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ASYMPTOTIC JOINT DISTRIBUTIONS OF FUNCTIONS OF THE ELEMENTS OF SAMPLE COVARIANCE MATRIX*

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1. Introduction

Several test statistics in multivariate analysis are based upon certain functions of the elements of the sample covariance matrix or sample correlation matrix. For example, the sample correlation coefficient, partial and multiple correlation coefficients, and various transformations of the sample correlation coefficients depend upon the elements of the sample covariance matrix and correlation matrix. The exact distributions of many of these statistics are quite complicated and so there is a need to derive asymptotic expressions of the above functions. In this paper, we consider asymptotic joint distributions of functions of the elements of the noncentral Wishart matrix and the associated noncentral correlation matrix.

In Section 2 of this paper, we derive the asymptotic joint distribution of certain functions of the elements of the noncentral Wishart matrix. The first term in the asymptotic expression is the multivariate normal density, whereas the second term involves partial derivatives of the multivariate normal density. Similar expressions are given for the case of the noncentral correlation matrix. The method used involves expanding the functions in terms of Taylor series and computing the characteristic functions. Olkin and Siotani (1976) obtained the first term in the asymptotic joint distribution of functions of the elements of the central correlation matrix. Siotani and Hayakawa (1964) obtained the first terms in the asymptotic joint distributions of the partial and multiple correlation coefficients by expressing them as functions of the elements of the central Wishart matrix. Konishi (1979) obtained the first two terms in the asymptotic joint distribution of various functions of the central correlation matrix; these results are special cases of the results given in this paper. In Section 3 of this paper, we studied the accuracy of the asymptotic expressions given in Section 2 for some special cases. Asymptotic expressions for the joint distributions of functions of the elements of the sample covariance matrix are given in Section 4 when the underlying distribution is not multivariate normal. Finally, applications of the results of this paper are discussed in Section 5.

2. Joint Distribution of the Functions of the Elements of the Noncentral Wishart Matrix

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ be distributed independently as multivariate normal with mean vectors $\underset{\sim}{\mu}_1, \ldots, \underset{\sim}{\mu}_n$ and covariance matrix $\Sigma = (\sigma_{ij})$. Then, the distribution of $S = \sum_{j=1}^{n} \underset{\sim}{X}_j \underset{\sim}{X}_j' = (S_{ij})$ is known to be the noncentral Wishart matrix with n degrees of freedom and $E(S/n) = \Sigma + (M/n) = \Omega$, with noncentrality parameter $M = \sum_{j=1}^{n} \underset{\sim}{\mu}_j \underset{\sim}{\mu}_j' = n(\nu_{ij})$. Now, let

$$T_i(S/n) = T_i(s_{11}, \ldots, s_{pp}, s_{12}, \ldots, s_{1p}, s_{23}, \ldots, s_{p-1,p})$$

for $i = 1, \ldots, k$, where $s_{ij} = S_{ij}/n$, are analytic functions in the neighborhood of $\Omega = (\omega_{ij})$. Also, let

$$a^{(i)}_{j_1 j_2} = \left(\frac{1+\delta_{j_1 j_2}}{2}\right) \frac{\partial}{\partial s_{j_1 j_2}} T_i(S/n)\Big|_{(S/n) = \Omega} \tag{2.1}$$

$$a^{(i)}_{j_1 j_2 \cdot j_3 j_4} = \left(\frac{1+\delta_{j_3 j_4}}{2}\right)\left(\frac{1+\delta_{j_1 j_2}}{2}\right) \frac{\partial^2}{\partial s_{j_3 j_4} \partial s_{j_1 j_2}} T_i(S/n)\Big|_{(S/n)=\Omega}$$

where δ_{hk} is given by

$$\delta_{hk} = \begin{cases} 1 & h=k \\ 0 & h \neq k. \end{cases}$$

The Taylor expansion of $T_i(S/n)$ about Ω is

$$T_i(S/n) = T_i(\Omega) + \sum_{j_1=1}^{p} \sum_{j_2=1}^{p} a^{(i)}_{j_1 j_2} ((S_{j_1 j_2}/n) - \omega_{j_1 j_2})$$

$$+ \frac{1}{2} \sum_{j_1, j_2, j_3, j_4}^{p} a^{(i)}_{j_1 j_2 \cdot j_3 j_4} ((S_{j_1 j_2}/n) - \omega_{j_1 j_2})((S_{j_3 j_4}/n) - \omega_{j_3 j_4})$$

$$+ \text{ higher order terms} \tag{2.2}$$

where $\sum_{j_1, j_2, \ldots, j_u}^{p}$ denotes the summation over all values of $j_1, j_2, \ldots, j_u$ varying from 1 to p.

