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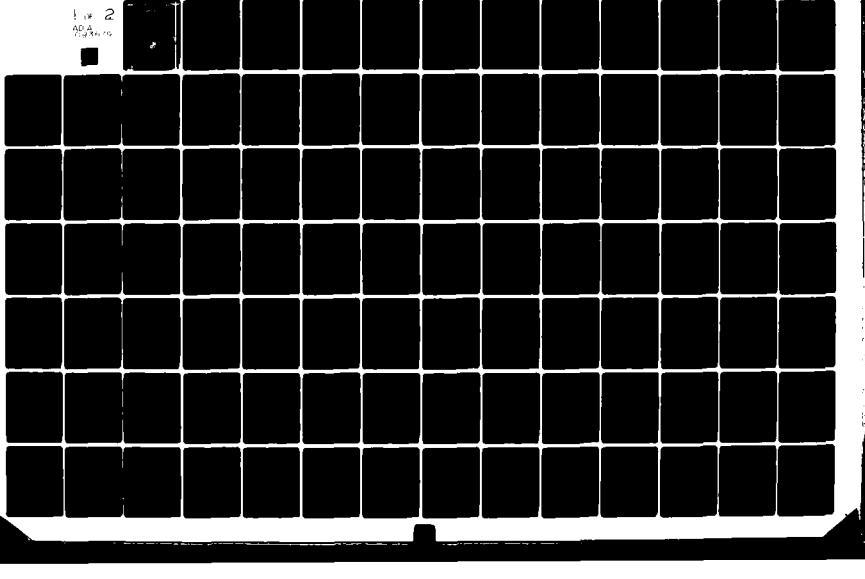

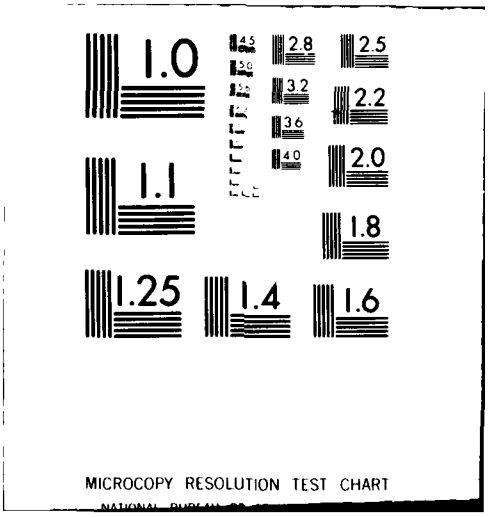
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A Tutorial on

Markov Renewal Theory

Semi-Regenerative Processes

and Their Applications

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July, 1980
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This is a tutorial paper on Markov renewal processes, semi-regenerative processes and their applications to many fields of interest. It was presented in the tutorial sessions of the O.R.S.A./T.I.M.S. meetings in Colorado Springs, November, 1980.

The paper reviews the basic ingredients of the processes discussed such as: structure of the process, semi-Markov kernels, Markov renewal functions, and integral equations of semi-regenerative processes. Considerable time is spent discussing several applications to combat line-of-sight, queueing, ...

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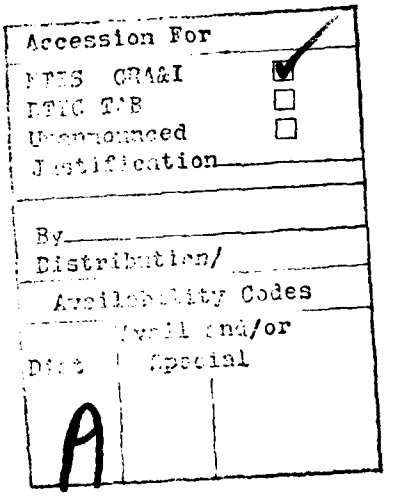
police medical emergency systems, disease modelling, power generator reliability and worker fatigue.

The bibliography of 21 items provides further readings for the beginner to pursue these topics.

This report is an interim report of on-going research. It may be amended, corrected or withdrawn, if called for, at the discretion of the author.



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Preface

Modelling stochastic systems is an art. As such one learns by doing. But the process of learning is long and has a number of steps preliminary to doing. One must learn some basic concepts of stochastic processes and their properties. Unfortunately, this step and its output have come to be called "theoretical" and seem to be viewed with a jaundiced eye as though they were unnecessary, irrelevant and an impediment to doing.

Except for the gifted few this process of learning must next include some familiarity with how others have used the theory to study important problems. This step I would call the study of models. It is not an end, nor a beginning, but an intermediate step to learning how to do. Unfortunately, most textbooks and many journal pages leave one with the impression that models previously developed somehow have an intrinsic value and if one knows enough models then applying the "theory" is just a plug-in exercise. This may be true in some fields but in operations research we know so little of the basic "science" of our processes that except in rather rare cases we do not have coherent models. All applications must cut, paste, extend, compress, rearrange or even develop models to make them do. The study of existing models is necessary but not sufficient to the doing of modelling. It is necessary to see how others have handled the problems of messy data (Unlike laboratory sciences, operations researchers almost always deal with messy data.), complexity of systems, systems that just do not satisfy assumptions of existing models and the like. But this process is not sufficient to doing modelling by oneself. My non-linearity is not your non-linearity. My dependence is not your dependence. Therefore, the final

step in learning how to model is to model.

Modelling, it seems to me, cannot be taught in the usual sense that we use that word. It is clear that it can be learned because many people have learned how to do it.

These notes try to reach the second of the three levels of modelling. That is, they try to expose very briefly, models of processes that others have developed. To understand how these models have been developed will require considerably more digging than we provide. Some of the studies we will mention occupy hundreds of pages of discussion. They cannot be summarized into a "how to do it manual" especially in our relatively few pages. But when all is said and done these examples are models. They are not "reality". They are not prescriptions or descriptions of "how to do it".

We start these notes with what we hope is a mildly surprising result obtained from a simulation of a very simple system. Unravelling the seeming mystery of the example takes us two rather long sections (II and III) to develop some structure for the random process underlying the example. It will turn out that the structure developed for that purpose (which has been in the research literature for about 25 years) has many other uses. In section IV some of these other uses are exposed. In particular we note, in very brief summary form, some of the uses to which the results of section II and III have been put. Here, we continually implore the reader to consult the source documents. Our discussion is intended simply to goad the reader into that literature. In no sense do we do the literature the service it deserves. Such is not our intention nor could it be done within the confines of space, time and our short span of interest in rehashing work that others have exposed

well. In section V we summarize where we have been and try to put the state-of-the-art in perspective in three pages. The bibliography is not complete in any dimension but a careful reading of the documents therein will take the reader more deeply into both theory and other applications.

During my years of teaching, I have had the priviledge of working with some very bright young men and women. To them is due credit for nearly everything in these notes. The errors are mine and I hope they will forgive me the distortions I made to their work as I tried to condense hundreds of pages to these few. Drs. Peter Cherry, Erhan Çinlar, Arthur Cole, Carlos Daganzo, Gilles D'Avignon, Atillio DiMarco, William Hall, Ralph King, Myun Lee, Gordon Swartzman, Burton Simon, James Solberg, and Thomas Vlach will recognize how deeply I am indebted to their work in these pages. Mr. Ziv Barlach did the simulation analysis noted therein and produced some of the analytic results with which we compared these results with the simulation. Dr. Robert D. Foley a former student and now valued colleague read this manuscript as did Dr. Jeffrey Hunter whose good nature was taken advantage of during his sabbatical leave from the University of Auckland, New Zealand at VPI & SU. Dr. Hunter was kind enough to share with me the thesis by Ms. Sim. Finally, what can I say about my right arm, Ms. Paula Myers? She typed, retyped and re-retyped these pages, always outwardly good humoredly and met impossible deadlines caused by my normal procrastination.

Ralph L. Disney

Blacksburg, Virginia
July, 1980

I. INTRODUCTION

1.0. Overview. Our purpose in these notes is to expose some concepts in stochastic modelling.

Since computer simulation is a widely used tool, we attempt to motivate our discussion with a problem that was simulated. The problem is very simple, well known to all operations researchers, has an analytic solution and is easy to simulate. However, we will see a result that at first glance can be perplexing. In particular a variance estimate made using the computer's output does not seem to be behaving "as it should". One can guess many reasons for the anomaly. The sample is not big enough. The simulation was not generating steady state results. These guys do not know how to simulate. While all of this may be true, the problem is much deeper than this. To understand and correct for it takes us far afield. By the time we return to the example (section 4.1.2) we have discussed a large body of knowledge (sections II and III). Since that knowledge is considerably more useful than our simplistic simulation problem, we expose several examples and a few real-life applications of it.

1.1. A Simple Problem. Let us begin with a very simple problem whose structure is well known to everyone in operations research. Consider the M/M/1/N queue. Suppose that we would like to determine, experimentally, the mean time between departures from such a queue. Such an endeavor is not too far fetched because in a queueing system this departure process might be the input to other queues that we wish to study or it might be the output of the system that we wish to control.

In many experimental studies one seeks not simply a point estimate of such a thing as a mean but rather one wants an interval estimate, perhaps a confidence interval. Therefore, let us require of our experimental procedure that we find the variance of the time between departures from the M/M/1/N queue. To make the matter more precise, let us choose N = 3 and several values for the usual traffic intensity $\rho = \lambda/\mu$.

Now we know that we probably should choose a large sample to gain precision in our estimates. Therefore, let us arbitrarily select 100 departure intervals (not because of any magic formula but simply because I did not have more money for computing).

One of my students, much more knowledged in simulation than I, simulated 100 steady state departure intervals from a M/M/1/3 queue. We then computed, using the standard tools of statistics, the mean and standard deviation for this sample.

A simple piece of stochastic modelling can give us the "true" mean for this data. The analysis might go as follows.

Consider the queue just after a departure has occurred. At that instant, the queue left behind is either 0 - the queue is empty or it is not 0 - there is at least someone in service.

Now if the queue is empty at a departure point (with probability $\pi_0(n)$) we must wait $1/\lambda$ time units on the average until the next arrival plus $1/\mu$ time units to service that arrival. If the queue is not idle at a departure time (with probability $(1 - \pi_0(n))$ then we need wait only $1/\mu$ time units on the average until the next departure. Thus, if $E[T_{n+1} - T_n]$ is the mean time between departures n and n+1 we have

$$(1.1.1) \qquad E[T_{n+1} - T_n] = \pi_0(n)\left(\frac{1}{\lambda} + \frac{1}{\mu}\right) + (1 - \pi_0(n))\,\frac{1}{\mu}$$

$$= \frac{\pi_0(n)}{\lambda} + \frac{1}{\mu}\,.$$

We have plotted the "true" value of (1.1.1) in figure 1.1.1 as a function of n. Note that the graph converges to (1.1.1) rather rapidly and for $n \geq 6$ the "time" dependent expectation (using $\pi_0(n)$) is very close to (1.1.1).

To find the standard deviation of the time between departures takes a bit of work but it can be done. We will return to this problem in section IV example 4.1.2. In figure 1.1.2 we have plotted both the estimated variance and the "true" variance again as a function of n. It is important to notice that the time dependent "true" variance is not converging to our computed steady state variance.

1.2. Observations. From an analytic theory of these processes (see section 4.1.2), these variance estimates should converge to their steady state values rather fast. Figure 1.1.2 shows they converge in 5-6 steps for reasonable measures of "closeness" which checks with the theory. The problem is, and the figures show this clearly, these variances are converging rapidly but to the wrong value. True the values are not much different but they are obviously different. Furthermore, they are converging to values less than the sample variances would indicate. This means that if one used the sample variance to produce confidence intervals these intervals would be too large or larger sized samples would be required. (One can, alternatively, produce examples where the sample variances are too small and things such as confidence intervals would

Figure 1.1.1
The Time Dependent
Mean Time Between Departures
for M/M/1/3 Queues

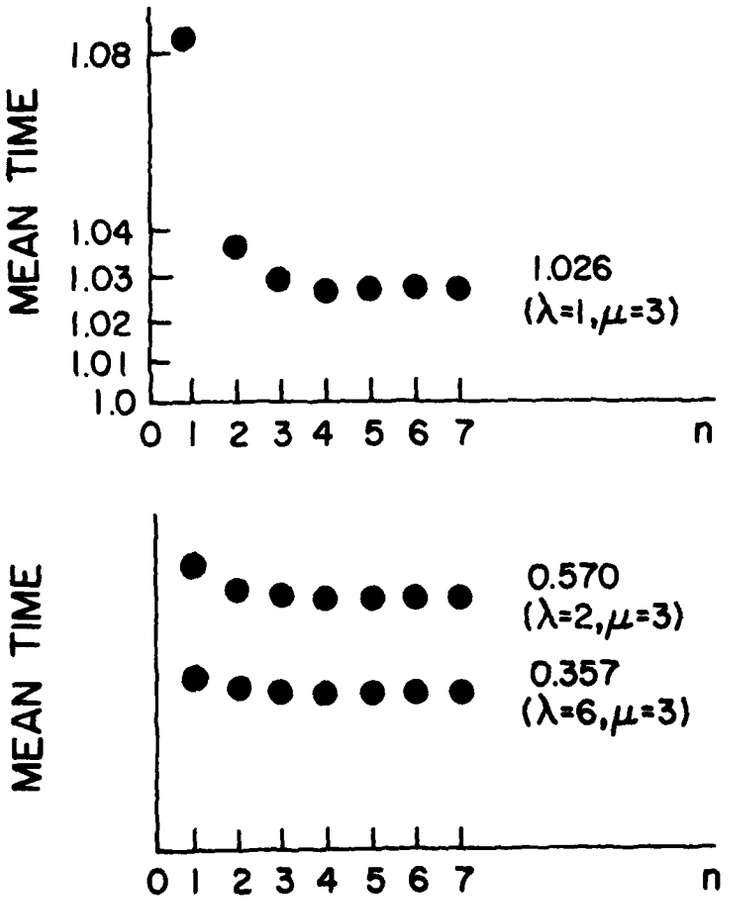
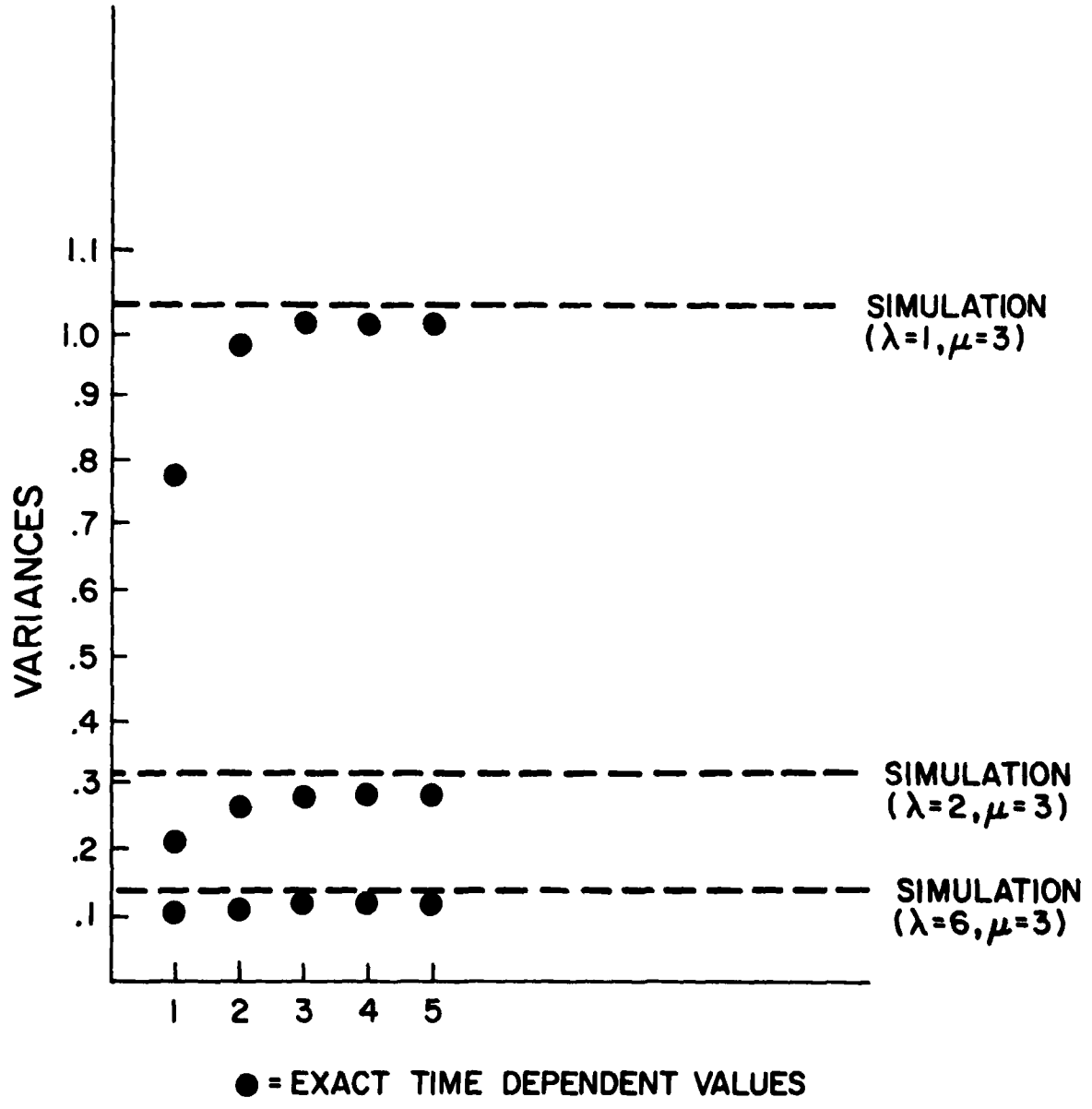


Figure I.1.2
Time Dependent Variances of Departures
from M/M/1/3 Queues



be too small.)

But there is more here than meets the eye. The differences shown in figure 1.1.2 are caused by correlations in the data produced. Thus, the observations are not independent and statistical tools that rely on independence are at best suspect for this simulation.

Knowing what the problem is here, of course, allows one to design simulations to eliminate it. Understanding what the problem is will take us awhile. We return to this problem in section IV.

II. MARKOV RENEWAL PROCESSES

2.0. Introduction. The trouble with our study in section 1 is that we used standard statistical tools to estimate a variance. These methods assume that the random variables giving us the data are mutually independent. (i.e., all subcollections of the random variables are collections of independent random variables.) But the intuitive argument leading up to formula (1.1.1) clearly indicates that this is not so. We consciously had to take into account whether the queue was idle or not at a departure point in order to compute formula (1.1.1). Thus, these interdeparture intervals must at least depend on the queue present at the beginning of the interval. In essence that was the fact we used when we found two conditional expectations, one depending on the queue being empty, the other depending on the queue not being empty. In this way we simply used the fact that

$$E_Y[E_X[X|Y]] = E[X],$$

a well known result.

We could have used a related argument to compute our variances. We did not because it would have spoiled the fun. More importantly, however, it would not have told us much about the probability structure of the departure process.

It appears that if we are to explain the probability structure of this departure process we must include in our structure some knowledge of the queue length process. Furthermore, because it seems that there is at least a first order correlation here (accounting for the difference between the variance as computed and that as calculated) we must account for at least the joint distribution of two consecutive intervals ($T_{n+2} - T_{n+1}$ and $T_{n+1} - T_n$, perhaps and maybe more). But since each such interval depends on the queue length at the start of the departure interval and since these queue lengths are Markov dependent in the M/G/1/N models, it appears that we need a model that at least allows for Markov dependent queue lengths and interdeparture intervals that at least allow for some dependency on these queue lengths. We will see how all of this comes about in section 4.1.2 but first we must develop a bit of theory of a rather useful random process.

