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OPTIMAL UPPER CONFIDENCE LIMITS FOR PRODUCTS OF POISSON PARAMETERS WITH APPLICATIONS TO THE INTERVAL ESTIMATION OF THE FAILURE PROBABILITY OF PARALLEL SYSTEMS

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ABSTRACT

The problem of obtaining optimal upper confidence limits for systems of independent parallel components is treated. Exact optimal upper confidence limits are obtained for an arbitrary number of components for specified failure combinations. For a small number of failures, bounds on the upper confidence limits are obtained. For an arbitrary number of failures an approximation is given which is justified numerically and asymptotically. The results of this paper are compared with the results given by Buehler (1957) and some numerical examples are presented.

AMS (MOS) Subject Classification: 62N05, 90B25.

Key Words: Lindstrom-Madden approximation; Optimal confidence bounds; Reliability; Series system.

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SIGNIFICANCE AND EXPLANATION

Parallel systems arise naturally in practice in engineering and physics. Therefore it is of substantial significance to be able to efficiently utilize data obtained on individual components for the purpose of obtaining an overall assessment of the reliability of the system. The methods of this paper may be employed for this purpose.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

OPTIMAL UPPER CONFIDENCE LIMITS FOR PRODUCTS OF POISSON PARAMETERS WITH APPLICATIONS TO THE INTERVAL ESTIMATION OF THE FAILURE PROBABILITY OF PARALLEL SYSTEMS

Bernard Harris and Andrew P. Soms*

1. Introduction and Summary

A problem of fundamental interest to practitioners in reliability is the statistical estimation of the reliability of a system using experimental data collected on subsystems. In this paper, the subsystem data available consists of a sequence of Bernoulli trials in which a "one" is recorded if the subsystem functions and a zero is recorded if the subsystem fails. Thus for each of the k subsystems composing the system, the data provided consists of the pair (n_i, Y_i), $i=1,2,\ldots,k$, where Y_i is binomially distributed (n_i, p_i). We assume that $Y_1, Y_2, \ldots, Y_k$ are mutually independent random variables.

The magnitude of interest in this problem is easily evidenced by the extensive literature devoted to it. In this regard, see the survey paper by Harris (1977) and Section 10.4 of the book by Mann, Schafer, and Singpurwalla (1974). In addition, the Defense Advanced Research Projects Agency has recently issued a Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability (1980).

Historically, the first significant work on this problem was produced by Buehler (1957). However, Buehler's method as described in that paper is difficult to implement computationally when $k>2$.

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In this paper, we examine the problem of obtaining upper confidence limits for products of Poisson parameters. This problem is studied by means of majorization methods and Schur-convexity, such as described in the book by Marshall and Olkin (1979). A significant application is the determination of confidence limits for the reliability of systems of k parallel subsystems, a fundamental problem in the statistical analysis of reliability.

2. Exact Solutions for Products of Poisson Parameters for Small Failure Combinations

Let $\tilde{X} = (X_1, X_2, \ldots, X_k)$ be independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$, $k \geq 2$, and let $h(\tilde{\lambda}) = \prod_{i=1}^{k} \lambda_i$. Let

$$g(\tilde{x}) = \prod_{i=1}^{k} (x_i + d) \quad , \quad 1<d<1.5 \quad , \quad x_i = 0,1,\ldots \tag{2.1}$$

and denote the ordered points in the range of $g(\tilde{x})$ by $j_1 < j_2 < \ldots < j_m < \ldots$. Define

$$A_i = \left\{ \tilde{x} \mid g(\tilde{x}) = j_i \right\} . \tag{2.2}$$

Since x_i, $i=1,2,\ldots,k$, takes on non-negative integral values, we regard it as desirable to have d in (2.1) only assume non-integer values. This has the effect of making the partition defined in (2.2) finer than would be the case if d were an integer.

It is easily verified that

$$a_n = \sup\left\{ h(\tilde{\lambda}) \mid \sum_{\tilde{x}_i \varepsilon A_i,\, i \leq n} f(\tilde{x}_i;\tilde{\lambda}) = \alpha \right\} \tag{2.3}$$

is a $(1-\alpha)$ upper confidence limit for $h(\tilde{\lambda})$, where

$$f(\tilde{x};\tilde{\lambda}) = e^{-\sum_{i=1}^{k} \lambda_i} \prod_{i=1}^{k} \frac{\lambda_i^{x_i}}{x_i!} \quad , \quad \lambda_i > 0, \quad x_i = 0,1,\ldots \ . \tag{2.4}$$

The proof is identical with that given in Harris and Soms (1980).

Note that if $\tilde{x}$ is fixed as $n_i \to \infty$, $i=1,2,\ldots,k$, then $a_n = \lim_{n \to \infty} \bar{q} \prod_{i=1}^{k} n_i$ where

$$\bar{q} = \sup\left\{ \prod_{i=1}^{k} q_i \,\Big|\, \sum_{\tilde{x}_i \in A_i, i \le n} \prod_{j=1}^{k} \binom{n_j}{x_{ij}} p_j^{n_j - x_{ij}} q_j^{x_{ij}} = \alpha \right\} .$$

Thus in practice $a_n / \prod_{i=1}^{k} n_i$ may be employed as an approximate $(1-\alpha)$ upper confidence limit for $\prod_{i=1}^{k} q_i$, $q_i = 1-p_i$. In this sense the methods of this paper can be used as approximations for estimating the reliability of parallel systems when independent binomially distributed data is obtained for each component.