Now, let

$$L_i = \sqrt{n}\ \{T_i(S/n) - T_i(\Omega)\}. \tag{2.3}$$

Using (2.2), we obtain the following expression for the joint characteristic function of $\underset{\sim}{L}' = (L_1, \ldots, L_k)$:

$$\psi(\underset{\sim}{t}) = E\{\exp(i \sum_{i=1}^{k} t_j L_j)\} \qquad (2.4)$$
$$= E_1(t) + E_2(t) + 0(n^{-1})$$

where $\underset{\sim}{t}' = (t_1, \ldots, t_k)$, and

$$E_1(\underset{\sim}{t}) = E\left\{\exp\left[\sqrt{n}\ i \sum_{g=1}^{k} \sum_{j_1, j_2}^{p} t_g\, a_{j_1 j_2}^{(g)} ((S_{j_1 j_2}/n) - \omega_{j_1 j_2})\right]\right\} \qquad (2.5)$$

$$= \exp\left[-i\sqrt{n}\ \mathrm{tr}\, B\,\Omega\right] \left|I - \frac{2iB\Sigma}{\sqrt{n}}\right|^{-\frac{1}{2}} \exp\left[i\,\mathrm{tr}\, M\left(I - \frac{2iB\Sigma}{\sqrt{n}}\right)^{-1} \frac{B}{\sqrt{n}}\right]$$

$$B = \sum_{i=1}^{k} t_i (a_{j_1 j_2}^{(i)}) = \sum_{i=1}^{k} t_i\ A^{(i)}$$

$$E_2(\underset{\sim}{t}) = E\left\{\frac{i\sqrt{n}}{2} \sum_{i=1}^{k} \sum_{j_1, j_2, j_3, j_4}^{p} t_i a_{j_1 j_2 j_3 j_4}^{(i)} ((S_{j_1 j_2}/n) - \omega_{j_1 j_2})((S_{j_3 j_4}/n) - \omega_{j_3 j_4})\right.$$

$$\left. \times \exp\left[i\sqrt{n} \sum_{i_1=1}^{k} \sum_{j_1, j_2}^{p} t_i a_{j_1 j_2}^{(i_1)} ((S_{j_1 j_2}/n) - \omega_{j_1 j_2})\right]\right\} \qquad (2.6)$$

Starting from (2.4), we obtain the following asymptotic expression for the joint characteristic function of $L_1, \ldots L_k$:

$$\Psi(\underset{\sim}{t}) = \exp(-\frac{1}{2}\, t'Q\, \underset{\sim}{t})\{1 + \frac{1}{2\sqrt{n}} \sum_{i=1}^{k} i\, t_i\, h_1 \tag{2.7}$$

$$+ \frac{1}{\sqrt{n}} \sum_{i_1,i_2,i_3}^{k} i\, t_{i_1} t_{i_2} t_{i_3} (h_2 + h_3) + (n^{-1})\}$$

where

$$Q = (Q_{i_1 i_2}), \quad Q_{i_1 i_2} = 2 \operatorname{tr} R^{(i_1)} R^{(i_2)} + 4 \operatorname{tr} R^{(i_1)} \psi^{(i_2)}$$

$$h_1 = \sum_{j_1,j_2,j_3,j_4}^{p} a^{(i)}_{j_1 j_2 \cdot j_3 j_4} (\sigma_{j_1 j_3} \omega_{j_2 j_4} + \sigma_{j_1 j_4} \omega_{j_2 j_3} + \sigma_{j_2 j_3} \nu_{j_1 j_4} + \sigma_{j_2 j_4} \nu_{j_1 j_3})$$

$$h_2 = \frac{4}{3} \operatorname{tr} R^{(i_1)} R^{(i_2)} R^{(i_3)} + 4 \operatorname{tr} R^{(i_1)} R^{(i_2)} \psi^{(i_3)} \tag{2.8}$$

$$h_3 = 2 \sum_{j_1,j_2,j_3,j_4}^{p} a^{(i_1)}_{j_1 j_2 \cdot j_3 j_4} (\Xi^{(i_2)}_{j_1 j_2} + T^{(i_2)}_{j_1 j_2})(\Xi^{(i_3)}_{j_3 j_4} + T^{(i_3)}_{j_3 j_4})$$

and

$$R^{(i)} = A^{(i)}\Sigma, \quad \psi^{(i)} = A^{(i)} M/n,$$

$$\Xi^{(i)} = \Sigma A^{(i)}\Omega, \quad T^{(i)} = \frac{M}{n} A^{(i)}\Sigma$$

and U_{ij} denotes the (i,j)th element of matrix $U = (U_{ij})$. By inverting the above characteristic function, we obtain the following expression for the joint density of $L_1,\ldots,L_k$:

$$f(L_1,\ldots,L_k) = N(\underset{\sim}{L},Q)\ \{1 + \frac{1}{2\sqrt{n}} \sum_{i=1}^{k} H_i(\underset{\sim}{L})\ h_1 \tag{2.9}$$

$$+ \frac{i}{\sqrt{n}} \sum_{i_1,i_2,i_3}^{k} H_{i_1 i_2 i_3}(\underset{\sim}{L})(h_2+h_3)+O(n^{-1})\}$$

where

$$N(\underset{\sim}{L},Q) = \frac{1}{(\sqrt{2\pi})^{k/2}|Q|^{\frac{1}{2}}} \exp\ (-\ \tfrac{1}{2}\ \underset{\sim}{L}'Q^{-1}\underset{\sim}{L})$$

$$H_{j_1}\ldots{}_{j_u}(\underset{\sim}{L})N(\underset{\sim}{L},Q) = (-1)^u \frac{\partial^u}{\partial L_{j_1}\ldots\partial L_{j_u}} N(\underset{\sim}{L},Q) \tag{2.10}$$