In the process of development, we will expose a class of random processes that are useful for modelling complex systems and which has the advantage of including most of the standard processes that are now popular for modelling. We will point out these connections. We will not be able to expose what we are about with rigor that the mathematics deserves. Whenever possible we will try to motivate and provide at least plausible heuristic arguments in defense of our assertions. The reader interested in the deeper aspect of the subject can begin with the excellent text: Çinlar, E., Introduction to Stochastic

Processes, Prentice-Hall, (1975) or the article by the same author (1975).

The topics we discuss have been known since 1954 due to the two papers Levy (1954) and Smith (1955). They became more widely known because of the papers by R. Pyke, (1961). However, they were being used as models of inventory and queueing problems a bit before Pyke's papers by Fabens, (1959). In fact, P. Finch (1959) studied the departure process of section I over 20 years ago and gave almost a complete account of what is going on in M/G/1 queues using some of the results we now develop.

2.1. A Markov Renewal Process. We start by defining a sequence of pairs of random variables. Let X_n be a random variable such that for each $n = 0,1,2,\cdots$, X_n takes values in some fixed, countable space E called the state space. If for some n, $X_n = j$, $j \in E$ we say the state of the process at the nth step is j. Let T_n be another random variable that for each $n = 0,1,2,\cdots$, takes values in the non negative real numbers, say R_+. Then the sequence of pairs $\{(X_n,T_n)\}$ is called a Markov Renewal process if

$$\Pr[X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_0,X_1,\cdots,X_n = i,\ T_0,T_1,\cdots,T_n]$$

(2.1.1)

$$= \Pr[X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_n = i].$$

We will take $T_0 = 0$ throughout our discussion and suppose $\Pr[X_0 = j]$ is given. If in addition the probabilities in (2.1.1) do not depend on n, then the process is homogeneous or has a stationary transition mechanism. In most modelling this is the assumption made and we will assume the homogeneity throughout our discussion (but see section 4.1.4). In this case we will identify the probabilities in (2.1.1) by $Q_{ij}(t)$ and call these the transition functions of the process. The

matrix $Q(t)$, for which $Q_{ij}(t)$ is the i,j element will be called the semi-Markov kernel of the process $\{(X_n, T_n)\}$.

2.2. An Interpretation. Equation (2.1.1) can be rewritten in either of two forms that helps one gain an intuitive feeling for how such processes could arise in natural phenomence. First note that $Q_{ij}(t)$ can be written in either of the forms

$$(2.2.1) \qquad Q_{ij}(t) = Pr[T_{n+1} - T_n \leq t | X_n = i] Pr[X_{n+1} = j | T_{n+1} - T_n = t,\ X_n = i]$$

or

$$(2.2.2) \qquad Q_{ij}(t) = Pr[X_{n+1} = j | X_n = i] Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i].$$

Then, if we think of this process as one evolving over "time" (n) by jumps, where a jump carries the process into some state in the space E, (2.2.1) implies that the time between the (n+1)st and nth jump, $T_{n+1} - T_n$, is a random variable whose distribution depends on the state the process is in at T_n (i.e., X_n). The next state to be visited, X_{n+1}, has a distribution dependent on both the state the process is in at T_n, and how long it remains in the state, $T_{n+1} - T_n$.

Thus, if one were to try to simulate this process on a computer, one would have to create values of two random variables at each iteration. By (2.2.1) one would have to generate a value for a non-negative random variable $T_{n+1} - T_n$. The distribution from which this value was generated would be the distribution associated with X_n. There would be as many distributions to draw from as there are elements in E. Having generated this value for $T_{n+1} - T_n$ one would then generate a value for the discrete random variable X_{n+1}. The distribution from

which this value was generated would depend on the previously chosen value of $T_{n+1} - T_n$ and the given X_n. Thus, to generate X_{n+1} one would have as many distributions to choose from as there are elements in $R_+ \times E$. While this is not physically feasible, it will turn out that many modelling problems are best understood if one thinks that this is how the application is generating its behavior. As we shall see later, this thought process is natural to understanding the departure phenomena of section I.

(2.2.2) is more natural for simulation. By that formula the states visited, X_n, X_{n+1}, simply form a Markov chain. The time between jumps, $T_{n+1} - T_n$, is a random variable that depends on both the current state, X_n, and the state to be visited next, X_{n+1}.

Thus, if one is to try to simulate this process using (2.2.2) one would first generate the states of a Markov chain using $\Pr\{X_{n+1} = j | X_n = i\}$ as the transition probability of a jump from state i to state j. Then knowing that this jump was made, one would generate a value of the continuous valued random variables, $T_{n+1} - T_n$, from a distribution whose probabilities are given by $\Pr\{T_{n+1} - T_n \leq t | X_{n+1} = j, X_n = i\}$. Of course, there would be as many such distributions to draw from as there are values in E x E.

2.3. Two Applications. Models such as those described in section 2.1 have been used to model road traffic flow in at least two studies. One study is a little easier to elucidate. We will discuss another model in section 4.1.7.1.

One supposes that there are two types of vehicles that travel a road, cars and trucks. One assigns

$$X_n = \begin{cases} 0, \text{ if the nth vehicle passing a point is a car,} \\ 1, \text{ if the nth vehicle passing a point is a truck.} \end{cases}$$

It is assumed that the sequence of cars and trucks passing a fixed observer forms a Markov chain with one step transition probabilities

$$\Pr[X_{n+1} = j | X_n = i] = p_{ij}, \; i,j \in E = \{0,1\}.$$

Then, given that the leader vehicle is of type i and the follower vehicle is of type j,

$$\Pr[T_{n+1} - T_n \leq t | X_{n+1} = j, \; X_n = i]$$

would give the probability distribution of headway measured in units of time or distance. One would expect that this headway depends on whether a car is being followed by a car or a truck and whether a truck is being followed by a car or a truck. Thus, there are four possible distributions for this headway distribution depending on i,j. The i,j elements of the matrix

$$\underset{\sim}{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

would give the probability that a vehicle of type j was following a vehicle of type i (i.e., $\Pr[X_{n+1} = j | X_n = i]$). The i,j elements of the matrix

$$\underset{\sim}{\hat{F}}(t) = \begin{bmatrix} F_{00}(t) & F_{01}(t) \\ F_{10}(t) & F_{11}(t) \end{bmatrix}$$

would give the probability distribution of the headway, when a vehicle of type j was following a vehicle of type i (i.e., $\Pr[T_{n+1} - T_n \leq t | X_{n+1} = j, X_n = i]$).

Hall (1969) has developed a Markov renewal model in his study of dual functioning ambulance systems. In his study, two types of calls are received by a police dispatcher. One type of call is for police assistance. The other type is for medical emergency assistance (i.e., ambulances). In his model, Hall assumes that the calling process is a Markov renewal process. He defines

$$X_n = \begin{cases} 0, \text{ if the nth call is for police assistance,} \\ 1, \text{ if the nth call is for an ambulance.} \end{cases}$$

Then the i,j element of the matrix

$$\underset{\sim}{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

is the one step transition probability for the "type of call" process (i.e., $p_{ij} = \Pr[X_{n+1} = j | X_n = i]$). The i,j element of the matrix

$$\hat{\underset{\sim}{F}}(t) = \begin{bmatrix} F_{00}(t) & F_{01}(t) \\ F_{10}(t) & F_{11}(t) \end{bmatrix}$$

then gives the probability distributions for the times between calls arriving to the dispatcher depending on the types of calls involved (i.e., $F_{ij}(t) = \Pr[T_{n+1} - T_n \leq t | X_{n+1} = j, X_n = i]$). We will discuss the Hall problem in more detail in section 4.1.5.

2.4. Some Properties of $Q_{ij}(t)$ and the Markov Renewal Process. The matrix $\underset{\sim}{Q}(t)$ has some useful properties.

(2.4.1) $$\lim_{t\to\infty} Q_{ij}(t) = Q_{ij}(\infty) = \Pr[X_{n+1} = j | X_n = i].$$

That is, the marginal distribution of X_{n+1} given X_n is the distribution that is associated with some Markov chain (i.e., that chain whose one step transition probabilities are given by $\Pr[X_{n+1} = j | X_n = i]$). We will call this Markov chain the underlying Markov chain. In many queueing applications of Markov renewal theory this underlying Markov chain is precisely the one obtained by embedding methods. In particular, $\{X_n\}$ is the embedded Markov chain for the M/M/1/3 queue of the problem in section I.

(2.4.2) $$F_i(t) = \sum_{j\in E} \Pr[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i]$$

$$= \Pr[T_{n+1} - T_n \leq t | X_n = i].$$

That is, the marginal distribution $F_i(t)$ is the probability distribution

function for the time spent in state i regardless of the next state to be visited. In most applications $F_i(t)$ is a proper distribution (but see section 4.1.3). In particular, $E[T_{n+1} - T_n | X_n = 0] = 1/\lambda + 1/\mu$ and $E[T_{n+1} - T_n | X_n \geq 1] = 1/\mu$ for the problem in section I.

$$F_{ij}(t) = \Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i] \tag{2.4.3}$$

$$= Q_{ij}(t)/Q_{ij}(\infty).$$

That is, $F_{ij}(t)$ is the distribution for the time spent in state i when it is known that the next transition is to state j. We will assume that $F_{ij}(t)$ is properly defined. Clearly, then $Q_{ij}(t) = F_{ij}(t)Q_{ij}(\infty)$. That is, $Q_{ij}(t)$ can be obtained in our previous examples from $Q_{ij}(t) = F_{ij}(t)p_{ij}$ where $p_{ij} = Q_{ij}(\infty)$. When $Q_{ij}(\infty) = 0$ for some i,j, $F_{ij}(t)$ can be chosen to be any distribution function.

We know from the discussion in section 2.2 that the $\{X_n\}$ sequence forms a Markov chain and from (2.4.1) the one step transition probabilities for this chain are given by $\lim_{t\to\infty} Q_{ij}(t)$. But what about the $\{T_{n+1} - T_n\}$ sequence? What are its properties? From (2.2.2) we know that in general $T_{n+1} - T_n$ depends on X_{n+1} and X_n. Therefore, it follows that

$$\Pr[T_{n+1} - T_n \leq t_{n+1} - T_n - T_{n-1} \leq t_n, \cdots, T_1 - T_0 \leq t_1 | X_{n+1}, X_n, \cdots, X_0] \tag{2.4.4}$$

$$= F_{X_{n+1},X_n}(t_{n+1}) F_{X_n,X_{n-1}}(t_n) \cdots F_{X_1,X_0}(t_1).$$

That is, the sequence $\{T_{n+1} - T_n\}$ is a sequence of random variables that are <u>conditionally independent</u> where the conditioning random variables are the states of the sequences $\{X_n\}$. The $T_{n+1} - T_n$, $n = 0,1,2,\cdots$, themselves are not

independent. In fact for each n, $T_{n+1} - T_n$ depends on all increments $T_{m+1} - T_m$ for $m < n$.

2.5. Other Processes as Markov Renewal Processes. Other processes that are commonly used for modelling can be thought of as special cases of the Markov renewal process. However, the special cases were developed first and an extensive theory exists for them. Because of this, when one knows that he is working with the special case, it is probably preferrable to use the special knowledge of that process. None-the-less, the more general theory of Markov renewal processes often leads to new insights, new interrelations, and interpretations, ties together many seeming disparate topics, and provides for a consistent framework that often allows easy generalization.

2.5.1. Renewal Processes. We know from the above discussion that $\{T_{n+1} - T_n\}$ is a sequence of dependent random variables. Those random variables are conditionally independent given $\{X_n\}$. Now suppose that the state space E contains just one element. Then in formula (2.4.4) every X_n takes only this one value or, equivalently, knowledge of the sequence $\{X_n\}$ is irrelevant to the conditional probabilities in (2.4.4) because there is only one sequence that the X_n could possibly take. Every X_n must take only the one value in E.

Consequently from (2.4.4) we obtain

$$(2.5.1.1) \qquad F_{X_{n+1},X_n}(t_{n+1})F_{X_n,X_{n-1}}(t_n)\cdots F_{X_1,X_0}(t) = F(t_{n+1})F(t_n)\cdots F(t_1).$$

In this case the sequence $\{T_{n+1} - T_n\}$ is not only a sequence of conditionally independent random variables, it is a sequence of mutually independent random variables. Furthermore, since we have assumed that $Q_{ij}(t)$ does not depend on

n, F does not depend on n in (2.5.1.1). Then in this case $\{T_{n+1} - T_n\}$ is a sequence of mutually independent, identically distributed, (obviously non-negative) random variables. But these are precisely the conditions necessary for a sequence of random variables to be a renewal sequence. That is, a renewal process is a Markov renewal process with only one state.

2.5.2. Markov Processes. It is well known (e.g., Çinlar, (1975), pp. 246-247) that if $\{Y(t)\}$ is a (regular) Markov process then one can identify two underlying processes that generate $\{Y(t)\}$. If we let T_n be the time of the nth jump of $Y(t)$ and let X_n be the state into which the process jumps at the nth jump, then for $\{(X_n,T_n)\}$

$$\Pr[X_{n+1} = j,\ T_{n+1} - T_n > t \mid X_n = i,\ X_{n-1},\cdots,X_0,\ T_n,\ T_{n-1},\cdots,T_0]$$

$$= p(i,j)\ \exp(-\lambda(i)t), \tag{2.5.2.1}$$

where $p(i,j) = \Pr[X_{n+1} = j \mid X_n = i]$ are the usual one step transition probabilities of the (jump) Markov chain $\{X_n\}$. Conversely, if we start with a pair $\{(X_n,T_n)\}$ with probability structure as in (2.5.2.1) then we can always construct a (regular) Markov process $\{Y(t)\}$.

If we use (2.2.2) we see that these Markov processes are special Markov renewal processes. The particularization comes about by requiring

$$1 - F_{ij}(t) = \Pr[T_{n+1} - T_n > t \mid X_{n+1} = j,\ X_n = i] = \exp(-\lambda(i)t).$$

That is, if one requires that the time between jumps in the Markov renewal process be exponentially distributed random variables with parameters depending only on the current state (X_n) and not on the next state to be visited, then

the Markov renewal process is a Markov process. In this way one can see that a Markov renewal process is more general than a Markov process in that one does not require the time between jumps in the process to be exponentially distributed in the Markov renewal case nor does one require these interjump time distributions to depend only on the current state of the process. These distributions may depend on both the current and next state in a Markov renewal process.

This latter property may be one reason for the appeal of the Markov renewal models for road traffic. One simply expects the headway distributions to depend on both the leader and follower not just the leader.

2.5.3. Markov Chains. If again in the structure of formula (2.2.2) we require that

$$\Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i] = \begin{cases} 0, & t < 1, \\ 1, & t \geq 1, \end{cases}$$

then the interjump times are always of length 1 or jumps occur at $1,2,\cdots,n\cdots$. In this case, the Markov renewal process is simply a Markov chain with one step transition probabilities given by $\Pr[X_{n+1} = j | X_n = i] = p_{ij}$.

2.6. The Markov Renewal Function. Before going much further we must develop some additional concepts.

First we have

$$Q_{ij}(t) = \Pr[X_{n+1} = j,\ T_{n+1} - T_n \leq t | X_n = i].$$

Now it follows from first principles that

$$\Pr[X_{n+2} = k,\ T_{n+2} - T_{n+1} \le y | X_{n+1} = j] \cdot$$

$$\Pr[X_{n+1} = j,\ T_{n+1} - T_n \le x | X_n = i]$$

$$= \Pr[X_{n+2} = k,\ X_{n+1} = j,\ T_{n+2} - T_{n+1} \le y,\ T_{n+1} - T_n \le x | X_n = i],$$

where the result follows since the event $\{X_{n+2} = k,\ T_{n+2} - T_{n+1} \le y\}$ does not depend on $T_{n+1} - T_n$ where X_{n+1} is given. Then by usual convolution arguments on the two random variables $T_{n+2} - T_{n+1}$, $T_{n+1} - T_n$ whose sum is $T_{n+2} - T_n$, we have

$$\Pr[X_{n+2} = k,\ X_{n+1} = j,\ T_{n+2} - T_n \le t | X_n = i]$$

$$= \int_0^t Q_{ij}(dx) Q_{jk}(t - x)$$

$$= \Pr[X_2 = k,\ X_1 = j,\ T_2 \le t | X_0 = i]$$

upon using the homogeneity property and the convention $T_0 = 0$. Then we can write

$$\Pr[X_2 = k,\ T_2 \le t | X_0 = i] =$$

$$\sum_{j \in E} \int_0^t Q_{ij}(dx) Q_{jk}(t - x).$$

In concert with the corresponding functions in Markov process theory these functions are denoted by $Q_{ik}^{(2)}(t)$ and are called the <u>two step transition functions</u>. By induction one can show that

$$\Pr[X_n = k,\ T_n \leq t \mid X_0 = i]$$

(2.6.1)

$$= \sum_{j \in E} \int_0^t Q_{ij}(ds) Q_{jk}^{(n-1)}(t-s) = Q_{ik}^{(n)}(t),$$

called the n step transition functions of the process.

If one defines the matrix $\underset{\sim}{Q}^{(n)}(t)$ to be that whose elements are $Q_{ik}^{(n)}(t)$ then (2.6.1) can be written in matrix form as

$$\underset{\sim}{Q}^{(n)}(t) = \int_0^t \underset{\sim}{Q}(ds)\underset{\sim}{Q}^{(n-1)}(t-s) = (\underset{\sim}{Q} * \underset{\sim}{Q}^{(n-1)})(t)$$

where the operation here on these matrices is that defined by equation (2.6.1). We will call this operation matrix convolution and denote it by the symbol *. By convention we define $\underset{\sim}{Q}^{(0)}(t)$ to be the identity matrix for all $t \geq 0$.

Now let us define two very useful random variables. Let

$$1_j(X_n) = \begin{cases} 1, \text{ if } X_n = j, \\ 0, \text{ otherwise.} \end{cases}$$

This function of the random variable X_n simply indicates whether $X_n = j$ or not. Similarly let

$$I_{[0,t]}(T_n) = \begin{cases} 1, \text{ if } T_n \in [0,t], \\ 0, \text{ otherwise.} \end{cases}$$

This function of T_n is again an indicator random variable that takes the value 1 if the nth transition falls in the fixed interval [0,t] and is 0 otherwise.

Clearly the product

$$1_j(X_n)I_{[0,t]}(T_n)$$

will take the value 1 if both $X_n = j$ and $T_n \in [0,t]$, and will take the value 0 otherwise. Then

$$\sum_{n=0}^{\infty} 1_j(X_n)I_{[0,t]}(T_n)$$

is simply a sum of 0's and 1's where 1 occurs each time $X_n = j$ in the (fixed) interval $[0,t]$. That is, the sum simply gives the number of times $X_n = j$ in the interval $[0,t]$.

Now define

$$R_{ij}(t) = E_i\left[\sum_{n=0}^{\infty} 1_j(X_n)I_{[0,t]}(T_n)\right]$$

where E_i denotes the conditional expectation $E[\cdot\,|X_0 = i]$. Then pass the expectation operator inside the sum (which can be proven to be a valid operation here). Recall that the expected value of a Bernoulli random variable (which the indicators are) is simply the probability that the random variable takes the value 1. Here this is equivalent to the probability that both $X_n = j$ and $T_n \in [0,t]$ which from (2.6.1) is just $Q_{ij}^{(n)}(t)$. Then one finds that

$$R_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t), \tag{2.6.2}$$

a result well known at least in renewal theory. $R_{ij}(t)$ is finite for all finite t, is right continuous for all $t \geq 0$ and is the expected number of visits to state j in $[0,t]$ for the process that starts at $T_0 = 0$ in state i.

