problem (6), the x-mesh should be fine enough to resolve the variations in the x-direction, of $\phi^{(0)}$, k_0 , k_1 and λ but not too fine in order to save computational effort.

The marching scheme for the distortion of the mean flow equation will be described in Section 3.2 and the correct specification of the boundary conditions will be discussed in Section 3.3. In the next section, we shall describe how we handle the eigenvalue problem (1).

3.1 Treatment of the Eigenvalue Problem

As we have mentioned, problem (1) is an eigenvalue problem; both $\phi^{(1)}$ and k_0 are to be determined. The approach taken by many people is to drop one boundary condition at one end of the computational interval and impose a normalization boundary condition at the other end, thus making the new problem inhomogeneous, allowing a solution to be easily computed for a given estimate of k_0 . An iterative scheme is then used to find the correct value of k_0 so that the dropped boundary condition is satisfied. The adjusting of k_0 can be done by Newton's method, for example. We have chosen another approach which, as discussed in Keller (1976), is to view k_0 as an extra unknown (in addition to ϕ) and introduce an extra equation

$$\frac{d k_0}{d y} = 0 \quad (3.3)$$

since k_0 is constant independent of y and an extra normalization boundary condition for $\phi^{(1)}$ such as

$$\phi_{yy}^{(1)}(y_1) = 1. \quad (3.4)$$

The "inflated" system becomes inhomogeneous because of (3.4) and is of the form suitable for PASVA3. In effect, both $\phi^{(1)}$ and k_0 are obtained through the use of Newton's method that is built-in in PASVA3. This greatly simplifies the computer programming in the actual implementation.

REDUCTION TO FIRST ORDER SYSTEMS

PASVA3 accepts only real first order systems of 2PBVP. However, the problems listed in Section 1 are all of higher orders and are complex in general. Therefore, these equations have to be reduced to equivalent real first order systems before PASVA3 can be applied. Except for the distortion of mean flow problem (6), this can be done rather straightforwardly. We shall present these reductions in this section.

Fundamental Mode Equation

If we define $\underline{u} = (\phi^{(1)}, \phi_y^{(1)}, \phi_{yy}^{(1)}, \phi_{yyy}^{(1)}, k_0)^T$, then the Orr Sommerfeld equation, together with (3.3), reduce to the following system of five first-order equations:

$$\frac{d\underline{u}}{dy} = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \alpha u_1 + \beta u_3 \\ 0 \end{bmatrix} \equiv \underline{f}(\underline{u}) \quad (3.5)$$

where

$$\begin{aligned} u_1 &= \phi^{(1)} \\ u_2 &= \phi_y^{(1)}, \dots \text{etc.} \end{aligned}$$

further,

$$\alpha = -iR(u_5 \bar{\psi}_y - \omega) u_5^2 - iR u_5 \bar{\psi}_{yyy} - u_5^4$$

and

$$\beta = iR(u_5 \bar{\psi}_y - \omega) + 2 u_5^2.$$

To reduce (3.5) to a real system, we introduce the real and imaginary parts of u as

$$u = \chi + i z, \quad (3.6)$$

where χ and z are real.

Then, (3.5) is equivalent to

$$\frac{d}{dy} \begin{bmatrix} \chi \\ z \end{bmatrix} = \begin{bmatrix} \text{Re } f(\chi + i z) \\ \text{Im } f(\chi + i z) \end{bmatrix} \equiv F(\chi, z) \quad (3.7)$$

where Re and Im denote the real and imaginary parts. This is a real first order system of order 10 to which PASVA3 can be directly applied.

The system (3.7) is nonlinear and PASVA3 requires the computation of the Jacobian of the right hand side F in (3.7). From (3.7) it follows that, in block form,

$$J = \frac{\partial \underline{f}}{\partial (\underline{y}, \underline{z})} = \begin{bmatrix} \frac{\partial \text{Re } \underline{f}}{\partial \underline{y}} & \frac{\partial \text{Re } \underline{f}}{\partial \underline{z}} \\ \frac{\partial \text{Im } \underline{f}}{\partial \underline{y}} & \frac{\partial \text{Im } \underline{f}}{\partial \underline{z}} \end{bmatrix}$$

Since $\underline{f}(\underline{u})$ is an analytic function in \underline{u} , each submatrix can be computed easily from

$$\frac{\partial \underline{f}}{\partial \underline{y}} = \frac{\partial \underline{f}}{\partial \underline{u}} \quad , \quad \frac{\partial \underline{f}}{\partial \underline{z}} = i \frac{\partial \underline{f}}{\partial \underline{u}}$$

Now, $\frac{\partial \underline{f}}{\partial \underline{u}}$ is easily computed from (3.5) as

$$\frac{\partial \underline{f}}{\partial \underline{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\gamma = \left\{ -i R \left[\bar{\psi}_y u_5^2 + 2 u_5 (u_5 \bar{\psi}_y - \omega) \right] - R \bar{\psi}_{yyy} - 4 u_5^3 \right\} u_1 \\ + \left[i R \bar{\psi}_y + 4 u_5 \right] u_3$$

Hence, the Jacobian of \underline{f} is given by

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_r & 0 & \beta_r & 0 & \gamma_r & -\alpha_i & 0 & -\beta_i & 0 & -\gamma_i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha_i & 0 & \beta_i & 0 & \gamma_i & \alpha_r & 0 & \beta_r & 0 & \gamma_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.8)$$

where the subscripts r and i on α , β , and γ denote the real and imaginary parts, respectively. In the actual implementation, the vector \underline{E} and the matrix J can be easily calculated as a function of y and z (i.e., y), using complex arithmetic if that is available.

Second Harmonic Equation

This equation is linear and its reduction to a first order system can be carried out in an analogous, but much simpler, way than for the fundamental mode equation. With $\psi^{(1)}$ and k_0 computed from the fundamental mode problem, the forcing function F_2 on the right hand side of equation can be easily calculated. Now if we define the vector

$$\underline{u} = \left(\phi^{(2)}, \phi_y^{(2)}, \phi_{yy}^{(2)}, \phi_{yyy}^{(2)} \right)^T,$$

where the notation $()^T$ denotes the transpose.

Then it is easily seen that the fourth-order equation for $\phi^{(2)}$ is equivalent to

$$\frac{du}{dy} = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \alpha u_1 + \beta u_3 + F_2 \end{bmatrix} \quad (3.9)$$

where

$$\alpha = -8i R (k_0 \bar{\psi}_y - \omega) k_0^2 - 2i R k_0 \bar{\psi}_{yyy} - 16 k_0^4 \quad (3.10)$$

and

$$\beta = 2i R (k_0 \bar{\psi}_y - \omega) + 8 k_0^2 \quad (3.11)$$

To further reduce (3.9) to a real system, we again define the real and imaginary parts of u as $u = y + i z$ and rewrite (3.9) as

$$\frac{d}{dy} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ \text{Re}(\alpha u_1 + \beta u_3 + F_2) \\ z_2 \\ z_3 \\ z_4 \\ \text{Im}(\beta u_1 + \beta u_3 + F_2) \end{bmatrix}$$

The Jacobian matrix is easily computed from (3.9) as

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_r & 0 & \beta_r & 0 & -\alpha_i & 0 & -\beta_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_i & 0 & \beta_i & 0 & \alpha_r & 0 & \beta_r & 0 \end{bmatrix} \quad (3.12)$$

where the subscripts r and i denote the real and imaginary parts, respectively, of α and β as defined in equation (3.10) and (3.11).

Adjoint Equation

Analogous to the previous two equations, the reduction of the adjoint equation can be carried out as follows. Defining

$$u = \left(\phi^{(1)*}, \phi_y^{(1)*}, \phi_{yy}^{(1)*}, \phi_{yyy}^{(1)*} \right)^T$$

we get

$$\frac{du}{dy} = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \alpha u_1 + \beta u_2 + \gamma u_3 \end{bmatrix}$$

where $\alpha = -i R(k_0 \bar{\psi}_y - \omega) k_0^2 - k_0^4$

and $\beta = 2 i k_0 R \bar{\psi}_{yy}$

$\gamma = i R(k_0 \tilde{\psi}_y - \omega) + 2 k_0^2$

(3.13)

Defining $u = v + i z$ as before, the real system is

$$\frac{d}{dy} \begin{bmatrix} v \\ z \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ \text{Re}(\alpha u_1 + \beta u_2 + \gamma u_3) \\ z_2 \\ z_3 \\ z_4 \\ \text{Im}(\alpha u_1 + \beta u_2 + \gamma u_3) \end{bmatrix} \quad (3.14)$$

and the Jacobian matrix is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_r & \beta_r & \gamma_r & 0 & -\alpha_i & -\beta_i & -\gamma_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha_i & \beta_i & \gamma_i & 0 & \alpha_r & \beta_r & \gamma_r & 0 \end{bmatrix} \quad (3.15)$$

using the same notation as before.

Notice that k_0 is a known quantity (obtained from the fundamental mode computation) in the above formulation, and thus the problem is linear. To make the problem inhomogeneous, a boundary condition at one end of

the computational interval is dropped, and an extra inhomogeneous normalization boundary condition is imposed at the other end of the interval. Since the fundamental and the adjoint problems have the same eigenvalue, the dropped boundary condition should be automatically satisfied. This condition should be checked after the computation. More details on boundary conditions will be discussed in Section 3.3.

3.2 Distortion of Mean Flow Equation

This problem is an initial and boundary value problem (IBVP) for a partial (rather than ordinary) differential equation. Since the equation is of parabolic type and the variation in the x-direction is not expected to be large, implicit schemes (e.g., Crank-Nicolson) are more efficient than explicit methods. Implicit methods, however, necessitate the solution of a linear algebraic system of equations at each x-station. PASVA3 can be used to solve this linear system, thus utilizing the service of its efficient elimination scheme. Even more important than this is the achievement of high order accuracies through the deferred correction procedure built-in in PASVA3. We shall next show how this method can be applied to the distortion of mean-flow equation and how PASVA3 can be used in conjunction with the method.

First notice that the right hand side F_1 in the problem (6) is real by definition and so is $\phi^{(0)}$. We first approximate the x-derivatives of $\phi^{(0)}$ by two-point centered-difference approximations, centered at $x_{n+1/2} = (x_n + x_{n+1})/2$ and also approximate the coefficients at $x_{n+1/2}$ by averaging. Specifically, we approximate the partial differential equation for $\phi^{(0)}$ by

$$\begin{aligned}
 & \left(k_{oi}^{n+1} \bar{\psi}_{yy}^{n+1} \phi^{(0)n+1} + k_{oi}^n \bar{\psi}_{yy}^n \phi^{(0)n} \right) \\
 & - \left(k_{oi}^{n+1} \bar{\psi}_y^{n+1} \phi_y^{(0)n+1} + k_{oi}^n \bar{\psi}_y^n \phi_y^{(0)n} \right) \\
 & + \frac{1}{2} \left(\bar{\psi}_{yx}^{n+1} \phi_y^{(0)n+1} + \bar{\psi}_{yx}^n \phi_y^{(0)n} \right) - \frac{1}{2\Delta x_n} \left(\bar{\psi}_{yy}^{n+1} + \bar{\psi}_{yy}^n \right) \left(\phi^{(0)n+1} - \phi^{(0)n} \right) \\
 & + \frac{1}{2\Delta x_n} \left(\bar{\psi}_y^{n+1} + \bar{\psi}_y^n \right) \left(\phi_y^{(0)n+1} - \phi_y^{(0)n} \right) - \frac{1}{2} \left(\bar{\psi}_x^{n+1} \phi_{yy}^{(0)n+1} + \bar{\psi}_x^n \phi_{yy}^{(0)n} \right) \\
 & - \frac{1}{2R} \left(\bar{\psi}_{yyy}^{(0)n+1} + \phi_{yyy}^{(0)n} \right) = \frac{1}{2} \left(F_1^{n+1} + F_1^n \right)
 \end{aligned}$$

where the superscripts n and $n+1$ correspond to the x -stations. Now multiplying by $2R$ and collecting terms with superscript $(n+1)$ on one side, we get

$$\phi_{yyy}^{(0)n+1} + a_3^{n+1} \phi_{yy}^{(0)n+1} + (a_2^{n+1} + a_1^{n+1}) \phi_y^{(0)n+1} + (a_4^{n+1} + a_0^{n+1}) \phi^{(0)n+1} = H^{n+1} \quad (3.16)$$

where

$$\begin{aligned}
 H^{n+1} = & - \phi_{yyy}^{(0)n} - a_3^n \phi_{yy}^{(0)n} - (-a_2^{n+1} + a_1^n) \phi_y^{(0)n} + (a_4^n + a_0^{n+1}) \phi^{(0)n} \\
 & - R [F_1^{n+1} + F_1^n] \quad (3.17)
 \end{aligned}$$

where the coefficients in (3.16) and (3.17) are given by

$$\left. \begin{aligned} a_0^{n+1} &= R \left(\bar{\psi}_{yy}^{n+1} + \bar{\psi}_{yy}^n \right) / \Delta x_n , \\ a_1^{n+1} &= - R \bar{\psi}_{yx}^{n+1} + 2 R k_{oi}^{n+1} \bar{\psi}_y^{n+1} , \\ a_2^{n+1} &= - R \left(\bar{\psi}_y^{n+1} + \bar{\psi}_y^n \right) / \Delta x_n , \\ a_3^{n+1} &= R \bar{\psi}_x^{n+1} \\ a_4^{n+1} &= - 2 R k_{oi}^{n+1} \bar{\psi}_{yy}^{n+1} . \end{aligned} \right\} \quad (3.18)$$

These coefficients are all known quantities since $\bar{\psi}$ and its derivatives are given quantities. The equation (3.16) (with appropriate boundary conditions) is in the form of a linear 2PBVP, with the right hand side function H^{n+1} known at the y-grid points at $x = x_n$. Thus, PASVA3 can be called on to solve this 2PBVP if we keep the same y-grid at $x = x_{n+1}$ as at $x = x_n$. Of course, in this fashion, we cannot exploit the adaptive mesh selection capability of PASVA3, unless we provide some accurate interpolation routine for H^{n+1} . As explained before, using the same y-grid for all x-stations is adequate for near-parallel flows and the computer programming is kept simple this way. On the other hand, high order accuracies (in y) can still be achieved by the deferred correction procedure in PASVA3, though the accuracy in the x-discretization is second order.