We proceed by showing that $g(\tilde{x})$ is a Schur-concave function and consequently

$$B_{\tilde{x}_o} = \left\{ \tilde{x} \,|\, g(\tilde{x}) \le g(\tilde{x}_o) \right\}$$

is a Schur-convex set (see Marshall and Olkin (1974), pp. 1189-90 and Nevius, Proschan and Sethuraman (1977), p. 264). The Schur-concavity of $g(\tilde{x})$ follows immediately by noting that

$$(x_1 - x_2)\left(\frac{\partial g(\tilde{x})}{\partial x_1} - \frac{\partial g(\tilde{x})}{\partial x_2} \right) \le 0 .$$

Define $F(\tilde{x}_o;\tilde{\lambda})$ by

$$F(\tilde{x}_o;\tilde{\lambda}) = \sum_{\tilde{x}_i \in B_{\tilde{x}_o}} f(\tilde{x}_i;\tilde{\lambda}) = P_{\tilde{\lambda}}(B_{\tilde{x}_o}) \tag{2.5}$$

and let

$$u(\tilde{x}_o;a) = \sup_{h(\tilde{\lambda})=a} F(\tilde{x}_o;\tilde{\lambda}) , \quad 0<a<1 . \tag{2.6}$$

Since the Poisson distribution has a monotone likelihood ratio, $u(\tilde{x}_o;a)$ is a strictly decreasing function of a for fixed x_o. Hence for every c, $0<c<1$, there is a unique a(c) such that

$$u(\tilde{x}_o;a(c)) = c . \tag{2.7}$$

Consequently, we also have that a_n (see (2.3)) is the solution in a of

$$u(\tilde{x}_o;a) = \alpha \, . \qquad (2.8)$$

(2.8) is established exactly as in Harris and Soms (1980).

The methodology to be employed is as follows. If $F(\tilde{x};\tilde{\lambda})$ is a Schur-concave function of $R_i = -\ln \lambda_i$, $i=1,2,\ldots,k$, then it follows that $u(\tilde{x}_o;a) = F(\tilde{x}_o;a^{1/k}\tilde{1})$, where $\tilde{1} = (1,1,\ldots,1)$, and then the solution in a of $u(\tilde{x}_o;a) = \alpha$ is an optimal upper confidence limit for $\prod_{i=1}^{k} \lambda_i$. This will entail verifying (for fixed $\tilde{x}_o$) that

$$(R_1-R_2)\left(\frac{\partial F(\tilde{x}_o;\tilde{\lambda})}{\partial R_1} - \frac{\partial F(\tilde{x}_o;\tilde{\lambda})}{\partial R_2}\right) \le 0 \qquad (2.9)$$

(see Marshall and Olkin (1974), p. 1190). Accordingly we have the following theorem.

Theorem 2.1: Let $g(\tilde{x}) = \prod_{i=1}^{k} (x_i+d)$, $1<d<1.5$, $k\ge 3$. Define $\tilde{0}_j$ as the j-vector all of whose components are zeros. Then let $x^{(1)} = \tilde{0}_k$, $x^{(2)} = (1,\tilde{0}_{k-1})$, $x^{(3)} = (2,\tilde{0}_{k-1})$, $x^{(4)} = (1,1,\tilde{0}_{k-2})$, $x^{(5)} = (3,\tilde{0}_{k-1})$, $x^{(6)} = (4,\tilde{0}_{k-1})$, $x^{(7)} = (2,1,\tilde{0}_{k-2})$ and $x^{(8)} = (5,\tilde{0}_{k-1})$. The set A_i defined by (2.2) is the point $x^{(i)}$ and the different permutations of its components, $i=1,2,\ldots,8$. Further, for $j=1,2,\ldots,7$, $F(x^{(j)};\tilde{\lambda})$ is Schur-concave in R_i, $i=1,2,\ldots,k$.

Proof: In the sense of the ordering given by (2.2), obviously $x^{(1)} < x^{(2)} < x^{(3)} < x^{(5)} < x^{(6)} < x^{(8)}$. Trivially, $d(2+d) < (1+d)^2$ and hence $(2+d)d^{k-1} = g(x^{(3)}) < (1+d)^2 d^{k-2} = g(x^{(4)})$. Similarly, since $1<d<1.5$, $(1+d)^2 < d(3+d)$ and hence $g(\tilde{x}^{(4)}) < g(\tilde{x}^{(5)})$. In the same way $g(x^{(6)}) < g(x^{(7)})$, $g(x^{(7)}) < g(x^{(8)})$, $g(x^{(8)}) < g(1,1,1,\tilde{0}_{k-3})$ and $g(x^{(8)}) < g(2,2,\tilde{0}_{k-2})$, establishing the first part of the conclusion.

In order to establish Schur-concavity, we must verify (2.9). Thus consider

$$F(\tilde{x}^{(\ell)};\tilde{\lambda}) = \sum_{\tilde{x}_j \varepsilon A_j, j \leq \ell} \prod_{i=1}^{k} e^{-\lambda_i} \lambda_i^{x_i^{(j)}} \Big/ x_i^{(j)}! \qquad (2.10)$$

where $\lambda_i = e^{-R_i}$. Define

$$G(\tilde{x}^{(\ell)};\tilde{R}) = \left(\frac{\partial F(\tilde{x}^{(\ell)};\tilde{\lambda})}{\partial R_1} - \frac{\partial F(\tilde{x}^{(\ell)};\tilde{\lambda})}{\partial R_2}\right) \Big/ e^{-\sum_{i=1}^{k} e^{-R_i}} . \qquad (2.11)$$

