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ON LIMITS OF MULTIVARIATE B-SPLINES

Wolfgang Dahmen and
Charles A. Micchelli

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ON LIMITS OF MULTIVARIATE B-SPLINES

Wolfgang Dahmen* and Charles A. Micchelli**

Dedicated to I. J. Schoenberg with admiration and esteem.

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ABSTRACT

In a definitive series of papers I. J. Schoenberg with H. B. Curry clarified the relationship between several diverse properties of distribution functions on R^1. They showed that a distribution function is a limit of B-spline distributions if and only if the reciprocal of its Laplace transform is in the Pólya-Laguerre class. When the distribution function corresponds to a density $\Lambda(x)$ Schoenberg showed that these properties are equivalent to Λ being a Pólya frequency function or that the convolution transform $\Lambda * h$ is variation diminishing.

The purpose of this paper is to extend some of these properties to a multivariate setting. The major tool in this investigation is a notion of multivariate B-spline which we have both studied earlier.

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*Institut für Angewandte Mathematik, Universität Bonn, West Germany.

**IBM Thomas J. Watson Research Center, Yorktown Heights, NY 10598.

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SIGNIFICANCE AND EXPLANATION

B-splines occupy a pivotal position in the theory of univariate spline functions. They offer a firm bases for many of the attractive theoretical and practical properties of splines. Therefore any advance in our knowledge concerning the multivariate B-spline should be of some value.

The purpose of this paper is to investigate another aspect of multivariate B-splines. We succeed in identifying those distribution functions which are limits of multivariate B-splines (Pólya distributions). This question was solved in one dimension by Curry and Schoenberg. Some properties of Pólya distributions on R^s are also discussed.

A surprising consequence of this work is the relationship of the reciprocal of the Laplace transform of our distribution functions and the class of really lineal entire functions studied more than 25 years ago by Motzkin and Schoenberg.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON LIMITS OF MULTIVARIATE B-SPLINES

Wolfgang Dahmen* and Charles A. Micchelli**

Dedicated to I. J. Schoenberg with admiration and esteem.

Section 1. INTRODUCTION

In a definitive series of papers I. J. Schoenberg [1,8,9] with H. B. Curry [1] clarified the relationship between several diverse properties of distribution functions on R^1. They showed that a distribution function is a limit of B-spline distributions if and only if the reciprocal of its Laplace transform is in the Pólya-Laguerre class. When the distribution function corresponds to a density $\Lambda(x)$ Schoenberg showed that these properties are equivalent to Λ being a Pólya frequency function or that the convolution transform $\Lambda * h$ is variation diminishing.

The purpose of this paper is to extend some of these properties to a multivariate setting. The major tool in this investigation is a notion of multivariate B-spline which we have both studied earlier [2,6].

Section 2 of this paper contains some preliminary material on entire functions of affine lineage. For the most part, we discovered these results before we became aware of a paper by Motzkin and Schoenberg [7] which treats the same subject. Our theorems are slightly stronger than those stated in Motzkin and Schoenberg. Moreover, we have patterned our proofs closely after the univariate case which differs from the presentation given in [7]. In Section 3, we identify all distribution functions which are limits of (multivariate) B-spline distributions. Section 4 is devoted to some of the structural properties of these density functions. Also, in this section, we extend

*Institüt für Angewandte Mathematik, Universität Bonn, West Germany.

**IBM Thomas J. Watson Research Center, Yorktown Heights, NY 10598.

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some important identities previously known only for the multivariate B-spline and truncated powers, [2,6]. The final section contains some results pertaining to the analog of Pólya frequency functions and variations diminishing convolution transforms on R^s.

Section 2. PRELIMINARIES

In this section, we develop the required background material concerning Pólya-Laguerre functions on $\mathbb{C}^s$.

Let $z \cdot \zeta = \sum_{j=1}^{s} z_j \bar{\zeta}_j$, $z = (z_1, \ldots, z_s)$, $\zeta = (\zeta_1, \ldots, \zeta_s) \in \mathbb{C}^s$ denote the inner product on $\mathbb{C}^s$.

Definition 1. A function $f(z)$, $z \in \mathbb{C}^s$ is in the Polya-Laguerre class E_s provided that

$$f(z) = e^{-Az\cdot z + \zeta^0 \cdot z} \prod_{j=1}^{m} (z^j \cdot z) \prod_{\nu=1}^{\infty} (1 + \zeta^j \cdot z) e^{-\zeta^j \cdot z}$$

where $\sum_{j=1}^{\infty} \|\zeta^j\|^2 < \infty$, $z^j, \zeta^j \in R^s$ and A is a positive semi-definite $s \times s$ matrix with real entries.

We will usually be dealing with $f \in E_s$ which are normalized so that $f(0) = 1$. For this class, the principal result which we will need is

Theorem 1. If $\zeta^{n,j} \in R^s$ and

$$f_n(z) = \prod_{j=1}^{n} (1 + \zeta^{n,j} \cdot z)$$

satisfies

$$\lim_{n\to\infty} f_n(ix) = f(ix), \quad x \in R^s$$

uniformly in $\|x\| \leq a$ for some a. Then f has an analytic extension to $\mathbb{C}^s$ which is in E_s and

$$\lim_{n\to\infty} f_n(z) = f(z)$$

uniformly on compact subsets of $\mathbb{C}^s$.