A reduction of (3.16) to first order system is necessary before PASVA3 can be applied. If we define

$$z = \left[\phi^{(0)n+1}, \phi_y^{(0)n+1}, \phi_{yy}^{(0)n+1} \right]^T$$

then (3.16) reduces to

$$\frac{dz}{dy} = \begin{bmatrix} z_2 \\ z_3 \\ -a_0^{n+1} z_1 - (a_2^{n+1} + a_1^{n+1}) z_2 - a_3^{n+1} z_3 + H^{n+1} \end{bmatrix} \quad (3.19)$$

and the Jacobian matrix is:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0^{n+1} & -(a_2^{n+1} + a_1^{n+1}) & a_3^{n+1} \end{bmatrix} \quad (3.20)$$

3.3 Boundary Conditions

The specification of the problems will not be complete without appropriate boundary and initial conditions. These conditions depend on the type of mean flow and the computational domain of a specific problem. In this section, we shall consider two kinds of flows: plane Poiseuille flow and Blasius flow over a flat plate.

3.3.1 Poiseuille Flow

Let the walls be at $y = \pm 1$. Then the computational domain could be either $[-1,1]$ or the half-width $[0,1]$ for solutions symmetric with respect to the mid-channel. The no-slip condition at the walls reduces to the condition that the stream function and its first derivative should be zero there.

For the domain $[0,1]$, the appropriate boundary conditions are, for the symmetric fundamental mode problem:

$$\left. \begin{aligned} \phi_y^{(1)}(0) &= \phi_{yyy}^{(1)}(0) = 0 \\ \phi^{(1)}(1) &= \phi_y^{(1)}(1) = 0 \end{aligned} \right\} \quad (3.21)$$

and the normalization boundary condition could be taken to be

$$\phi^{(1)}(0) = 1 \quad (3.22)$$

Then the second harmonic mode is anti-symmetric about $y = 0$, and the appropriate boundary conditions are

$$\phi^{(2)}(0) = \phi_{yy}^{(2)}(0) = 0 \quad (3.23)$$

$$\phi^{(2)}(1) = \phi_y^{(1)}(1) = 0 . \quad (3.24)$$

The boundary conditions for the adjoint problem are the same as for the fundamental mode problem.

For the distortion of mean flow, the appropriate boundary conditions on $\phi^{(0)}$ should be:

$$\left. \begin{aligned} \phi^{(0)}(0) &= \phi_{yy}^{(0)}(0) = 0 \\ \phi_y^{(0)}(1) &= 0 . \end{aligned} \right\} \quad (3.25)$$

For the perturbed mean flow in a channel, we can specify either (i) no change in the total volume flow through the channel or (ii) no change in the pressure drop between two stations, $x = x_1$ and x_2 say, along the channel. For the case (i) the boundary condition is

$$\phi^{(0)}(x,1) = 0 \quad (3.26)$$

and then the partial differential equation for $\phi^{(0)}$ contains a non-zero function $P(x)$ (negative pressure gradient) whose value is to be determined so that the last mentioned condition for $\phi^{(0)}$ is satisfied. For the case (ii) the boundary condition for $\phi^{(0)}$ is

$$\phi^{(0)}(x,1) = \phi_1, \text{ independent of } x \quad (3.27)$$

and the value of $P(x)$ in the PDE is determined to satisfy the above boundary condition everywhere along the channel. The value of ϕ_1 is to be determined so that the imposed pressure condition is satisfied:

$$\int_{x_1}^{x_2} P(x) dx = 0 \quad (3.28)$$

Only in the temporal problem, in which the mean-flow perturbation is independent of x , the last condition reduces to

$$P(x) = 0 \quad (3.29)$$

3.3.2 Blasius Flow

At the wall, the no-slip boundary conditions become

$$\phi(0) = \phi'(0) = 0 \quad (3.30)$$

where ϕ is any of the functions $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(1)*}$ and $\phi^{(0)}$. However, the boundary conditions at the other end of the computational interval (i.e., $y = y_N$), which is physically of the form

$$\lim_{y \rightarrow \infty} \phi(y) = 0, \quad (3.31)$$

requires more careful treatment.

First of all, the computational interval should really be semi-infinite, but for practical reasons, a finite interval must be used instead. Second, the perturbations vanish at infinity. The fundamental mode equation, when evaluated outside the boundary layer has four linearly independent basic solutions, two of which decay while the other two grow, all exponentially. We want our computed solution at the finite end point to be composed of only the two decaying basic solutions in order to satisfy condition (3.31). This "outgoing wave" condition can be derived quite easily for the Orr-Sommerfeld problem. The following brief description is adapted from Keller (1976).

Consider a n th-order linear 2PBVP of the form

$$\frac{du}{dy} = A(y) u \quad (3.32)$$

on the semi-infinite interval $y \geq 0$, where the limit

$$\lim_{y \rightarrow \infty} A(y) = A_{\infty} \quad (3.33)$$

is assumed to exist.

Let the eigenvalues of A_{∞} be denoted by λ_j , let \underline{z}_j and \underline{r}_j be the independent left and right eigenvectors of A_{∞} corresponding to λ_j ; that is,

$$\left. \begin{aligned} \underline{z}_j^T A_{\infty} &= \lambda_j \underline{z}_j^T, \\ A_{\infty} \underline{r}_j &= \lambda_j \underline{r}_j. \end{aligned} \right\} \quad (3.34)$$

Then

$$(\lambda_1 - \lambda_j) \underline{z}_j^T \underline{r}_1 = \underline{z}_j^T A_{\infty} \underline{r}_1 - \underline{z}_j^T A_{\infty} \underline{r}_1 = 0$$

and hence

$$\left. \begin{aligned} \xi_j^T \xi_i &= 0 & \text{if } i \neq j, \\ &\neq 0 & \text{if } i = j. \end{aligned} \right\} \quad (3.35)$$

Now every solution of $du/dy = A_\infty u$ has the form

$$u(y) = \sum_{j=1}^n a_j \xi_j e^{\lambda_j y} \quad (3.36)$$

To ensure that the solution decays at ∞ , we have to require $a_j = 0$ for those j 's with $\text{Re } \lambda_j \geq 0$. By the use of (3.35), this condition can be expressed as:

$$\lim_{y \rightarrow \infty} \xi_j^T u(y) = 0, \quad \text{if } \text{Re } \lambda_j \geq 0.$$

Therefore, the proper boundary condition for the finite problem, with right end point b , should be

$$\xi_j^T u(b) = 0 \quad \text{if } \text{Re } \lambda_j \geq 0 \quad (3.37)$$

Note that the often-used boundary condition of setting some components of $u(b)$ to zero is inaccurate if the desired solution decays very slowly, unless the value for b is taken to be large enough, which is obviously undesirable for efficiency considerations.

For the fundamental mode problem, if we define

$$u = \left[\phi^{(1)}, \phi_y^{(1)}, \phi_{yy}^{(1)}, \phi_{yyy}^{(1)} \right]^T,$$

then we have

$$\frac{du}{dy} = A(y) u,$$

where

$$A(y) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & \beta & 0 \end{bmatrix}$$

and

$$\alpha = -i R (k_0 \bar{\psi}_y - \omega) k_0^2 - i R k_0 \bar{\psi}_{yyy} - k_0^4$$

$$\beta = i R (k_0 \bar{\psi}_y - \omega) + 2 k_0^2 .$$

Since

$$\lim_{y \rightarrow \infty} \bar{\psi}_y(y) = 1 \quad \text{and} \quad \lim_{y \rightarrow \infty} \bar{\psi}_{yyy}(y) = 0$$

for Blasius flow,

$$A_{\infty} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_{\infty} & 0 & \beta_{\infty} & 0 \end{bmatrix}$$

where

$$\alpha_{\infty} = -i R (k_0 - \omega) k_0^2 - k_0^4$$

$$\beta_{\infty} = i R (k_0 - \omega) + 2 k_0^2$$

The eigenvalues of A_{∞} are easily calculated to be

$$\lambda_1 = k_0, \quad \lambda_2 = -k_0, \quad \lambda_3 = \gamma, \quad \lambda_4 = -\gamma \quad (3.38)$$

where

$$\gamma = \sqrt{k_0^2 + i R (k_0 - \omega)}, \quad \text{and } \text{Re}(k_0), \text{Re}(\gamma) > 0 .$$

The corresponding left eigenvectors are

$$\begin{aligned} \tilde{e}_1 &= \begin{bmatrix} -k_0 \gamma^2 \\ -\gamma^2 \\ k_0 \\ 1 \end{bmatrix}, \quad \tilde{e}_2 = \begin{bmatrix} k_0 \gamma^2 \\ -\gamma^2 \\ -k_0 \\ 1 \end{bmatrix}, \quad \tilde{e}_3 = \begin{bmatrix} -k_0^2 \gamma \\ -k_0^2 \\ \gamma \\ 1 \end{bmatrix}, \quad \tilde{e}_4 = \begin{bmatrix} k_0^2 \gamma \\ -k_0^2 \\ -\gamma \\ 1 \end{bmatrix} \quad (3.39) \end{aligned}$$

The eigenvalues λ_1 and λ_3 have a positive real part. The corresponding left eigenvectors should be used in the boundary condition (3.37).

In terms of the original dependent variable, $\phi^{(1)}$ and with $D \equiv d/dy$, condition (3.37) reduces to

$$\left. \begin{aligned} (D^2 - k_0^2) (D + \gamma) \phi^{(1)} &= 0 \\ (D^2 - \gamma^2) (D + k_0) \phi^{(1)} &= 0 \end{aligned} \right\} \quad \text{at } y = y_N. \quad (3.40)$$

As Keller pointed out, these boundary conditions seem to have rarely been used other than in shooting methods for the Orr-Sommerfeld problems, although related form such as

$$\phi^{(1)} = \phi_y^{(1)} = 0 \quad \text{at } y = y_N \quad (3.41)$$

or

$$\left. \begin{aligned} (D + \gamma) \phi^{(1)} &= 0 \\ (D + k_0) \phi^{(1)} &= 0 \end{aligned} \right\} \quad \text{at } y = y_N \quad (3.42)$$

have been used. Grosch and Orszag (1977) have found that conditions like (3.41) and (3.42) can perform rather poorly.

In our formulation where k_0 is treated as an extra unknown as shown in Section 5.1, an extra normalization boundary condition like

$$\phi_{yy}^{(1)}(0) = 1$$

have to be imposed. This completes the description of the boundary conditions for the fundamental mode problem.

For the adjoint problem, it can be easily verified that the matrix A_∞ is the same as for the fundamental mode problem (i.e., the Orr-Sommerfeld problem is self-adjoint at ∞), and hence the same boundary conditions (3.40) should be used.

Boundary conditions for the second harmonic equation and the distortion of mean flow equation can be derived in a similar fashion. The presence of forcing terms in the differential equation requires a slightly different treatment.

4. NUMERICAL RESULTS

The numerical procedure described in the previous sections can be applied to very general classes of hydrodynamic stability problems. Our main interest in the present report is to compute the solution for a particular case of boundary-layer stability over a flat plate, and compare the numerical results to those computed by Murdock (1977) using a totally different approach. However, in order to have some feeling for the performance and accuracy of the present numerical procedure, we also applied the procedure to two cases where published results are available.

4.1 Plane Poiseuille Flow (Reynolds and Potter (1967))

Computations were performed for the case of $R = 5772.12$, $\omega = 0.2694879$.^{*} This corresponds to the critical point on the neutral curve. Reynolds and Potter (1967) also computed solutions for this case, but from a temporal stability standpoint, in which R and k_0 are given and ω (complex, in general) is computed as the eigenvalue of the Orr-Sommerfeld problem. But for points on the neutral curve, ω is purely real so the temporal solution is identical to the spatial stability solution obtained by specifying a real value for ω and computing the generally complex k_0 as the eigenvalue. The wavenumber k_0 used by Reynolds and Potter is 1.02071 and the frequency ω as computed by them is $0.2694879 + 0i$. Therefore, we solved the case with $\omega = 0.2694879$ (real) as input, and the eigenvalue k_0 as computed by us is

$$1.0207099 + 4.6032093 \times 10^{-7} i ,$$

which agrees with Reynolds and Potter's result to all significant digits given.

^{*} Note that our normalization differs from theirs by a factor of 3/2.