Letting $\tilde{R} = (R_1,\ldots,R_k)$, we obtain

$$G(\tilde{x}^{(1)};\tilde{R}) = (e^{-R_1} - e^{-R_2}) ,$$

$$G(\tilde{x}^{(2)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\left(\sum_{i=1}^{k} e^{-R_i}\right) ,$$

$$G(\tilde{x}^{(3)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\left(\sum_{i=3}^{k} e^{-R_i} + \sum_{i=1}^{k} \frac{e^{-2R_i}}{2}\right) ,$$

$$G(\tilde{x}^{(4)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\left(\sum_{i=1}^{k} \frac{e^{-2R_i}}{2} + \sum_{i<j}^{k} e^{-R_i-R_j}\right) ,$$

$$G(\tilde{x}^{(5)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\left(\sum_{i=3}^{k} \frac{e^{-2R_i}}{2} + \frac{e^{-R_1-R_2}}{2} + \sum_{i<j,(i,j)\neq(1,2)}^{k} e^{-R_i-R_j} + \sum_{i=1}^{k} \frac{e^{-3R_i}}{3!}\right) ,$$

$$G(\tilde{x}^{(6)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\left(\sum_{i=3}^{k} \frac{e^{-2R_i}}{2} + \frac{e^{-R_1-R_2}}{2} + \sum_{i<j,(i,j)\neq(1,2)}^{k} e^{-R_i-R_j} + \sum_{i=3}^{k} \frac{e^{-3R_i}}{3!} - \frac{e^{-2R_1-R_2} + e^{-R_1-2R_2}}{3!} + \sum_{i=1}^{k} \frac{e^{-4R_i}}{4!}\right) ,$$

and

$$G(\tilde{x}^{(7)};\tilde{R}) = (e^{-R_1} - e^{-R_2})\Big(\frac{e^{-2R_1-R_2}}{3} + \frac{e^{-R_1-2R_2}}{3} + \sum_{3\le i\le j} e^{-R_i-R_j}$$

$$+ \sum_{i=3}^{k} \frac{e^{-3R_i}}{3!} + \sum_{i=1}^{k} \frac{e^{-4R_i}}{4!} + \sum_{i\ne j,(i,j)\ne(1,2)\text{ or }(2,1)} \frac{e^{-2R_i-R_j}}{2}\Big).$$

Now $R_1 > R_2$ implies $e^{-R_1} < e^{-R_2}$ and thus (R_1-R_2) and $(e^{-R_1} - e^{-R_2})$ have opposite signs. Hence it follows that $F(\tilde{x}^{(i)};\tilde{\lambda})$, i=1,2,3, 4,5,7 is Schur-concave in R_i. The verification that $F(\tilde{x}^{(6)};\tilde{\lambda})$ is Schur-concave may be accomplished by letting k=2, $e^{-R_2} = ce^{-R_i}$, $c>0$, and examining the discriminant.

To show that (2.9) need not be positive for all $\tilde{x}_o$, consider $k = 2$ and $x_o = (7,0)$. Then

$$G(\tilde{x}_o;\tilde{R}) = (e^{-R_1} - e^{-R_2})\Big(\frac{e^{-6R_1} + e^{-6R_2}}{6!}$$

$$- \frac{e^{-4R_1-R_2}+e^{-3R_1-2R_2}+e^{-2R_1-3R_2}+e^{-R_1-4R_2}}{5!}$$

$$- \frac{e^{-3R_1-R_2}+e^{-2R_1-2R_2}+e^{-R_1-3R_2}}{4!} + \frac{e^{-2R_1-R_2}+e^{-R_1-2R_2}}{3}\Big)$$

and this is Schur-convex near $e^{-R_1} = e^{-R_2} = 4$.

Buehler provided an extensive discussion of this problem for the ordering function determined by the product of the upper confidence limits for the individual components. In particular, he provided some numerical tabulations for k=2. Asymptotically Buehler's ordering function is given by

$$g_B(\tilde{x}) = \prod_{i=1}^{k} (x_i+z_\alpha, x_i^{1/2}) \;,$$

where [illegible] = $1-(1-\alpha)^{1/k}$, z_α satisfies $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(x)$ is the standard normal cumulative distribution function. It is easy to see that $g_B(x)$ is Schur-concave (see, e.g., Marshall and Olkin (1974), p. [illegible]).

3. Bounds on Confidence Limits

In this section we employ majorization techniques described in Proschan and Sethuraman (1977) and Nevius, Proschan and Sethuraman (1977) to obtain bounds for a_n. Throughout this section we assume only that the ordering function $g(\tilde{x})$ is strictly increasing in each component and Schur-concave and thus the set $B_{\tilde{x}_0}$ will be Schur-convex (see the discussion immediately preceding (2.5)).

In order to proceed, we need the preliminary results established below.

<u>Theorem 3.1</u>: Let c and a be given with $c>ka^{1/k}$ and consider the set $A(a,c)$ of vectors $\tilde{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_k)$, $\lambda_i \geq 0$, such that

$$\prod_{i=1}^{k} \lambda_i = a \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = c \ . \tag{3.1}$$

Let $S_j = \max_{\tilde{\lambda}\varepsilon A(a,c)} \sum_{i=1}^{j} \lambda_i$. Then there is a unique $\lambda^* \varepsilon A(a,c)$ of the form $\lambda_i = M_j$, $1\leq i\leq j$, $\lambda_i = m_j$, $j+1\leq i\leq k$, $M_j>m_j$, $S_j = jM_j$.

