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STOCHASTIC CONVERGENCE UNDER NONLINEAR TRANSFORMATIONS ON METRIC SPACES

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ABSTRACT

This paper considers stochastic convergence properties of real-valued measurable mappings defined on separable metric spaces. A general L_p convergence theorem is presented for mappings of sequences of random elements converging setwise and in probability. A number of the implications of this theorem with respect to the properties of systems defined on separable metric spaces are discussed.

I. Introduction

Consider a system with a given input and the corresponding output. If a sequence of inputs converged to that particular input, it would often be of interest to know when the corresponding sequence of outputs converged to the particular output. We will be concerned with this problem in a stochastic framework. In an early work in this area, Wong and Zakai [1] considered several convergence properties of systems whose inputs are derivatives of sequences of random processes converging in various modes to the Wiener process, thereby characterizing (in the limit) the "white-noise" response of certain types of systems. More recently, Sussman [2] has developed a general approach to this problem by demonstrating and exploiting the continuity of a class of mappings on C[0,1] which are defined in terms of Lamperti's extension [3] of the domain of definition of certain differential equations to all continuous (and not only differentiable) functions.

It is the purpose of this paper to present some results which complement those of Sussman. In particular, we consider the output convergence properties of Borel measurable (but not necessarily continuous) mappings defined on separable metric spaces. In Section II, general results are presented which explore the L_p convergence of the outputs of measurable systems when subjected to sequences of input random quantities converging in setwise and in-probability modes. These results are cast in the framework of systems whose domains are separable metric spaces. In Section III, a practical example, involving quantized feedback, is given of a system defined on C[a,b] which is Borel measurable but which is discontinuous with respect to the uniform topology. In Section IV we comment on continuous systems.

II. General Development and Main Results

Let (S,ρ) be a separable metric space and let $\mathscr{A}$ be the σ-algebra in S generated by the closed sets. Let $(\Omega, \mathscr{F}, P)$ be a probability space. An S-valued random variable will be a measurable function from $(\Omega,\mathscr{F})$ to $(S,\mathscr{A})$. Let X be an S-valued random variable, and let μ denote the measure induced on $\mathscr{A}$ by X, that is, for $A \in \mathscr{A}$, $\mu(A) = P\{X \in A\}$. Similarly, let $\{X_n;\ n=1,2,\ldots\}$ be a sequence of S-valued random variables with corresponding measures μ_n induced on $\mathscr{A}$. We note in passing that $\rho(X_n,X)$ is a real-valued random variable [4,p.225]; but if (S,ρ) is not separable, this need not be true. The random variables X_n are said to converge to X <u>in probability</u> if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{\rho(X,X_n) > \varepsilon\} = 0.$$

The measures μ_n are said to converge to μ <u>setwise</u> if, for any element A of $\mathscr{A}$,

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \quad .$$

Let $\mathscr{B}$ denote the Borel sets on $\mathbb{R}$. Consider a measurable function $k:(S,\mathscr{A}) \to (\mathbb{R},\mathscr{B})$ and an S-valued random variable Y. Then $k(Y) = k\circ Y$ is a real-valued random variable. We say that k(Y) belongs to L_p $(p \geq 1)$ if

$$\int_\Omega |k[Y(\omega)]|^p\, P(d\omega) < \infty \ .$$

If $k(Y) \in L_p$, we define the L_p norm as

$$\| k(Y) \| = \left[\int_\Omega |k[Y(\omega)]|^p\, P(d\omega) \right]^{1/p} \ .$$

In this section we will be interested in a sequence of S-valued random variables X_n that converge to X in such a way that $g(X_n)$ converges to g(X) in L_p, where g is a measurable function. The next theorem addresses this situation. Various consequences of the theorem are then investigated in the remainder of this section.

<u>Theorem 1</u>: Assume that $X_n \to X$ in probability and that $\mu_n \to \mu$ setwise. Suppose g is a measurable function from $(S,\mathscr{A})$ to $(\mathbb{R},\mathscr{B})$ such that g(X) and

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$g(X_n)$ belong to L_p. Then $g(X_n) \to g(X)$ in L_p if, and only if, $\| g(X_n)\| \to \| g(X)\|$.

Proof: Since $|\| g(X_n)\| - \| g(X)\| | \le \| g(X_n) - g(X)\|$, we see the necessity of $\| g(X_n)\| \to \| g(X)\|$.