As in renewal theory the deriviative $R_{ij}(dt)$ of $R_{ij}(t)$, when it exists,

can be given a useful interpretation. One can think of $R_{ij}(dt)$ as the probability that some transition into state j occurred in the interval $[t, t + dt]$ for the process that started in state i at $T_0 = 0$. This interpretation provides heuristic justification for several of our later results.

We define $\underset{\sim}{R}(t)$ to be the matrix whose i, j element is $R_{ij}(t)$. $\underset{\sim}{R}(t)$ is called the Markov renewal function. Then (2.6.2) can be written as

$$\underset{\sim}{R}(t) = \sum_{n=0}^{\infty} \underset{\sim}{Q}^{(n)}(t). \tag{2.6.3}$$

From this it follows that

$$\underset{\sim}{R}(t) = \underset{\sim}{I} + \underset{\sim}{Q}(t) + \underset{\sim}{Q}^{(2)}(t) + \cdots,$$

$$(\underset{\sim}{Q} * \underset{\sim}{R})(t) = \underset{\sim}{Q}(t) + \underset{\sim}{Q}^{(2)}(t) + \cdots .$$

Thus one has that $\underset{\sim}{R}(t)$ satisfies the equation

$$\underset{\sim}{R}(t) = \underset{\sim}{I} + (\underset{\sim}{Q} * \underset{\sim}{R})(t) \tag{2.6.4}$$

or in component form

$$R_{ik}(t) = \begin{cases} 1 + \sum_{j \in E} \int_0^t Q_{ij}(ds) R_{jk}(t-s), & i = k, \\ \sum_{j \in E} \int_0^t R_{ij}(ds) Q_{jk}(t-s), & i \neq k. \end{cases} \tag{2.6.5}$$

where $\underset{\sim}{I}$ is the identity matrix. That is $\underset{\sim}{R}(t)$ is a solution to the integral equations (2.6.4). It is not clear (and in general not true), however, that $\underset{\sim}{R}(t)$ is the unique solution to this equation.

In those cases where $\underset{\sim}{Q}(t)$ is a finite matrix we can define $\underset{\sim}{Q}^*(\alpha)$ to be the matrix whose elements are the Laplace-Stieltjes transform of the corresponding elements of $\underset{\sim}{Q}(t)$. If we define $\underset{\sim}{R}^*(\alpha)$ similarly for $\underset{\sim}{R}(t)$, then for finite state processes (2.6.4) leads to the useful result

$$\underset{\sim}{R}^*(\alpha) = (\underset{\sim}{I} - \underset{\sim}{Q}^*(\alpha))^{-1}, \tag{2.6.6}$$

where $\underset{\sim}{I}$ is the usual identity matrix. For finite matrices this inverse is unique. However, if $\underset{\sim}{Q}(t)$ is not finite the inverse may not be unique.

2.7. The Markov Renewal Equation. If we let $\underset{\sim}{f}$, $\underset{\sim}{g}$ be column vectors whose elements $f_i(t)$ and $g_i(t)$ respectively are non-negative functions, bounded on finite intervals, then the equation

$$\underset{\sim}{f} = \underset{\sim}{g} + \underset{\sim}{Q} * \underset{\sim}{f}$$

or in component form

$$f_i(t) = g_i(t) + \sum_{k \in E} \int_0^t Q_{ij}(ds) f_k(t-s), \quad i \in E$$

is called a Markov renewal equation. In the special case where E has just one element, this is the well known renewal equation. For most applications given $\underset{\sim}{Q}$ and $\underset{\sim}{g}$ this equation has a unique solution $\underset{\sim}{f}$ given by:

$$\underset{\sim}{f}(t) = (\underset{\sim}{R} * \underset{\sim}{g})(t).$$

The solution is unique in the renewal case but there are exceptions in the Markov renewal case. The reader should consult Çinlar (1975) carefully here. In component form this solution is

$$f_i(t) = \sum_{k \in E} \int_0^r R_{ik}(ds) g_k(t-s). \tag{2.7.1}$$

Proving that this is a solution is easy, by substitution. Proving it is not unique requires more work. Very roughly, if the Markov renewal process never stops or if the transition functions $Q_{ij}(b)$ are uniformly bounded away from 1 for some $b > 0$ then the above solution is unique. One of these two conditions is almost always satisfied in modelling applications so we will assume that the Markov renewal equation has a unique solution.

Notice in particular that if we let $\underset{\sim}{f}_i(t)$ be any column of $\underset{\sim}{R}(t)$ then equation (2.6.4) implies that $\underset{\sim}{R}(t)$ satisfies a Markov renewal equation where $\underset{\sim}{f}_i(t)$ is a column of $\underset{\sim}{R}(t)$ and $\underset{\sim}{g}_i(t)$ is the corresponding column of $\underset{\sim}{I}$. Thus one could have reversed our discussion by starting with (2.6.4) and proving that (2.6.3) was the unique solution by the results of this section.

Interest in the Markov renewal equation and its solution lies not only in the form of (2.7.1) and its computations but also in the limit ($t \to \infty$) of this solution. A complete discussion of this limiting behavior is not possible here. For our future purposes we can say that if $\underset{\sim}{g}_i(t)$ is a proper probability distribution for each i and if this function is Riemann integrable then the solution to the Markov renewal equation has a unique limit. Both of these conditions exist in our applications.

This limit, when it exists may be computed as follows. Consider the underlying Markov chain $\{X_n\}$. Assume it has a positive stationary vector (i.e., a solution of the equations $\pi = \pi Q(\infty)$). Such is always the case at least if $\{X_n\}$ is recurrent, aperiodic as is well known from Markov chain theory. Let

$$m(j) = E_j[T_1]$$

be the mean time spent in state j initially. Then

$$\lim_{t\to\infty} f_j(t) = \lim_{t\to\infty} \sum_{k\in E} \int_0^t R_{jk}(ds) g_k(t-s)$$

(2.7.2)

$$= \frac{\sum_{k\in E} \pi(k) \int_0^\infty g_k(s)ds}{\sum_{i\in E} \pi(i)m(i)} .$$

2.8. Summary. In this section we have presented the bare minimum knowledge of Markov renewal processes and a very few properties. This background is sufficient for building some useful models (but not 4.1.4) and exploring some old problems that heretofore seemed intractable. We need one more construction in section III then we will be ready to expose some of the usefulness of our methods.

III. SEMI-REGENERATIVE PROCESSES

3.0. Introduction. In many models of stochastic processes it is common to find that the process has certain times at which the future behavior of the process is independent of the past. The process in a sense renews or regenerates itself at such times. For example, in a Poisson process every instant of time is a renewal or a regeneration time for the time between jumps. This is so well known that the phenomena is given a name. It is called the "forgetfulness" property. Indeed, it is this forgetfulness that makes the Poisson extremely useful in stochastic modelling.

But there are also processes for which there are some points at which the process regenerates itself but not at every $t \geq 0$. For example, the times of entry of customers to an empty queue are such points in G1/G/1 queues. Recent work in the statistical analysis of stochastic processes relies on the existence of such points (the so called "regenerative method").

But notice that if one observes a queue length process (call it $\{Z(t)\}$) for an M/G/1 queue, times at which departures occur are in general not points at which $\{Z(t)\}$ regenerates itself. The development of the queue length process after a departure depends very much on the size of the queue at that departure point. Such is true for the queue in section I as was noted. Therefore, the concept of regeneration while extremely important is not sufficiently general for some applications. In the remainder of this section, therefore, we will develop another concept that can be called "semi-regeneration". It will follow from our discussion that all random processes that are regenerative are also semi-regenerative. The converse is not true.

3.1. Semi-Regenerative Processes. To generalize this concept of regeneration we start by defining a random process, say $\{Z(t)\}$, and a random time, say T. Then if the event $\{T \leq t\}$ depends on $\{Z(s)\}$ only for those $s \leq t$ we say T is a stopping time for the process $\{Z(t)\}$. Such stopping times occur rather often in stochastic models. In Markov chain models the times at which the process enters some fixed state j for the first time are stopping times. In fact the times at which the process enters j for the kth time are stopping times. The times at which j is left for the first time is a stopping time. But note that the time at which j is left for the last time is not a stopping time. This latter result follows since one must know about the behavior of the chain for all

future time (after each exit) to know whether this exit is the last one or not.

Now suppose $\{Z(t): t \geq 0\}$ is a random processes with state space F. (We are rather imprecise here.) Suppose further that $\{(X_n,T_n)\}$ is a Markov renewal process. Suppose $\{Z(t)\}$ and $\{(X_n,T_n)\}$ have the following properties:

(3.1.1) For each $n = 0,1,2,\cdots,T_n$ is a stopping time for $\{Z(t)\}$;

(3.1.2) X_n is determined by the events $\{Z(s): s \leq T_n\}$;

(3.1.3) for each $n = 0,1,2,\cdots$, $m \geq 1$, $s_1 \leq s_2 \leq \cdots \leq s_m$ and positive function f on F^m,

$$E_i[f(Z(T_n+s_1), Z(T_n+s_2),\cdots,Z(T_n+s_m)\,|\,Z(u): u \leq T_n]$$

$$= E_j[f(Z(s_1), Z(s_2),\cdots,Z(s_m)\,|\,X_n = j].$$

Then $\{Z(t): t \geq 0\}$ is called a <u>semi-regenerative process</u> and $\{T_n\}$ are called <u>semi-regeneration epochs or times</u>. Conditions 1 and 2 are rather straight forward. Condition 3 requires a bit of explanation. It is saying two different things. First of all f is some function defined on the m-dimensional space F^m. For example, f could be a cost function, as is often the case in inventory applications. f could also be an indicator function in which case the expectations in (3.1.3) are statements about probabilities. The important thing to recognize is that the left hand side of (3.1.3) is an expectation over states of the process <u>after</u> the stopping time T_n whereas the right side is an expectation over the future of the process <u>after</u> time 0. Thus, the right hand side of (3.1.3) is a re-initialized version of the left hand side with the re-initialization occurring at T_n.

But there is more here. $\{Z(u): u \leq T_n\}$ is the history of the $\{Z(t)\}$ process

up to the time T_n, and X_n is the state of the Markov renewal process at T_n. Then (3.1.3) is also claiming that the "future" of $\{Z(t)\}$ after T_n is independent of the "past" of $\{Z(t)\}$ before T_n if the "present" (at T_n) state of the $\{(X_n,T_n)\}$ process is known.

Thus, very roughly, a semi-regenerative process is one that has associated with it a Markov renewal process and has the properties that: the T_n of the associated Markov renewal process are stopping times for $\{Z(t)\}$; X_n depends only on $\{Z(u): u \leq T_n\}$; at each stopping time T_n, the $\{Z(t)\}$ process regenerates itself just as though it had started in the state of the Markov renewal process existing at T_n (i.e., had started in X_n). At the semi-regeneration point "the future of the process and the past of the process are independent if the current state X_n is given".

If E contains just one point then as we have noted (section 2.5.1), $\{T_{n+1} - T_n\}$ is a renewal sequence and $\{X_n\}$ plays no role in (3.1.1)-(3.1.3). Thus, in this special case we could redefine the $\{Z(t)\}$ by requiring:

(3.1.1.a) T_n be stopping times for $\{Z(t)\}$;

(3.1.1.b) irrelevant;

(3.1.1.c) for each $n = 0,1,2,\cdots$, $m \geq 1$, $s_1 \leq s_2 \leq \cdots \leq s_m$ and positive function f on F^m;

$$E[f(Z(T_n + s_1), Z(T_n + s_2),\cdots,Z(T_n + s_m))|Z(u), u \leq T_n],$$

$$= E[f(Z(s_1), Z(s_2),\cdots,Z(s_m))].$$

In this case $\{Z(t)\}$ is called a <u>regenerative process</u>. We will not pursue the topic but such processes have applications in many areas. Much of what we

discuss later in this section is also true for regenerative processes by restricting the results to state spaces with one element.

3.2. Some Examples of Semi-Regenerative Processes. We will give a few examples of semi-regenerative processes here. Some of these examples we hope the reader has encountered in other contexts. Other examples are given in section IV. First we note that every regenerative process is a semi-regenerative process and every (regular, jump) Markov process is a semi-regenerative process.

3.2.1. Forward Recurrence Times. Let $\{Y_n: n = 0,1,2,\cdots\}$ with $Y_0 = 0$ be a sequence of independent, identically distributed, non-negative random variables. Such things are called renewal sequences. They occur in reliability theory where Y_n is taken to be the lifetime of the nth replacement of a part. Y_1 is the original part's lifetime and it is assumed that the original part is put into operation at time 0. In queueing theory such sequences occur when Y_n is taken to be the time between the n and n - 1 arrival to the queue. In Markov process theory Y_n is the time between the n and n - 1 visit to some fixed state.

Now let

$$T_n = \sum_{j=0}^{n} Y_j.$$

Then T_n is the time at which the event of interest occurs for the nth time. For example, T_n is the time of the nth replacement in reliability theory or the time of the nth arrival in queueing theory.

A process of some interest is defined by

$$Z(t) = T_{n+1} - t, \quad T_n \leq t < T_{n+1}.$$

Then for each t, Z(t) is the random time from t until the next event occurs. In renewal theory this is called a forward recurrence time. A picture of a sample path of {Z(t)} is informative. (See figure 3.2.1.) The things to observe are teh Z(t) jumps upward at each T_n by an amount $Y_n = T_n - T_{n-1}$. It then decreases linearly with slope - 1 until it hits 0 (at T_n) then jumps again. Since the $\{Y_n\}$ sequence is a sequence of independent, identically distributed random variables, the heights of the jumps have these properties.

Now it is rather obvious that $\{T_n\}$ is a sequence of stopping times for {Z(t)}. One needs only look at the above picture to tell if, for example, t_3 has occurred by some fixed t or not. In our picture it is obvious by looking at {Z(t)} up to t that indeed T_3 has occurred before t. We need no other information than the paths of {Z(t)} up to t to partition those paths into ones where T_3 occurs before t and those where such does not happen. Thus $\{T_n\}$ is a sequence of stopping times for {Z(t)}. Furthermore, at each T_n a part fails in the reliability application or an arrival occurs in the queueing application. That is, associated with each T_n there is only one thing that can happen. Hence the associated $\{X_n\}$ sequence has only one state. Thus, as we argued in section 2.5.1 the Markov renewal sequence $\{(X_n,T_n)\}$ is a renewal sequence. Obviously property 2 of our semi-regenerative process definition is satisfied.

Finally, since $\{Y_n\}$ is a sequence of independent, identically distributed random variables any function of Y_n after T_n is independent of that function of Y_n prior to T_n. Since there is only one value of X_n condition 3 of our semi-

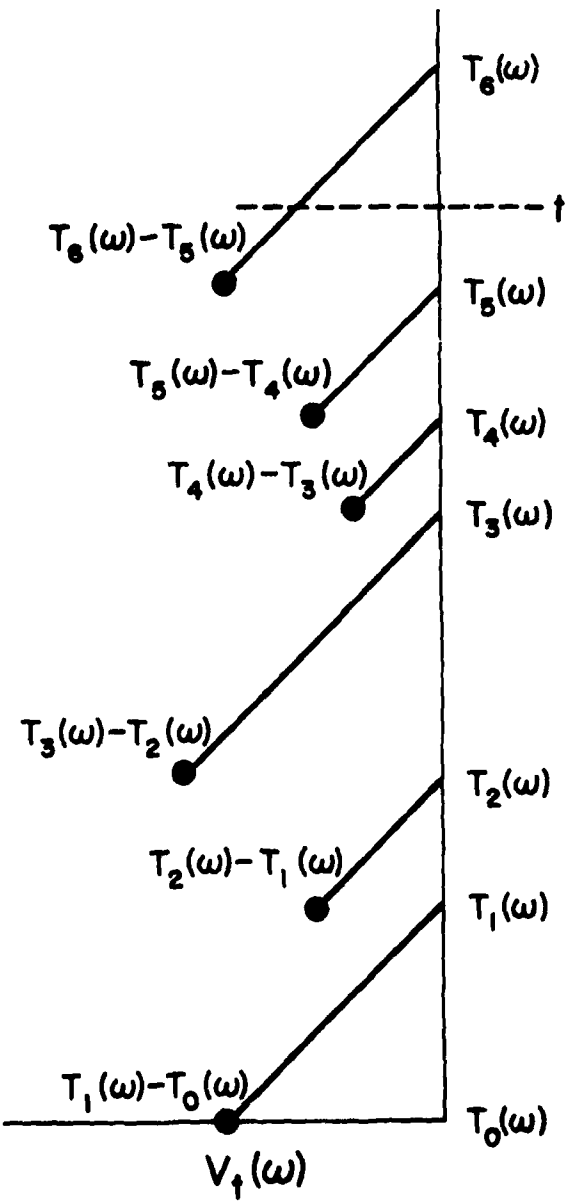


Figure 3.2.1
A Forward Recurrence Time Sample Path
for the Sample Point ω

regenerative process definition is trivially satisfied.

Thus we conclude that $\{Z(t)\}$, the forward recurrence time process of a renewal process is a semi-regenerative process. In fact since this semi-regenerative process does not even depend on $\{X_n\}$ (since there is only one value X_n can take for every n) it is a regenerative process. That is it satisfies conditions (3.1.1.a)-(3.1.1.c).

3.2.2. The Birth Process. Let us define a simple random process, called a pure birth process, as follows. Let $E = \{0,1,2,3,\cdots\}$, $X_0 = 0$,

$$X_{n+1} = X_n + 1, \quad n = 0,1,2,\cdots .$$

and

$$\Pr[T_{n+1} - T_n \leq t | X_n = i,\ X_{n+1} = j] = 1 - e^{-\lambda(i)t}$$

$\{(X_n,T_n)\}$ is a Markov renewal process according to (2.2.1). $\{X_n\}$ is a transient Markov chain and $T_{n+1} - T_n$ depends on the state X_n through the parameter $\lambda(i)$. We assume

$$\sum_{i=0}^{\infty} 1/\lambda(i) = \infty$$

so that the increments in $\{T_n\}$ remain finite with probability 1. (e.g., the process does not "explode" in finite time.)

Define

$$Y(t) = X_n, \quad T_n \leq t < T_{n+1}.$$

Then $\{Y(t)\}$ is a process that has almost all sample paths that are step functions. At each t, Y(t) expresses the total number of jumps that have

occurred since t = 0. {Y(t)} is a semi-regenerative process but it is not regenerative.

If further we define $Z(t) = Y(t + T_n) - Y(T_n)$ then {Z(t)} counts the number of jumps in {Y(t)} over the interval $[T_n, T_n + t]$. Then {Z(t)} is semi-regenerative and is regenerative if and only if $\lambda(i) = \lambda$.

3.2.3. The Time Dependent Queue Length in the M/G/1 Queue. Let us now take a somewhat more complicated example of a semi-regenerative process. Consider the M/G/1 queue. Let T_n be the time of the nth departure from this queue and let X_n be the number of customers left behind by the nth departure. If we let S_n be the service time of the nth customer and I the idle time of the server when the queue is empty then we have obviously

$$T_{n+1} - T_n = \begin{cases} S_{n+1}, & \text{if } X_n > 0, \\ I + S_{n+1}, & \text{if } X_n = 0. \end{cases}$$

Therefore,

$$\Pr[T_{n+1} - T_n \leq t | X_n]$$

is completely determined by the service time distribution (say H(t)) if $X_n > 0$ and is given by the convolution of an idle time distribution (which is exponential for the M/G/1 queue) and the service time distribution if $X_n = 0$.