<u>Proof</u>: The condition $c>ka^{1/k}$ is a consequence of the arithmetic-geometric mean inequality and insures that $A(a,c)$ is non-trivial for $k\geq 3$. If $k=2$, there is only one solution of (3.1) with $\lambda_1>\lambda_2$, and hence the Theorem is trivially true. Consequently, suppose $k\geq 3$. Then for fixed j, (3.1) requires that any solution of the required type satisfy

$$jM_j + (k-j)m_j = c \ , \quad M_j^{\,j}\, m_j^{\,k-j} = a$$

and hence setting $m_j = (c-jM_j)/(k-j)$, we consider

$$f_j(M) = M^j[(c-jM)/(k-j)]^{k-j}, \quad 1\le j\le k-1, \quad 0\le M\le c/j \ . \qquad (3.2)$$

Note that $f_j(0) = f_j(c/j) = 0$, and

$$f_j'(M) = (c-Mk)\left(\frac{jM^{j-1}}{k-j}\right)\left(\frac{c-jM}{k-j}\right)^{k-j-1} . \qquad (3.3)$$

Thus, $f_j(M)$ is increasing for $0\le M<c/k$ and decreasing otherwise, further $f_j(c/k) = (c/k)^k > a$. Hence there is exactly one solution M_j of $f_j(M)=a$ with $M_j>c/k$, and therefore $M_j>m_j$.

Now assume that for some j, $1\le j\le k-1$, the vector $\lambda^* = (\lambda_1^*,\lambda_2^*,\ldots,\lambda_k^*)$ with $\sum_{i=1}^{j}\lambda_i^* = S_j$ is not of the form $(\underbrace{M_j,\ldots,M_j}_{j}, \underbrace{m_j,\ldots,m_j}_{k-j})$. Then let $\bar\lambda_{1j} = S_j/j$ and $\bar\lambda_{2j} = (c-S_j)/(k-j)$. Define $\lambda_j' = (\lambda_{j1}',\lambda_{j2}',\ldots,\lambda_{jk}')$ by $\lambda_{ji}' = \bar\lambda_{1j}$, $1\le i\le j$, $\lambda_{ji}' = \bar\lambda_{2j}$, $j+1\le i\le k$. Since the geometric mean of a set of positive numbers whose sum is fixed is a maximum when they are all equal, we have $\prod_{i=1}^{k}\lambda_{ji}' > a$. Now λ_j' is of the required form, however, from (3.2) and (3.3), $\prod_{i=1}^{k}\lambda_{ji}' > a$ implies that there is another solution of the required form with $\lambda_i > S_j/j$, $1\le i\le j$, contradicting the maximality of S_j.

From (2.5) and (2.6), we can write

$$u(\tilde x_o;a) = \sup_{\prod_{i=1}^{k}\lambda_i=a} P_{\tilde\lambda}(B_{\tilde x_o}) = \sup_{c}\ \sup_{\sum_{i=1}^{k}\lambda_i=c,\ \prod_{i=1}^{k}\lambda_i=a} P_{\tilde\lambda}(B_{\tilde x_o}) \ . \qquad (3.4)$$

We state now the main result of this section, using Theorem 3.1.

<u>Theorem 3.2</u>: Let $v_1 = M_1$, $v_i = iM_i-(i-1)M_{i-1}$, $2\le i\le k-1$, $v_k = c - \sum_{i=1}^{k-1} v_i$, where M_i is specified by Theorem 3.1. Then

$$u(\tilde{x}_o;a) \leq \sup_c P_{\tilde{v}}(B_{\tilde{x}_o}) \ . \qquad (3.5)$$

Proof: Since $\sum_{i=1}^{j} v_i = S_j$, $1 \leq j \leq k-1$, $\sum_{i=1}^{k} v_i = c$, $\tilde{v}$ majorizes every λ with $\sum_{i=1}^{k} \lambda_i = c$, $\prod_{i=1}^{k} \lambda_i = a$ (Theorem 3.1). Then (3.5) follows, since if $\tilde{\lambda}_1$ majorizes $\tilde{\lambda}_2$, then for any Schur-convex set A, $P_{\tilde{\lambda}_1}(A) \geq P_{\tilde{\lambda}_2}(A)$ (Proschan and Sethuraman (1977) and Nevius, Proschan and Sethuraman (1971), p. 264 and pp. 267-9).

The vector $\tilde{v}$ may be interpreted as the best vector that majorizes all vectors $\tilde{\lambda}$ such that $\sum_{i=1}^{k} \lambda_i = c$ and $\prod_{i=1}^{k} \lambda_i = a$. More specifically, there is no vector $\tilde{w} \neq \tilde{v}$ such that $\tilde{v}$ majorizes $\tilde{w}$ and $\tilde{w}$ majorizes all $\tilde{\lambda}$ satisfying the two conditions given above.

The following is a suggested method for employing Theorem 3.2. Find a_d such that

$$\alpha = F(\tilde{x}_o; a_d^{1/k} \tilde{1}) \ .$$

Next calculate the smallest a, say a_m, such that $\sup_c P_{\tilde{v}}(B_{\tilde{x}_o}) \leq \alpha$. If $a_m = a_d$, this is the exact solution. Otherwise $a_d < a_m$ and $\sup_c P_{\tilde{v}}(B_{\tilde{x}_o}) < \alpha$ (here $a = a_m$) and the solution a_n satisfies $a_d \leq a_n \leq a_m$. The vector $\tilde{v}$ may be calculated by any of a variety of numerical techniques. In the numerical examples presented here, interval bisection was employed.