Poiseuille, $R=5772.12, \omega=.26949$

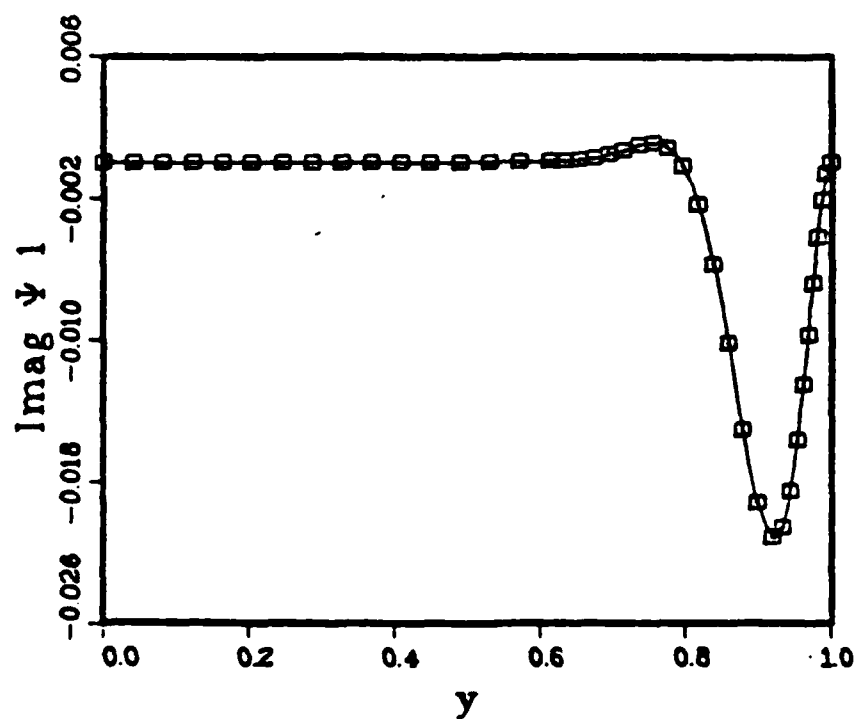
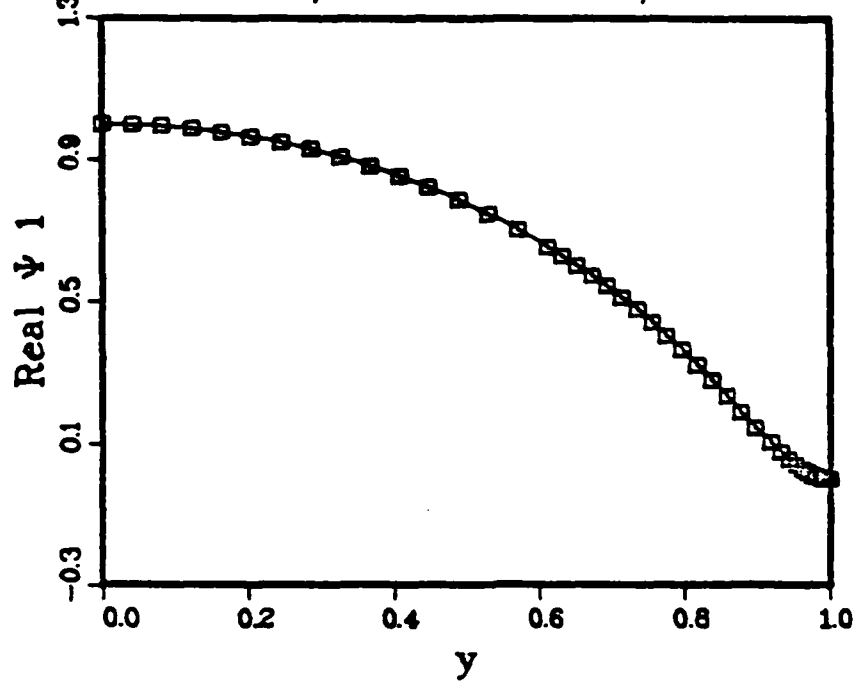


Figure 4.1 Plane Poiseuille Flow, Numerical Results

Poiseuille, $R=5772.12, \omega=.26949$

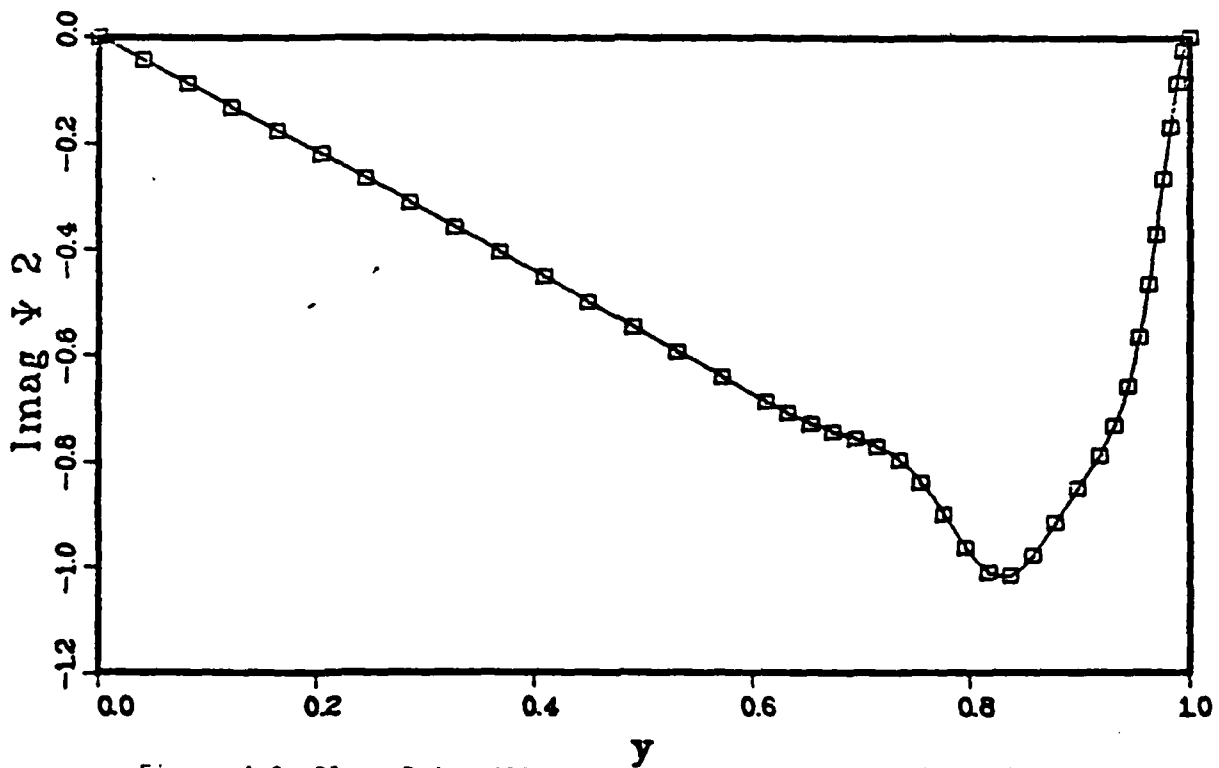
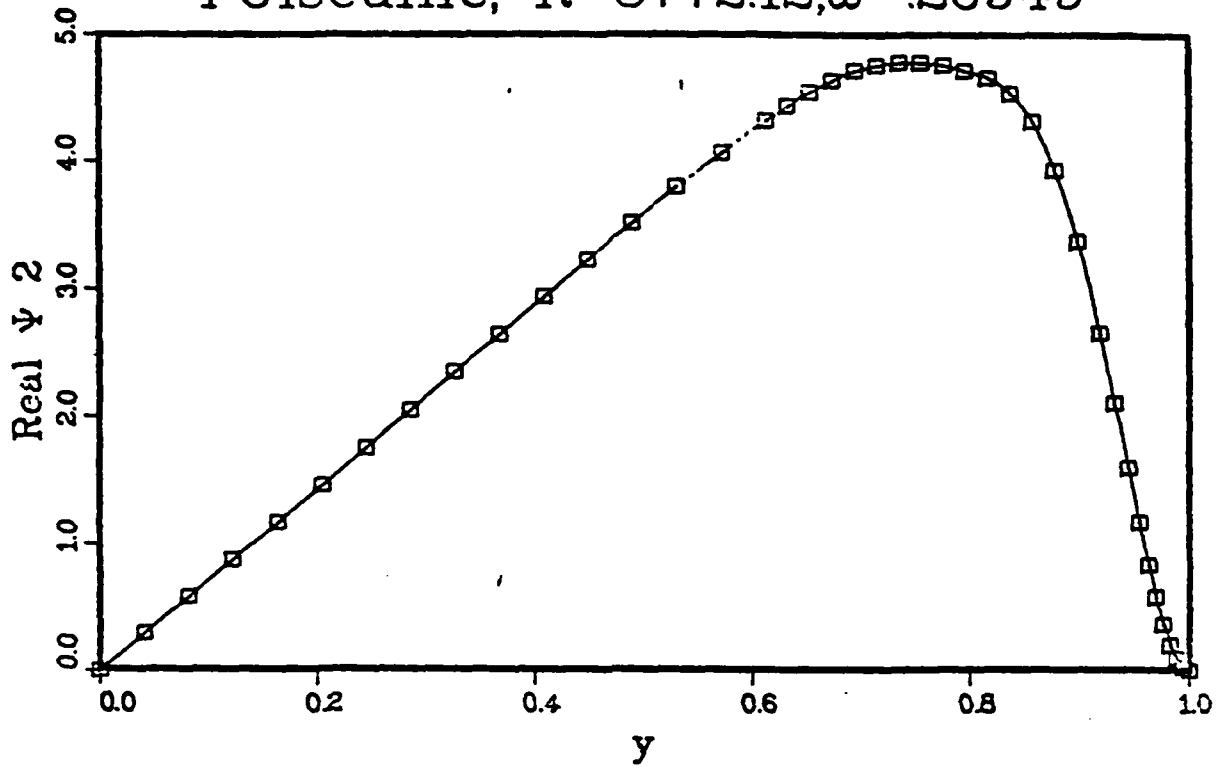


Figure 4.2 Plane Poiseuille Flow, Numerical Results (Cont.)

Poiseuille, $R=5772.12, \omega=.26949$

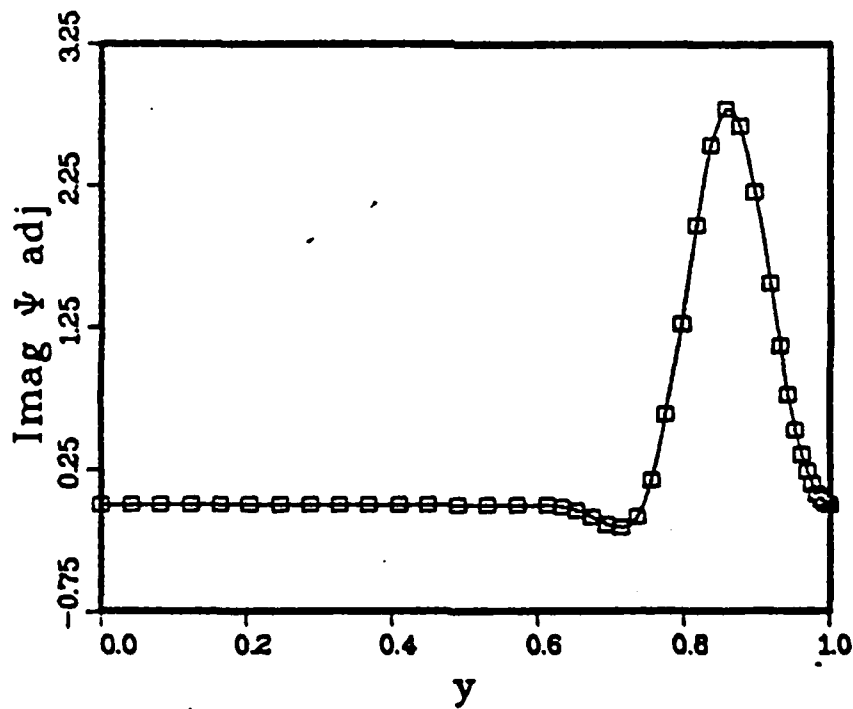
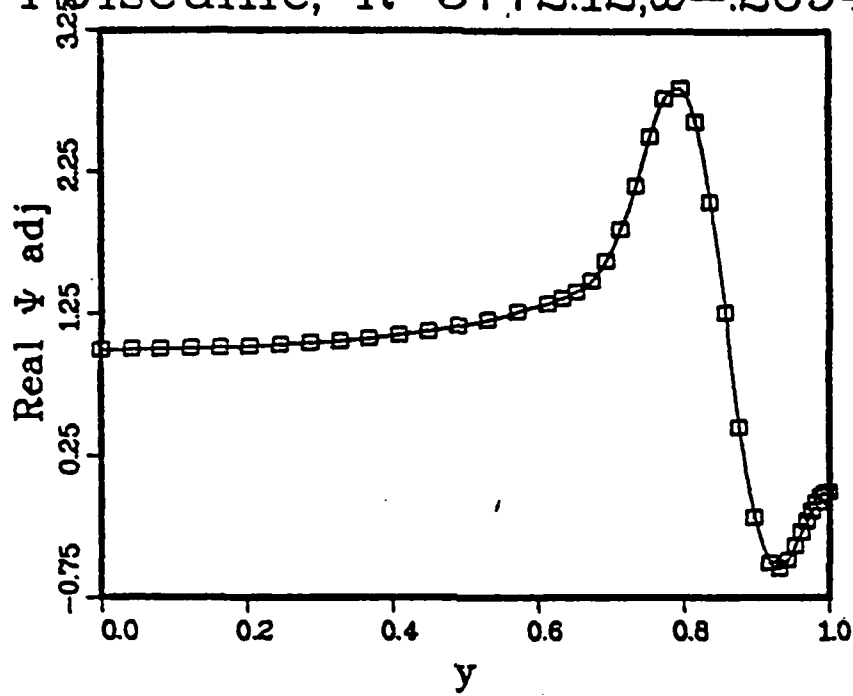


Figure 4.3 Plane Poiseuille Flow, Numerical Results (Cont.)