Furthermore,

$$X_{n+1} = X_n + A(T_{n+1} - T_n) - 1, \quad X_n > 0$$

$$= A(T_{n+1} - T_n), \quad \text{if } X_n = 0,$$

where $A(T_{n+1} - T_n)$ is a Poisson distributed random variable with parameter $\lambda(T_{n+1} - T_n)$. Thus,

$$\Pr[X_{n+1} = j | T_{n+1} - T_n, \quad X_n = i]$$

is simply the probability that there are $j - i + 1$ arrivals during the given interval $T_{n+1} - T_n$ of length, say, t, if $X_n > 0$ and is simply the probability that there are j arrivals over the interval if $X_n = 0$. Since the arrival process is a Poisson process, these probabilities are completely known. Therefore, the process $\{(X_n, T_n)\}$ is a Markov renewal process according to (2.2.1). We will exploit this result in section 4.1.2.

Now define the random process $\{Z(t)\}$ by

$$Z(t) = X_n, \text{ for } T_n \leq t < T_{n+1}.$$

Then $\{Z(t)\}$ is the continuous time queue length process. It is easy to see that $\{T_n\}$ is a sequence of stopping times for $\{Z(t)\}$. (A picture of a sample path is some help.) X_n depends only on $\{Z(t)\}$ for $t \leq T_n$, obviously. However, it is clear here that the future of $\{Z(t)\}$ after T_n depends on how many people are in the queue at T_n (i.e., depends on X_n). For example, the queue length at any time after T_n and before T_{n+1} depends on whether the queue was empty at T_n or not. Nonetheless conditions (3.1.1)-(3.1.3) are seen to be satisfied and we conclude that the time dependent queue length process for the M/G/1 queue is a semi-regenerative process.

3.2.4. A Disease Model. In a study undertaken some years ago of a serious disease, one form of the disease was modelled as a Markov renewal process. At that time it was medically acceptable to assume that the disease

progressed in stages. Thus we can define X_n to be the stage of the disease entered the nth time it changes stage. T_n can be taken as the time of the nth change of stage. It was assumed that the pair $\{(X_n,T_n)\}$ was a Markov renewal process. Of particular interest in tracking the disease is a three-tuple $\{(Y(t), V(t), U(t))\}$ random process where

$$Y(t) = X_n, \quad T_n \leq t < T_{n+1},$$

$$V(t) = T_{n+1} - t, \quad T_n \leq t < T_{n+1},$$

$$U(t) = t - T_n, \quad T_n \leq t < T_{n+1}.$$

For this model, Y(t) gives the stage of the disease at time t, V(t) gives the time until the next stage is reached and U(t) is the length of time the current stage has been occupied. It is not difficult to show that this $\{(Y(t), V(t), U(t))\}$ is a semi-regenerative process whose associated Markov renewal process is $\{(X_n,T_n)\}$. A new wrinkle, that we must by-pass, is that in this process the underlying Markov chain $\{X_n\}$ has an absorbing state.

3.2.5. The Minimal Semi-Markov Process. The random process defined by

$$Y(t) = X_n, \; T_n \leq t < T_{n+1}, \quad t \geq 0,$$

where $\{(X_n,T_n)\}$ is a Markov renewal process, is a semi-regenerative process. The process $\{Y(t)\}$ is called the minimal semi-Markov process associated with $\{(X_n,T_n)\}$.

3.3. Some Properties of Semi-Regenerative Processes. Conditions (3.1.1)-(3.1.3) turn out to be useful for computing probabilities for semi-regenerative

processes. For if we are concerned about the occurrence of some event at some future time, t, we may partition the probability into two cases: Either the time t occurs before semi-regeneration epoch T_1 or it does not. Therefore, the probability of the event can be composed by considering two mutually exclusive cases: $t < T_1$, $t \geq T_1$. The first case will take a bit of work to obtain probabilities but the second case is made easier than it would appear to be. For if T_1 is a semi-regeneration epoch, the process loses all memory of its past except for the state of the $\{X_n\}$ process occupied at T_1. Furthermore, the semi-regenerative process has probability laws after T_1 that are replicas of those laws if the process had started in X_n at time 0. That is, except for rescaling time the probability laws are invariant to shifts of the time scale from T_1 to 0 (assuming the state is X_n at both times, of course). Thus, for example, if $f(t)$ is a probability distribution for some event for $t \geq T_1$ then $f(t - T_1)$ is the probability for this same event for $t \geq 0$. This independence and time shift property are enormously useful as shown below.

Let $\{Z(t)\}$ be a semi-regenerative process with state space F, with underlying Markov renewal process $\{(X_n, T_n)\}$, with state space E, whose semi-Markov kernel is $\underset{\sim}{Q}(t)$ and whose Markov renewal function is $\underset{\sim}{R}(t)$. Let $A \subset F$. Let $h_i(A,t) = \Pr[Z(t) \in A, T_1 > t | X_0 = i]$. That is, $h_i(A,t)$ is the probability that the semi-regenerative process is in set A at t and the first regeneration point has not yet been reached, given $X_0 = i$. Let $f_i(A,t) = \Pr[Z(t) \in A | X_0 = i]$, $t \geq 0$. That is, $f_i(A,t)$ is the total probability that the semi-regenerative process is in set A for any $t \geq 0$ given that $X_0 = i$. Then we have

$$f_i(A,t) = h_i(A,t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) f_k(A, t-s). \tag{3.3.1}$$

We can give a heuristic argument to explain (3.3.1) as follows. Starting with $X_0 = i$, $Z(t) \in A$ only two ways depending on whether $t \geq T_1$ or not. If $t < T_1$ then the first semi-regenerative point has not been reached. Thus, $\Pr[Z(t) \in A, T_1 > t | X_0 = i] = h_i(A,t)$ simply by definition. On the other hand if $t \geq T_1$ then by t the first semi-regeneration point has been reached. At that point (say, $T_1 = s$) the process $\{X_n\}$ jumps from state i where it started to some state k. By the semi-regeneration property (3.1.3) the future probability behavior of $\{Z(t)\}$ depends only on this new state k and given this k,

$$\Pr[Z(t) \in A | X_s = k, T_1 = s] = \Pr[Z(t-s) \in A | X_0 = k]$$

$$= f_k(A, t-s), \; k \in E, \; A \subset F.$$

Since the intermediate state k and the time it is first entered is of no interest to us for the probability in (3.3.1) we can sum out k and integrate out s. Then (3.3.1) is simply a Markov renewal equation as defined in section 2.7.

Furthermore, from equations (2.7.1) and (2.7.2) we know the solution to (3.3.1) for all t and the limiting value for $t \to \infty$. In this sense the entire time path of $\Pr[Z(t) \in A | X_0 = i]$ is known. Of course, we can take any initial probability vector for $\Pr[X_0 = i]$. Thus, from first principles, $\Pr[Z(t) \in A]$ is completely determined for all $t \geq 0$ and $A \subset F$.

3.4. Summary. Let us quickly summarize where we stand before moving to some examples and applications of the theory of semi-regenerative processes. If we know (can prove) that a random process $\{Z(t)\}$ is a semi-regenerative process (satisfies conditions (3.1.1) to (3.1.3)) then it is a simple matter, in

principle, to computer the entire time path of the probability of {Z(t)}. One first constructs a Markov renewal equation as (3.3.1). Then immediately one has from equations (2.7.1) or (2.7.2) the sought-for probabilities.

If E is large, one may need computer assistance to perform the necessary operations unless $\underset{\sim}{Q}(t)$ has some structure that can be exploited. If E is infinite, as occurs in most queueing applications, then one probably must rely on computer assistance and numerical analysis. If $\underset{\sim}{Q}(t)$ is infinite with special structure, that structure may be exploitable. However numbers are obtained from the above results, one must recall that the time dependent solution corresponding to A = {j} and Pr[Z(t) = j] for the M/M/1 queue, for example, (probably the simplest of all queues to study) is given by an infinite number of Bessel functions (each of which is an infinite series). Nothing in any theory says the answer will be simple or "in closed form" or in terms of elementary functions. However, if the problem being modelled is important enough then the effort necessary to get numerical answers may be worthwhile. If the problem is not that important gross approximations, perhaps to $\underset{\sim}{Q}(t)$, may be good enough to get usable answers.

IV. EXAMPLES AND APPLICATIONS

4.0. Introduction. In section I we presented some results to show a difference between a variance computed from a simulation study of the departure process from an M/M/1/3 queue and the exact variance. While some of those differences were not absolutely large there are some that are relatively large (25% difference or more). In this section we will return to that problem to see where these differences are coming from. At the same time we will have

accumulated enough information to push that problem further. We will present these two examples as 4.1.2 and 4.1.3. Section 4.1.2 is a nice example of how Markov renewal processes arise in queueing theory. Thus, it will serve to exemplify the contents of sections II and III. Example 4.1.3 is a nice extension of 4.1.2 to show how semi-regenerative processes arise. In that section we will note how the time dependent solution to any M/G/1 queue can be expressed. This will illustrate our remarks at the end of section III that conceptually the structure of the M/G/1 queue is not difficult. The difficulty lies elsewhere - namely in computational procedures to get numbers.

The main purpose of example 4.1.1 is merely to show how information, useful to applications can be derived from our results. The example is trivial but it has been used as part of a large combat model.

In example 4.1.4 we analyze the disease problem of section 3.2.3 somewhat further. In section 4.1.5 we will discuss a few aspects of the Hall study (section 2.3) but that complete study required nearly 300 pages to discuss in depth when it was originally presented so we cannot hope to reproduce all of it. In section 4.1.6 we will present a quick review of a study of equipment reliability due to DeMarco. In section 4.1.7 we will present a quick review of a few other applications. We must beg the reader to examine the source documents. Some of them require several hundred pages to completely expose the underlying processes and their analysis.

For those interested in pursuing these topics and their applications futher, it is difficult to say how to begin. Research and applications have been going on for at least twenty years (25 years if one dates the Smith and Levy work as the start of the field). There is a large literature on the

theory and applications of Markov renewal processes but it is diffused through the world's applied probability literature. In this sense there is no "home" for these topics.

Real-life-applications literature is probably in worse shape than the theory literature with respect to what is openly available. Private reports, industrial studies, government reports, university theses and dissertations and the like would be major sources of real life applications. But little of this is ever published and often what is published is a skeleton of the true work. (The Hall work of nearly 300 pages ended up as a 4 page paper in a probability journal - hardly a process intended to expose the interesting application underlying the 4 pages of theory.) Even the computing literature which has produced an enormous number of papers purportedly relevant to real life applications of these topics to computer modelling seldom supports the model with the requisite data, statistical analysis, parameter estimation, etc. called for in a real life application.

Thus, the reader is warned that finding theoretical studies concerning Markov renewal processes takes some digging into a diffused literature, but it can be done. Finding real life applications done by others that one can study is not only difficult but probably must be done outside the normal channels of journal communications.

4.1. Examples and Applications. In this section we will present examples of various aspects of section II-III. Several of these are "real life" applications chosen from our experience. Thus, they are a biased view of "real life" applications. We chose them because we know them, not for any other reason.

4.1.1. Visibility. The example of this section is of no particular importance. It is small enough to use many of the ideas of the previous section with which to compute and we shall do that. We do note that in a model for military combat a model such as this was used originally to provide some insights into weapons used against targets travelling over rough terrain. The basic idea here was that the target could be in one of two states. It was either visible or not. One question posed was whether the target was visible at t or not. Another question posed was how long was the target visible when it was visible.

Without attempting to get deeply into the larger model, let us simply study the visibility process. Thus, let

$$X_n = \begin{cases} 0, \text{ if the target is not visible at the nth transition,} \\ 1, \text{ otherwise.} \end{cases}$$

Because it is not possible to tell if a visible target changes state to the same visible state or an invisible target changes state but to the same invisible state, we take

$$p_{ij} = Pr[X_{n+1} = j | X_n = i] = \begin{cases} 0, \text{ if } i = j, \\ 1, \text{ if } i \neq j. \end{cases}$$

That is the matrix $\underset{\sim}{P}$ whose elements are p_{ij} has the form

$$\underset{\sim}{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

That is, our model assumes that the target can be in only one of two states. (i.e., $E = \{0,1\}$). If at any time it is in state i it must at the next change of states go to $j \neq i$.

Then to try to get a model of terrain conditions we define T_n to be the time that the target changes state for the nth time. (A more useful assumption might be to measure "time" in terms of range. There are other useful addenda one could give.) We assume

$$\Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i] = F_{ij}(t)$$

reflects the terrain. If the terrain has natural cover features we might expect that on the average the target remains hidden for long periods of time. This would be reflected through $F_{01}(t)$. If the terrain were open with little chance of cover then we would expect that on the average the target was visible for long periods. This would be reflected in the $F_{10}(t)$.

Then the basic process of interest can be taken as the Markov renewal process $\{(X_n,T_n)\}$ with $E = \{0,1\}$, semi-Markov kernel $\underset{\sim}{Q}(t)$ as below and at $T_0 = 0$ we take $\Pr[X_0 = 0] = 1$. Then we have

$$\begin{aligned} Q_{ij}(t) &= \Pr[X_{n+1} = j,\ T_{n+1} - T_n \leq t | X_n = i], \\ &= \Pr[X_{n+1} = j | X_n = i] \Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i], \\ &= (1 - I(i,j)) F_{ij}(t). \end{aligned}$$

The coefficient here is 0, if $i = j$ and is 1, otherwise. Therefore,

$$(4.1.1.1) \qquad \underset{\sim}{Q}(t) = \begin{bmatrix} 0 & F_{01}(t) \\ F_{10}(t) & 0 \end{bmatrix}.$$

A Markov renewal process with this $\underset{\sim}{Q}(t)$ structure is sometimes called an alternating renewal process. Notice that if we are given that we are in state i, the next state is completely determined. So in the $F_{ij}(t)$ notation the subscript j is superfluous. (It must be $\neq i$.) Thus,

$$\Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i] = \Pr[T_{n+1} - T_n \leq t | X_n = i].$$

Our notation is overdone here. It does have the virtue of exposing the theory of sections II and III so we keep it.

Now

$$Q_{ij}^{(2)}(t) = \Pr[X_2 = j,\ T_2 \leq t | X_0 = i],$$

$$= \Pr[X_{n+2} = j,\ T_{n+2} - T_n \leq t | X_n = i],$$

by our homogeneity assumption. And by (2.6.1)

$$Q_{00}^{(2)}(t) = \sum_{j \in E} \int_0^t Q_{0j}(ds) Q_{j0}(t-s),$$

$$(4.1.1.2) \qquad = \int_0^t Q_{00}(ds) Q_{00}(t-s) + \int_0^t Q_{01}(ds) Q_{10}(t-s).$$

But $Q_{00}(ds) = 0$ by (4.1.1.1). So

$$Q_{00}^{(2)} = (F_{01} * F_{10})(t),$$

using our symbolism * to denote a convolution. Similarly

$$Q_{01}^{(2)}(t) = 0$$

since either $Q_{11}(t)$ or $Q_{00}(t)$ is zero when one uses (4.1.1.2) to compute.

Then it follows from these results that if the process starts in state 0, the ensuing entrances to state 0 ("invisibility") form a sequence $\{S_n\}$ and $\{S_{n+1} - S_n\}$ is a sequence of independent, identically distributed random variables with

(4.1.1.3) $$\Pr[S_{n+1} - S_n \leq t] = (F_{01} * F_{10})(t).$$

That is the process of successive entrances to state 0 (for the process starting in state 0) is a renewal process with intervals distributed as in (4.1.1.3).

Suppose the target is invisible at $t = 0$. We would be interested in knowing many things about it at some future time. For example we might want to know the probability that the target is visible at some time t. More importantly in order to attack it when it is visible, it is important to know how long it will remain visible. Considering that it takes time to lay a weapon on a target, if the target is not visible for a long enough time we simply cannot destory it.

To get at such a problem as this, define a process $\{Y(t)\}$ so that

$$Y(t) = X_n, \text{ if } T_n \leq t \leq T_{n+1}.$$

Then for each t, Y(t) simply tells us that the target is or is not visible. Also define

$$Z(t) = T_{n+1} - t, \; T_n \leq t < T_{n+1}.$$

Then for each t, Z(t) tells us how long the target will remain in whatever state it occupies at t. We will not stop here to prove that $\{(Y(t), Z(t))\}$ is a semi-regenerative process with $F = \{0,1\} \times R_+$. It is.

For $A = \{j\} \times \{(y,\infty)\} \subset F$, define

$$G_i(j,y,t) = Pr[Y(t) = j,\ Z(t) > y | X_0 = i], \tag{4.1.1.4}$$

$$h_i(j,y,t) = Pr[Y(t) = j,\ Z(t) > y,\ T_1 > t | X_0 = i]. \tag{4.1.1.5}$$

(4.1.1.4) contains the information we want. For if $j = 1$, then that formula will yield the probability that the target is visible at t and will remain so for more than another y minutes given that originally the target was invisible. (4.1.1.5) is simply the initial function of the $\{(Y(t),Z(t))\}$ as required by formula (3.3.1). But $h_i(j,y,t)$ can be easily found from the basic $\{(X_n,T_n)\}$ process. For example, given that $X_0 = 0$ we will have $Y(t) = j$, with probability 1 if $j = 0$ and with probability 0 otherwise when $T_1 > t$. Furthermore, $Z(t) > y$ and $T_1 > t$ if and only if there has been no change of state before t (and thus at t, $Y(t) = 0$) and there will be no change of state for more than y time units after t. In general,

$$Pr[Z(t) > y,\ T_1 > t | X_0 = i] = Pr[T_1 > t + y | X_0 = i] = 1 - F_i(t+y)$$

where $1 - F_i(t+y) = 1 - \sum_{j \in E} Q_{ij}(t)$. Altogether then,

$$Pr[Y(t) = j,\ Z(t) > y,\ T_1 > t | X_0 = i] = I(i,j)[1 - F_i(t+y)].$$

Then from (3.3.1) we put the pieces together as

$$G_i(j,y,t) = I(i,j)[1 - F_i(t+y)] + \sum_{k\epsilon E} \int_0^t Q_{ik}(ds)G_k(j,y,t-s)$$

and from (2.7.1) we have

(4.1.1.6) $$G_i(j,y,t) = \int_0^t R_{ij}(ds)(1 - F_j(t+y-s)).$$

Here $R_{ij}(t)$ is the i,j term of the Markov renewal function of the underlying $\{(X_n,T_n)\}$ process. In general the $R_{ij}(t)$ function is difficult to compute. But from the very special structure of this $\underset{\sim}{Q}(t)$ matrix it is rather simple for this problem.

$$\underset{\sim}{Q}^*(\alpha) = \begin{bmatrix} 0 & F_{01}^*(\alpha) \\ F_{10}^*(\alpha) & 0 \end{bmatrix}$$

where

$$F_{ij}^*(\alpha) = \int_0^\infty e^{-\alpha t} F_{ij}(dt).$$

And using (2.6.6), it is easy to see that

(4.1.1.7) $$\underset{\sim}{R}^*(\alpha) = \begin{bmatrix} \dfrac{1}{1 - F_{01}^*(\alpha)F_{10}^*(\alpha)} & \dfrac{F_{01}^*(\alpha)}{1 - F_{01}^*(\alpha)F_{10}^*(\alpha)} \\ \dfrac{F_{10}^*(\alpha)}{1 - F_{01}^*(\alpha)F_{10}^*(\alpha)} & \dfrac{1}{1 - F_{01}^*(\alpha)F_{10}^*(\alpha)} \end{bmatrix}.$$

At this point we can proceed in several ways. We can formally invert

(4.1.1.7). Once the functions $F_{ij}(t)$ are given (as in our example below), of course, R(t) can be found by inverting $\underset{\sim}{R}^*(\alpha)$ term by term and then using (4.1.1.6).