Example 1: Let k = 5, a = 25, c = 15. Then the 4 vectors $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ of Theorem 3.1 are

$$\tilde{\lambda}_1 = (9.9660,\ 1.2585,\ 1.2585,\ 1.2585,\ 1.2585)$$

$$\tilde{\lambda}_2 = (6.2004,\ 6.2004,\ .8664,\ .8664,\ .8664)$$

$$\tilde{\lambda}_3 = (4.6696,\ 4.6696,\ 4.6696,\ .4955,\ .4955)$$

and $$\tilde{\lambda}_4 = (3.7172,\ 3.7172,\ 3.7172,\ 3.7172,\ .1309) \ ,$$

from which $\tilde{v}$ is determined to be

$$\tilde{v} = (9.9660,\ 2.4349,\ 1.6079,\ .8601,\ .1309) .$$

Note that in the above example $v_1 \geq v_2 \geq \ldots \geq v_k$. This in fact is always true, as the following theorem establishes.

Theorem 3.3: For $\tilde{v}$ defined by Theorem 3.2, we have $v_1 \geq v_2 \geq \ldots \geq v_k$.

Proof: It follows immediately that $v_1 \geq v_2$, since $M_1 \geq M_2$. Consider therefore v_j, $j \geq 2$. $v_j \geq v_{j+1}$, $j=2,3,\ldots,k-1$ holds if and only if

$$jM_j-(j-1)M_{j-1} \geq (j+1)M_{j+1}-jM_j$$

or

$$jM_j \geq ((j+1)M_{j+1}+(j-1)M_{j-1})/2 ,$$

where $M_k = c/k$ (satisfying the condition $S_k = c = kM_k$ of Theorem 3.1).

Let $\tilde{\lambda}_{A_j} = (1-\alpha_j)\tilde{\lambda}_{j-1} + \alpha_j\tilde{\lambda}_{j+1}$, $j=2,3,\ldots,k-1$, where $\alpha_j = (1/2)+(1/(2j))$ and

$$\tilde{\lambda}_j = (\lambda_{j1},\lambda_{j2},\ldots,\lambda_{jk})$$

and

$$\lambda_{ji} = M_j , \quad 1 \leq i \leq j, \quad \lambda_{ji} = m_j , \quad j+1 \leq i \leq k .$$

It follows that

$$\prod_{i=1}^{k} \lambda_{A_j,i} \geq a ,$$

since $\sum_{i=1}^{k} \ln x_i$ is a concave function of $x_1,\ldots,x_k$. Now let $\lambda_{B_j,i} = \left(\sum_{i=1}^{j} \lambda_{A_j,i}\right)/j$, $i=1,2,\ldots,j$, $\lambda_{B_j,i} = \left(\sum_{i=j+1}^{k} \lambda_{A_j,i}\right)/(k-j)$, $i=j+1,\ldots,k$. Then

$$\prod_{i=1}^{j} \lambda_{B_j,i} \geq \prod_{i=1}^{j} \lambda_{A_j,i} , \quad j=1,2,\ldots,k-1 .$$

Thus, using the properties of M_i in Theorem 3.1,

$$jM_j \geq (j-1)[(1-\alpha_j)M_{j-1}+\alpha_j M_{j+1}] + (1-\alpha_j)m_{j-1} + \alpha_j M_{j+1} ,$$

yielding

$$jM_j \geq ((j+1)M_{j+1}+(j-1)M_{j-1})/2 + (1-\alpha_j)m_{j-1} ,$$

which establishes the theorem.

To illustrate the techniques of this paper, we compare numerical values obtained by the above method with those given in the examples from Mann, Schafer and Singpurwalla (1974, p. 505). From now on we assume d = 1.1.

Example 2: For $\tilde{x}_o = (1,2,1)$ we obtain $a_d = a_n = 20.56$ for $\alpha = .10$. In Mann, Schafer and Singpurwalla, an AO non-randomized confidence bound of 20.7 is obtained.

Example 3: Let $\tilde{x}_o = (2,3,5)$, $\alpha = .10$. Then we obtain $a_d = 135.46$. A summary of computer calculations which establishes $135.46 \leq a_n \leq 142.46$ is given below in Table 1. With the exception of the likelihood-ratio value of 133 and the AO non-randomized confidence bound of 129, all the other confidence bounds given in Mann, Schafer and Singpurwalla exceed the upper bound of 142.46. For k=3 it is possible to do a direct computer tabulation of $u(\tilde{x}_o;a)$. This gives $a_n = 135.46$, the diagonal value.

Insert Table 1 here.

The two examples below are for four and five component systems for which there are no comparable numerical examples available.

Example 4: Let $\tilde{x}_o = (2,2,2,2)$ and $\alpha = .10$. Then $a_d = a_n = 150.63$.

Example 5: Let $\tilde{x}_o = (2,2,2,2,2)$ and $\alpha = .10$. Then $a_d = 429.69$. A summary of the computer calculations which establish $429.69 \leq a_n \leq 435.69$ is provided in Table 2.

Insert Table 2 here.

As $a_d^{1/k}$ increases, the difference between a_d and a_n becomes wider. Thus the techniques of Section 3 are more useful for small x_o, or equivalently, small $a_d^{1/k}$. For example, for $\tilde{x}_o = (5,5,5)$, $a_d = 387.18$, and it is not practical to compute a_m because it is much bigger than a_d. However, direct tabulation of $u(\tilde{x}_o;a)$ reveals once more that $a_d = u(\tilde{x}_o;a)$. A justification of why $a_d \tilde{=} u(\tilde{x}_o;a)$ for large $a_d^{1/k}$ is given in the Appendix. This, together with the results of Section 2, suggests very strongly that for all practical purposes $a_d = a_n$.