This particular computation was done on a half-depth domain of $[0,1]$ starting with 50 uniformly spaced grid points. PASVA3 added 31 more points and took about 2 seconds* to achieve an estimated relative error of 10^{-7} in $\psi^{(1)}$ and all its derivatives up to the third. As discussed before, PASVA3 gives an estimate for the error

$$\max |u_i(\text{computed}) - u_i(\text{exact})|, y \in [y_1, y_N]$$

for each component of the computed solution u . In this report, the estimated relative error is defined as

$$\frac{\max |u_i(\text{computed}) - u_i(\text{exact})|}{\max |u_i(\text{computed})|}$$

The initial guess for the $\psi^{(1)}$ and its derivatives are simply constants and not close to the exact solutions at all. The initial guess for k_0 was unity. PASVA3 encountered no difficulties with convergence to the correct eigenvalues and eigenfunctions. The computed $\phi^{(1)}$, $\phi^{(2)}$, and $\phi^{(1)*}$ are plotted in Figures (4.3)-(4.3) where we show only every other grid point. It is seen that PASVA3 does indeed put more grid points in places where the functions change the most. These plots are indistinguishable from those computed by Reynolds and Potter.

4.2 Linear Stability of Blasius Flow (Jordinson (1970))

We also computed the stability solutions for Blasius flow over a flat plate and compared them to three cases computed by Jordinson. Jordinson's three cases, after transformation to our dimensionless variables (in our computation we scale lengths by the distance from the leading edge whereas Jordinson scales by the displacement thickness), are tabulated in Table-4.1.

* The computer was a CDC Cyber 176.

Table 4.1.

Case	R	ω	k_{or}	k_{oi}
1	38125.69	14.717	34.994	0.8964
2	120765.14	24.254	62.1799	-0.3837
3	336356.86	37.815	104.0077	-1.9211

Our computed results are tabulated in Table 4.2.

Table 4.2

Case	k_{or}	k_{oi}	y_{max}	η_{max}	N	Approximate Relative Error	
						k_{or}	k_{oi}
1	34.98617	0.905057	0.05	9.76	99	10^{-6}	10^{-4}
2	62.17120	-0.374498	0.03	10.43	95	10^{-6}	10^{-4}
3	104.01196	-1.907428	0.015	8.7	97	10^{-6}	10^{-4}

All computations were started with 20 uniformly spaced grid points and y_{max} is chosen large enough so that it is effectively outside the boundary layer thickness. The execution time for each case was about 1.5 seconds. Jordinson used a finite difference technique with 80 uniformly spaced grid points and he claimed an accuracy of about five decimal places which is equivalent to a relative error of about 10^{-5} for the three cases above. But on comparison of the results shown in Tables 4.1 and 4.2, we see that our results, in general, agree with Jordinson's to about the same accuracy for k_{or} but our k_{oi} 's differ by much more.

Direct comparison with Jordinson's eigenfunctions is not possible because we do not know how he normalizes his eigenfunction. Without the availability of an exact solution, it is not possible to conclude which computation is more accurate. However, the availability of reliable error estimates in our computation certainly gives us more confidence in the estimated accuracy of our computed results, especially when viewed in conjunction with the accuracy of our Poiseuille flow calculations in Section 4.1. It is precisely in such situations when exact solutions are not easily available that reliable error estimates like those computed by PASVA3 are invaluable in assessing the accuracy of the computed results.

4.3 Nonlinear Stability of Blasius Flow (Murdock (1977))

The main goal of our present effort is to compute the growth of disturbances as they propagate downstream from the stable region through the neutral point to the unstable region. Murdock (1977) computed the growth of a large amplitude Tollmien-Schlichting wave for the case: $R = 10^5$, $\omega = 13.19$ from $x = 1.0$ to $x = 2.2$ in our dimensionless variables. His approach was to solve the time-dependent "parabolized" Navier-Stokes equation with the Tollmien-Schlichting wave as initial condition. In order to compare with his results, we computed for this case the quantities $\phi^{(1)}$, $\phi^{(1)*}$, $\phi^{(2)}$, $\phi^{(0)}$, $\phi_x^{(1)}$, k_{ox} , k_i , λ , $\phi_1^{(1)}$, and $\phi_2^{(1)}$, on a finite difference grid with 96 non-uniform mesh points ($N = 96$), 13 x-stations ($L = 13$) uniformly spaced between $x = 1.0$ and 3.4. y_N was chosen to be 0.04 so that $n_{\max} \approx 12.6$. The mesh in y was determined by solving the Orr-Sommerfeld problem with initially 20 uniformly spaced grid points in the y -direction. The estimated relative error in the numerical solutions as computed by PASVA3 are all less than 10^{-3} and, in most cases, less than 10^{-4} . To give an idea of the efficiency, the execution times are summarized in Table 4.3.

		CPU seconds per x-station
$\phi^{(1)}$	nonlinear	≤ 1
$\phi^{(2)}$	linear	$\lesssim .25$
$\phi_x^{(1)}$	linear	$\lesssim .25$
$\phi_1^{(1)}$	linear	$\lesssim .25$
$\phi_2^{(1)}$	linear	$\lesssim .25$
$\phi^{(1)*}$	linear	$\approx .3$
$\phi^{(0)}$	linear	$< .1$
λ, k_1		$< .1$

Table 4.3

The results of our computation for $\omega = 13.19$ are summarized in Table 4.4. λ_r and λ_i vary somewhat irregularly at a few points near $x = 1$ and this behavior probably was caused by our choice of $\phi^{(0)} \equiv 0$ (correction to the mean flow) at $x = 0^*$.

$R \times 10^{-5}$	k_{or}	k_{oi}	k_{1r}	k_{1i}	$\lambda_r \times 10^3$	$\lambda_i \times 10^3$
1.0	36.67	1.079	.119	-.666	-1.22	-.15
1.2	36.32	.398	.151	-.523	-1.26	-1.70
1.4	36.08	-.059	.162	-.422	-.97	-1.24
1.6	35.91	-.360	.164	-.349	-.67	-1.84
1.8	35.80	-.545	.164	-.294	-.49	-2.05
2.0	35.71	-.642	.161	-.252	-.20	-2.67
2.2	35.65	-.668	.160	-.218	+.15	-3.27
2.4	35.60	-.639	.159	-.191	.73	-4.19
2.6	35.56	-.562	.158	-.168	1.56	-5.17
2.8	35.53	-.447	.156	-.148	2.76	-6.34
3.0	35.49	-.298	.155	-.131	4.29	-7.44
3.2	35.45	-.120	.155	-.115	6.12	-8.44
3.4	35.41	+.084	.155	-.101	8.02	-9.14

Table 4.4 Stability Parameters ($\omega = 13.19$)

* We made this choice in order to compare our computation with Murdock's (see next section).

For infinitesimal disturbances, k_{0r} and k_{0i} are the wave number and the damping rate in local parallel mean flow. k_{1r} and k_{1i} represent the corrections to the wave number and amplification rate resulting from the slow variation of the mean flow with the streamwise distance. If we define the corrected amplification rate by $-k_{0i} + k_{1r}$ as in Saric and Nayfeh (1975), the neutral Reynolds number (corresponding to $-k_{0i} + k_{1r} = 0$) is decreased by 5.4% from that for the local parallel flow ($k_{0i} = 0$), in close agreement with Saric-Nayfeh's result.

$\lambda (= \lambda_r + i\lambda_i)$ is the Landau constant which is the measure of nonlinear effects on the stability. The second-harmonic component, $\psi^{(2)}$, and the correction to the mean flow, $\psi^{(0)}$, contribute nearly equally to the constant λ .

With the values of k_0 , k_1 , λ computed for a range of x , the "amplitude" $A(x)$ of the basic component may be computed from the differential equation

$$\frac{dA}{dx} = [i k_0(x) + k_1(x)] A + \lambda(x) |A|^2 A$$

The solution is given by

$$|A(x)| = A_0 p(x) / [1 - A_0^2 q(x)]^{1/2}$$

$$A(x) = |A(x)| \exp \left\{ i \int_{x_0}^x (k_{0r} + k_{1i} + \lambda_i |A|^2) d\xi \right\}$$

where

$$p(x) = \exp \left\{ \int_{x_0}^x [-k_{1r}(\xi) + k_{1i}(\xi)] d\xi \right\}$$

$$q(x) = 2 \int_{x_0}^x \lambda_r(\xi) p^2(\xi) d\xi .$$

$$A_0 = |A(x_0)| .$$

Thus, if $A_0^2 |q(x)| \ll 1$, we get

$$\frac{1}{A_0} |A(x)| \doteq p(x)$$

which is the linear result discussed earlier. The nonlinear effects reduces the amplitude in the region where $q(x) < 0$, and increases the amplitude where $q(x) > 0$. When $A_0^2 q(x)$ becomes unity, $|A(x)|$ becomes infinite. A similar instability "burst" was reported by Hocking, Stewartson and Stuart (1972) in their study of infinitesimal three-dimensional wave packet in a fully developed plane Poiseuille flow at a slightly supercritical Reynolds number.* In a slowly changing boundary-layer flow, $q(x)$ is also a slowly changing function of x which is negative for low Reynolds number (based on x) region and becomes positive beyond a certain Reynolds number. It, however, is a universal function for a fixed initial station x_0 in a given undisturbed boundary layer, and once computed it may be used to estimate the "burst" location as A_0 is varied.

The amplitude functions $p(x)$ and $q(x)$ for $\omega = 13.19$ are given in Table 4.5. The last column gives the value of A_0 for which the corresponding x is the singularity of $A(x)$. The numerical value for A depends on how the eigenfunctions are normalized. In our computation we arbitrarily chose $\phi''(0) = 1$, and $|\phi'(y)|_{\max}$ (corresponding to $A_0 = 1$ at $x = 1$) is about 0.15×10^{-2} . Hence, though $A_0 = 0(10)$ seems large, the streamwise velocity fluctuation is only $O(3\%)$. Near the burst, the amplitude becomes so large and the weakly nonlinear theories become invalid. If we assume that truly nonlinear processes leading to boundary layer transition take place in a short distance around the instability burst predicted by the present theory, then we can use the burst location as an estimate for the transition location replacing the e^n -criterion. For this purpose, we have to compute $q(x)$ for a range of fre-

* This instability "burst" should not be confused with the instability burst observed experimentally in boundary layers. The "burst" in the present case simply means the break-down of weakly nonlinear solution.

x	p(x)	q(x)*100	A ₀
1.	1.	0	*
1.05	.96	-.012	*
1.1	.93	-.023	*
1.15	.91	-.034	*
1.2	.89	-.044	*
1.25	.88	-.054	*
1.3	.88	-.063	*
1.35	.88	-.072	*
1.4	.89	-.08	*
1.45	.9	-.087	*
1.5	.92	-.094	*
1.55	.94	-.101	*
1.6	.96	-.107	*
1.65	.99	-.113	*
1.7	1.02	-.119	*
1.75	1.05	-.125	*
1.8	1.09	-.131	*
1.85	1.13	-.137	*
1.9	1.17	-.142	*
1.95	1.22	-.147	*
2.	1.27	-.151	*
2.05	1.32	-.153	*
2.1	1.38	-.155	*
2.15	1.44	-.154	*
2.2	1.5	-.152	*
2.25	1.56	-.147	*
2.3	1.63	-.138	*
2.35	1.69	-.125	*
2.4	1.76	-.106	*
2.45	1.83	-.079	*
2.5	1.91	-.044	*
2.55	1.98	2E-03	212.7
2.6	2.05	.061	40.6
2.65	2.13	.134	27.27
2.7	2.2	.227	21.01
2.75	2.27	.34	17.15
2.8	2.34	.478	14.46
2.85	2.41	.644	12.46
2.9	2.48	.842	10.9
2.95	2.55	1.075	9.65
3.	2.61	1.346	8.62
3.05	2.66	1.659	7.76
3.1	2.72	2.018	7.04
3.15	2.76	2.425	6.42
3.2	2.81	2.882	5.89
3.25	2.84	3.389	5.43
3.3	2.87	3.946	5.03
3.35	2.89	4.553	4.69
3.4	2.91	5.206	4.38

Table 4.5. Nonlinear Stability Amplitude Functions

Since the consensus in the transition research community is that three-dimensional disturbances are more important than two-dimensional disturbances in pre-transition nonlinear regions, a more pressing task is to extend the present theory to weakly nonlinear three-dimensional disturbances.

Comparison with Murdock's Computation

Murdock (1977) investigated the nonlinear effects on the stability of boundary layer over a flat plate, by numerically integrating the non-steady Navier-Stokes equations. The computations are carried out in the Reynolds number range $10^5 \leq U_\infty x/\nu \leq 2.5 \times 10^5$. At the upstream boundary, $U_\infty x/\nu = 10^5$, the condition is the superposition of Blasius steady boundary-layer solution and a time-periodic solution of the temporal Orr-Sommerfeld equation. He presents the results for dimensionless frequency $\omega x_1/U_\infty = 13.19$ and amplitudes* $A = 0.001$ and 0.08 .

Our result, corresponding to Murdock's $A = 0.001$, is shown in Figures 4.4a,b. We determined the initial amplitude by matching his results at the initial station. Since he used the temporal solution of the Orr-Sommerfeld equation at the upstream boundary while our solution is the spatial solution (the frequency ω is real and the wave number k is complex), a perfect match was not possible and we used the arithmetic average of two values obtained by matching at $y\sqrt{U_\infty/2\nu x} = 0.2$ and 1 . In the case corresponding to Murdock's $A = 0.001$, the nonlinear effects are negligible and the amplitude is given by the solution of

$$\frac{1}{|A|} \frac{d|A|}{dx} = -k_{oi} + k_{1r}$$

* The meaning of A is not clearly defined in the paper, since he failed to state how the Orr-Sommerfeld solution is normalized.