Alternatively, one can view (4.1.1.6) as the convolution of two functions and using the convolution properties of Laplace-Stieltjes transforms, the transform of the solution can be obtained immediately from (4.1.1.6) and (4.1.1.7).

Because $\underset{\sim}{R}(t)$ has some independent interest, we will proceed to our example, following, along the first of these two paths.

An example may make the manipulations involved here more apparent.

Example.

$$F_{01}(t) = F_{10}(t) = 1 - e^{-bt}.$$

Then

$$F_{01}^*(\alpha) = \frac{b}{b+\alpha} = F_{10}^*(\alpha).$$

Then it is easy to see that

$$\underset{\sim}{R}^*(\alpha) = \frac{1}{1 - b^2/(b+\alpha)^2} \begin{bmatrix} 1 & \frac{b}{b+\alpha} \\ \frac{b}{b+\alpha} & 1 \end{bmatrix}$$

from (4.1.1.7). This matrix of transforms is not hard to invert. One does it term by term to obtain

$$R_{00}(t) = R_{11}(t) = \begin{cases} 1, \text{ if } t = 0, \\ 1 + \frac{bt}{2} - \frac{1}{4}(1 - e^{-2bt}), \ t > 0. \end{cases}$$

$$R_{01}(t) = R_{10}(t) = \frac{bt}{2} + \frac{1}{4}[1 - e^{-2bt}], \ t \geq 0.$$

It should be noted that $R_{ii}(t)$ will always have a jump at the origin of size 1 since we have defined this function on the closed interval [0,t] and assumed a jump occurs at t = 0 (i.e., $T_0 = 0$). Thus, the expected number of visits to state i is always at least one if $X_0 = i$.

Then from (4.1.1.6) we have after a bit of algebraic manipulation

$$\text{(a)} \quad G_0(0,y,t) = \frac{1}{2} e^{-by} + \frac{1}{2} e^{-2bt - by}$$

(4.1.1.8)

$$\text{(b)} \quad G_0(1,y,t) = \frac{1}{2} e^{-by} - \frac{1}{2} e^{-2bt - by} .$$

In interpreting (4.1.1.8) we have that, if the target is hidden initially, it will be hidden at t and will remain hidden for more than y more time units with a probability given by (a). On the other hand, if it is hidden at t = 0, it will be visible at t and will remain visible for more than y time units with probabilities given by (b). One notes that

(4.1.1.9) $$G_0(0,y,t) + G_0(1,y,t) = e^{-by}, \ y \geq 0$$

as it should for this is simply the probability that the time until the next change of state is greater than y no matter what state is next occupied. Since state occupancy times are identical exponentially distributed random variables

(4.1.1.9) is a consequence of the forgetfulness property.

If in (4.1.1.8) we let $y \to 0$ then

$$G_i(j,0,t) = \Pr[Y(t) = j | X_0 = i].$$

Therefore this limit is the time dependent state probabilities for $\{Y(t)\}$. If then we let $t \to \infty$, we obtain the "steady state" probabilities for $\{Y(t)\}$. Of course these probabilities are each 1/2 because of the symmetry we have built into the problem.

Since

$$G_0(1,y,t) = \Pr[Y(t) = 1,\ Z(t) > y | X_0 = 0] = \frac{1}{2} e^{-by} - \frac{1}{2} e^{-2bt - by}$$

and

$$G_0(1,0,t) = \Pr[Y(t) = 1 | X_0 = i] = \frac{1}{2} (1 - e^{-2bt}),$$

we have

$$\Pr[Z(t) > y | Y(t) = 1,\ X_0 = 0] = \frac{1}{2} e^{-by}(1 - e^{-2bt})/\frac{1}{2}(1 - e^{-2bt})$$

$$= e^{-by}, \quad \text{for all } t \geq 0.$$

Finally, since $\Pr[X_0 = 0] = 1$,

$$\Pr[Z(t) > y | Y(t) = 1] = e^{-by}, \quad \text{for all } t \geq 0.$$

Of course, this is expected because of the forgetfulness of the exponential distribution. The point is that the left hand side is one of the sought for probabilities. It is the probability that the target remains visible for more than y more minutes given it is now visible. The simple result follows

from the very special assumptions made about $F_{ij}(t)$.

Because of the simplicity built into this example in an attempt to provide a model that easily exposes the concepts of sections II and III, the problem can be solved many (and easier) ways than we have done. The $\{Y(t)\}$ process is a Markov process and those topics can be used here. The $\{(X_n,T_n)\}$ process is an alternating renewal process and those methods can be used here.

4.1.2. Departures from the M/G/1/N Queue. Let us return to the example of section I to see why our variance estimates differed from the "true" variances. There, recall, we were interested in an M/M/1/3 queue and its departure process and in particular its mean and variance. We have seen in section 1.1 that one can use simple arguments to obtain the mean value (formula (1.1.1)). The variance is a different matter.

To start let us define

X_n = the queue length left behind by the nth departing customer, $n = 1,2,\cdots$.

T_n = the time of the nth departure.

While our results will be true for many queue disciplines (but not all) we will fix our attention on a first in - first out discipline.

The increment $(T_{n+1} - T_n)$ is the time between the $n+1$ and n departing customer. Clearly, this increment satisfies the identity

$$T_{n+1} - T_n = \begin{cases} S_{n+1}, & \text{if } X_n > 0 \\ I + S_{n+1}, & \text{if } X_n = 0. \end{cases} \tag{4.1.2.1}$$

That is, the time between two consecutive departures may be one service time

(of customer n+1 which we have denoted as S_{n+1}). This occurs if the queue left behind by the nth departure is at least 1. On the other hand, this inter-departure time will be the total time spent awaiting the next customer (I) and then serving him (S_{n+1}) if the nth departure leaves the system empty.

The identity 4.1.2.1 has been known as one of those folk theorems in queueing for many years. However, its exploitation awaited a development of Markov renewal theory.

Now, from that identity, we have immediately

$$(4.1.2.2) \qquad \Pr[T_{n+1} - T_n \leq t | X_n = j] = 1 - e^{-\mu t}, \; j = 1,2,\cdots,$$

where the right hand side is simply the probability that a service time is less than or equal to t. (Remember in M/M/1 queues service times are identically distributed. n plays no major role here.)

Also,

$$(4.1.2.3) \qquad \Pr[T_{n+1} - T_n \leq t | X_n = 0] = \int_0^t [1 - e^{-\lambda(t-s)}] \, e^{-\mu s} ds$$

which simply says that $T_{n+1} - T_n \leq t$ given $X_n = 0$ if and only if $I + S_n \leq t$. The integral on the right is the convolution of the distributions of I (an exponentially distributed random variable with parameter λ) and S_n (an exponentially distributed random variable with parameter μ). As is usual in queueing theory we have assumed the arrival process and service time process are independent processes.

We further know from first principles of queueing theory that $\{X_n\}$ is a Markov chain and

$$(4.1.2.4)\qquad \Pr[X_{n+1}=j \mid X_n=0,\ T_{n+1}-T_n=t]=\frac{(\lambda t)^j e^{-\lambda t}}{j!},\ \text{for } j=0,1,2,$$

$$(4.1.2.5)\qquad \Pr[X_{n+1}=j \mid X_n=i,\ T_{n+1}-T_n=t]=\frac{(\lambda t)^{j-i+1} e^{-\lambda t}}{(j-i+1)!},\ 2>j\geq i-1,\ i>0.$$

Since N = 3, the maximum queue that can exist at a point of departure is 2. Note, therefore, that this $\{(X_n,T_n)\}$ does not give, directly, the queue length process that one would obtain from the usual Markov chain analysis of this problem for all $t \geq 0$.[1]

$$(4.1.2.6)\qquad \Pr[X_{n+1}=2 \mid X_n=i,\ T_{n+1}-T_n=t]=\sum_{j=2}^{\infty}\frac{(\lambda t)^{j-i+1} e^{-\lambda t}}{(j-i+1)!},\ i>0.$$

$$(4.1.2.7)\qquad \Pr[X_{n+1}=2 \mid X_n=0,\ T_{n+1}-T_n=t]=\sum_{j=2}^{\infty}\frac{(\lambda t)^{j} e^{-\lambda t}}{j!}.$$

Then putting (4.1.2.4)-(4.1.2.7) together with (4.1.2.2), (4.1.2.3) we obtain the semi-Markov kernel as

$$(4.1.2.8)\qquad Q_{ij}(t)=\begin{cases} 0, & \text{if } i,j=(2,0)\\ \int_0^t \frac{(\lambda z)^{j-i+1}}{(j-i+1)!} e^{-\lambda z}(\mu e^{-\mu z})\,dz, & (i,j)\in\{(1,0),(1,1),(2,1)\}\\ \int_0^t \frac{(\lambda z)^{j}}{j!} e^{-\lambda z}[1-e^{-\lambda(t-z)}](\mu e^{-\mu z})\,dz, & (i,j)\in\{(0,0),(0,1)\}\\ \sum_{k=2}^{\infty}\int_0^t \frac{(\lambda z)^{k-i+1}}{(k-i+1)!} e^{-\lambda z}(\mu e^{-\mu z})\,dz, & (i,j)\in\{(1,2),(2,2)\}\\ \sum_{k=2}^{\infty}\int_0^t \frac{(\lambda z)^{k}}{k!} e^{-\lambda z}[1-e^{-\lambda(t-z)}](\mu e^{-\mu z})\,dz, & (i,j)=(0,2) \end{cases}$$

[1] See the discussion on state spaces in section 4.1.3.

In this way we can establish our first result as:

The $\{(X_n, T_n)\}$ process is a Markov renewal process on $E = \{0,1,2\}$ with semi-Markov kernel $\underset{\sim}{Q}(t)$ whose elements are given by (4.1.2.8).

The identity 4.1.2.1 establishes that $T_{n+1} - T_n$ depends only on X_n. Formulas (4.1.2.4)-(4.1.2.7) establish that X_{n+1} depends only on X_n and $T_{n+1} - T_n$. Formula (2.2.1) establishes that $\{(X_n, T_n)\}$ is a Markov renewal process. That the transitions functions are as given requires the arguments that produced (4.1.2.8).

The Markov chain $\{X_n\}$ is the usual Markov chain embedded at departure points. From the theory of queueing (or directly by letting $t \to \infty$ in 4.1.2.8) we find the one step transition probabilities for the $\{X_n\}$ process to be given by the Markov matrix:

$$\underset{\sim}{P} = \begin{bmatrix} \frac{1}{1+\rho} & \frac{\rho}{(1+\rho)^2} & \frac{\rho^2}{(1+\rho)^2} \\ \frac{1}{1+\rho} & \frac{\rho}{(1+\rho)^2} & \frac{\rho^2}{(1+\rho)^2} \\ 0 & \frac{1}{1+\rho} & \frac{\rho}{1+\rho} \end{bmatrix} .$$

Since we need $\pi(n)$ below it is necessary to determine $\underset{\sim}{P}^n$. This can be done easily for this problem using an eigenvalue analysis. We have done that. The eigenvalues are respectively

$$\lambda_0 = 1, \ \lambda_1 = \rho/(1+\rho)^2, \ \lambda_2 = 0,$$

The eigenvector matrices are

$$S = \begin{bmatrix} 1 & -\rho^2 & 0 \\ 1 & -\rho^2 & -\rho \\ 1 & (1+\rho) & 1 \end{bmatrix}; \quad S^{-1} = \begin{bmatrix} \frac{1}{1+\rho+\rho^2} & \frac{\rho}{1+\rho+\rho^2} & \frac{\rho^2}{1+\rho+\rho^2} \\ -\frac{(1+\rho)}{\rho(1+\rho+\rho^2)} & \frac{1}{(1+\rho+\rho^2)} & \frac{1}{(1+\rho+\rho^2)} \\ 1/\rho & -1/\rho & 0 \end{bmatrix}.$$

That is,

$$P^n = S\,\Lambda^n\,S^{-1}$$

for S, Λ, and S^{-1} given above. From this one finds that $\pi(n)$ converges to its steady state values of $\frac{1}{1+\rho+\rho^2}(1,\rho,\rho^2)$ as $b_i\lambda_1^n \to 0$, $i=0,1,2$. The b_i are respectively $\frac{(1+\rho)}{1+\rho+\rho^2}$, $\frac{-\rho}{1+\rho+\rho^2}$, $\frac{-\rho^2}{1+\rho+\rho^2}$.

Now

$$(4.1.2.9)\quad F^{(n)}(t) = \Pr[T_{n+1} - T_n \le t] = \sum_{i\in E}\sum_{j\in E} \Pr[X_{n+1} = j,\ T_{n+1} - T_n \le t \,|\, X_n = i]\Pr[X_n = i$$

Thus if we take $\pi_i(n) = \Pr[X_n = i]$ to be the n step state probabilities for the underlying Markov chain $\{X_n\}$ ($\pi(n) = \pi(0)P^n$), (4.1.2.9) can be written in matrix notation as

$$(4.1.2.10)\qquad \underset{\sim}{F}^{(n)}(t) = \underset{\sim}{\pi}(n)\underset{\sim}{Q}(t)\underset{\sim}{U},\ n = 1,2,\cdots$$

where $\underset{\sim}{\pi}(n)$ is the (row) vector whose elements are $\pi_i(n)$, $\underset{\sim}{Q}(t)$ is the semi-Markov

kernel of elements (4.1.2.8) and $\underset{\sim}{U}$ is a (column) vector of 1's.

Furthermore,

$$F^{(n)}(t_1,t_2) = Pr[T_{n+2} - T_{n+1} \leq t_2,\ T_{n+1} - T_n \leq t_1]$$

$$= \sum_{j\epsilon E} \sum_{k\epsilon E} Pr[X_{n+2} = j,\ X_{n+1} = k,\ T_{n+2} - T_{n+1} \leq t_2,\ T_{n+1} - T_n \leq t_1].$$

or

$$F^{(n)}(t_1,t_2) = \underset{\sim}{\pi}(n)\underset{\sim}{Q}(t_1)\underset{\sim}{Q}(t_2)\underset{\sim}{U}, \quad n = 1,2,\cdots . \tag{4.1.2.11}$$

The probability distribution for any two intervals k transitions apart is given in matrix form by

$$F^{(n)}(t_1,t_k) = \underset{\sim}{\pi}(n)\underset{\sim}{Q}(t_1)\underset{\sim}{P}^{k-1}\underset{\sim}{Q}(t_k)\underset{\sim}{U}. \tag{4.1.2.12}$$

Then from (4.1.2.10)-(4.1.2.12) one can determine the means, variances and autocovariance functions for any n. From this one finds

$$\text{a.}\quad E[T_{n+1} - T_n] = \frac{\pi_0(n)}{\lambda} + \frac{1}{\mu},$$

$$\text{b.}\quad Var[T_{n+1} - T_n] = \frac{\pi_0(n)}{\lambda^2}\,[2 - \pi_0(n)] + \frac{1}{\mu^2}, \tag{4.1.2.13}$$

$$\text{c.}\quad Cov[T_{n+2} - T_{n+1},\ T_{n+1} - T_n] = \frac{\pi_0(n)}{\lambda^2}\,[1 - \lambda E(T_{n+1} - T_n)],$$

for the case M/M/1/3. Graphs of (c) are given on the following pages. Covariances of lag 2 and 3 are given in table 4.1.2.1 for a few values of ρ.

It is instructive to note that the covariance of lag 1 (4.1.2.13c) is always negative for the M/M/1/N queues ($0 < N < \infty$). Therefore the variance

Figure 4.1.2.1
Time Dependent Covariance
Lag 1 - M/M/1/3 Queue Departure
Process

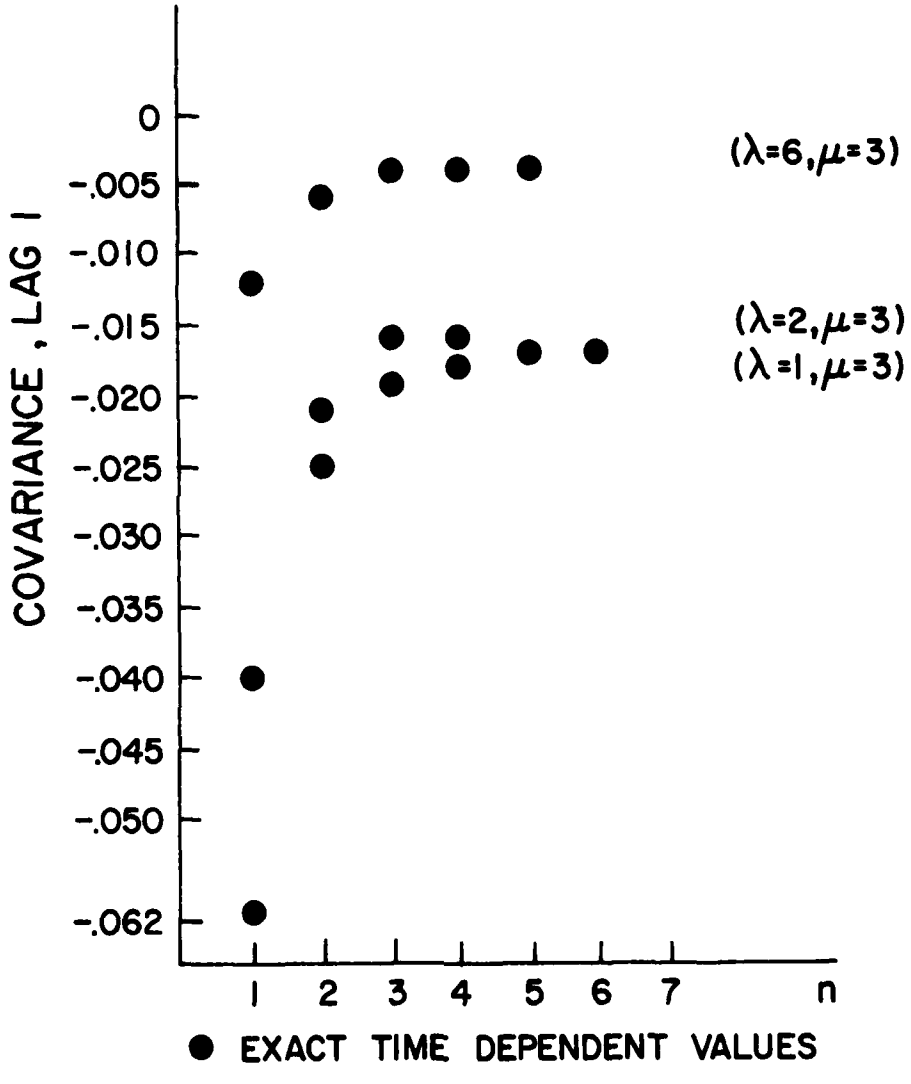


Table 4.1.2.1

Covariances of Lag 2 and 3
for the M/M/1/3 Queue

Lag	Covariance	$\lambda = 1,\ \mu = 3$	$\lambda = 2,\ \mu = 3$	$\lambda = 6,\ \mu = 3$
2	$-\mu\lambda/(\mu^2 + \mu\lambda + \lambda^2)^2$	-.01775	-.01662	-.00453
3	$-(\mu\lambda)^2/(\mu^2 + \mu\lambda + \lambda^2)^2 (\lambda + \mu)^2$	-.00332	-.00398	-.00100

estimated under the assumption of independence (as was computed in section I from the simulation) overestimates the true variance. The overestimate can be relatively large. In the class of Erlang server queues, it can be shown (Disney and de Morais (1976)) that the covariances can be positive, negative or zero depending on λ/μ, N and the Erlang parameter. Therefore, variances estimated under the assumption of independence can underestimate, overestimate or correctly estimate the "true" variance depending on these parameters: The key point to be made, however, is that the departure process from the M/M/1/N queue is a Markov renewal process. This Markov renewal process has $\{T_{n+1} - T_n\}$ as a sequence of independent, identically distributed random variables in only 4 cases (Disney, et al. (1973)). Terms in this sequence are dependent even in the steady state except for those 4 cases. Thus, any statistical analysis which relies on independence of the random variables is inappropriate to the analysis of this sequence. The problem is caused by dependence not by distributional assumptions.