Remarks: Note that Tables 1 and 2 are virtually linear in their behavior in the neighborhood of the solution. This suggests that solutions are obtainable by interpolation and then one should subject them to verification.

The calculations described above utilized two short FORTRAN programs for 2-10 components. Listings are obtainable from the authors.

4. Comparisons with Buehler's Tables

In order to provide an illustration of the performance of $g(\tilde{x}_o) = \prod_{i=1}^{k} (x_i+d)$, $1<d<1.5$, when compared with the tables given by Buehler (1957), we chose d=1.1, k=2. For k=2, the values of a_n and a_d coincided for both the ordering based on $g(\tilde{x})$ and Buehler's ordering and further were for all practical purposes equal for the two different orderings.

In Table 3 we give Buehler's upper confidence limit, Buehler's diagonal value and the exact upper confidence limit and diagonal value corresponding to g, denoting them by a_{nB}, a_{dB}, a_{ng} and a_{dg}, respectively. These values are provided for all failure combinations from (0,0) to (5,5) for α=.1.

Insert Table 3 here

An examination of Table 3 shows that differences between the four alternatives presented are small for the specific example (k=2, α=.1).

5. Concluding Remarks

In this paper a procedure for obtaining bounds on an optimal upper confidence limit for the failure probability of a parallel system is given. The procedure employs the theory of majorization and is valid for an arbitrary number of components and gives the exact answer or narrow bounds when the observed number of failures is small for each component. In addition, numerical and asymptotic justification is given for using a_d as an approximation to a_n. Tables of a_d are in preparation for moderate numbers of failures for 3, 4 and 5 components and will be available in the near future.

Appendix

Theorem A1: Let X_{1i}, $1 \le i \le k$, be independent identically distributed normal random variables with means λ and variances λ. Let X_{2i}, $1 \le i \le k$, be independent normally distributed random variables with means τ_i and variances τ_i, where $\tau_i = \lambda + O(\lambda^c)$, $c < 1$, as $\lambda \to \infty$, and $\prod_{i=1}^{k} \tau_i = \lambda^k$. Let β be given, $0 < \beta < 1$, let a be a specified positive real number, let $Z_1 = \prod_{j=1}^{k} (X_{1j} + a)$, $Z_2 = \prod_{j=1}^{k} (X_{2j} + a)$ and let $d(\lambda)$ satisfy

$$P[Z_1 \le d(\lambda)] = \beta. \tag{A.1}$$

Then as $\lambda \to \infty$,

$$\beta - P[Z_2 \le d(\lambda)] = \begin{cases} O[(\ell n\ \lambda)^{1.5}\lambda^{-1}] & , \quad c \le 0, \\ O[\lambda^{c-1}] & , 0 < c < 1. \end{cases} \tag{A.2}$$

Proof: Throughout, let ϕ and Φ denote the density and distribution function of the standard normal. Clearly,

$$P[Z_1 \le d(\lambda)] - P[Z_2 \le d(\lambda)] = \underset{\{\tilde{x}:\ \prod_{j=1}^{k} (x_j + a) \le d(\lambda)\}}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}} (f_1(\tilde{x}) - f_2(\tilde{x}))d\tilde{x}, \tag{A.3}$$

where $\tilde{x} = (x_1, x_2, \ldots, x_k)$, f_1 is the probability density function of $X_{11}, X_{12}, \ldots, X_{1k}$ and f_2 is that of $X_{21}, X_{22}, \ldots, X_{2k}$. Now

$$P[X_{1j} \geq -a,\ j = 1,2,\ldots,k] \geq \left(1 - \frac{\lambda^{1/2}}{(\lambda+a)} \phi(\frac{\lambda+a}{\lambda^{1/2}})\right)^k \tag{A.4}$$

and

$$P[X_{2j} \geq -a,\ j = 1,2,\ldots,k] \geq \prod_{j=1}^{k} \left(1 - \frac{\tau_j^{1/2}}{\tau_j+a} \phi\left(\frac{\lambda_j+a}{\tau_j^{1/2}}\right)\right) . \tag{A.5}$$

Consequently, for λ sufficiently large, there exists a constant $m > 0$ such that

$$P[X_{ij} \geq -a,\ j = 1,2,\ldots,k] \geq 1 - e^{-m\lambda} ,\ i = 1,2. \tag{A.6}$$

Then, for $i = 1,2$,

$$P[Z_i \leq d(\lambda)] = P[Z_i \leq d(\lambda), X_{ij} \geq -a,\ j = 1,2,\ldots,k]$$

$$+ P[Z_i \leq d(\lambda), \bigcup_{j=1}^{k} (X_{ij} < -a)],$$

and therefore

$$P[Z_i \leq d(\lambda)] - P[Z_i \leq d(\lambda), X_{ij} \geq -a,\ j = 1,2,\ldots,k] \leq e^{-m\lambda}. \tag{A.7}$$

Next, we calculate

$$P[Z_1 \leq d(\lambda), X_{1j} \geq -a,\ j = 1,2,\ldots,k] - P[Z_2 \leq d(\lambda), X_{2j} \geq -a,\ j = 1,2,\ldots,k].$$