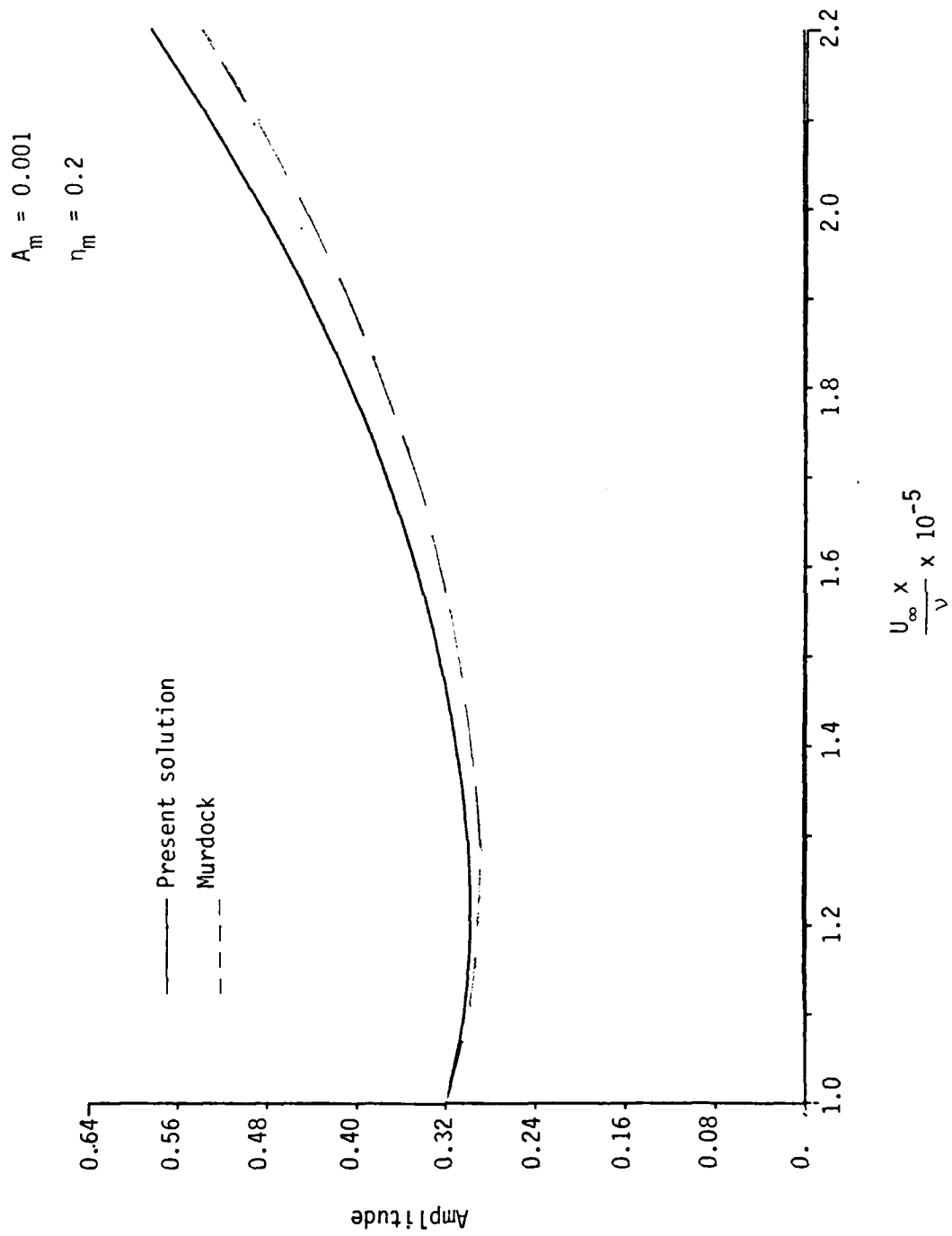


Figure 4.4a Normalized Amplitude of Streamwise Fluctuations

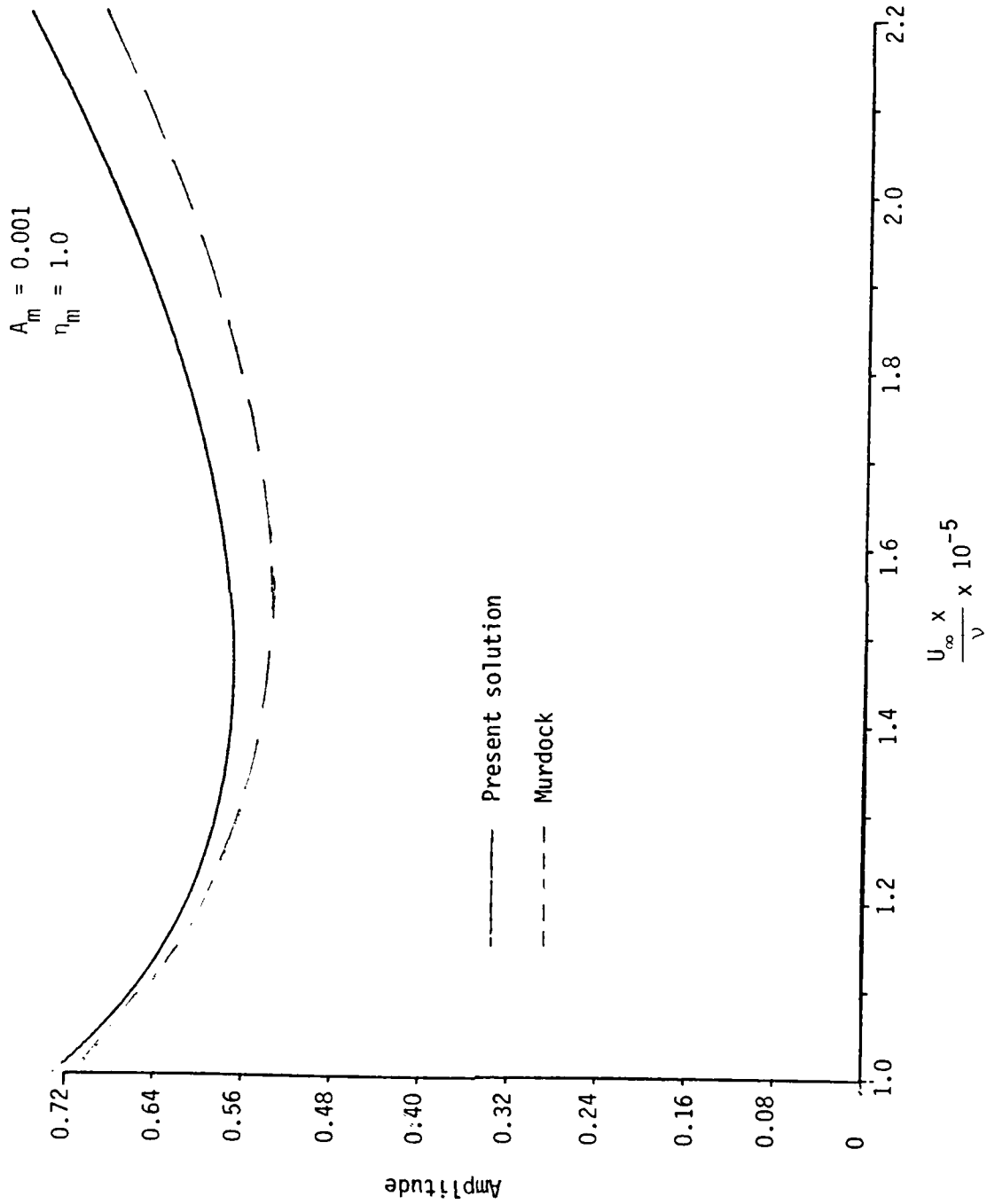


Figure 4.4b Normalized Amplitude of Streamwise Fluctuations

Figure 4.4a shows the variation of the amplitude of streamwise component of perturbation velocity along $y = 0.2\sqrt{2\nu x/U_\infty}$. The agreement between our result and Murdock's computation is only fair, and we feel that the difference is caused by the imperfect matching of the boundary condition at $x = 0$.

The comparison in the large amplitude case ($A = 0.08$ in Murdock) turned out to be more complicated. Murdock simply increased the amplitude of the Orr-Sommerfeld solution without introducing any nonlinear contributions at $x = 1$, whereas, in our formulation presented so far, nonlinear contributions are automatically introduced when we increase the amplitude of the fundamental mode (see Figure 4.5). This is especially apparent in the second-harmonic component. The amplitude of the second harmonics starts from 0 at $x = 1$ in Murdock's calculation, while our solution would give non-zero value proportional to A^2 . In order to match Murdock's initial condition, we have to construct unforced boundary-value solution for the second harmonic (2ω component) that will cancel our forced second-harmonic component. The free solution is formally given by

$$\sum_n a_n \phi_{E,n}^{(2)}(y) \exp(i k_{2,n} x - 2i\omega t)$$

where $\phi_{E,n}^{(2)}$ and $k_{2,n}$ are the eigenfunctions and the eigenvalues of Orr-Sommerfeld equation corresponding to the frequency 2ω . The expansion coefficients a_n are to be determined from the boundary condition at $x = 1$:

$$\sum_n a_n \phi_{E,n}^{(2)}(y) = -\phi^{(2)}(y)$$

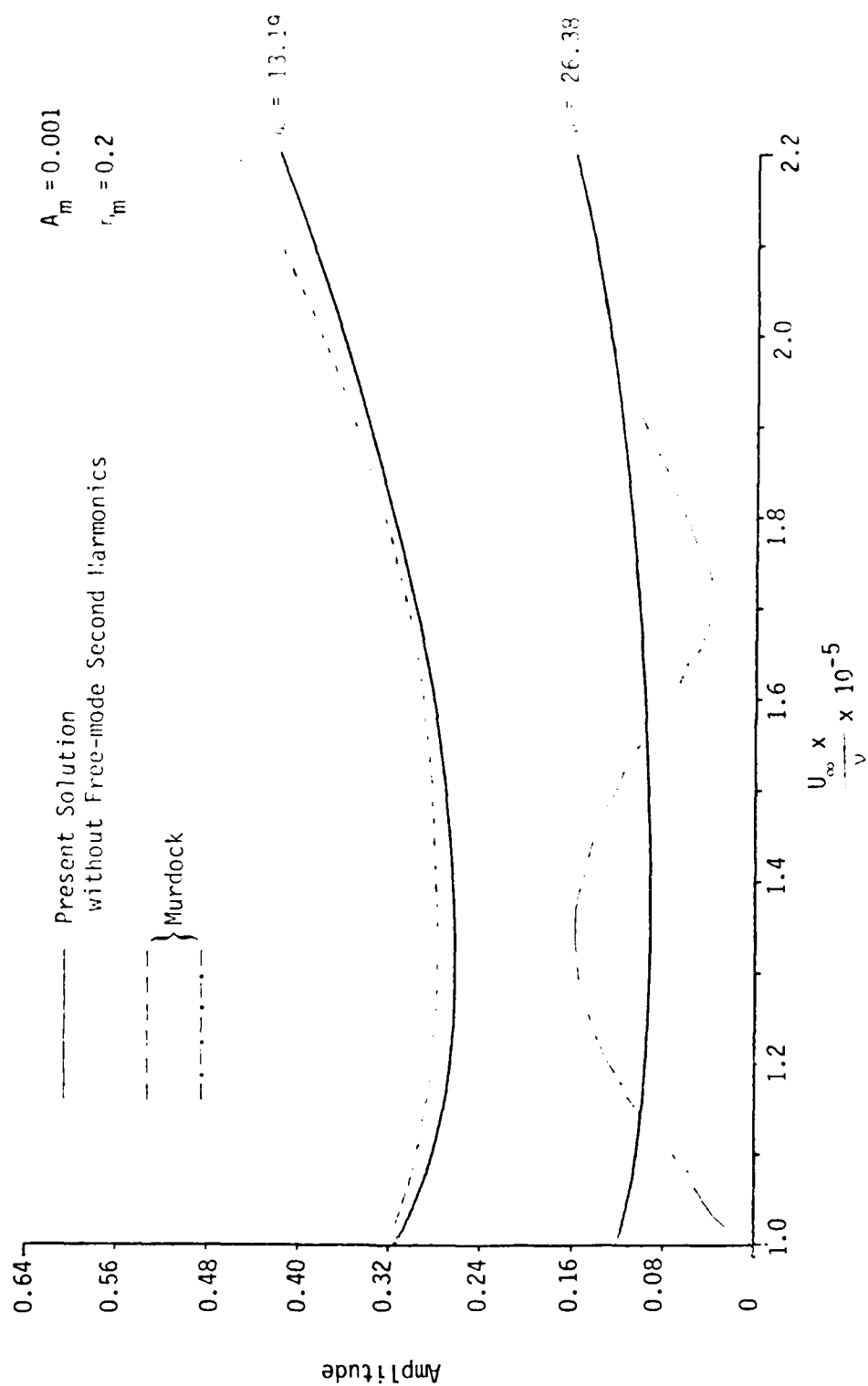


Figure 4.5a Normalized Amplitude of Streamwise Perturbations

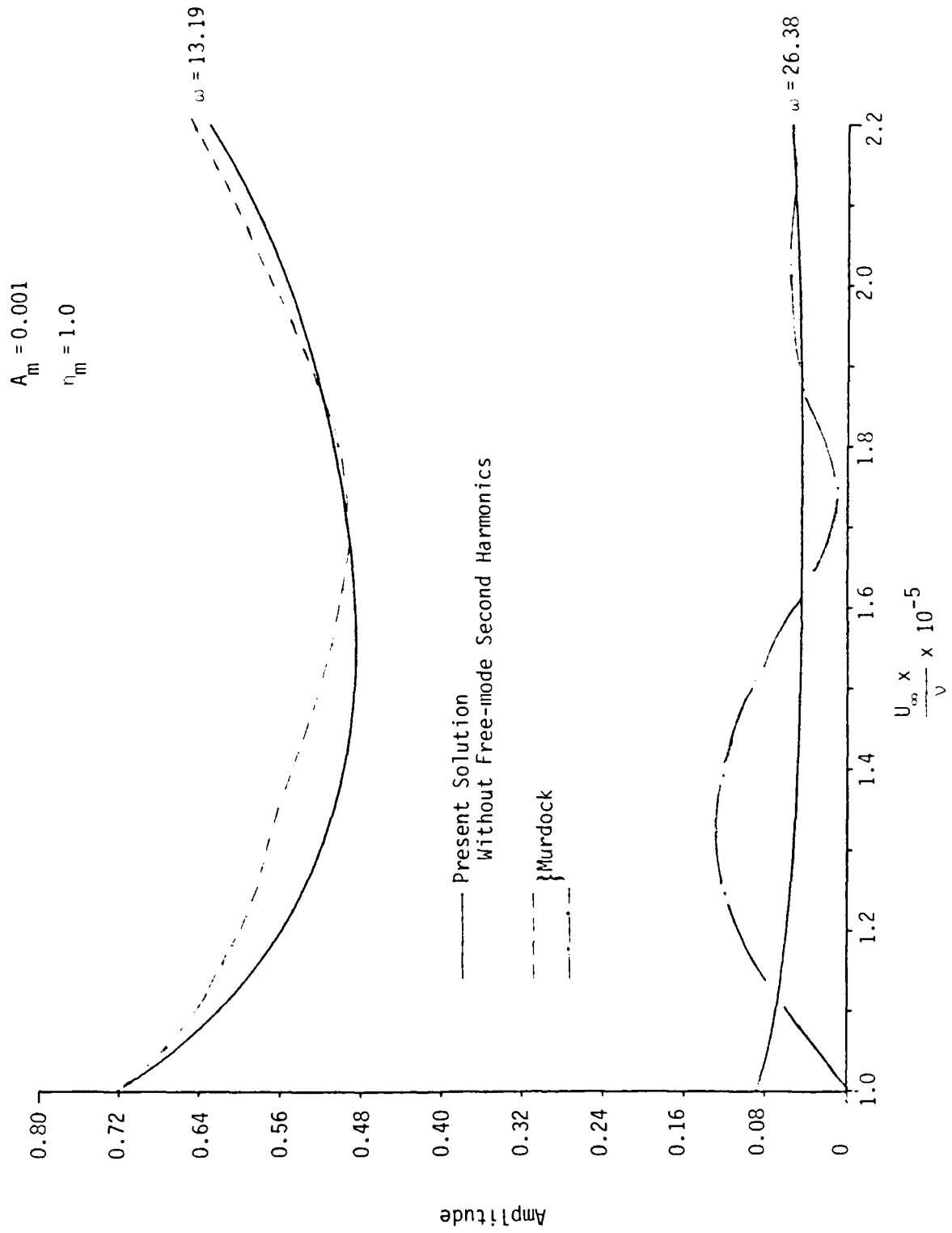


Figure 4.5b Normalized Amplitude of Streamwise Fluctuations

where $\phi^{(2)}(y)$ is the forced second harmonic at $x = 1$. Using the orthogonality condition derived in the appendix, we determine

$$a_n = \int_0^\infty \phi_{E,n}^{(2)*} B \phi^{(2)} dy / \int_0^\infty \phi_{E,n}^{(2)*} B \phi_{E,n}^{(2)} dy$$

where $\phi_{E,n}^{(2)*}$ are the adjoint eigensolution and B is the operator defined by

$$B = 4 k_{2,n} (D^2 - k_{2,n}^2) + iR [U(D^2 - 3k_{2,n}^2) + 4\omega k_{2,n} - U']$$

In our calculation, we include only the first mode since we believe it is the least damped solution. We calculated also the correction to the second-harmonic eigenvalues due to the mean-flow variation.

x	k_{2r}	k_{2i}	k_{21r}	k_{21i}	$2k_{or}$
1.	65.2	.28	.39	-.39	73.3
1.2	65.0	.82	.40	-.28	72.6
1.4	64.8	1.70	.41	-.185	72.2
1.6	64.6	2.84	.43	-.101	71.8
1.8	64.1	4.17	.46	-.004	71.6
2.0	63.4	5.64	.49	+.135	71.4
2.2	62.3	7.15	.46	.37	71.3
2.4	60.7	8.39	.18	.61	71.2
2.6	58.8	8.96	-.23	.48	71.1
2.8	57.3	8.96	-.30	.19	71.0
3.0	56.2	8.80	-.22	.04	71.0
3.2	55.3	8.63	-.15	-.23	70.9
3.4	54.6	8.50	-.09	-.47	70.8

Table 4.6 Second-Harmonic Eigenvalues

We present the computed eigenvalues in Table 4.6. (k_{2r}, k_{2i}) is the eigenvalue for 2ω frequency in local parallel flow, and (k_{21r}, k_{21i}) is the correction due to the mean flow variation. The spatial amplification rate of the second harmonics is given by $-k_{2i} + k_{21r}$, and near $x = 1$ the second-harmonic eigenmode is amplified because of the mean-flow variation, while the local-parallel-flow result is damped second harmonics. The last column is twice the wave number of the fundamental (frequency ω) eigenmode. As we expect, k_{2r} is close to but not identical to $2k_{0r}$.

We denote this free-mode second harmonic component by

$$\text{Re} \left\{ A_2(X) \phi_E^{(2)}(y, X) \exp [i\theta_2(X) - 2i\omega t] \right\}$$

Then, for some distance downstream of $x = 1$, before it is damped out, the amplitude of the free-mode second harmonics will be of the same order of magnitude as the forced-mode second harmonics, and its beating with the basic fundamental component will produce an additional contribution to the fundamental component which will be of the form:

$$\text{Re} \left\{ \tilde{A}(X) A_2(X) \phi_4^{(1)}(y, X) \exp [i(\theta_2 - \tilde{\theta}_1) - i\omega t] \right\}$$

The differential equation for $\phi_4^{(1)}$ is the following nonhomogeneous Orr-Sommerfeld equation:

$$\begin{aligned} & [\partial_y^2 - (k_2 - \tilde{k}_0)^2]^2 \phi_4^{(1)} - iR \left\{ [(k_2 - \tilde{k}_0)U(y) - \omega][\partial_y^2 - (k_2 - \tilde{k}_0)^2] \phi_4^{(1)} \right. \\ & \quad \left. - (k_2 - \tilde{k}_0) U_{yy}(y) \phi_4^{(1)} \right\} \\ & = -iR \left[\tilde{k}_0 \phi_{Ey}^{(2)} (\partial_y^2 - \tilde{k}_0^2) \tilde{\phi}^{(1)} + k_2 \phi_E^{(2)} (\partial_y^2 - k_2^2) \tilde{\phi}_y^{(1)} \right. \\ & \quad \left. - k_2 \tilde{\phi}_y^{(1)} (\partial_y^2 - k_2^2) \phi_E^{(2)} - \tilde{k}_0 \tilde{\phi}^{(1)} (\partial_y^2 - k_2^2) \phi_{Ey}^{(2)} \right] \end{aligned}$$

with boundary conditions

$$\phi_4^{(1)}(0) = \phi_{4y}^{(1)}(0) = 0 \quad \text{and} \quad \phi_4^{(1)} \rightarrow 0 \quad \text{as} \quad y \rightarrow 0 .$$

Our results, including the contributions from the free-mode second harmonics, are compared with Murdock's computation in Figure 4.6. Since the wave numbers for the forced and free-mode second harmonics are not equal, the amplitude of the second harmonics is modulated with x variation. Since we did not include all modes of the eigenfunction for the second harmonics, we do not cancel the forced second harmonics perfectly at $x = 1$, but the second harmonic modulation is very well reproduced except for a slight phase shift. Because of the above-mentioned difference in the wave numbers, the computed amplitude of the fundamental component also exhibits small amplitude modulation until the free-mode second harmonics is damped out. Except for this modulation, our results for the fundamental component agree fairly well with Murdock's computations.

The longitudinal component of the perturbation velocity at $\eta_m = y\sqrt{U/2\nu x} = 0.2$ at a fixed t is shown in Figure 4.7 together with Murdock's result. The agreement between the two computations is good, and especially the phase relationship between the fundamental component and the second-harmonics is very well reproduced by the present theory.

From these results, we conclude that, in the range of x covered in Murdock's computation, the present nonlinear stability theory, which includes the fundamental and second-harmonic components with the corrections for the slow variation of the undisturbed laminar boundary layer flow with x , provide an adequate description of the evolution of small but finite amplitude perturbations introduced into the boundary layer, compared with direct numerical solution of the Navier-Stokes equations. Even though the computation of eigensolutions at several stations along the developing boundary layer is quite involved, once the eigenvalues and eigenfunctions are computed, these can be used for any amplitude within the limitation of the theory, and provide a valuable "tool" for a parametric study in the transition problem.