Now assume that $\| g(X_n)\| \to \| g(X)\|$. For any bounded continuous function h mapping S to $\mathbb{R}$, we have via the triangle inequality

$$\| g(X_n) - g(X)\| \le \| g(X_n) - h(X_n)\| + \| h(X_n) - h(X)\| + \| h(X) - g(X)\| .$$

Let ε be an arbitrary positive number. Since S, with the topology generated by ρ, is a normal topological space, we know [5,p.198] that there exists a bounded continuous function h such that

$$\| h(X) - g(X)\| < \varepsilon/4.$$

Since h is continuous, $h(X_n) \to h(X)$ in probability. It then follows from the boundedness of h and the dominated convergence theorem [5,pp.124-125] that $\| h(X_n) - h(X)\| < \varepsilon/4$ for n sufficiently large.

For the remaining term, we have that

$$\| g(X_n) - h(X_n)\|^p = \int_S |g(x) - h(x)|^p \mu_n(dx).$$

Since h is bounded, there exists a finite number K such that

$$|g(x) - h(x)|^p \le 2^{p-1}|g(x)|^p + K \quad \text{for all } x \in S.$$

Since $\| g(X_n)\| \to \| g(X)\|$ and μ_n is setwise convergent to μ, we have [6,p.232] that

$$\int_S |g(x) - h(x)|^p \mu_n(dx) \to \int_S |g(x) - h(x)|^p \mu(dx) . \tag{1}$$

Therefore, if n is sufficiently large, we have that

$$| \, \| g(X_n) - h(X_n)\| - \| g(X) - h(X)\| \, | < \varepsilon/4.$$

Putting the three inequalities together, we have that if n is sufficiently large,

$$\| g(X_n) - g(X)\| < \varepsilon .$$

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In the above proof, notice that if g is bounded then the setwise convergence can be invoked [6, p.232] to result in (1). Thus when g is bounded there is no need to assume that $\| g(X_n)\| \to \| g(X)\|$. This result is given as the following corollary.

Corollary 1: Assume that $X_n \to X$ in probability and that $\mu_n \to \mu$ setwise. Let g be a bounded measurable function from $(S,\mathscr{A})$ to $(\mathbb{R},\mathscr{B})$. Then $g(X_n) \to g(X)$ in L_p.

In some cases the convergence in probability is necessary. For example, if $(S,\mathscr{A}) = (\mathbb{R},\mathscr{B})$ and g is any strictly monotonic function, then convergence in probability is a necessary condition in Theorem 1.

If μ is purely atomic with only one atom, that is, if there is a point $x \in S$ such that $P\{X = x\} = 1$, then convergence in probability of X_n to X is equivalent to weak convergence [4,p.25]. However, weak convergence is implied by setwise convergence [4,pp.11-12]. We summarize this observation as a corollary.

Corollary 2: If there exists a point $x \in S$ such that $P\{X = x\} = 1$, then the condition of convergence in probability can be omitted in Theorem 1.

Let x be a point in S such that $P\{X = x\} = 1$, and suppose that $g(x) = y$. For example, x and y might be the nominal input-output pair for a system represented by the function g. However, the system input might be subject to disturbances, so that the actual input is N. We see from the preceding two corollaries that if g is bounded, then the output g(N) is close to y, in an L_p sense, if

$$\sup_{A \in \mathscr{A}} |\tilde{\mu}(A) - I_A(x)|$$

is sufficiently small, where $\tilde{\mu}$ is the measure induced on $\mathscr{A}$ by N and I represents the indicator function.

Notice that if $\mu_n \not\to \mu$ setwise, then there exists a set $A \in \mathscr{A}$ such that $\mu_n(A) \not\to \mu(A)$. Letting g be the indicator function of the set A, we see that $\| g(X_n)\| = \mu_n(A)$ and $\| g(X)\| = \mu(A)$. Since $\mu_n(A) \not\to \mu(A)$, we see from Theorem 1 that $g(X_n) \not\to g(X)$ in L_p. This result is given as the following corollary.

Corollary 3: If $\mu_n \not\to \mu$ setwise, then there always exists a bounded measurable function $g:(S,\mathscr{A}) \to (\mathbb{R},\mathscr{B})$ such that $g(X_n) \not\to g(X)$ in L_p.