Now

$$P[Z_1 \leq d(\lambda), X_{1j} \geq -a,\ j = 1,2,\ldots,k | X_{1j} = x_j,\ j = 2,3,\ldots,k]$$

$$= P\left[X_{11} \leq \frac{d(\lambda)}{\prod_{j=2}^{k} (x_j+a)} - a\right] \tag{A.8}$$

$$= \Phi\left(\left(\frac{d(\lambda)}{\prod_{j=2}^{k} (x_j+a)} - a - \lambda\right)/\lambda^{1/2}\right) = \Phi(b_{\tilde{\lambda}}) .$$

Therefore

$$P[Z_1 \le d(\lambda), X_{1j} \ge -a,\ j = 1,2,\ldots,k]$$
$$= \int_{-a}^{\infty}\int_{-a}^{\infty}\cdots\int_{-a}^{\infty} \Phi(b_{\tilde{\lambda}})\, g_1(x_2,x_3,\ldots,x_k)\,dx_2 dx_3\ldots dx_k\ , \tag{A.9}$$

where $g_1(x_2,x_3,\ldots,x_k)$ is the probability density function of $X_{12},X_{13},\ldots,X_{1k}$. From (A.6), we have that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Phi(b_{\tilde{\lambda}})\, g_1(x_2,x_3,\ldots,x_k)\,dx_2 dx_3\ldots dx_k$$
$$- \int_{-a}^{\infty}\int_{-a}^{\infty}\cdots\int_{-a}^{\infty} \Phi(b_{\tilde{\lambda}})\, g_1(x_2,x_3,\ldots,x_k)\,dx_2 dx_3\ldots dx_k \le e^{-m\lambda}. \tag{A.10}$$

Hence we will estimate the first expression on the left hand side of (A.10). Similarly, for Z_2 we will consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Phi\left[\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(x_j+\tau_j)} - a - \tau_1\right)/\tau_1^{1/2}\right] g_2(x_2,x_3,\ldots,x_k)\,dx_2 dx_3\ldots dx_k\ , \tag{A.11}$$

where $g_2(x_2,x_3,\ldots,x_k)$ is the probability density function of $X_{22},X_{23},\ldots,X_{2k}$. In the first integral in (A.10), let $(y_i - \lambda)/\lambda^{1/2} = u_i$ and in (A.11) let $(y_i - \tau_i)/\tau_i^{1/2} = u_i$, $i = 2,3,\ldots,k$, obtaining

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi\left(\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(x_j+a)}-a-\lambda\right)/\lambda^{1/2}\right)g_1(x_2,x_3,\ldots,x_k)\,dx_2dx_3\ldots dx_k$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi\left(\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(x_j+a)}-a-\tau_1\right)/\tau_1^{1/2}\right)g_2(x_2,x_3,\ldots,x_k)\,dx_2dx_3\ldots dx_k \tag{A.13}$$

$$=\int_{-M}^{M}\int_{-M}^{M}\cdots\int_{-M}^{M}\left\{\Phi\left(\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(\lambda^{1/2}x_j+\lambda+a)}-a-\lambda\right)/\lambda^{1/2}\right)\right.$$

$$\left.-\Phi\left(\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(\tau_j^{1/2}x_j+\tau_j+a)}-a-\tau_1\right)/\tau_1^{1/2}\right)\right\}\left(\prod_{j=2}^{k}\phi(x_j)\right)dx_2dx_3\ldots dx_k$$

$$+R_M,$$

where $M=(2\,\ell n\,\lambda)^{1/2}$ and $R_M\le 4\,\dfrac{(k-1)e^{-M^2/2}}{(2\pi)^{1/2}M}=O(\lambda^{-1})$.

Using $d(\lambda)=\lambda^k-k_d(\lambda)\lambda^{k-1/2}$, $k_d(\lambda)=O(1)$,

$$\left(\frac{d(\lambda)}{\prod_{j=2}^{k}(\lambda^{1/2}x_j+\lambda+a)}-a-\lambda\right)/\lambda^{1/2}=(\lambda^{1/2}-k_d(\lambda))\prod_{j=2}^{k}(1+x_j\lambda^{-1/2}+a\lambda^{-1})^{-1}$$

$$-\lambda^{1/2}-a\lambda^{-1/2}\;.$$

Since $|x_i|\le M$, we have

$$(1+x_i\lambda^{-1/2}+a\lambda^{-1})^{-1}=1-x_i\lambda^{-1/2}+(-a+x_i^2)\lambda^{-1}+O((\ell n\,\lambda)^{1.5}\lambda^{-1.5}).$$

Thus

$$(\lambda^{1/2} - k_d(\lambda)) \prod_{j=2}^{k} (1 + x_j\lambda^{-1/2} + a\lambda^{-1})^{-1} - \lambda^{1/2} - a\lambda^{-1/2}$$

$$= -\sum_{i=2}^{k} x_i - k_d(\lambda) - ka\lambda^{-1/2} + k_d(\lambda)(\sum_{i=2}^{k} x_i)\lambda^{-1/2} + (\sum_{i=2}^{k} x_i^2)\lambda^{-1/2} \tag{A.14}$$

$$+ (\sum_{2\le i<j}\sum x_i x_j)\lambda^{-1/2} + O((\ell n\ \lambda)^{1.5}\lambda^{-1}) .$$

Similarly, using $\tau_1 = \lambda^k / \prod_{j=2}^{k} \tau_j$, $\tau_j/\lambda = 1 + O(\lambda^{c-1})$,