4.1.3. The Time Dependent M/G/1/N Queues.[1] In this section we will study the queue length process of an M/M/1/3 queue. Because of the computations involved this is about as far as we wish to push our hand computation talents. But one should be aware that the exercise here is just that - an exercise. The methods illustrated by the exercise apply to the M/G/1/N queue in general. Çinlar (1975) pursues the topic more generally for the M/G/1 case.

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[1]There's a rather large literature on queueing applications. The reader interested in this special case might consult [16], [17], [18], [19], and [20] in the Bibliography to start.

We have seen in section 4.1.2 that for the M/M/1/3 queue there is an embedded Markov renewal process $\{(X_n,T_n)\}$ on $E = \{0,1,2,\}$ where X_n is the queue length left behind by the nth departure and T_n is the time of the nth departure. The semi-Markov kernel $\underset{\sim}{Q}(t)$ was given by (4.1.2.8).

Let

$$Z(t) = X_n,\ T_n \leq t < T_{n+1}.$$

There is a question here of some interest. Since $N = 3(< \infty)$, $X_n \neq N$ for any n except possibly n = 0. That is, the queue can never be full at a departure point (almost universally we consider the queue just after a departure.) Thus, in the limit the model $\{X_n\}$ gives us no direct information about the queue being full. The state $X_n = 3$ is a transient state. (In fact once state 3 is left after n = 0, if it starts, there, it can never be reentered in the $\{X_n\}$ process. If it does not start in state 3, the process can never be there.)

However, if $X_n = 2$ for some n then Z(t) may be 3 (if an arrival occurs before the next departure). Thus the continuous time process may occupy state 3 even though the embedded process never does after the process is started. Thus, we take $F = \{0,1,2,3\}$.

Then $\{Z(t)\}$ is the time dependent queue length process. It is easy to check conditions (3.1.1)-(3.1.3) to see that $\{Z(t)\}$ is a semi-regenerative process. Then if we let $A = \{j\}$ for $j = 0,1,2,3$ we have

$$(4.1.3.1)\quad h_i(j,t) = \begin{cases} e^{-\lambda t}, & (i,j) = (0,0), \\ \int_0^t \lambda e^{-\lambda(t-s)} e^{-\mu s} \frac{(\lambda s)^{j-1}}{(j-1)!} e^{-\lambda s} ds, & (i,j) \in \{(0,1),(0,2)\}, \\ \sum_{k=2}^{\infty} \int_0^t \lambda e^{-\lambda(t-s)} e^{-\mu s} \frac{(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s} ds, & (i,j) = (0,3), \\ \frac{(\lambda t)^{j-i} e^{-\lambda t}}{(j-1)!} e^{-\mu t}, & (i,j) \in \{(1,1),(1,2),(2,2)\}, \\ \sum_{k=3-i}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} e^{-\mu t}, & (i,j) \in \{(1,3),(2,3)\}, \\ e^{-\mu t}, & (i,j) = (3,3), \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $i = j = 0$, *then* the queue is empty and the first departure (after the previous departure) has not yet occurred if and only if there has been no arrival in $[0,t]$. This accounts for the first term in (4.1.3.1). On the other hand, $i = 2$, $j = 3$ then at T_n the queue has 2 customers and at t (which occurs before the next departure) the queue has 3 customers if and only if <u>at least one</u> customer arrives in $[0,t]$ and the server does not finish the customer he began at T_n.

The queue length process $\{Z(t)\}$ is a semi-regenerative process with state space $F = \{0,1,2,3\}$ and the underlying Markov renewal process $\{(X_n,T_n)\}$ of section 4.1.2. For the $\{Z(t)\}$ process one has

$$h_i(j,t) = \Pr[Z(t) = j,\ T_n > t \mid X_0 = i]$$

as given in (4.1.3.1). Furthermore, the continuous time probability paths

are given by

$$\Pr[Z(t) = j] = \sum_{k \in E} \int_0^t R_{ik}(ds) h_k(j, t-s)$$

where $\underset{\sim}{R}(t)$ is the Markov renewal function associated with the semi-Markov kernel $\underset{\sim}{Q}(t)$ of equation (4.1.1.8). It is most easily found from formula (2.6.6). Formula 2.7.1 and the arguments in this section preceeding that result produce this result.

For this process the embedded Markov chain $\{X_n\}$ on $E = \{0,1,2\}$ has for its steady state solution the solution to the equations

$$\underset{\sim}{\pi} = \underset{\sim}{\pi}\underset{\sim}{P}$$

where $\underset{\sim}{P}$ has been given on p. 47. These steady state solutions are given by

$$\pi_0 = 1/(1 + \rho + \rho^2),$$

$$\pi_1 = \rho/(1 + \rho + \rho^2), \tag{4.1.3.2}$$

$$\pi_2 = \rho^2/(1 + \rho + \rho^2).$$

Time dependent solutions for $\{X_n\}$ can be found from a standard eigenvalue analysis or by raising $\underset{\sim}{P}$ to its various powers. The results needed for this analysis were given on pp. 47-48. We will not go through the details here.

The limiting values of the $\{Z(t)\}$ process can be found from formula (2.7.2) using (4.1.3.2) for the values of π_j in that formula with the obvious modification needed. For this example, $h_i(j,t)$ is directly Reimann integrable so (2.7.2) can be used. Nothing new is gained by going through these manipulations. However, the curious reader might be interested in

comparing the $\{X_n\}$ and $\{Z(t)\}$ limiting values.

This problem can be solved other ways, of course. But our methods have the virtue of exposing the structure of the problem and showing vividly where the difficulties are coming from in the computation of time dependent solutions. The matrix of distributions for $Pr[Z(t) = j, T_1 > t|X_0 = i]$ causes some problems but it has some redeeming features. Rather the computational problems are coming primarily from the convolutions of $\underset{\sim}{Q}(t)$ necessary to generate $\underset{\sim}{R}(t)$. Whether these problems can be surmounted by a careful numerical analysis or not is beyond our ken. The important observation is that we have pinpointed the problem that ultimately must be tackled. The same problem arises in the study of time dependent solutions for the state probabilities for M/G/1/N and G1/M/1/N queues. However, these computational problems exist no matter how one tries to solve the time dependent problem. As always these problems satisfy the well known conservation of difficulty law.

4.1.4. A Disease Model. In the disease model of section 3.2.3 we started by assuming the disease could be modelled as a Markov renewal process. There was little hard data at that time on which to base any assumption about the stochastic nature of the evolution of the process. The assumption of Markov renewalness was made simply because we thought it would be a good first approximation and because it seemed clear from discussion with people more knowledged in the process than we as operations research people were that sojourn times in the various states of the disease depended on both the current state and the next state to be visited. If later data proved the process was Markov we had not lost time or analytic capability. If later data showed that the process was not a Markov renewal process, the Markov renewal assumption at

least directed our attention to what data was needed in what form - a not inconsiderable accomplishment by itself.

At the time of this study there was some question as to whether the disease was regressive or not (i.e., could return to a state it had already visited). Models of both regressive and non-regressive diseases were studied. We will discuss here only the non-regressive model because of its simpler stochastic structure.

Because lifetimes are finite we consider the disease as starting in some state 0 which can be taken to be "the disease absent". The disease may then pass through its several stages which we will simply call stages 1,2. (In the real study there were more than these two stages but to keep the discussion simple we will keep just two.) At any stage (0,1,2) the disease can enter an absorbing state 4 to denote death from causes other than this disease. From stage 2, the disease may also enter an absorbing state, state 3, to denote death from this disease.

The ultimate aim of the study was to try to determine when and how often a person should be examined for this disease and its progress over time. For our present purposes, we will simply look at a few properties of the model of the disease. The other questions take us too far afield and require the development of too much more machinery than we can accomodate in these notes.

To set up the problem formally, let $\{(X_n,T_n)\}$ be a Markov renewal process on the state space $E = \{0,1,2,3,4\}$. Let $X_n = j$ if, at the nth change of state, the disease enters state j. Transitions occur from j to $j + 1$ or 4 (the disease is not regressive). State 3,4 are absorbing states. Let T_n be the time of the nth transition. $T_0 = 0$, $X_0 = 0$ are the initial conditions to imply

that one is born free of this disease. (This is not a crucial assumption. We could just as well take $T_0 = 0$, $\Pr[X_0 = j] = p_j$.)

Then, we let

$$\Pr[X_{n+1} = j | X_n = i] = p_{ij}, \; i,j \in E,$$

and let $\underset{\sim}{P}$ be the matrix of these p_{ij}. Then $\underset{\sim}{P}$ has the form

$$\underset{\sim}{P} = \begin{bmatrix} 0 & p_{01} & 0 & 0 & p_{04} \\ 0 & 0 & p_{12} & 0 & p_{14} \\ 0 & 0 & 0 & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that by our modelling procedure $\underset{\sim}{P}$ is upper triangular. There are two absorbing states and transitions are as shown under the assumption that the disease is not regressive.

Then, let

$$F_{ij}(t) = \Pr[T_{n+1} - T_n \leq t | X_{n+1} = j, \; X_n = i], \; i,j \in E.$$

Then, using (2.2.2) we have that the Markov renewal process $\{(X_n, T_n)\}$ has semi-Markov kernel given by

$$\underset{\sim}{Q}(t) = \begin{bmatrix} 0 & p_{01}F_{01}^{(t)} & 0 & 0 & p_{04}F_{04}^{(t)} \\ 0 & 0 & p_{12}F_{12}^{(t)} & 0 & p_{14}F_{14}^{(t)} \\ 0 & 0 & 0 & p_{23}F_{23}^{(t)} & p_{24}F_{24}^{(t)} \\ 0 & 0 & 0 & F_{33}^{(t)} & 0 \\ 0 & 0 & 0 & 0 & F_{44}^{(t)} \end{bmatrix} .$$

Except for the last 2 rows, we can interpret the non-zero elements here as telling us that at the nth change of state, the disease moved into state i. In at most t time units it will change state again and thence will move into state j. The last two rows require a bit of discussion. These rows correspond to the absorbing states 3,4. From either of these states transitions to any other state is impossible. This explains the 0's in the row except for the diagonal term. Transitions from the state to itself occur at each step with probability 1 (as is true for any Markov chain model with absorbing states). However, the time from first entry to these states to the next transition is finite with probability 0. Thus $F_{ij}(t) = 0$ for all finite $t \geq 0$ and $F_{33}(\infty) = F_{44}(\infty) = 1$. So, in effect, once states 3 or 4 are reached the process stops.

For diagnostic purposes and treatment one would like to know for any $t \geq 0$, the stage of the disease, how long the disease has been in that stage and how much longer that stage will be occupied. These three variables are not independent so knowledge of the probability of the triplet will be more informative than knowledge of the marginal probabilities of each. Therefore we define

$$Y(t) = X_n, \text{ if } T_n \leq t < T_{n+1},$$

(4.1.4.1) $$U(t) = t - T_n, \text{ if } T_n \leq t < T_{n+1},$$

$$V(t) = T_{n+1} - t, \text{ if } T_n \leq t < T_{n+1}.$$

(It should be noted, as is well known in renewal thoery, $V(t) + U(t) = T_{n+1} - T_n$ does not have the distribution $F_{ij}(t)$. Thus, for example, one cannot directly use $V(t) + U(t)$ to estimate $F_{ij}(t)$ from data.) Let $Z(t) = (Y(t), U(t), V(t))$. Then $\{Z(t)\}$ is a three-tuple valued random process on the state space $E \times R_+ \times R_+$. This process carries the information we want.

Now it is clear that $\{T_n\}$ are stopping times for $\{Z(t)\}$. (For example, see section 3.2.1.) Furthermore, X_n is completely determined by $Z(t)$ for $t \leq T_n$. In fact X_n is simply $Y(T_n)$. That $\{Z(t)\}$ satisfies condition 3 of section 3.1 is not as clear. We will give a heuristic argument that it does. Notice first of all that at each T_n, $U(T)_n = 0$ and $V(T)_n$ is the time until the next transition starting in state X_{T_n}. Furthermore, $Y(t)$ for all $t \geq T_n$ has probability laws determined by X_{T_n} by (4.1.5.1). Thus, $(Y(t), U(t), V(t))$ is probabilistically determined by X_{T_n} for $t > T_n$. Since the $\{(X_n, T_n)\}$ process is homogeneous, these probabilities do not depend on n so we can take $n = 0$. Thus, condition 3 of section 3.1 is valid and $\{Z(t)\}$ is a semi-regenerative process.

Let $A = \{Y(t) = j, U(t) > x, V(t) > y\}$ denote the event "the disease at time t is in state j, it has been in this state more than x time units and it will remain there more than another y time units." Let

$$f_0(A,t) = \Pr[Z(t) \in A | X_0 = 0].$$

Let

$$h_0(A,t) = \Pr[Z(t) \in A, T_1 > t | X_0 = 0].$$

Then

$$(4.1.4.2) \qquad h_0(A,t) = I(0,j)I_{(x,\infty)}(t)\Pr[T_1 > t + y | X_0 = 0].$$

We can explain (4.1.4.2) as follows. The condition $T_1 > t$ means that the disease has not changed states by t. Therefore, for Z(t) to be in A it must have started in A (i.e., $X_0 = 0$) and not moved out of that state (hence, $J = 0$). No other circumstances have $T_1 > t$. Thus, here, $Z(t) \in A$ only if $A = \{0, U(t) > x, V(t) > y\}$. But there's more. Z(t) will be in A only if $t \in (x,\infty)$. The indicator functions $I(0,j)$, $I_{(x,\infty)}(t)$ correspond to the probabilities for these observations. And finally, $V(t) > y$ only if $T_1 > t + y$. Let

$$g_0(t + y) = \Pr[T_1 > t + y | X_0 = 0].$$

then

$$g_0(t + y) = 1 - \sum_{j \in E} \Pr[X_1 = j,\ T_1 \leq t + y | X_0 = 0].$$

Thus, we have

$$\Pr[Z(t) \in A,\ T_1 > t | X_0 = 0].$$

Then since {Z(t)} is a semi-regenerative process we have from (3.3.1)

$$f_0(A,t) = h_0(A,t) + \sum_{k \in E} \int_0^t Q_{0k}(ds) f_k(A, T - s),$$

and from (2.7.1)

$$f_i(A,t) = \sum_{k \in E} \int_0^t R_{ij}(ds) h_k(A, t - s)$$

where $\underset{\sim}{R}$ is the Markov renewal function for $\underset{\sim}{Q}(t)$.

Because of the special structure of $\underset{\sim}{Q}(t)$, the computation of $\underset{\sim}{R}(t)$ is simple. In fact, because of the several assumptions made here (primarily the lack of a regression in the disease), there are several other ways this problem could have been solved. Because of the structure of the problem and the ability to easily generalize it to account for more states, different transitions or regression we think it is preferrable to other models.

Then $h_i(A,t)$ is Riemann integrable (it is a probability function) and hence if necessary one can use (2.7.2) to determine the limit of $f_0(A,t)$ for $t \to \infty$. Of course, because $\{X_n\}$ is an absorbing Markov chain this process eventually ends in states 3 or 4 with probability 1. However, for disease control it might be important to know which of these absorbing states has the larger probability of eventual entry so there is some value in computing the limiting probability.

There is nothing new to be gained by going through more detailed calculations here. The intent of the example is simply to show how one could model a process as a Markov renewal process and use the results for controlling the process. Details of calculation have been provided in previous examples.

It should be mentioned that there was some argument that this particular disease did not have a stationary transition mechanism. Case studies were inconclusive. Some seemed to support the assumption of homogeneity, some did not.

Therefore, this disease was also modelled as a full, two dimensional Markov model with (X_n, T_n) as before, but now it was assumed that

$$\Pr[X_{n+1} = j,\ T_{n+1} \leq t | X_n = i,\ X_{n-1} \cdots X_0,\ T_n = y,\ T_{n-1}, \cdots, T_0]$$

$$= \Pr[X_{n+1} = j,\ T_{n+1} \leq t | X_n = i,\ T_n = y].$$

In this way the next stage of the disease and the time at which that stage is reached (not the interval between the nth and (n + 1)st jump) depends on the current stage and the time at which that stage was reached. This model is a Markov model on E x R_+. While it is appealing at first glance, one wonders how enough data would ever be collected to estimate the above probabilities so as to validate the model.

The principle here seems to be that we are concerned with modelling. The model that cannot conceivably be validated (not necessarily with today's technology, however,) may well be useless. In short, the cry for "realistic" models can often be detrimental to modelling.

There is one more afterword of some importance here and in other models. In this model one must obtain some estimates of $Q_{ij}(t)$ either directly or by estimating p_{ij} and $F_{ij}(t)$. At the time this research was done such estimates were not available. Therefore, one could proceed parametrically by proposing likely p_{ij} and $F_{ij}(t)$ to observe the sensitivity of the {Z(t)} probabilities. If these probabilities (or more likely moments obtained from these probabilities) are not sensitive to reasonable values of these parameters then it does not seem reasonable to spend effort getting precise estimates for the purpose of this model. If on the other hand the derived information is sensitive to the parameters then one must have precise estimates. In that case the model is useful in determining what data to collect and analyze. There is no virtue in collecting "all possible data" in the hopes that it will be

useful. It often is not and the data collection and analysis study then is not only useless but expensive. That is, it seems to us that a researcher of real life problems should have some model in his mind before extensive, costly data collection is begun. Of course, one often needs a preliminary study to suggest such a model.

4.1.5. Police Emergency Calls. Nearly every community in the country must supply public medical emergency service for its citizens. How a locality does this varies. In Detroit in 1968 such service was provided through the police force.

In the Detroit system of 1968 police precincts, which act as nearly autonomous units, had a mixture of sedans and station wagons used as police vehicles. The sedans were always used for the normal duties of police work such as patrolling. These sedans were called "squad cars". The station wagons manned by police served a dual role. Their normal duty was to act in a squad car capacity. However, these station wagons had some medical emergency equipment so that if necessary they could act as an "ambulance". Whenever a medical emergency arose in a precinct, a citizen could get ambulance assistance from these police ambulances.

For many reasons, the city decided to take a look at this dual functioning police-ambulance system in 1968. The results of our part of that study are contained in a long report (Hall (1968)) that contains the details of the following analysis, it's use in a rather sophisticated analytic model which we cannot reproduce here and the conclusions drawn from this model. The purpose of the following section is to show how the foregoing study of Markov renewal processes was used in one place (it was actually used in several places) in

our study of the dual functioning ambulance system in Detroit in 1968.

After a rather extensive period of observation of the system and many discussions with personnel intimately aware of the details of the system it was possible to develop a crude model depicting the major components of the system. The precinct contained a number of squad cars (the exact number is not important to our later discussion) and one or two ambulances (again exact numbers are not important here). Telephone calls triggered a dispatcher to dispatch a vehicle to some location in the precinct. Based on information available from the call the dispatcher sent the nearest squad car (sedan or "ambulance") for police assistance or an ambulance to the call for medical emergency assistance. The intuitive rule used by the dispatcher was primarily to send to a call for police assistance the vehicle assigned to patrol the area from whence the call arrived. (These areas were called "beats.") In the event the normal vehicle was not available (perhaps it was on another call), the nearest available vehicle in the precinct was dispatched. In this part of the system squad cars and ambulances were indistinguishable.