$$A_m = 0.08$$

$$\eta_m = 0.2$$

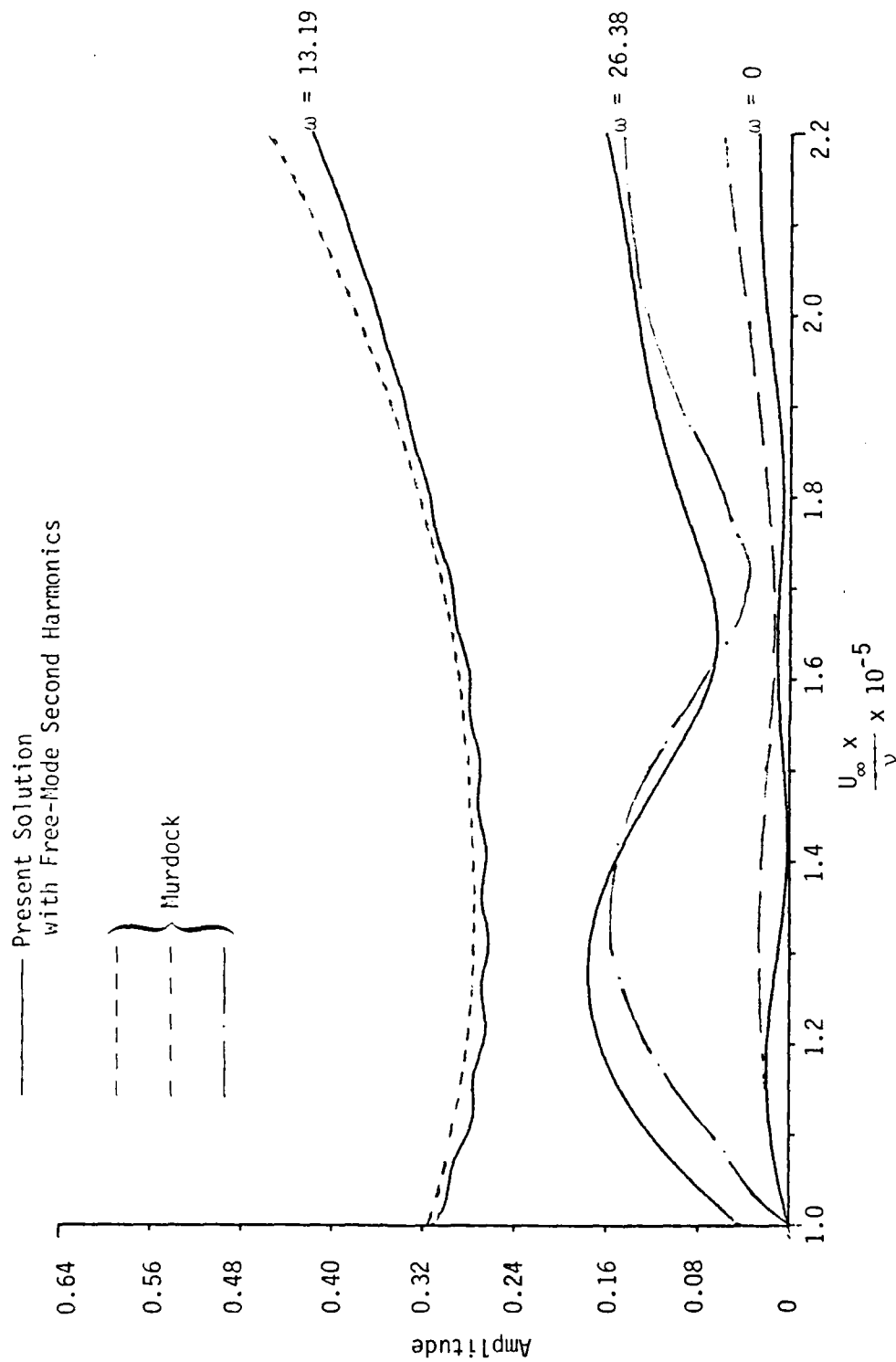


Figure 4.6a Normalized Amplitude of Streamwise Perturbations

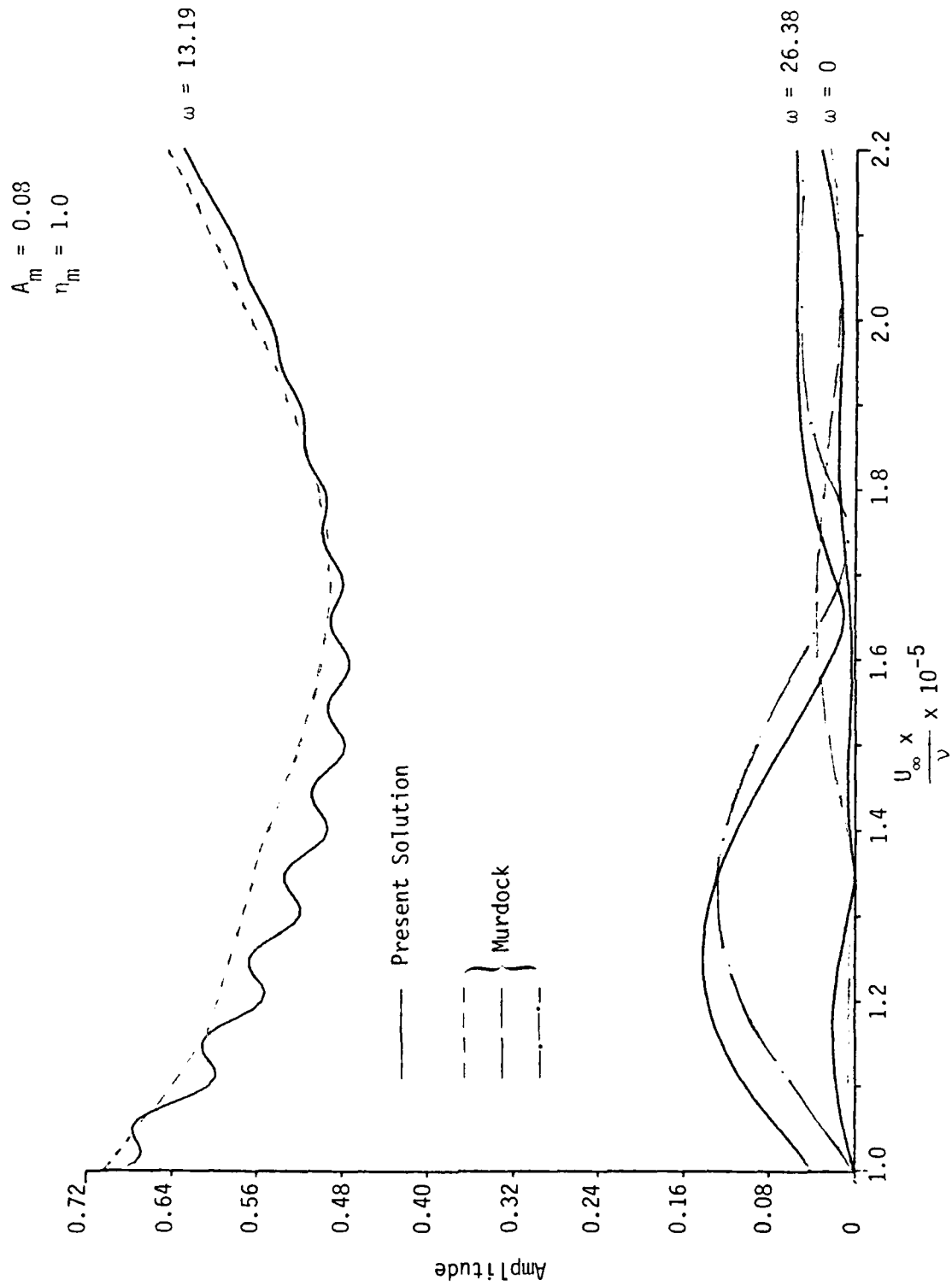


Figure 4.6b Normalized Amplitude of Streamwise Perturbations

$$A_m = 0.08$$
$$\eta_m = 0.2$$

— Present Solution

- - - - - Murdock

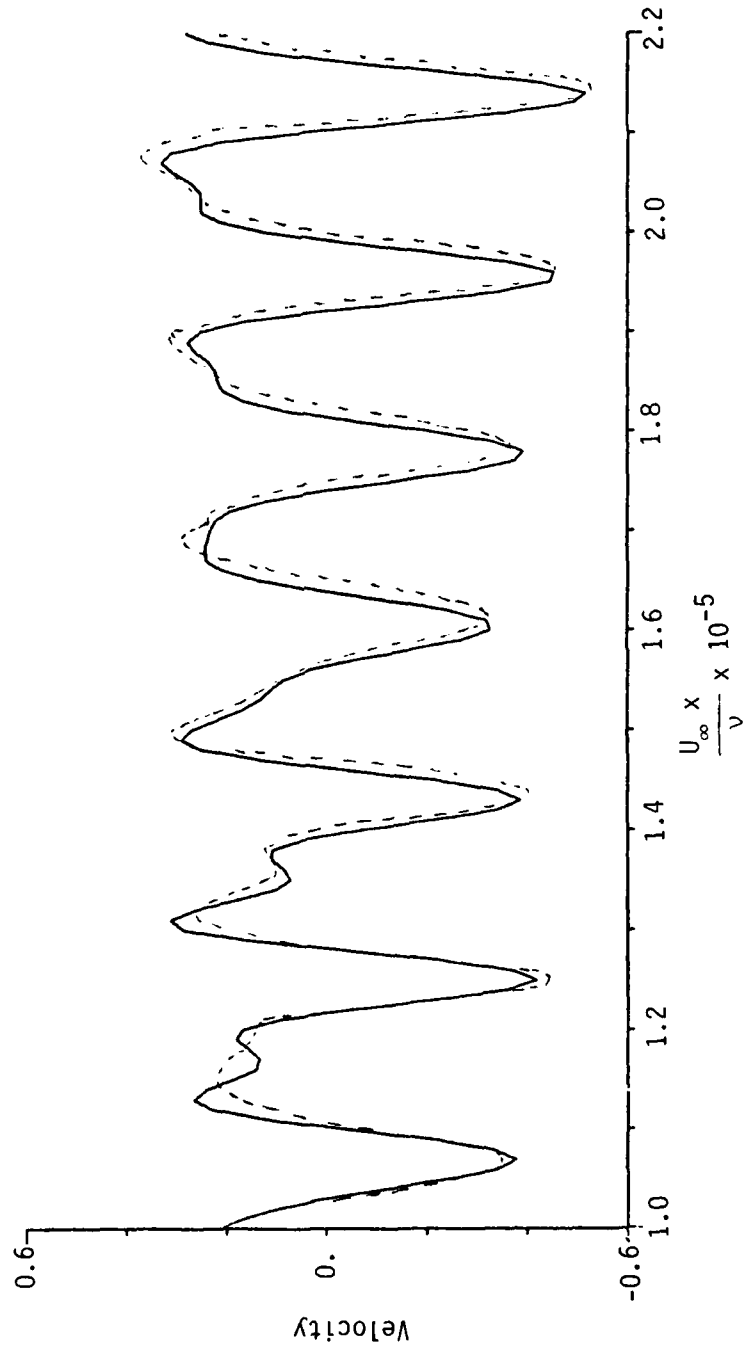


Figure 4.7 Perturbation Velocity at $\eta_m = 0.2$

As an example, we extended our computation to $x = 3.4$ beyond $x = 2.2$, and the results are shown in Figure 4.8. In this example, we have $A_0 = 18.1$ and the instability burst occurs at $x = 2.73$. It would have been very interesting and valuable if Murdock had extended his computation into this range and obtained the flow field near the instability burst from the numerical solution of the Navier-Stokes equations.