The correlation matrix is given by $R = S_0^{-\frac{1}{2}} S\ S_0^{-\frac{1}{2}} = (r_{ij})$ where $S_0 = \text{diag.}(S_{11},\ldots,S_{pp})$. Now, let

$$G_j(R) = G_j(r_{12},r_{13},\ldots,r_{1p},r_{23},\ldots,r_{2p},\ldots,r_{p-1,p}) \tag{2.11}$$

be an analytic function in the neighborhood of $\Omega_0^{-\frac{1}{2}}\ \Omega\ \Omega_0^{-\frac{1}{2}} = P^* = (\rho^*_{ij})$ where $\Omega_0 = \text{diag.}(\omega_{11},\ldots,\omega_{pp})$. If we denote $r_{k_1k_2}$ by $T_{k_1k_2}(S/n)$, Eq. (2.11) can be written as

$$G_j(R) = G_j(T_{12}(S/n),\ldots,T_{p-1,p}(S/n)) = (G \circ T)_j(S/n). \tag{2.12}$$

Now, let

$$c^{(j)}_{k_1k_2} = \frac{\partial}{\partial r_{k_1k_2}} G_j(R) \Big|_{R = P^*}$$

$$c^{(j)}_{k_1k_2\cdot k_3k_4} = \frac{\partial^2}{\partial r_{k_3k_4}\,\partial r_{k_1k_2}} G_j(R) \Big|_{R = P^*}$$

in Eq. (2.1). Then we obtain

$$a^{(j)}_{j_1j_2} = \left(\frac{1+\delta_{j_1j_2}}{2}\right)\left(\frac{\partial}{\partial s_{j_1j_2}}\right)(G\cdot T)_j(S/n)\Big|_{(S/n)=\Omega} \quad (2.13)$$

$$= \frac{1}{2}\sum_{k_1\neq k_2} c^{(j)}_{k_1k_2}\,\zeta^{(k_1k_2)}_{j_1j_2},$$

$$a^{(j)}_{j_1j_2\cdot j_3j_4} = \frac{1}{4}(1+\delta_{j_3j_4})(1+\delta_{j_1j_2})\frac{\partial^2}{\partial s_{j_3j_4}\,\partial s_{j_1j_2}}(G\cdot T)_j(S/n)\Big|_{(S/n)=\Omega}$$

$$= \sum_{k_1<k_2}\sum_{k_3<k_4} c^{(j)}_{k_1k_2\cdot k_3k_4}\,\zeta^{(k_1k_2)}_{j_1j_2}\,\zeta^{(k_3k_4)}_{j_3j_4}$$

$$+ \sum_{k_1<k_2} c^{(j)}_{k_1k_2}\,\zeta^{(k_1k_2)}_{j_1j_2\cdot j_3j_4} \quad (2.14)$$

where

$$\zeta^{(k_1k_2)}_{j_1j_2} = \begin{cases} \dfrac{1}{2\sqrt{\omega_{k_1k_1}\,\omega_{k_2k_2}}} & j_1 = k_1, j_2 = k_2 \text{ or } j_1 = k_2, j_2 = k_1 \\[2ex] \dfrac{-\rho^*_{k_1k_2}}{2\omega_{k_1k_1}} & j_1 = j_2 = k_1 \\[2ex] \dfrac{-\rho^*_{k_1k_2}}{2\omega_{k_2k_2}} & j_1 = j_2 = k_2 \\[2ex] 0 & \text{otherwise} \end{cases} \quad (2.15)$$

$$
\zeta^{(k_1k_2)}_{j_1j_2.j_3j_4} = \begin{cases} \dfrac{-1}{4\,\omega^{1/2}_{k_1k_1}\,\omega^{3/2}_{k_2k_2}} & \text{for any 3 values of } j_1, j_2, j_3, j_4 \text{ equal to } k_2 \\[2ex] \dfrac{-1}{4\,\omega^{3/2}_{k_1k_1}\,\omega^{1/2}_{k_2k_2}} & \text{for any 3 values of } j_1, j_2, j_3, j_4 \text{ equal to } k_1 \\[2ex] \dfrac{\rho^*_{k_1k_2}}{4\,\omega_{k_1k_1}\,\omega_{k_2k_2}} & \text{for } j_1=j_2=k_1,\ j_3=j_4=k_2 \\ & \text{or } \ j_1=j_2=k_2,\ j_3=j_4=k_1 \\[2ex] \dfrac{3\rho^*_{k_1k_2}}{4\,\omega^2_{k_ik_i}} & \text{for all } j_1=j_2=j_3=j_4=k_i,\ i=1,2 \\[2ex] 0 & \text{otherwise} \end{cases}
$$

Substituting (2.13) and (2.14) in Eq. (2.9) we obtain the asymptotic density for functions of the elements of correlation matrix.

3. An Empirical Study on the Accuracy of the Asymptotic Expressions

In this section, we study the accuracy of the asymptotic expressions derived in Section 2 for some special cases.