Corollary 3 illustrates the importance of setwise convergence in the present case. We note in passing that convergence with probability one, that is,

$$P\{\limsup \rho(X_n,X) = 0\} = 1 ,$$

does not imply setwise convergence. For example, let $(S,\mathscr{A}) = (\mathbb{R},\mathscr{B})$ and let X = 0 with probability one and $X_n = 1/n$ with probability one. Then $X_n \to X$ with probability one. In this case

$$\mu(B) = \begin{cases} 1 & \text{if } 0 \in B \\ 0 & \text{if } 0 \notin B \end{cases}$$

and

$$\mu_n(B) = \begin{cases} 1 & \text{if } 1/n \in B \\ 0 & \text{if } 1/n \notin B . \end{cases}$$

Letting $B = (0,\infty)$ we see that $\mu_n(B) = 1$ for all n but $\mu(B) = 0$.

Let $\mathcal{T}$ denote the topology on S generated by ρ. If

$$\sup_{G \in \mathcal{T}} |\mu_n(G) - \mu(G)| \to 0 ,$$

then it follows that $\mu_n \to \mu$ setwise. That is, suppose $A \in \mathcal{A}$. Then, since probability measures on S are regular [4,pp.7-8], there is a sequence $\{G^n_m\}_{m=1}^{\infty}$ of open sets containing A such that

$$\lim_{m\to\infty} \mu_n(G^n_m) = \mu_n(A) .$$

By diagonalization we can construct a sequence of open sets G'_m containing A such that

$$\lim_{m\to\infty} \mu_n(G'_m) = \mu_n(A), \quad n = 1,2,...$$

and

$$\lim_{m\to\infty} \mu(G'_m) = \mu(A).$$

We then have

$$|\mu_n(A) - \mu(A)| \leq |\mu_n(A) - \mu_n(G'_m)|$$
$$+ |\mu_n(G'_m) - \mu(G'_m)|$$
$$+ |\mu(A) - \mu(G'_m)| .$$

Letting $m \to \infty$, we get that

$$|\mu_n(A) - \mu(A)| \leq \sup_{G \in \mathcal{T}} |\mu_n(G) - \mu(G)| \to 0 .$$

Similarly, if

$$\sup_{F:(S- \ )\in \mathcal{T}} |\mu_n(F) - \mu(F)| \to 0 ,$$

then it follows that $\mu_n \to \mu$ setwise.

Let g be a measurable function from $(S,\mathcal{A})$ to $(\mathbb{R},\mathcal{B})$ and let D_g denote the set of discontinuity points of g. We note that $D_g \in \mathcal{A}$ [4,pp.225-226]. It follows from [4,p.31] that if g is bounded and $\mu(D_g) = 0$, then the weak convergence of μ_n to μ implies that

$$\int_S g(x)\mu_n(dx) \to \int_S g(x)\mu(dx).$$

We recall that $X_n \to X$ in probability implies that $\mu_n \to \mu$ weakly [4,p.26] and thus there is no need to assume setwise convergence. This observation is summarized in the following corollary.

Corollary 4: Assume that $X_n \to X$ in probability, and let g be a bounded measurable function from $(S,\mathcal{A})$ to $(\mathbb{R},\mathcal{B})$ such that $\mu(D_g) = 0$. Then $g(X_n) \to g(X)$ in L_p.

Let λ be any measure on $\mathcal{A}$ such that μ and μ_n (for n = 1,2,...) are absolutely continuous with respect to λ. Such a measure always exists, for example,

$$\lambda(A) = \tfrac{1}{2}\mu(A) + \sum_{n=1}^{\infty} 2^{-n-1}\mu_n(A) \ ; \ A \in \mathcal{A} .$$

Let f and f_n, respectively, be the Radon-Nikodym derivatives of μ and μ_n with respect to λ. The following result aids in characterizing setwise convergence.

Scheffé's Theorem [4,pp.223-224]: If $f_n(x) \to f(x)$ a.e.[λ], then

$$\sup_{A \in \mathcal{A}} |\mu(A) - \mu_n(A)| = \tfrac{1}{2}\int_S |f(x) - f_n(x)|\lambda(dx) \to 0.$$

Consider for the moment the case where the μ_n are absolutely continuous with respect to μ, and let λ be equal to μ. Then by Scheffé's Theorem, if $f_n(x) \to 1$ a.e.[μ], the μ_n converge setwise to μ. As an example, let S be C[a,b], the space of all continuous real-valued functions defined on [a,b], and let ρ be the uniform metric. The separability of (S,ρ) is easily seen by considering the class of all polynomials with rational coefficients. Assume that μ is Wiener measure. There has been considerable effort expended in establishing the absolute continuity of certain measures with respect to Wiener measure, and in characterizing the resulting Radon-Nikodym derivatives. This theory has been fairly well developed for random processes of the diffusion type (see, for example, [7]). It is straightforward to construct examples of sequences of random processes of the diffusion type such that the resulting Radon-Nikodym derivatives converge to unity. Thus Theorem 1 or Corollary 1 can be invoked to exhibit discontinuous functional transformations such that the outputs due to random processes of the diffusion type converge in L_p to the output due to a Wiener input as the diffusion type processes converge to a Wiener process.