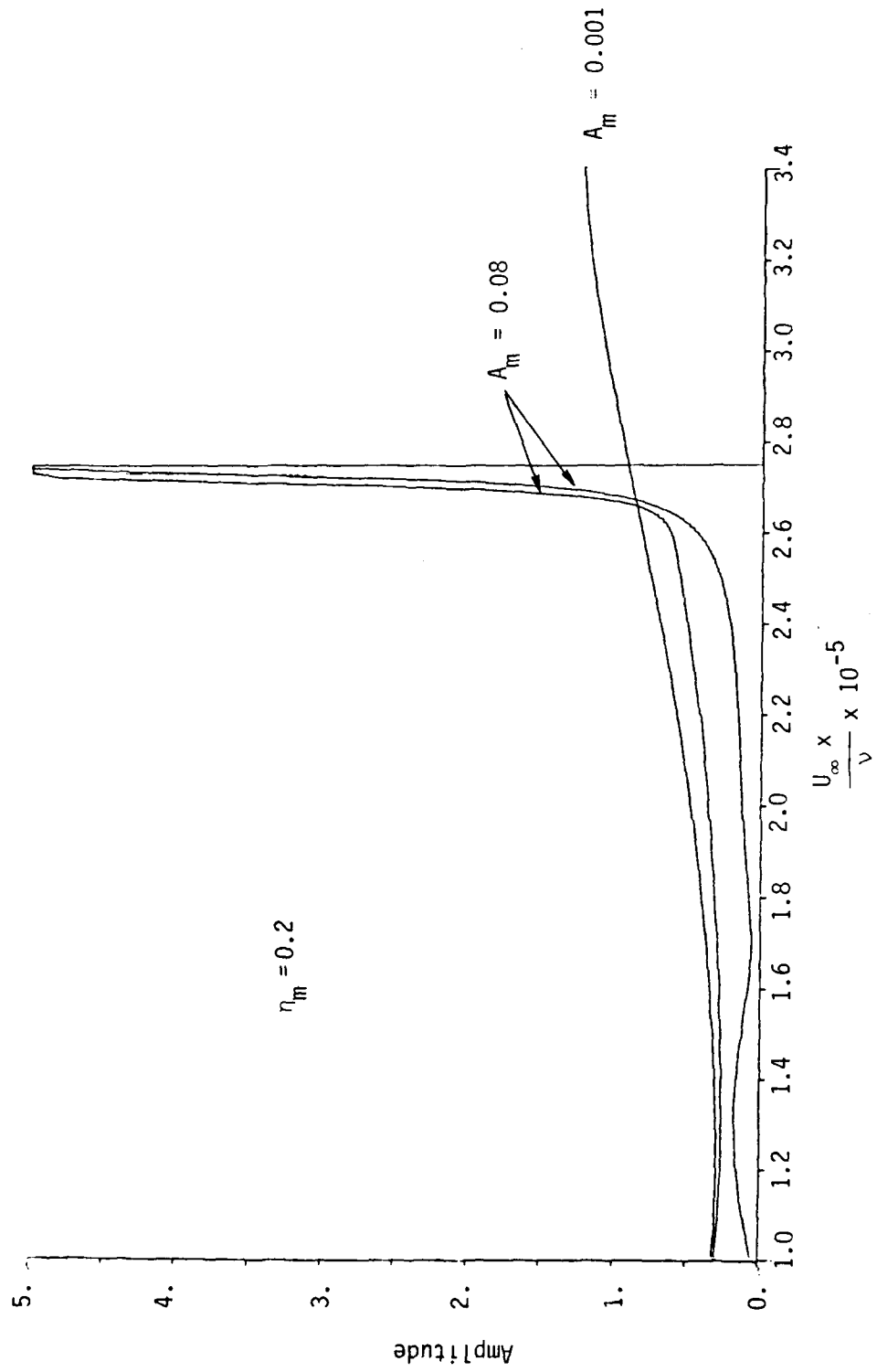


Figure 4.8a Normalized Amplitude of Streamwise Fluctuations in Extended Domain

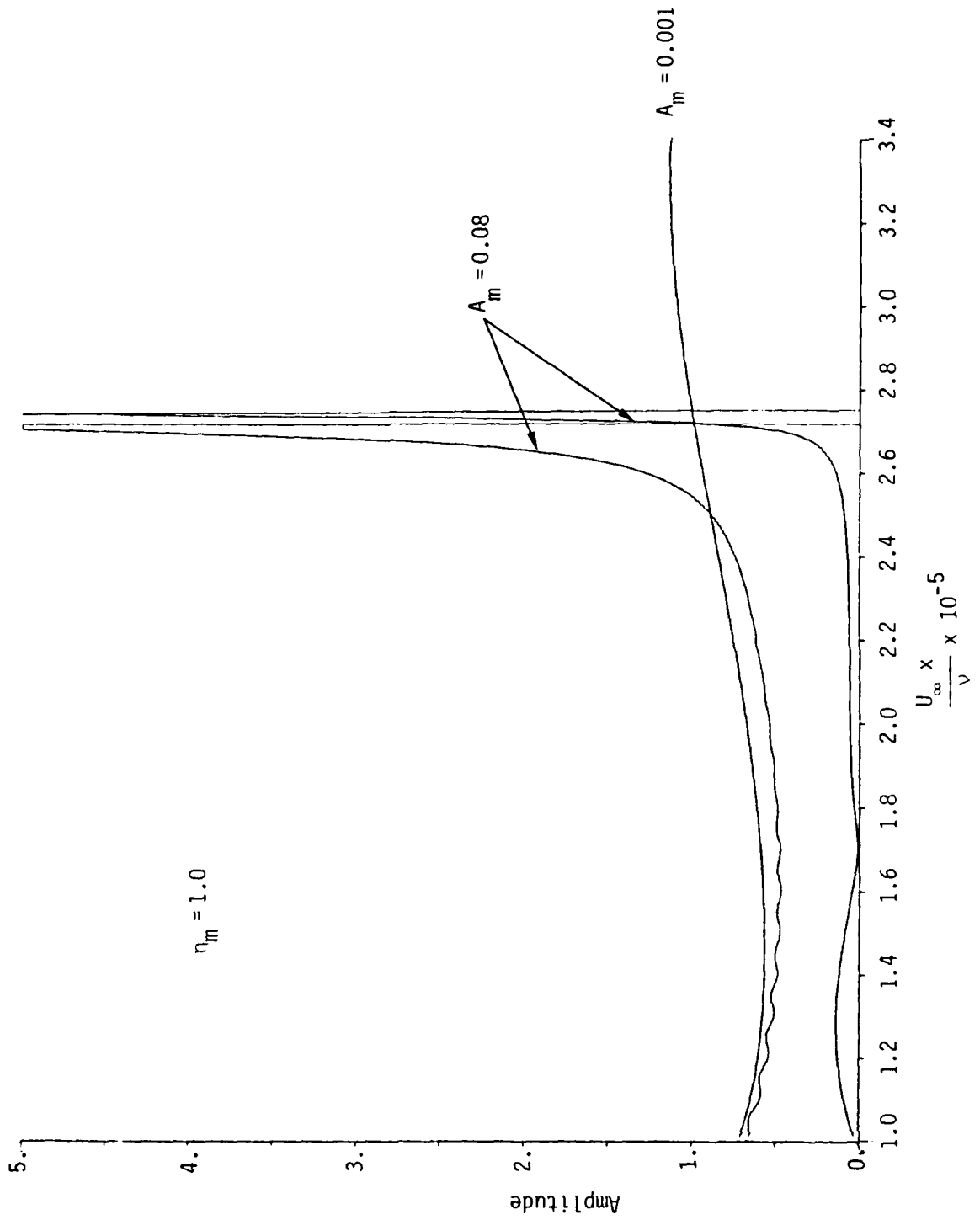


Figure 4.8b Normalized Amplitude of Streamwise Fluctuations in Extended Domain

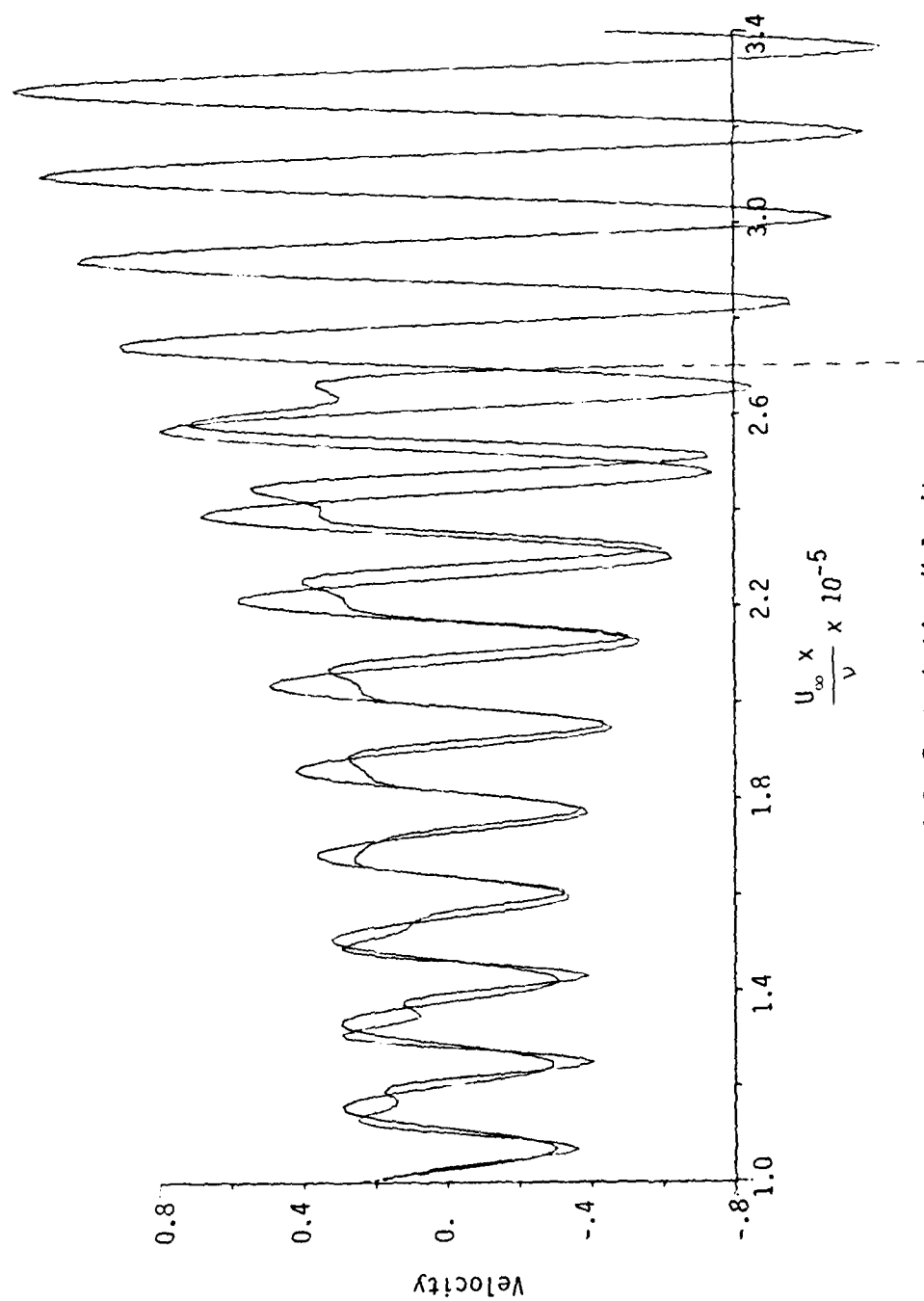


Figure 4.9 Perturbation Velocity

5. CONCLUSIONS

We have formulated a weakly nonlinear stability theory with the assumption that the amplification rate is also weak. The theory takes into account the slow variation of the undisturbed mean flow with the stream-wise distance and includes the fundamental and second-harmonic components. A computation was carried out for one frequency and the results compare favorably with the numerical solution of the Navier-Stokes equation by Murdock.

Theory predicts that the nonlinearity has a stabilizing effect at low Reynolds numbers, but becomes destabilizing beyond a threshold Reynolds number and leads to a break-down of the solution. The location of the singularity depends on the initial amplitude of the disturbance, and this location may be used as a criterion for the laminar-to-turbulence transition replacing the commonly used e^n criterion which is independent of the initial amplitude. The validity of this new criterion should be established by carrying out the computation for other frequencies and the prediction compared with the transition location measured on a flat plate as the free-stream turbulence level is changed.

The present theory may readily be extended to non-Blasius two-dimensional boundary layers.

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APPENDIX

Expansion of Arbitrary Function in Series of Spatial Mode
Eigenfunctions of Orr-Sommerfeld Equation

The Orr-Sommerfeld equation in the linear stability theory of parallel flow is

$$(D^2 - k^2)^2 \phi - i R [(kU - \omega)(D^2 - k^2)\phi - kU''\phi] = 0 \quad (1)$$

$$D \equiv \frac{d}{dy}$$

with boundary conditions

$$\begin{aligned} \phi = D\phi = 0 & \quad \text{at } y = 0 \\ \phi, D\phi, \dots \rightarrow 0 & \quad \text{as } y \rightarrow \infty \end{aligned} \quad (1-a)$$

when R and ω are real constants, the problem is called a spatial-mode solution and the eigenvalue k may be a complex number. Presumably an infinite number of eigenvalues (discrete and possibly continuous) and eigensolutions exist, and the problem under consideration in this note is how one can expand an arbitrary function of y in series (and integral if the eigenvalue is continuum) of eigenfunctions. This problem was considered and the formalism of the expansion was given by Schensted (1961) for the temporal mode, in which R, k are real and ω is a complex eigenvalue. Since the coefficients of equation (1) are polynomials of the eigenfunction k , the expansion formalism for the spatial mode is not as simple as in the temporal mode and, to our knowledge, it has not been given in the literature.

Langer (1923) states that the system of the form

$$u^{(n)}(x) + p_1(x, \rho)u^{(n-1)}(x) + \dots + p_n(x, \rho)u(x) = 0$$

$$\sum_{\ell=0}^{n-1} \left\{ \alpha_{j\ell} u^{(\ell)}(a) + \beta_{j\ell} u^{(\ell)}(b) \right\} = 0 \quad (j=1, 2, \dots, n)$$

with the coefficient $p_n(x, \rho)$ a polynomial of degree n in ρ may be written in the form

$$u_j'(x) = \sum_{\ell=1}^n \left\{ p_{j\ell}(x)\rho + q_{j\ell}(x) \right\} u_\ell(x)$$

$$\sum_{\ell=1}^n \left\{ \sigma_{j\ell} u_\ell(a) + \tau_{j\ell} u_\ell(b) \right\} = 0 \quad (j=1, 2, \dots, n)$$

The Orr-Sommerfeld equation falls into this class of system.

Birkhoff and Langer (1923) developed the theory of a system of n ordinary linear differential equations of the first order containing a parameter and subject to the homogeneous boundary conditions and, in particular, discussed the formal development of a vector of arbitrary functions into a series of the eigensolutions.

We are indebted to Professor D. Cohen of Caltech (also a Consultant at Dynamics Technology) for directing our attention to the Langer and Birkhoff-Langer papers and for his suggestions in this particular part of our problem.

In order to develop the expansion formalism, we consider the following system of first-order equations which is equivalent to the fourth-order equation given in equation (1). Let

$$\begin{cases} z_1 = \phi \\ z_2 = \phi' - k\phi = z_1' - kz_1 \\ z_3 = \phi'' - k^2\phi = z_2' + kz_2 \\ z_4 = \phi''' - k\phi'' - k^2\phi' + k^3\phi = z_3' - kz_3 \end{cases} \quad (2)$$

Then equation (1) becomes

$$z_4' + kz_4 = iR [(kU-\omega)(\phi'' - k^2\phi) - kU'\phi]$$

Therefore, equation (1) is equivalent to the system:

$$\begin{cases} z_1' = kz_1 + z_2 \\ z_2' = -kz_2 + z_3 \\ z_3' = kz_3 + z_4 \\ z_4' = -iRkU''z_1 + iR(kU-\omega)z_3 - kz_4 \end{cases} \quad (3)$$

The crucial feature of the above system is that the equations are linear in k .

In matrix notation the system (3) is written as

$$\underline{z}' = (kA+B)\underline{z} \quad (4)$$

where

$$\underline{z} = [z_1, z_2, z_3, z_4]^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -iRU'' & 0 & iRU & -1 \end{bmatrix} \quad (4-a)$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -iR\omega & 0 \end{bmatrix} \quad (4-b)$$

The boundary conditions are

$$\left. \begin{array}{ll} z_1 = z_2 = 0 & \text{at } y = 0 \\ \underline{z} \rightarrow 0 & \text{as } y \rightarrow \infty \end{array} \right\} \quad (4-c)$$

We introduce the adjoint solution $\underline{x} = [x_1, x_2, x_3, x_4]$ that satisfies

$$\underline{x}' = -\underline{x} [kA+B] \quad (5-a)$$

$$\left. \begin{array}{ll} x_3 = x_4 = 0 & \text{at } y = 0 \\ \underline{x} \rightarrow 0 & \text{as } y \rightarrow \infty \end{array} \right\} \quad (5-b)$$

The reason for this boundary condition at $y = 0$ becomes clear in the subsequent development.

Now, suppose that \underline{z}_m corresponds to an eigenvalue k_m and \underline{x}_n corresponds to another eigenvalue k_n ; namely

$$\underline{z}_m' = (k_m A+B) \underline{z}_m$$

$$\underline{x}_n' = -\underline{x}_n (k_n A+B)$$

By cross-multiplying and adding the products, we obtain

$$(\tilde{x}_n \tilde{z}_m)' = (k_m - k_n) \tilde{x}_n A \tilde{z}_m$$

By integrating with respect to y from 0 to ∞ , we get

$$\begin{aligned} (k_m - k_n) \int_0^{\infty} \tilde{x}_n A \tilde{z}_m dy &= [\tilde{x}_n \tilde{z}_m]_0^{\infty} \\ &= [x_{n1} z_{m1} + x_{n2} z_{m2} + x_{n3} z_{m3} + x_{n4} z_{m4}]_0^{\infty} \end{aligned}$$

The integrated terms vanish by the boundary condition imposed on \tilde{x} . Therefore, we obtain the ortho-normality condition

$$\int_0^{\infty} \tilde{x}_n A \tilde{z}_m dy = \delta_{nm} \quad (6)$$

Eigenfunction Expansion:

If an arbitrary function \tilde{z} is expanded in series of \tilde{z}_n :

$$\tilde{z} = \sum a_n \tilde{z}_n \quad (0 \leq y < \infty) \quad (7)$$

The coefficient a_n can be determined by multiplying by $\tilde{x}_m A$ and integrating with respect to y from 0 to ∞ .

$$\int_0^{\infty} \tilde{x}_m A \tilde{z} dy = \sum_n a_n \int_0^{\infty} \tilde{x}_m A \tilde{z}_n dy = A_m$$

Hence

$$a_m = \int_0^{\infty} x_m A z dy \quad (8)$$

Transformation to Single-Equation Formalism:

The relation between the solution of equation (1), ϕ , and z is given in equation (2). Written for each component, equation (5) is

$$\left. \begin{aligned} x_1' &= kx_1 + iRkU'' x_4 \\ x_2' &= -x_1 + kx_2 \\ x_3' &= -x_2 - kx_3 - iR(kU-\omega)x_4 \\ x_4' &= -x_3 + kx_4 \end{aligned} \right\}$$

By eliminating x_1, x_2, x_3 , we get

$$(D^2 - k^2)^2 x_4 - D^2 [iR(kU-\omega)x_4] - iR [k^2(kU-\omega) + kU''] x_4 = 0$$

with

$$\begin{aligned} x_4 &= x_4' = 0 & \text{at } y = 0 \\ x_4 &= x_4' = \dots = 0 & \text{at } y = \infty \end{aligned}$$

This is the adjoint equation directly obtainable from equation (1). If we denote the adjoint solution by ϕ^* , we get the following transformation:

$$\begin{cases} x_1 = -D^3\phi^* + kD^2\phi^* + D[k^2 + iR(kU-\omega)]\phi^* - [k^3 + iRk(kU-\omega)]\phi^* \\ x_2 = D^2\phi^* - [k^2 + iR(kU-\omega)]\phi^* \\ x_3 = -D\phi^* + k\phi^* \\ x_4 = \phi^* \end{cases}$$

If $f(y)$ is the first component of a function z to be expanded, then the other components are given by the relation in equation (2):

$$\begin{cases} z_1 = f \\ z_2 = Df - kf \\ z_3 = D^2f - k^2f \\ z_4 = D^3f - kD^2f - k^2Df + k^3f \end{cases}$$

Therefore, the formula given by equation (8) becomes

$$a_m = \int_0^\infty \left\{ (4k_m + iRU) D^2f - [4k_m^3 + iR(3k_m^2U - 2k_m\omega + U'')] \right\} f \phi_m^* dy$$

The normalization relation is

$$\int_0^\infty \left\{ (4k_m + iRU) D^2\phi_m - [4k_m^3 + iR(3k_m^2U - 2k_m\omega + U'')] \right\} \phi_m^* dy = 1$$

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