We will first study the accuracy of the approximation for the distribution of the noncentral chi-square. Let $y_i = S_{ii}/\sigma_{ii}$ for $i=1,2,\ldots,p$. Then y_1 is distributed as the noncentral chi-square distribution with n degrees of freedom and with the noncentrality parameter $\gamma = n\nu_{11}/\sigma_{11}$. Now let

$$\exp(-\gamma/2)\sum_{j=0}^{\infty}\frac{\gamma^j}{j!\;2^{(n/2)+2j}\Gamma((n/2)+j)}\int_0^u x^{(n/2)+j-1}\exp(-x/2)dx = \beta \tag{3.1}$$

and u is given by

$$\frac{1}{2^{n/2}\Gamma(n/2)}\int_0^u \exp(-x/2)\;x^{(n/2)-1}dx = 0.95. \tag{3.2}$$

The left-side of (3.1) is the probability integral of y_1. Also the left-side of (3.1) is equivalent to the left-side of (3.2) when $\gamma=0$. Table 1 given below compares the exact values of β with the corresponding values obtained by using the asymptotic expression (2.9).

TABLE 1

Comparison of the Asymptotic Expression with Exact Expression for the Noncentral Chi-Square Distribution

n	u	γ	$O(1)$	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
5	11.071	0.	.9726	-.0358	.9368	.95
		16.47	.1163	-.0091	.1072	.10

TABLE 1 (Continued)

n	u	γ	O(1)	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
		2.67	.7727	.0245	.7972	.80
10	18.307	0.	.9684	-.0260	.9424	.95
		20.53	.1132	-.0082	.1050	.10
		3.71	.7820	.0159	.7979	.80
20	31.41	0	.9644	-.0186	.9458	.95
		26.13	.1104	-.0071	.1033	.10
		5.18	.7880	.0105	.7985	.80
30	43.773	0.	.9623	-.0153	.9470	.95
		30.38	.1089	-.0064	.1025	.10
		6.31	.7906	.0083	.7988	.80

The values in the column "O(1)" give the values of β when the first term in the asymptotic expression (2.9) is taken whereas the column "$O(1)+O(n^{-\frac{1}{2}})$" gives the values of β when the first two terms in (2.9) are taken. The column "$O(n^{-\frac{1}{2}})$" gives the contribution of the second term in (2.9) to the value of β. The exact values given in the last column are taken from Owen (1962).

We will compare the asymptotic expansion given by (2.9) for the distribution of y_1+y_2 with the corresponding exact expression. When $\rho = \rho_{12} = 0$ the distribution of y_1+y_2 is the noncentral chi-square with 2n degrees of freedom and with $\gamma_1+\gamma_2$ as the noncentrality parameter where $\gamma_1 = n\nu_{11}/\sigma_{11}$ and $\gamma_2 = n\nu_{22}/\sigma_{22}$. When $\rho_{12} \neq 0$, the distribution of y_1+y_2 is the same as the distribution of a quadratic form in the noncentral case. Now, let

$$P[y_1 + y_2 \leq u | \rho, \gamma_1, \gamma_2] = \beta \qquad (3.3)$$

where

$$P[y_1 + y_2 \le u | \rho=0, \gamma_1=\gamma_2=0] = 0.95 \tag{3,4}$$

In Table 2, the entries in the column "$O(1)+O(n^{-\frac{1}{2}})$" represent the approximate value of β when the first two terms in the asymptotic expression (2.9) are taken. The entries in the columns "$O(1)$" and "$O(n^{-\frac{1}{2}})$" represent respectively the contribution of the first term and the second term of the above approximate value of β. The entries in the column "exact values" are computed by Monte Carlo methods using the IMSL subroutine GGNSM. In computing the simulated values, 5000 trials are performed.

TABLE 2

Comparison of the Asymptotic Expression with Exact Expression for the Distribution of Sum of Correlated Chi-Square Variables

n	u	ρ	γ_1	γ_2	$O(1)$	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
10	31.41	0.	0.	0.	.9644	-.0186	.9458	.95
		.5	0.	0.	.9467	-.0230	.9237	.9364
		.5	4.	2.	.7353	.0238	.7591	.7574
		.9	0.	0.	.9100	-.0192	.8908	.9032
		.9	4.	2.	.7092	.0310	.7432	.7362
25	67.505	0.	0.	0.	.9600	-.0119	.9481	.95
		.2	0.	0.	.9570	-.0125	.9444	.9454
		.8	0.	0.	.9142	-.0126	.9016	.9018
		.5	4.	2.	.5730	.0289	.6019	.5960

We now discuss the distribution of the ratio $F_0 = y_1/y_2$. The distribution of F_0 is known (Bose (1935) for $\gamma_1=\gamma_2=0$ to be

$$f(F_0) = \frac{2^n(1-\rho^2)^{n/2}\Gamma[\frac{1}{2}(n+1)]\ F_0^{n-1}(1+F_0^2)}{\sqrt{\pi}\ \Gamma(\frac{1}{2}n)[1+(1+F_0^2)^2 - 4\rho^2F_0^2]^{(n+1)/2}} \tag{3.5}$$

Finney (1938) showed that

$$\Pr[F_0 \leq t^2] = 1 - I_x(\tfrac{1}{2}n, \tfrac{1}{2}n) \tag{3.6}$$

where

$$x = \tfrac{1}{2}\left[1 - \frac{(t-t^{-1})}{\{(t+t^{-1})^2 - 4\rho^2\}^{\frac{1}{2}}}\right]$$

$$I_x(a,b) = \frac{1}{\beta(a,b)} \int_0^x y^{a-1}\,(1-y)^{b-1}\,dy.$$

Krishnaiah et al. (1965) obtained an alternative expression for $\Pr[F_0 \leq 0]$ when $\gamma_1=\gamma_2=0$ and also gave tables for this distribution. In Table 3, we give the values of $\Pr[F_0 \leq u|\rho,\gamma_1,\gamma_2] = \beta$ by using the asymptotic expressions (2.9), where u is the 75% critical value of the central F distribution with (n,n) degrees of freedom. In the last column, the entries with * are obtained by using the formula (3.6) and the remaining entries in this column are obtained by simulation for 5000 trials.