If the call arriving to the dispatcher was for an ambulance, the nearest ambulance was dispatched. This meant that the beat normally patrolled by that ambulance, when acting as a squad car, was unpatrolled during the time the vehicle was acting as an ambulance.

The driving force of this system was the sequence of calls received by the dispatcher. Rather than delve into the details of the dispatcher himself and our models of him or the process of vehicle dispatching and response and our models of them, we will limit ourselves here only to one aspect of the

process of demand - the time sequence of calls.

To study this sequence, data were obtained on over 5000 calls received by the dispatcher in one precinct in one month. Because these calls are immediately recorded by the dispatcher and punched on a clock to record time and date, this data was about as clean as one can expect to get from any real life system. (Such was not the case in our study of other parts of this system.) One problem with the data was that it included a holiday which was obviously not a typical day. The analysis tried to correct for that.

By collecting data for one month only, in one precinct, of course, it is not possible to discuss the behavior of the system over the year. (e.g., one would expect data collected in summer months to be rather different from data collected in winter months and our study cannot reflect that difference.) Furthermore, you may recall that 1968 may not have been a typical year for police work in Detroit. Our data cannot reflect that. Therefore, whether the detailed numbers of the following discussion are valid for every month of every year or every precinct is a moot point. The analysis of the data and the overall model, however, are probably useful for more general studies and the following is presented in that spirit.

The total calls received by the dispatcher were first split into police calls and ambulance calls. Each series was analyzed separately to start. We will pursue the ambulance data briefly in the following. The police data exhibited somewhat different behavior that would require a much longer discussion than this example deserves as an example of Markov renewal modelling.

First the total number of ambulance calls per day were plotted for the period of study (see figure 4.1.5.1) and then were broken into three sub series

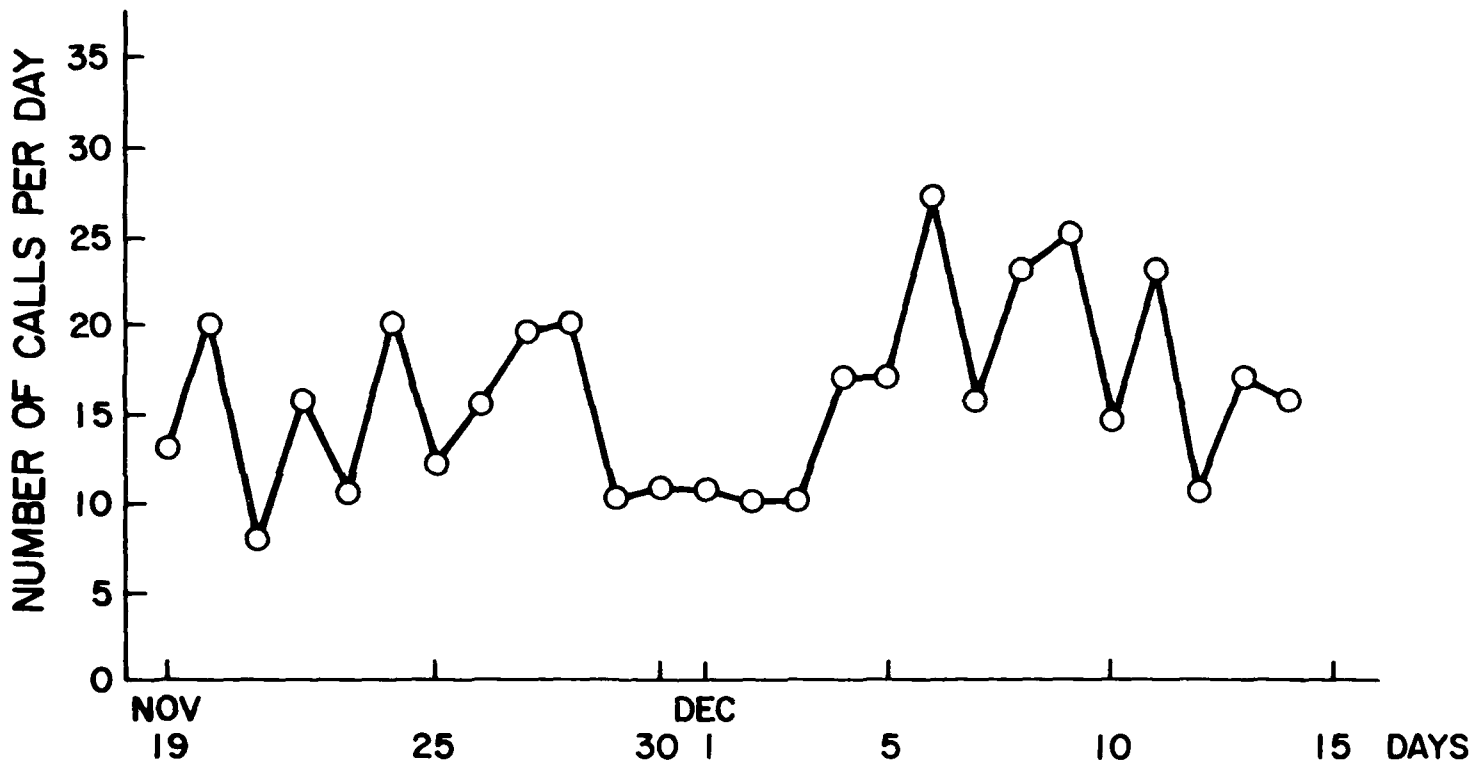

Figure 4.1.5.1
Number of Ambulance Calls per Day

for each of the three shifts of police work. (See figure 4.1.5.2.) No obvious trends were observed nor were there any noticeable shift to shift variations. (The police data differed here.)

Based on the above preliminary analysis, times between ambulance calls were computed and subjected to a battery of tests available in a statistical package for time series analysis then available for computer analysis. This package computed for us the usual means and variances of the interevent times. More importantly for modelling, however, it computed auto-covariances. We computed these for lags up to 100 and, using the usual large sample theory of these covariances, we came to the conclusion that the ambulance data were from a renewal process. (i.e., the covariances were all essentially 0.) This result is not surprising. One expects emergencies to occur "randomly" over time and therefore exhibit properties of renewal processes. But this does not mean that calls for ambulances in the police system (which is what is under observation here) should obey such a process. Road accidents involving multiple victims might well introduce non-renewal properties into the process of calls for ambulances. What the data seemed to indicate is that, if such properties exist for real they occur so seldom that over the course of the entire data record we had, they could not be exposed by the methods used in the data analysis.

To double check the correlation results the periodogram of the data was estimated. Again, using standard statistical tools and the available computer routine, this analysis supported the renewal assumption of the correlation study. (See table 4.1.5.1.) Based on these results and our intuition we proceeded under the assumption that the ambulance call process was a renewal

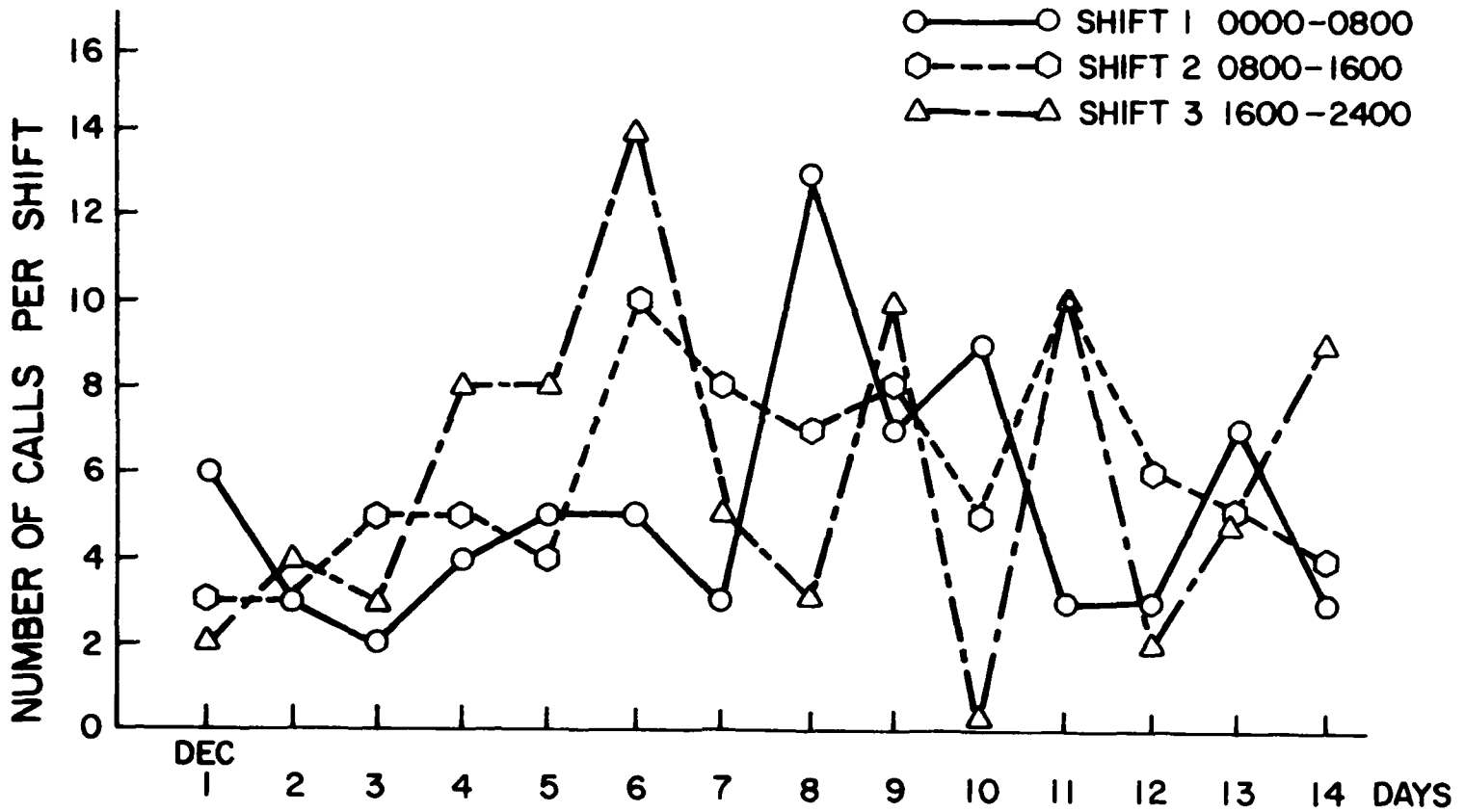

Figure 4.1.5.2
Number of Ambulance Calls per Shift

Table 4.1.5.1

Test of Renewal Hypothesis Based
on Sample Periodogram

Statistic	Value	.05 Significance
Upper-Sided Kolmogorov-Smirnoff	.694	1.224
Lower-Sided Kolmogorov-Smirnoff	.866	1.224
Two-Sided Kolmogorov-Smirnoff	.866	1.358
Moran	.818	2.492

process.

Since an ordinary renewal process is characterized by a single probability function, the remaining task was to analyze the data to estimate this function. There are many tests for goodness of fit of data to distributions. We used five and found that for the precinct under consideration an assumption that the data satisfied the hypothesis of a Poisson process had to be rejected. (i.e., the interevent times form a renewal process - which we accepted - and these interevent times are exponentially distributed - which we had to reject at this point. (See table 4.1.5.2.)) The Moran test (one of the five goodness of fit tests failed by the data) has high power against a gamma distribution alternate hypothesis, so an attempt was made to fit a gamma distribution to the data. This attempt was successful using different methods to test the goodness of fit. (We also used a hypothesis test that is designed to be powerful against a log-normal alternate hypothesis.) The gamma density function is given by the formula

$$f(x) = \left(\frac{k}{\mu}\right)^{k} x^{k-1} e^{-kx/\mu}/\Gamma(k),\ k > 0,\ \mu > 0,\ x > 0.$$

Based on all of the above analysis then we used maximum likelihood estimators to estimate the requisite parameters of the gamma distribution. Table 4.1.5.3 gives the maximum likelihood estimates of k and μ. Then we had a complete picture of the arrival of telephone calls to the dispatcher for medical emergency service. But we were not out of the woods yet.

The objection was raised that even though the calls for ambulance assistance seemed to be a renewal process with gamma distribution intervals there was no evidence that this process and the process of calls for police

Table 4.1.5.2

Test of Negative Exponential Distribution
of Times Between Ambulance Calls

Statistic	Value	.05 Significance
Upper-Sided Kolmogorov-Smirnoff	.361	1.224
Lower-Sided Kolmogorov-Smirnoff	1.799*	1.224
Two-Sided Kolmogorov-Smirnoff	1.799*	1.358
Anderson-Darling	3.137*	2.492
Moran	624.715*	451

*Significant at .05.

Table 4.1.5.3

Maximum Likelihood Estimates
of Parameters for Gamma Model
of Time Between Ambulance Calls

$\hat{\mu}$	101.2
$\hat{k}$	.77
Coefficient of Variation ($\hat{k}^{-1/2}$	1.14

assistance were independent processes. Thus, so it was argued, we could not model the demand on the dispatcher as the superposition to two independent processes. Since the quality of service of the ambulance system could be compromised by the time spent by ambulances in responding to police calls, this objection is non-trivial to the entire study. Indeed, so it was argued, it was not unlikely to find calls for police service closely followed by calls for ambulance service. People can get hurt and need an ambulance when a crime is committed, so went the argument.

Thus, most of the previous analysis must be redone. It does give information about the ambulance demand process. Therefore, any new model must be consistent with it. But it, by itself, is a marginal study of the demand process (i.e., considers only ambulances) when what is required is a study of the joint process (i.e., ambulance and police calls).

To model this joint process we proceeded as follows. Let

$$X_n = \begin{cases} 0, \text{ if the nth call is for an ambulance,} \\ 1, \text{ if the nth call is for police assistance.} \end{cases}$$

Let $T_{n+1} - T_n$ be the time between the nth and (n + 1)st call no matter what types these are. Now consider the pair $(X_{n+1}, T_{n+1} - T_n)$. We assume that the sequence of pairs here forms a Markov renewal process with state space $E = \{0,1\}$.

The Markov renewal model did support the previous data analysis on the ambulance calls, provided a somewhat simpler structure to compute with and provided some new insights into the demand process. Based on that reasoning we thought the Markov renewal model was a "good" model.

Based on the previously collected data we estimated

$$p_{ij} = Pr[X_{n+1} = j | X_n = i],\ i,j \in E.$$

From m_{ij}/m where m_{ij} is the number of transitions from i to j (type i call followed by a type j call) and m is the total number of transitions. (See table 4.1.5.4.) These p_{ij} estimates were then tested against a model wherein the state process formed a Bernoulli process (i.e., X_{n+1} is independent of X_n) by using a χ^2 test of independence. It was found that the sequence of call types was a Bernoulli process from this test. That is, we modelled the $\{X_n\}$ process as one with $Pr[X_n = j | X_{n-1} = i] = Pr[X_n = j] = p_j$.

Then to estimate

$$F_{ij}(t) = Pr[T_{n+1} - T_n \leq t | X_{n+1} = j,\ X_n = i],$$

the available data was split into four pieces corresponding to the possible values of X_{n+1}, X_n. Visual inspection suggested that each of these four distribution functions was an exponential distribution. Tests of goodness of fit were applied to each set of data again using the five goodness of fit tests available in the statistical package. (See table 4.1.5.5.) It was found that $F_{00}(t)$, $F_{01}(t)$, $F_{10}(t)$ passed all five tests of the exponential fit but $F_{11}(t)$ failed all but one. This caused us to investigate the process whereby police calls follow police calls and consider a more detailed model of that process (i.e., a compound Poisson process), but that analysis is not relevent here.

From this analysis we are able to conclude that the joint police - ambulance demand system could be modelled rather well as a Markov renewal process as

Table 4.1.5.4

Frequency of Transitions
between Types of Calls

	0	1
0	42	373
1	373	4352

Table 4.1.5.5

Tests for Negative Exponential
Distribution of Times Between Calls

	0	1
0	.571(a)	.163
	1.029(b)	.604
	1.029(c)	.604
	1.494(d)	.966
	2.833(e)	40.435
1	1.048	.319
	.988	1.933*
	1.048	1.933*
	1.879	5.144*
	65.881	227.196*

* (a) using the Kolmogorov-Smirnoff upper-sided test.
(b) using the Kolmogorov-Smirnoff lower-sided test.
(c) using the Kolmogorov-Smirnoff two-sided test.
(d) using the Anderson-Darling test.
(e) using the Moran test.

* Significant at the .05 level.

described. The conditional intercall times were exponential and the parameters of these could be estimated from the data. (See table 4.1.5.6.) The $F_{11}(t)$ did not fit the data well but the discrepancies could be explained plausibly and a model built for that sequence alone. At that point we were quite satisfied that we had as good a model as our data would support and one that was preferrable to the more common assumption that the two sequences of calls were independent or Poisson or whatever. (Notice that the model is a rather special Markov renewal process. $Pr[X_{n+1} = j | X_n = i] = p_j$ independent of i but $Pr[T_{n+1} - T_n | X_n, X_{n+1}]$ depends on both X_n and X_{n+1}. Thus, even though $F_{ij}(t)$ is exponential, the calling process is not a Markov process.)

As an aside, due to theoretical research of ours performed five years before this study, we were able to study the entire system as a queue with a Markov renewal arrival process. The dispatching rules used by the dispatcher (and many he never thought of) could be modelled as a stochastic process that accounted for the type of arrival and the "state" of the vehicles. In this way a model that was purely analytic was developed and massaged. Results had to obtained numerically but, in spite of the size of the problem and its seeming complexity, it could be studied analytically. The entire study is a nice commentary on the interplay of theory, modelling, real life problems, and statistics. It certainly seems to undercut the "theory - application" dichotomy. Without the previous theory, this application would probably never have been made.

4.1.6. The DiMarco Study (1972). In his study, DeMarco considers a 10 year history of power outages in a system composed of 5 - 200mw steam generating units. Because the boiler unit and generator created nearly all

Table 4.1.5.6

Maximum Likelihood Estimates
of Mean Times Between Calls

	0	1
0	14.33 mins.	16.20 mins.
1	23.08 mins.	14.55 mins.*

* The times themselves here are probably not Poisson distributed.

power outages over this period he concentrates his study on these two units. The modelling process consists of a study of the available data for boiler down times and running times, turbine down times and running times, a combined model for the system's down time and a study of the system's capability to meet the demand placed on the system. We will discuss only a few aspects of the modelling of the boiler outage process.

After eliminating some obvious and explainable discrepencies in his data, DeMarco was left with 1111 outage duration intervals and 774 running duration intervals. A study of the outage duration intervals shows that such intervals can be attributed to one of three types, say type a, b or c. Running time is taken to be the length of time that the boiler system is operating at the full capacity of 200 mw. Thus, while the system is actually operating in some of the outage states (say, a, b), it is not up to capacity and is considered to be in an "outage" state.

This system then can be in one of four states; running, outage a, outage b, outage c. As the system operated every outage state was followed by a return to the running state.