$(\tau_j/\lambda)^{1/2} = 1 + O(\lambda^{c-1})$, $j = 1,2,\ldots,k$, $|x_i| \le M$, we have

$$\left(\frac{d(\lambda)}{\prod_{j=2}^{k} (\tau_j^{1/2} x_j + \tau_j + a)} - a - \tau_1 \right) / \tau_1^{1/2}$$

$$= (\tau_1/\lambda_1)^{1/2} \left[(\lambda^{1/2} - k_d(\lambda)) \prod_{j=2}^{k} (1 + x_j\tau_j^{-1/2} + a\tau_j^{-1})^{-1} - \lambda^{1/2} \right] - a\tau_1^{-1/2}$$

$$= -\sum_{i=2}^{k} x_i - k_d(\lambda) - ka\lambda^{-1/2} + k_d(\lambda)(\sum_{i=2}^{k} x_i)\lambda^{-1/2} + (\sum_{i=2}^{k} x_i^2)\lambda^{-1/2} \tag{A.15}$$

$$+ (\sum_{2\le i<j}\sum x_i x_j)\lambda^{-1/2} + O(\lambda^{c-1}) + O((\ell n\ \lambda)^{1.5}\lambda^{-1}).$$

Combining (A.14) and (A.15) with (A.7), (A.9), (A.10) and (A.11) establishes the theorem.

For $c < \frac{1}{2}$ standard weak convergence arguments show that

$$\lim_{\lambda\to\infty} (\beta - P[Z_2 \le d(\lambda)]) = 0.$$

In this case Theorem A1 provides additional information by specifying the rate of convergence.

By standardizing the first expression in (A.13) and applying the dominated convergence theorem the following result can be obtained.

<u>Theorem A2</u>: Let X_{1i}, $1 \leq i \leq k$, be independent identically distributed normal random variables with means λ and variances λ. Let X_{2i}, $1 \leq i \leq k$, be independent normally distributed random variables with means τ_i and variances τ_i, where $\tau_i = \lambda + O(\lambda^c)$, $c < 1$ and let β, Z_1, Z_2 and $d(\lambda)$ be specified as in Theorem A1. Then

$$\lim_{\lambda \to \infty}(\beta - P[Z_2 \leq d(\lambda)) = 0.$$

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1. Summary of Calculations Used to Obtain the Upper Bound for a_n in Example 3

a	$\tilde{v}$			$\sup_c P_{\tilde{v}}(B_{\tilde{x}_o})$
135.46	13.0680	4.7283	1.7108	.1101
136.46	13.0867	4.7409	1.7173	.1086
137.46	13.1053	4.7532	1.7240	.1071
138.46	13.1239	4.7656	1.7305	.1057
139.46	13.1423	4.7780	1.7370	.1042
140.46	13.1607	4.7902	1.7435	.1028
141.46	13.1789	4.8024	1.7500	.1014
142.46	13.3299	4.8057	1.7325	.9999

2. Summary of Calculations Used to Obtain the Upper Bound for a_n in Example 5

a	$\sup_c P_{\tilde{V}}(B_{\tilde{x}_o})$
429.69	.1016
430.69	.1013
431.69	.1010
432.69	.1007
433.69	.1004
434.69	.1001
435.69	.0998

3. Comparison of Exact and Diagonal Buehler's Values, a_{nB} and a_{dB}, Respectively, with the Exact and Diagonal Values a_{ng} and a_{dg}, Respectively, Corresponding to $g(\tilde{x})$

x_1	x_2	a_{nB}	a_{dB}	a_{ng}	a_{dg}
5	5	60.7	60.70	60.70	60.70
5	4	51.8	51.89	51.89	51.89
5	3	41.2	41.21	41.22	41.21
5	2	31.9	31.91	31.91	31.90
5	1	23.3	23.34	23.34	23.34
5	0	12.3	12.32	12.32	12.32
4	4	44.3	44.40	44.40	44.40
4	3	35.7	35.73	35.74	35.73
4	2	27.2	27.23	27.23	27.23
4	1	18.8	18.77	18.77	18.76
4	0	9.05	9.05	9.05	9.05
3	3	28.9	28.89	28.89	28.89
3	2	22.0	22.04	22.04	22.03
3	1	15.1	15.08	15.08	15.08
3	0	8.24	8.24	8.24	8.24
2	2	16.8	16.80	16.80	16.79
2	1	11.8	11.85	11.85	11.85
2	0	5.59	5.59	5.59	5.59
1	1	7.09	7.08	7.08	7.08
1	0	3.86	3.78	3.78	3.78
0	0	1.33	1.33	1.33	1.33

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The problem of obtaining optimal upper confidence limits for systems of independent parallel components is treated. Exact optimal upper confidence limits are obtained for an arbitrary number of components for specified failure combinations. For a small number of failures, bounds on the upper confidence limits are obtained. For an arbitrary number of failures an approximation is given which is justified numerically and asymptotically. The results of this paper are compared with the results given by Buehler (1957) and some numerical examples are presented.

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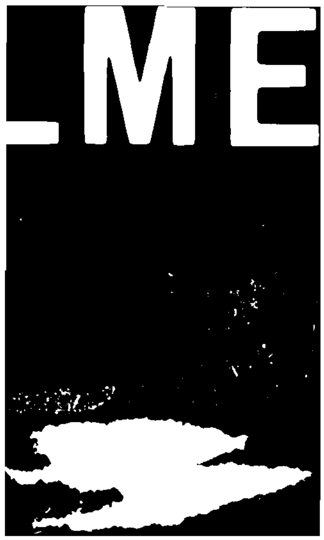
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