TABLE 3

Comparison of the Asymptotic Expression with Exact Expression for the Distribution of the Ratio of Correlated Chi-square Variables

n	u	ρ	γ_1	γ_2	O(1)	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
24	1.3214	0.	0.	0.	.7844	-.0370	.7474	.75 *
		.2	0.	0.	.7892	.0373	.7519	.7543*
		.5	0.	0.	.8183	-.0385	.7798	.7816*
		.8	0.	0.	.9053	-.0356	.8697	.8675*
		0.	12.	12.	.7981	-.0378	.7604	.7566
		.8	12.	12.	.8444	-.0389	.8055	.7942

n	u	ρ	γ_1	γ_2	O(1)	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
40	1.2397	0.	0.	0.	.7758	-.0272	.7486	.75*
		.2	0.	0.	.7804	-.074	.7530	.7544*
		.5	0.	0.	.8093	-.0285	.7808	.7818*
		.8	0.	0.	.8968	-.0272	.8696	.8683*
		0.	20.	20.	.7893	-.0278	.7615	.7564
		.8	20.	20.	.8352	-.0290	.8062	.8038

Next, we study the accuracy of the asymptotic expression for the case of the distribution of the sample correlation coefficient $r_{12} = r$. When $\gamma_1 = \gamma_2 = 0$. The distribution of r was first found by Fisher (1915), and Hotelling (1953) has expressed the distribution in terms of the hypergeometric function. When $\rho \neq 0$, the distribution of r is complicated and the cumulative distribution of r has been tabulated by David (1938). Pillai (1946) suggested the transformations $g(r) = (r-\rho)/(1-r\rho)$ that renders the distribution $\frac{g(r)\sqrt{n-2}}{\sqrt{1-(g(r))^2}}$ close to the Student's t distribution. In Table 4 we compare the asymptotic expressions for

$$\Pr\{r \leq u \mid \rho, \gamma_1, \gamma_2\} \quad \text{and} \quad \Pr\{g(r) \leq u \mid \rho, \gamma_1, \gamma_2\}$$

with the corresponding values. The exact values with * are taken from the tables of David (1938) and the remaining exact values are obtained by simulation with 5000 trials.

TABLE 4

Comparison of the Asymptotic Expression with Exact Expression for the Distributions of Functions of the Sample Correlation Coefficient

n	ρ	u	γ_1	γ_2	Stat.	O(1)	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
49	.5	.5	0.	0.	r	.5000	-.0142	.4858	.4856*

TABLE 4 (Continued)

n	ρ	u	γ_1	γ_2	Stat.	$O(1)$	$O(n^{-\frac{1}{2}})$	$O(1)+O(n^{-\frac{1}{2}})$	Exact
49	.5	0.	0.	0.	$g(r)$	.5000	-.0142	.4358	.4856*
		.5	24.5	12.25	r	.8748	.0106	.8854	.8818
		.0	24.5	12.25	$g(r)$	.8948	-.0088	.8860	
		.5	24.5	24.5	r	.9235	.0123	.9358	.9370
		0.	24.5	24.5	$g(r)$	.9438	-.0077	.9361	
		.35	0.	0.	r	.0808	.0156	.0964	.0966*
		-.1818	0.	0.	$g(r)$	.1016	-.0063	.0952	
		.35	24.5	12.25	r	.4486	-.0064	.4422	.4486
		-.1818	24.5	12.25	$g(r)$	.4491	-.0068	.4422	
		.35	24.5	24.5	r	.5568	-.0044	.5524	.5586
		-.1818	24.5	24.5	$g(r)$	.5573	-.0049	.5524	
24	.8	.7	0.	0.	r	.0868	.0349	.1217	.1183*
		-.2273	0.	0.	$g(r)$	.1328	-.0175	.1152	
		.7	12.	6.	r	.8341	.0208	.8549	.8434
		-.2273	12.	6.	$g(r)$	.8800	-.022	.8576	
		.7	12.	12.	r	.9105	.0295	.9400	.9370
		-.2273	12.	12.	$g(r)$	.9600	-.0180	.9420	
		.55	0.	0.	r	.0003	.0022	.0025	.0098*
		-.4464	0.	0.	$g(r)$	.0144	-.0030	.0114	
		.55	12.	6.	r	.3870	-.0069	.3802	.3680
		-.4464	12.	6.	$g(r)$	.3924	-.0124	.3800	
		.55	12.	12.	r	.5534	-.0057	.5478	.5388
		-.4464	12.	12.	$g(r)$	.5547	-.0069	.5477	