III. A Measurable System which is not Continuous

As noted in Section II, the setwise input convergence required for the output convergence of all measurable systems is somewhat stronger than the input convergence required for the output convergence of continuous systems. Thus, the question arises as to whether or not there are any meaningful examples of systems which are measurable but not continuous. In this section we present one such system, modeling quantized feedback, which might arise in any of a number of applications.

As before, let C[a,b] denote the space of continuous real-valued functions defined on the real interval [a,b] with the uniform topology. Define the function $g:\mathbb{R} \to \mathbb{R}$ by

$$g(x) = \text{sgn}(x) \triangleq \begin{cases} 1; & \text{if } x \geq 0 \\ -1; & \text{if } x < 0 \end{cases} ;$$

and, for each positive integer n, define $g_n:\mathbb{R} \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} \mathrm{sgn}(x) & ; \text{ if } x \notin (-1/n,0) \\ 2n(x+1/2n); & \text{if } x \in (-1/n,0) \end{cases} .$$

Note that, for each n,

$$|g_n(x)-g_n(y)| \le |x-y|\cdot 2n$$

so that g_n is Lipschitz continuous on $\mathbb{R}$. Thus, for each n,

$$x_t^n = \int_a^t g_n(x_s^n)ds + y_t \quad ; \quad a \le t \le b \qquad (2)$$

defines a continuous mapping from C[a,b] to C[a,b] (as in [5], [6]). Suppose n is a positive integer; define Δ_t on [a,b] by $\Delta_t = (x_t^n - x_t^{n+1})$. Note that Δ_t is continuously differentiable on [a,b] and that $\Delta_a = 0$. Consider the set $E \triangleq \{t \in [a,b] | \Delta_t < 0\}$. Either E is empty or $c \triangleq \inf(E) < b$. Suppose the latter is true. Then we must have $\Delta_c = 0$ and $\Delta_c' = d\Delta_t/dt|_{t=c} < 0$. But

$$\Delta_c' = [g_n(x_c^n) - g_{n+1}(x_c^{n+1})]$$
$$= [g_n(x_c^n) - g_{n+1}(x_c^n)] \ge 0$$

since $g_n \ge g_{n+1}$. This is a contradiction and E must be empty; thus $\Delta_t > 0$ for all $t \in [a,b]$. This implies that x_t^n is a decreasing sequence. Note that

$$\sup_{a \le t \le b} |x_t^n| \le (b-a) + \sup_{a \le t \le b} |y_t| < \infty$$

for each positive integer n. Thus, there is a function $x = \{x_t; t \in [a,b]\}$ such that $x^n \downarrow x$. It is easily seen that we must then also have $(g_n \circ x_n) \downarrow (g \circ x)$, and the monotone convergence theorem [6, p.227] implies that

$$\lim_{n\to\infty} \int_a^t g_n(x_s^n)ds = \int_a^t g(x_s)ds \; ; \; t \in [a,b] .$$

We thus have

$$x_t = \int_a^t g(x_s)ds + y_t \quad ; \quad t \in [a,b] . \qquad (3)$$

For each positive integer n and for each $t \in [a,b]$, (2) defines a continuous mapping from C[a,b] to $\mathbb{R}$. Thus, since $x_t = \lim_{n\to\infty} x_t^n$, (3) defines a Borel measurable mapping (see [6, p.223]) from C[a,b] to $\mathbb{R}$ (for each $t \in [a,b]$). This mapping is discontinuous for any $t > a$, however, which is seen by considering the two inputs $y^1 = \{\varepsilon; t \in [a,b]\}$ and $y^2 = \{-\varepsilon; t \in [a,b]\}$ and letting $\varepsilon \to 0$.

Note that (3) is a model for a first-order system with a hard limiter (one-bit quantizer) in the feedback loop. Note also that the development of the above paragraph is easily generalized to include higher-level quantizers and other systems which can be written as limits of continuous mappings.