A lengthy and careful statistical analysis of the available data indicated that the length of time the boiler was out depended only on the outage state. Running times appeared to be a sequence of independent and identically distributed random variables. However, if one considered only the sequence of outage states, eliminating temporarily the intervening running states, it appeared that this sequence was a Markov process. Thus, one had the interesting result that if X_n = the state of the boiler at the nth change of state (including the running state), $\{X_n\}$ was a second order

Markov process. That is $\{X_n\}$ had transition probabilities of the form

$$Pr[X_{n+1} = j | X_n = i, X_{n-1} = k].$$

Having to specifically account for this running state interuption to the outage states is a nuisance. To eliminate the problem, DiMarco defines three running states (call them 4, 5, 6) so that every outage of type a returns the system to running state 4, every outage of type b returns the system to running state 5, etc. In this way, the state process of the boiler system is a Markov chain on E = {1,2,3,4,5,6}. Thus, $\{X_n\}$ on this state space has one step transition probability matrix in the form:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ P_{11} & P_{12} & P_{13} & 0 & 0 & 0 \\ P_{21} & P_{22} & P_{23} & 0 & 0 & 0 \\ P_{31} & P_{32} & P_{33} & 0 & 0 & 0 \end{bmatrix} .$$

Having previously established that the lengths of the running times and the lengths of the outage times are each sequences of independent random variables and the several sequences are independent of each other, one can now estimate the probability densities of these interval's lengths. After a lengthy study of the data it was concluded that: these intervals can all be fit by hyper-exponential distributions with a reasonable degree of acceptability. The parameters of these hyper-exponentials are given

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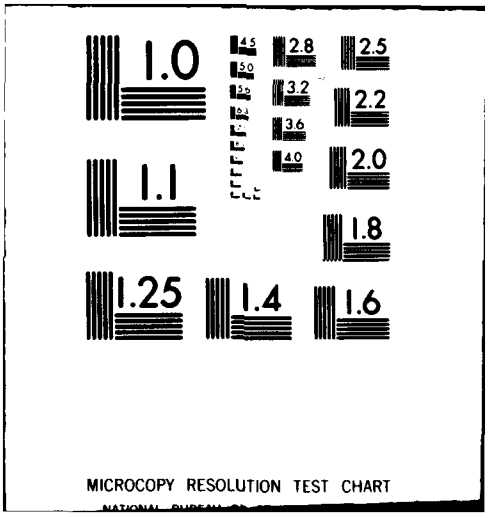
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2.2
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2.8
3.2
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4.0
4.5
5.0
MICROCOPY RESOLUTION TEST CHART

in the paper. They are of little concern to us here.

It is interesting to note, however, that the hyper-exponential density function has a decreasing failure rate. This implies, for example, the seemingly implausible condition that the longer the boiler goes without a breakdown, the less likely it is to breakdown in the near future! DiMarco argues that because of the complexity of these systems, it is unusual to repair them correctly the first time. Therefore, for example, if the boiler has been operational for a long period of time it is likely that it was repaired correctly and therefore will continue to operate. If on the other hand it was not repaired correctly (no matter how long it took to repair it) then it is likely to fail shortly after coming out of repair.

In summary, then, the DiMarco model of the boiler portion of the steam generators is a six state Markov renewal process whose requisite parameters and distributions are estimated from the data. The model has an interesting structure so that one has a degree of simplification and need not use the full theory of Markov renewal process. That is, the model has the structure

$$\Pr[X_{n+1} = j,\ T_{n+1} - T_n \leq t | X_n = i]$$

$$= \Pr[T_{n+1} - T_n \leq t | X_n = i]\ \Pr[X_{n+1} = j | X_n = i].$$

The sojourn times in state i do not appear to depend on the next state to be visited as is possible in the full Markov renewal structure. (See formula 2.2.2 for a comparison.) This makes for a simplification in the analysis of the model. But the hyper-exponential form of the running time and outage time distributions (i.e., the $\Pr[T_{n+1} - T_n \leq t | X_n = i]$) still does not allow for a

model as simple as a continuous time Markov model. (i.e., these sojourn times are not exponentially distributed.) A Markov model appears to be feasible here but to develop it would require a more elaborate state space (probably E x {1,2}, at least). Whether this elaboration would lead to a simpler analysis than DiMarco gives to the Markov renewal model is a moot point.

4.1.7. Other Applications. There are many other applications that have been made of Markov renewal process thoery. We can briefly mention a few here. The reader is invited to consult the noted literature for details.

4.1.7.1. The Daganzo (1975) Study. Road traffic has long been an area of application for random process theory. We briefly mentioned a study of road traffic as a Markov renewal process whose states were identified with leaders and followers in section.

Daganzo used a three state Markov renewal process to study the behavior of traffic on a two lane road. However, he identifies these states with fast moving vehicles and slow moving vehicles. In this way he obtains three states identified with fast moving vehicles that are moving fast (i.e., unimpeded in a platoon) slow moving vehicles (alone or in a platoon) and fast moving vehicles moving slowly (i.e., impeded in a platoon).

His study concentrates on the formation and dissipation of platoons of traffic, passing characteristics, road capacity and the other usual measures of road traffic. Of particular interest in the study is the assumption that vehicles have physical dimensions (most previous studies assumed vehicles were points in a point process). For this reason one can have (and Daganzo studies) the formation of platoons not only by fast vehicles catching up

with slow vehicles but by one platoon extending so as to encounter another platoon. In this way, the author can examine road traffic in medium and heavy traffic - a study rather different from older models of vehicles as points, light traffic conditions, platoons that do not interact and leader - follower models.

The author then observed vehicular flow on a two lane highway and collected data on 295 vehicles to estimate the parameters of his model. He also collected data on 476 other vehicles to test the predictive value of his finished model. The results of this part of the analysis show that this Markov renewal model of road traffic is superior to the then existing models. Indeed, this model is shown to be a generalization of two other existing models so that by particularizing the parameters of the model one can obtain other, previously developed models which seem to be adequate for light traffic conditions but inadequate for moderate to heavy traffic.

Furthermore, the Daganzo model seemed to fare better than some other models which are also Markov renewal models but on different state spaces. The author concludes that his model, while still in need to improvement, seems to produce more realistic results than models that existed at the time.

4.1.7.2. The Lee (1979) Study. The study of fatigue has been an ongoing enterprise in industrial engineering almost since the time of Taylor. In a recently completed study, Lee attempts to build a stochastic model of the physiologic processes that produce fatigue and to study the interaction of fatigue inducing work and lengths of rest periods. His model, which attempts to include many of the results found by many empirical studies, is much too detailed to summarize here. The reader interested in the application of Markov

renewal theory to a classical problem in industrial engineering is invited to consult this important study. To the best of our knowledge this study is about the first to attempt a modelling of the underlying physiologic processes. Most previous studies had been statistical and empirical.

4.2. Other Studies. In an unpublished thesis, Sim (1976) briefly reviews the theory of Markov renewal processes and then provides an annotated bibliography of applications of the theory to studies such as queues and Geiger counters as well as to "real-life" applications. We simply list here the areas where the applications have appeared. The Sim thesis should be consulted for precise references and source documents.

a. Movement of Coronary Patients in a Hospital.

b. Clinical Trials with Acute Leukemia Patients.

c. Screening for Chronic Disease (with applications to cancer).

d. Human Reproduction.

e. Description of Sleep Patterns.

f. Behavior Sequences.

g. Social Mobility.

h. Time Shared Computer Models.

It is interesting to note in this listing that the use of a Markov renewal model seems to be prevalent in the social and biological sciences. We have not noticed the same incursion in engineering and operations research.

A few other applications (items [16]-[21] in the bibliography) have come to our attention. We would be interested in obtaining references to other applications not noted in the references in our bibliography.

V. SUMMARY AND AFTERWORD

5.0. Summary. In the brief space available to us, but really because I lost enthusiasm for saying more, we have tried to accomplish two things. First we have tried to present an encapsulated version of some of the main ideas of Markov renewal theory. That version is neither a primer nor a treatise on the topic. Second we have tried to expose some of the areas of application of this theory. In those discussions we have kept the manipulations to a minimum so as to expose the ideas rather than dazzle the audience with our erring ability to use mathematics. Thus, for example, we assumed that hiding places in a terrain were distributed as a Poisson process in example 4.1.1. Clearly, that is a silly assumption. But our interest was not to avoid silliness as much as to show that Markov renewal theory had been applied and how.

In section IV we tried to expose some of the areas where the theory developed in sections II and III had been applied. Most of these applications have been "real life" in the sense that the researchers built a Markov renewal model based on some existing phenomena, used data collected on the system to estimate parameters of the model, exercised the model to obtain information about the phenomena and used this information to recommend policy, predict outcomes or gain new insights into previously little understood phenomena. In the appropriate cases, it appeared that models built on Markov renewal process theory were more appropriate than models based on the more often used methods of renewal theory or Markov process theory. That is, we think we have demonstrated that such models are useful to a host of modelling activities.

5.1. Afterword. The theory of Markov renewal processes is nearly complete. Literature on the topic is, as is most applied probability, spread over the relevant journals of the world. There is no one home for the topic. Material on the topic is beginning to reach the textbooks though it has been in the research literature for over 20 years. Applications of the theory to topics such as queueing and reliability have been going on for nearly 20 years, not always recognized as such. Indeed, there is reason to believe that the applications to queueing have driven queueing into new areas. Conversely, queueing problems have provided an impetus for the further development of the theory of Markov renewal processes. Modern queueing theory is a much different animal than that exposed in most currently available textbooks.

There is still much to be done at the theory-application interface. We have seen that the computation of most quantities of interest rely on the Markov renewal function. Except for special structure on the semi-Markov kernel, this function is difficult to compute. (Of course, if one only wants steady state behavior, this function need not be computed.) Whether one can develop efficient computational algorithms to help in these computations or can develop a theory of approximations is an open question. Research work at present is pursuing both of these topics.

Statistical properties of the processes are discussed and summarized in the Sim paper. There is a need to extend, unify and make available to potential users this knowledge.

Finally, we need more experience in the use of these topics for real life modelling. In those cases where they have been used for models they seem to perform well. But there are potential areas of application where they have

not been used and areas where they have been used but are still inadequate for modelling the processes.

Bibliography

The first 9 references here are basic to the theory of Markov renewal processes or present some aspect of the topic referenced in the notes. The list is far from complete. Reference [1] is a good elementary introduction to the topic for those whose background includes a study of Markov processes and renewal theory. Reference [2] is a textbook that includes a lengthy discussion of all of these topics and much more.

[1] Çinlar, E. (1975), "Markov Renewal Theory", Mgmt. Sci., 21, 727-752.

A nice discussion of Markov renewal processes on finite state spaces. The references provide the reader with an introduction into the mathematical literature of the topic and several applications and examples not included in our bibliography.

[2] Çinlar, E (1975), Introduction to Stochastic Processes, Prentice-Hall, Englewood Cliffs, N.J.

Chapter 10 of the book is one of the few textbook discussions of Markov renewal processes. The book provides proofs of most of our assertions as well as additional examples. The book is non-measure theoretic and is suitable for a first graduate text in random processes for operations researchers.

[3] Disney, R. L., Farrell, R. L., and de Morais, P. R. (1973), "A Characterization of M/G/1/N Queues with Renewal Departure Processes", Mgmt. Sci., 19, 1222-1228.

A rather complete discussion of departure processes from M/G/1/N($N \leq \infty$) queues. The process is shown to be a Markov renewal process. It is shown that in 4 special cases (and only those 4 cases) this Markov renewal process has $\{T_{n+1} - T_n\}$ as a sequence of i.i.d. random variables.

[4] Disney, R. L. and de Morais, P. R. (1976), "Covariance Properties of the Departure Process of $M/E_k/1/N$ Queues", Transactions (A.I.I.E.), 8, 169-175.

A thorough study of the indicated covariances of lag 1 including graphical results of some of these. The bibliography includes most of the work done on the problem up to about 1970.

[5] Fabens, A. J. (1961), "The Solution of Queueing and Inventory Models by Semi-Markov Processes", J. Roy. Stat. Soc., Ser. B., 23, 113-127, (with a correction note in same journal 25, 455-456.)

The first article that we know that explicitly uses Markov renewal theory in the study of processes of interest to operations research.

[6] Finch, P. D. (1959), "The Output Process of the Queueing System M/G/1", J. Roy. Stat. Soc., Ser. B., 21, 375-380.

An almost complete description of the departure process from M/G/1 queues. The paper implicitly uses some properties of Markov renewal theory.

[7] Levy, P. (1954), "Processus semi-Markoviens", Proc. Int. Congr. Math. (Amsterdam), 3, 416-426.

[8] Pyke, R. (1961), "Markov Renewal Processes: Definitions and Preliminary Properties (Part I)"; "Markov Renewal Processes with Finitely Many States (Part II)", Ann. Math. Stat., 32, 1231-1242 (Part I), 1243-1259 (Part II).

Along with [7] and [9] this is one of the pace setting articles. It was "the" reference until rather recently.

[9] Smith, W. L. (1955), "Regenerative Stochastic Processes", Proc. Roy. Soc. London, Ser. A., 232, 6-31.

Along with [7] is "the" historically basic work in the field.

The following papers are dissertations. Hall was completed in the School of Business Administration. Daganzo was completed in the Department of Civil Engineering. The other two were completed in the Department of Industrial and Operations Engineering. All 4 were done at the University of Michigan. Some of the work on these papers has appeared in journals. We are not aware that there were any articles derived from these works that fully expose the scope of the modelling activity. For that reason we note here the basic dissertation rather than derived papers. We understand that copies are available from University Microfilms International, 300 N. Zeeb Rd., Ann Arbor, Michigan.

[10] Daganzo, C. (1975), Two Lane Road Traffic: A Stochastic Model, 136 pp. (incl. appendix and references).

[11] DiMarco, A. (1972), The Intermediate Term Security Assessment of a Power Generating System, 238 pp. (incl. appendix and references).

[12] Hall, W. K. (1969), A Queueing Theoretic Approach to the Allocation and Distribution of Ambulances in an Urban Area, 262 pp. (incl. appendix and references).

[13] Lee, M. W. (1979), A Stochastic Model of Muscle Fatigue in Frequent, Strenuous Work Cycles, 307 pp. (incl. appendix and references).

The following reference was supplied by Dr. Jeffrey Hunter, Dept. of Mathematics, University of Auckland, Auckland, New Zealand. We understand that copies can be obtained from him.

[14] Sim, D. A. (1976), Semi-Markov Processes and Their Applications, 158 pp. (incl. references).

The following two papers were briefly referenced in the notes. However, they contain several references not included above. Both were reasonably up to date at time of their publication but much new work has appeared since then. We are unaware of more current surveys of either the theory or applications of Markov renewal processes.

[15] Cheong, C. K., deSmit, J. H. A., and Teugels, J. L. "Notes on Semi-Markov Processes" (Part II: Bibliography), Discussion paper 7207, C.O.R.E., Universite Catholique de Louvain.

[16] Disney, R. L. (1975), "Random Flow in Queueing Networks: A Review and Critique", Transactions (A.I.I.E.), 7, 268-288.

We will not try to give a complete lising of queueing applications. The following three entries are concerned with queueing applications.

[17] Cole, A. E., Jr. (1979) Airport Curbside Operations: A Queueing Model with Blocking, Doctoral dissertation submitted to the Department of Civil Engineering, University of Michigan, Ann Arbor. 196 pp. (incl. appendix and references).

An interesting application of a multidimensional Markov process to study a queue of vehicles at the curbside of airports. Data were collected and analyzed at several major airports to validate the model. Readers might be interested in the sampling methods used.

[18] Powers, J. E. and Lackey, R. T. (1975), "Interactions in Ecosystems: A Queueing Approach to Modelling", Math. Biosci., 25, 81-90.

An interesting use of a queueing model to study a predator-prey interaction.

[19] Purdue, P. (1975), "Stochastic Theory of Compartments: An Open, Two Compartment, Reversible System with Independent Poisson Arrivals", Bull. Math. Biol., 37, 269-275.

One of several articles by this author on the use of queueing theory in the study called "compartment models" in biology.

The following 2 papers are a bit off of the beaten path for operations researchers. They do illustrate that the foregoing topics are not solely in the domain of operations research.

[20] Çinlar, E., Bazant, Z., and Osman, H. (1977), "Stochastic Processes for Extrapolating Concrete Creep", J. Engr. Mech. Div. of ASCE, 103, 1069-1088.

A very nice application of Markov renewal processes to a problem describing some properties of creep in concrete.

[21] Rao, C. R. and Kshirsagar, A. M. (1978), "A Semi-Markovian Model for Predator-Prey Interactions", Biometrics, 34, 611-619.

Other reprints in the Department of IEOR, Virginia Polytechnic Institute and State University, Applied Probability Series.

7801 Equivalences Between Markov Renewal Processes, Burton Simon

7901 Some Results on Sojourn Times in Acyclic Jackson Networks, B. Simon and R. D. Foley

7906 Markov Processes with Imbedded Markov Chains Having the Same Stationary Distribution, Robert D. Foley

7922 The M/G/1 Queue with Instantaneous Bernoulli Feedback, Ralph L. Disney, Donald C. McNickle and Burton Simon

7923 Queueing Networks, Ralph L. Disney, revised August, 1980

8001 Equivalent Markov Renewal Processes, Burton Simon

8006 Generalized Inverses and Their Application to Applied Probability Problems, Jeffrey J. Hunter

8007 A Tutorial on Markov Renewal Theory, Semi-Regenerative Processes, and Their Applications, Ralph L. Disney

8008 The Superposition of Two Independent Markov Renewal Processes, W. Peter Cherry and Ralph L. Disney

8009 A Correction Note on "Two Finite M/M/1 Queues in Tandem: A Matrix Solution for the Steady State", Ralph L. Disney and Jagadeesh Chandramohan

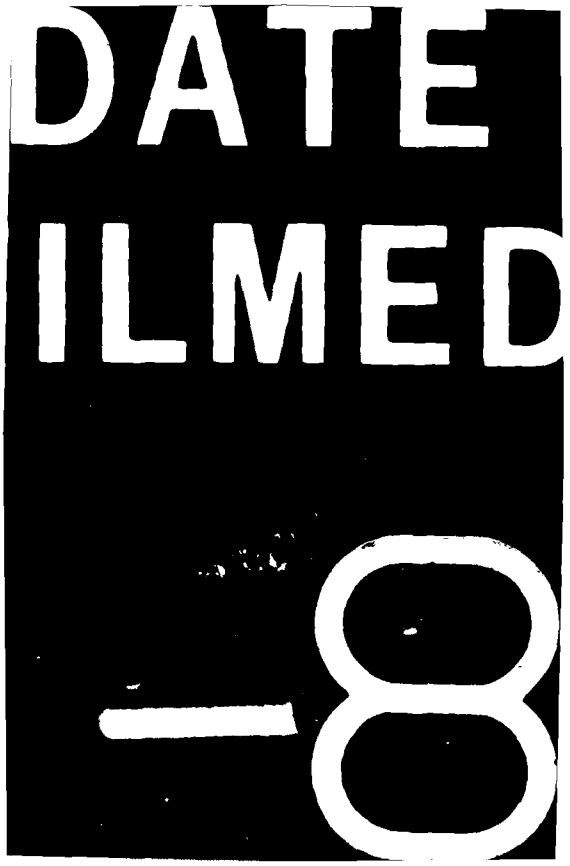
8010 The M/G/1 Queue with Delayed Feedback, Robert D. Foley

8011 The Non-homogeneous M/G/∞ Queues, Robert D. Foley

8012 The Effect of Intermediate Storage on Production Lines with Dependent Machines, Robert D. Foley

8015 Some Conditions for the Equivalence between Markov Renewal Processes and Renewal Processes, Burton Simon and Ralph L. Disney

8016 A Simulation Analysis of Sojourn Times in a Jackson Network, Peter C. Kiessler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