4. Asymptotic Distributions of Functions of the Elements of the Sample Covariance Matrix for Nonnormal Populations

Let $X = [\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n]$ where the $p \times 1$ random vectors $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are distributed independently. Also, let $S = XX' = (S_{ij})$ be the sample sums of squares and cross-products matrix such that $E(S/n) = \Omega = (\omega_{ij})$, $s_{ij} = S_{ij}/n$ and

$$Y = \sqrt{n}\left(\frac{S}{n} - \Omega\right) = (y_{ij})$$

In addition, let the functions

$$T_i(S/n) = T_i(s_{11}, \ldots, s_{pp}, s_{12}, \ldots, s_{1p}, s_{23}, \ldots, s_{p-1,p})$$

be analytic in the neighborhood of Ω for $i=1,2,\ldots,k$. The Taylor expansion of $L_i = \sqrt{n}\,(T_i(S/n) - T_i(\Omega))$ about Ω is

$$L_i = \sum_{j_1,j_2} a^{(i)}_{j_1j_2}\, y_{j_1j_2} + \frac{1}{2\sqrt{n}} \sum_{j_1,j_2} \sum_{j_3,j_4} a^{(i)}_{j_1j_2\cdot j_3j_4}\, y_{j_1j_2}\, y_{j_3j_4}$$

$$+ O(n^{-1}).$$

Hence

$$\mu_i = E(L_i) = \frac{1}{2\sqrt{n}} \sum_{j_1,j_2} \sum_{j_3,j_4} a^{(i)}_{j_1j_2\cdot j_3j_4}\, \kappa(j_1j_2, j_3j_4) + O(n^{-1})$$

$$\mu_{ij} = E(L_iL_j) = \sum_{j_1,j_2} \sum_{j_3,j_4} a^{(i)}_{j_1j_2}\, a^{(j)}_{j_3j_4}\, \kappa(j_1j_2, j_3j_4) + O(n^{-1})$$

$$\mu_{ij\ell} = E(L_iL_jL_\ell) = \sum_{j_1,j_2} \sum_{j_3,j_4} \sum_{j_5,j_6} a^{(i)}_{j_1j_2}\, a^{(j)}_{j_3j_4}\, a^{(\ell)}_{j_5j_6}\, \kappa(j_1j_2, j_3j_4, j_5j_6)$$

$$+ O(n^{-1}) \qquad (4.1)$$

where

$$\kappa(j_1j_2, j_3j_4) = E(y_{j_1j_2} y_{j_3j_4})$$

$$\kappa(j_1j_2, j_3j_4, j_5j_6) = E(y_{j_1j_2} y_{j_3j_4} y_{j_5j_6}).$$

Now, let $\kappa^{(i)}_{j_1j_2 \cdots j_u}$ denote the cumulant of order u for $\underset{\sim}{X}_i$ where $j_1, j_2, \ldots, j_u = 1, \ldots, p$. Then

$$\kappa(j_1j_2, j_3j_4) = \overline{\kappa_{j_1j_2j_3j_4}} + \sum^{4} \overline{\kappa_{j_1}\kappa_{j_2j_3j_4}} + \overline{(\kappa_{j_1j_3}\kappa_{j_2j_2}} + \overline{\kappa_{j_1j_4}\kappa_{j_2j_3})} + \sum^{4} \overline{\kappa_{j_1j_3}\kappa_{j_2j_4}} \tag{4.2}$$

$$\begin{aligned} &\kappa(j_1j_2, j_3j_4, j_5j_6) \\ &= \frac{1}{\sqrt{n}} \overline{\kappa_{j_1j_2j_3j_4j_5j_6}} + \sum^{6} \overline{\kappa_{j_1j_2j_3j_4j_5}\kappa_{j_6}} + \sum^{10} \overline{\kappa_{j_1j_2j_3}\kappa_{j_4j_5j_6}} \\ &+ \sum^{12} \overline{\kappa_{j_1j_2j_3j_5}\kappa_{j_4j_6}} + \sum^{12} \overline{\kappa_{j_1j_2j_3j_5}\kappa_{j_4}\kappa_{j_6}} + \sum^{48} \overline{\kappa_{j_1j_2j_3}\kappa_{j_4j_5}\kappa_{j_6}} \\ &+ \sum^{8} \overline{\kappa_{j_1j_3j_5}\kappa_{j_2}\kappa_{j_4}\kappa_{j_6}} + \sum^{8} \overline{\kappa_{j_1j_4}\kappa_{j_2j_5}\kappa_{j_3j_6}} + \sum^{24} \overline{\kappa_{j_1j_4}\kappa_{j_2j_5}\kappa_{j_3}\kappa_{j_6}} \end{aligned} \tag{4.3}$$

The expressions under "———" represent the average values over n samples. For instance:

$$n \, \overline{\kappa_{j_1j_3} \, \kappa_{j_2j_4}} = \sum_{i=1}^{n} \kappa^{(i)}_{j_1j_3} \, \kappa^{(i)}_{j_2j_4}$$

The summations in Eqs. (4.2) and (4.3) are over the possible ways of grouping the subscripts and the number of terms resulting is written over $\sum$. Eqs. (4.2) and (4.3) coincide with the expressions of Kaplan (1952) when $\underset{\sim}{X}_1, \ldots \underset{\sim}{X}_n$ are identically distributed.