IV. A Note on Continuous Systems on C[a,b]

In this section we discuss briefly two applications of the output convergence approach to continuous systems. As noted in Section II, if the system of interest is a continuous mapping to $\mathbb{R}$, then the requirement of setwise convergence can be dropped from the hypothesis of the theorem. Moreover, we note that, in the continuous case, convergence in probability of the input implies convergence in probability of the output regardless of whether or not the output L_p norms converge. Also, if the input converges almost surely, then the output of a continuous system will converge almost surely.

Consider the system defined for $\{y_t; t \in [a,b]\} \in C[a,b]$ by

$$x_t = \int_a^t h(x_s)ds + y_t \; ; \; t \in [a,b] . \qquad (4)$$

This equation defines a continuous mapping from C[a,b] to C[a,b] provided h is Lipschitz continuous and satisfies other mild conditions (see [2]). Thus, if a sequence $\{Y^n; n=1,2,...\}$ of random processes defined in C[a,b] converges almost surely to a process Y, then the corresponding outputs to (4) $(\{X^n; n = 1,2,...\})$ will converge almost surely to the random process defined by

$$X_t = \int_a^t h(X_s)ds + Y_t \quad ; \; t \in [a,b], \qquad (5)$$

where (5) is as interpreted by Sussman [2]. Note that Wiener's construction of the Wiener process on $[0,\pi]$ is given by (see [8])

$$W_t = tg_0/\sqrt{\pi} + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} \sqrt{2/\pi}\, \sin(kt) g_k/k$$

where $\{g_k; k = 0,1,2,...\}$ are independent standard Gaussian random variables. Moreover, the sum on the right is almost surely uniformly convergent on $[0,\pi]$. Thus the sums

$$W_t^N = tg_0/\sqrt{\pi} + \sum_{n=1}^{N} \sum_{k=2^{n-1}}^{2^n-1} \sqrt{2/\pi}\, \sin(kt) g_k/k \; ;$$

$t \in [0,\pi]$, N = 1,2,..., define a sequence of random processes in $C[0,\pi]$ converging almost surely in the uniform topology to a Wiener process. Therefore, we see that we can approximate the response of (4) to a Wiener process by computing the response to a sum of sinusoids.

In a related application suppose that we again have a continuous system of the form (4) which, under ideal operating conditions, is subjected to a deterministic input $\{S_t; t \in [a,b]\} \in C[a,b]$. Since the system is continuous we are guaranteed that, if the actual input is corrupted by noise, i.e., if $\{y_t = (S_t + N_t); t \in [a,b]\}$ for some C[a,b] random process $\{N_t; t \in [a,b]\}$, then the output will still be close in probability to the ideal output as long as $P\{\sup_{a \le t \le b} |N_t| \ge \varepsilon\}$ is small for large ε.

For example, if $\{N_t;\ t \in [a,b]\}$ is a Wiener process on $[a,b]$ with zero mean and autocorrelation function $\{E\{N_t N_s\} = \alpha^2 \min(t,s);\ (t,s) \in [a,b] \times [a,b]\}$, then [4, p.80]

$$P\{ \sup_{a \le t \le b} |N_t| > \varepsilon\} =$$

$$\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)} \exp\left[-\pi^2\alpha^2(2k+1)^2/8\varepsilon^2\right]$$

which is arbitrarily close to zero for small enough α. Thus, the system represented by (4) is stable with respect to an additive Wiener process and the corresponding differential equation is stable with respect to additive "white noise". Note that, in view of Corollaries 1 and 2, any bounded measurable system is stable in this sense with respect to perturbations in the neighborhoods generated by setwise convergence since a single continuous function $\{S_t\ ;\ t \in [a,b]\}$ is an atom in $C[a,b]$.

V. Summary and Conclusions

In this paper we have considered the output convergence of systems which might be discontinuous when viewed as mappings on function spaces. In a sense, the results of this paper complement those of Wong and Zakai [1] and Sussman [2] in that they treat a class of systems slightly more general than the continuous systems considered in these earlier works. However, we see from Section II that, not surprisingly, a stronger form of input convergence is required for measurable systems to converge than is required for continuous systems. The in-probability convergence is not a particularly strong form of convergence, but, as noted in Section II, the setwise convergence is somewhat stronger than the usual types of weak convergence of measures. However, the relationship of setwise convergence to the Kolmogorov variational distance provides adequate means for verifiability. The practicality of considering discontinuous systems is seen in Section III, where it is seen that a very simple and commonplace system cannot be treated by results specific to continuous systems. We note that a wide class of systems have properties similar to those of the system derived in Section III.

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