So the cumulants of $\underset{\sim}{L} = (L_1, \ldots, L_k)$ are

$$K_i = u_i$$

$$K_{ij} = u_{ij} + O(n^{-1}) \tag{4.4}$$

$$K_{ij\ell} = u_{ij\ell} - (u_i u_{j\ell} + u_j u_{i\ell} + u_\ell u_{ij}) + O(n^{-1})$$

for $i, j, \ell = 1, \ldots, k$. An equivalent equation (see Kendall and Stuart (1961))

$$\kappa(j_1 j_2, j_5 j_6)\kappa(j_3 j_4, j_7 j_8) + \kappa(j_1 j_2, j_7 j_8)\kappa(j_3 j_4, j_5 j_6)$$

$$= - \kappa(j_1 j_2, j_3 j_4)\kappa(j_5 j_6, j_7 j_8) + O(n^{-1}) \tag{4.5}$$

is used in calculating $u_i u_{j\ell}$, $u_j u_{i\ell}$, $u_\ell u_{ij}$ in Eq. (4.4).

The approximated characteristic function of $\underset{\sim}{L}$ is

$$E[\exp(i\, \underset{\sim}{t}'\, \underset{\sim}{L})] = \exp(-\tfrac{1}{2}\, \underset{\sim}{t}\, Q\, \underset{\sim}{t} + i \sum_{i=1}^{k} t_i K_i$$

$$+ i^3 \sum_{i,j,\ell}^{k} \frac{1}{\delta(i,j,\ell)} t_i t_j t_\ell K_{ij\ell}) + O(n^{-1})$$

$$= \exp(-\tfrac{1}{2}\, \underset{\sim}{t}\, Q\, \underset{\sim}{t}) \{1 + i \sum_{i=1}^{k} t_i K_i \tag{4.6}$$

$$+ i^3 \sum_{i,j,}^{k} \frac{1}{\delta(i,j,\ell)} t_i t_j t_\ell K_{ij\ell}\} + O(n^{-1})$$

where $Q = (K_{ij})$ and

$$\delta(i,j,\ell) = \begin{cases} 3! & ; \quad i = j = \ell \\ 2! & ; \quad \text{any two values of } i,j,\ell \text{ are equal} \\ 1 & ; \quad i \neq j \neq \ell \end{cases}$$

Inverting Eq. (4.6), we obtain the following asymptotic expression for the joint density of $L_1,\ldots,L_k$:

$$f(L_1,\ldots,L_k) = N(\underset{\sim}{L},Q) \{1 + \sum_{i=1}^{k} H_i(\underset{\sim}{L}) K_i + \sum_{i,j,\ell}^{k} \frac{1}{\delta(i,j,\ell)} H_{ij\ell}(\underset{\sim}{L}) K_{ij\ell}\} + O(n^{-1}) \tag{4.7}$$

where $N(\underset{\sim}{L},Q)$, $H_i(\underset{\sim}{L})$ and $H_{ij}(\underset{\sim}{L})$ are as defined in Eq. (2.10).

5. Applications of the Distributions of Functions of the Elements of the Covariance Matrix and Correlation Matrix

In this section, we discuss some applications of the results of Section 2 in simultaneous tests of hypotheses on the elements of the covariance matrix and correlation matrix.

Let $M=0$, $H_{ij}: \sigma_{ij}=\sigma_{0ij}$, $A_{ij}: \sigma_{ij}\neq\sigma_{0ij}$, $H^*_{ij}: \rho_{ij}=\rho_{0ij}$ and $A^*_{ij}: \rho_{ij}\neq\rho_{0ij}$ We will first discuss the problem of testing the hypotheses H_{ij} simultaneously against the alternative hypotheses A_{ij}. In this case, the hypothesis H_{ij} is accepted if

$$\begin{aligned} b_1 &\leq s_{ij}-\sigma_{0ij} \leq a_1, \quad \text{for } i<j \\ b_2 &\leq \frac{s_{ii}}{\sigma_{0ii}} \leq a_2 \end{aligned} \tag{5.1}$$

for $i,j=1,2,\ldots,p$, where

$$\begin{aligned} &\Pr[b_1 \leq s_{ij}-\sigma_{0ij} \leq a_1,\ i<j=1,2,\ldots,p,\ b_2 \leq \frac{s_{ii}}{\sigma_{0ii}} \leq a_2, \\ &\quad i=1,\ldots,p\,|\,H_1] = (1-\alpha) \end{aligned} \tag{5.2}$$

and $H_1 = \bigcap_{i\leq j} H_{ij}$. For practical purposes, we may choose the constants a_1 and b_1, a_2 and b_2 such that $a_1= -b_1$ and $a_2= 1/b_2$. We can propose similar procedures for testing the hypotheses H_{ij} against one-sided alternatives.

Next, consider the problem of testing the hypotheses $H_{11},\ldots,H_{pp}$ simultaneously against $A_{11},\ldots,A_{pp}$. In this case, we accept H_{ii} if

$$b_2 \leq \frac{s_{ii}}{\sigma_{0ii}} \leq a_2$$

and reject it otherwise where

$$P[b_2 \leq \frac{s_{ii}}{0ii} \leq a_2;\ i=1,\ldots,p \mid \bigcap_{i=1}^{p} H_{ii}] = (1-\alpha). \quad (5.3)$$

Also, consider the problem of testing the hypotheses H^*_{ij}(i j=1,2,...,p) simultaneously against the alternatives A^*_{ij} where H^*_{ij}: $\rho_{ij} = \rho_{0ij}$ and A^*_{ij}: $\rho_{ij} \neq \rho_{0ij}$. In this case, we accept or reject H^*_{ij} according as

$$(r_{ij}-\rho_{0ij})^2 \lesseqgtr c_\alpha$$

where

$$\Pr[(r_{ij}-\rho_{0ij})^2 \leq c_\alpha \cdot\ i<j=1,2,\ldots,p \mid H^*] = (1-\alpha)$$

and $H^* = \bigcap_{i<j} H^*_{ij}$.

The results of Section 2 are useful in computing approximate values of the critical values associated with the above tests. The results of Section 2 are also useful in finding approximate critical values of the tests of Krishnaiah (1975) for testing the hypothesis $\sigma_{11}=\ldots=\sigma_{pp}$

against different alternatives when the correlation matrix is known as well as the procedure of Roy and Bargmann (1958) for testing the hypothesis that the covariance matrix of multivariate normal is diagonal.

In the applications discussed above, we assumed that the matrix $S = (S_{ij})$ is the central Wishart matrix. But situations arise where the model itself is not correct. For example, if we assume that $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are distributed independently and identically as multivariate normal with a common mean vector $\underset{\sim}{\mu}$ and covariance matrix Σ, then S is the central Wishart matrix when $S = \sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}.)(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}.)'$ and $n\bar{\underset{\sim}{X}}. = \sum_{j=1}^{n} \underset{\sim}{X}_j$. But, if the mean vectors are given by $E(\underset{\sim}{X}_j) = \underset{\sim}{\mu}_j$, then S is the noncentral Wishart matrix. So, the results given in Section 2 for the case of the noncentral Wishart matrix and noncentral correlation matrix are useful in studying the robustness of several test procedures on the elements of Σ if the assumption of common mean vector for $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ is violated.

Next, consider the model,

$$X_{1j} = u_j + \delta_j$$

$$X_{2j} = v_j + \epsilon_j$$

when $u_j = \alpha + \beta v_j$, $(j=1,2,\ldots,n)$ and α, β are unknown constants. Also, we assume that v_j's are distributed independently and identically as normal with a common mean μ and variance σ^2. In addition, ϵ_j's and δ_j's are distributed as normal and

$$E(\epsilon_j) = E(\delta_j) = 0$$

$$\text{cov}(\epsilon_j, \delta_j) = 0, \ \text{cov}(v_j, \epsilon_j) = \text{cov}(v_j, \delta_j) = 0$$

$$\text{Var}(\epsilon_j) = \sigma_\epsilon^2, \ \text{Var}(\delta_j) = \sigma_\delta^2.$$

We also assume that the random vectors $(X_{1j}, X_{2j}, u_j, v_j, \epsilon_j, \delta_j)$ are distributed independent of each other for different values of j. When $\lambda = \sigma_\delta^2/\sigma_\epsilon^2$ is known, the maximum likelihood estimate of β is known (see Kendall and Stuart (1973), Chapter 29) to be

$$\hat{\beta} = \frac{(S_{11} - \lambda S_{22}) + \{(S_{11} - \lambda S_{22})^2 + 4\lambda S_{12}^2\}^{\frac{1}{2}}}{2S_{12}}$$

where $S = (S_{ij}) = \sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{X}.)(\underset{\sim}{X}_j - \bar{X}.)'$, $\underset{\sim}{X}_j' = (X_{1j}, X_{2j})$, and $n\bar{X}. = \sum_{j=1}^{n} \underset{\sim}{X}_j$. The results in Section 2 of this paper are useful in obtaining an asymptotic expression for the distribution of $\hat{\beta}$.

Now, let $\underset{\sim}{Y}_j' = (Y_{1j}, Y_{2j})$, $(j = 1, 2, \ldots, n)$ be distributed independently as a bivariate normal with mean vector (μ_{1j}, μ_{2j}) and covariance matrix $\Sigma = \sigma^2 I$ where I is an identity matrix. Also, let $\mu_{1j} = \alpha + \beta\mu_{2j}$. Then, the maximum likelihood estimate $\hat{\hat{\beta}}$ of β is known (e.g., see Anderson (1976)) to be

$$\hat{\hat{\beta}} = \frac{S_{11} - S_{22} + \{(S_{11} - S_{22})^2 + 4S_{12}^2\}^{\frac{1}{2}}}{2S_{12}}$$

where $S = (S_{ij}) = \sum_{j=1}^{n} (\underset{\sim}{Y}_j - \underset{\sim}{\bar{Y}}.)(\underset{\sim}{Y}_j - \underset{\sim}{\bar{Y}}.)'$ and $n\underset{\sim}{\bar{Y}}. = \sum_{j=1}^{n} \underset{\sim}{Y}_j$.

Since S is distributed as the noncentral Wishart matrix, an asymptotic expression for the distribution of $\hat{\hat{\beta}}$ can be detained as a special case of the results of Section 2. Here, we note that Kunimoto (1980) has recently obtained an asymptotic expression for the distribution of $\hat{\hat{\beta}}$.

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In this paper, the authors give asymptotic expressions for the joint distribution of the functions of the elements of the sample covariance matrix and sample correlation matrix in the noncentral cases when the underlying distribution is multivariate normal. Accuracy of the above expressions is also studied. Also, asymptotic expressions are given for functioning of the elements of the sample covariance matrix for nonnormal populations. Finally, some applications of the above results are discussed.

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