PROOF. Choose any $y \in R^s$ with $\|y\| > 0$. Then letting $a' = a\|y\|$ we have

$$\lim_{n\to\infty} f_n(ity) = f(ity)$$

uniformly in $|t| \le a'$, $t \in R^1$. Using Theorem 3.4, Hirshman and Widder [4], p. 46 we conclude that the convergence above holds for all $z \in \mathbb{C}$ and $f(zy) \in E_1$. Hence

$$f(zy) = e^{-a(y)z^2+b(y)z} \prod_{j=1}^{\infty} (1 + c_j(y)z)e^{-c_j(y)z}$$

where $a(y) \ge 0$, $c_j(y)$, $b(y) \in R^1$ and $\Sigma c_j^2(y) < \infty$. An easy calculation shows that

$$\nabla f_n(0)\cdot y = \sum_{j=1}^{n} \zeta^{n,j}\cdot y$$

and

$$(\nabla f_n(0)\cdot y)^2 - y\cdot\nabla f_n(0)\cdot y = \sum_1^n (\zeta^{n,j}\cdot y)^2 .$$

Similarly, we have

$$\nabla f(0)\cdot y = b(y)$$

and

$$(\nabla f(0)\cdot y)^2 - y\cdot\nabla f(0)y = \Sigma c_j^2(y) + 2a(y) .$$

Since all the derivatives of $f_n(zy)$ at the origin converge to the respective derivatives of $f(zy)$ we conclude that $\lim_{n\to\infty} \sum_1^n \zeta^{n,j} = \zeta^0$ exists and $b(y) = \zeta^0\cdot y$. Note that the zeros of $f_n(zy)$ are $(\zeta^{n,j}\cdot y)^{-1}$. Now, by Hurwitz's theorem the zeros of $f(zy)$ are limits of zeros of f_n. Clearly, the cluster values of $\{\zeta^{n,j}\cdot y\}$ for a fixed y are on one hand equal to $\{c_j(y)\}$ and on the other hand are $\{\zeta^j \cdot y\}$ where $\{\zeta^j\}$ are the cluster values of $\{\zeta^{n,j}\}$ in R^s. Therefore we may take $c_j(y) = \zeta^j \cdot y$ and so we see that $0 \le a(y) = Ay \cdot y$ for some $s \times s$ real matrix A. Also, since

$$\infty > \Sigma c_j^2(y) = \Sigma(\zeta^j \cdot y)^2$$

for all $y \in R^s$ we have $\Sigma \|\zeta^j\|^2 < \infty$ which establishes that $f \in E_s$.

So far we have shown that $f_n(z) \to f(z)$ pointwise on R^s. Using the standard normal family argument, the uniform convergence on compact subsets $\mathbb{C}^s$ will follow directly from the uniform boundedness of f_n. To bound f_n we use an inequality in [4], page 44 which yields

$$|f_n(z)| \le \prod_{j=1}^{n} (1 + |\zeta^{n,j} \cdot z|) \le e^{|p_n| + 4q_n}$$

where $p_n = \Sigma \zeta^{n,j} \cdot z$, $q_n = \Sigma(\zeta^{n,j} \cdot z)^2$. Clearly both p_n, q_n are uniformly bounded on any compact set of $\mathbb{C}^s$. Thus f_n is also bounded there as well and so the proof is complete.

There is an obvious converse to Theorem 1 which says that any $f \in E_s$ with $f(0) = 1$ is a limit of polynomials of the form

$$P(z) = \prod_{j=1}^{n} (1 + \zeta^{n,j} \cdot z), \quad \zeta^{n,j} \in R^s .$$

A proof of this fact follows easily from the observation that

$$e^{\zeta \cdot z} = \lim_{n \to \infty} \left(1 + \frac{\zeta \cdot z}{n}\right)^n .$$

The identification of E_s as the set of all limits of "lineal" polynomials is due to Motzkin and Schoenberg [7].

Below we describe a multivariate version of the one-sided univariate Pólya-Laguerre functions. These functions will also be used in the next section.

Definition 2. We will say that $W \subseteq R^s$ is an admissible wedge if the following conditions hold:

a) $ty \in W$ *for all* $t > 0$, if $y \in W$

b) *The smallest linear space of* R^s *containing* W *is of dimension* s

c) The set $\{zy : |z| \leq 1,\ z \in \mathbb{C},\ y \in W\}$ *is a uniqueness set for entire functions on* $\mathbb{C}^s$.

The polar set of W is defined as

$$W^0 = \{x : x \in R^s,\ y \cdot x > 0, \text{ for all } y \in W\} .$$

Example 1. $R_+^s = \{y : y = (y_1, \ldots, y_s),\ y_j > 0\}$ is an admissible wedge which is self-polar.

Definition 3. We will call P *a* W *- lineal polynomial if it has the form*

$$P(z) = \prod_{j=1}^{N} \zeta^j \cdot z$$

where $\zeta^j \in W^0$. *Any function which is a limit (uniform on every compact subset of* $\mathbb{C}^s$*) of* W *- lineal polynomials is called a* W *- the lineal entire function.*

Our intention is to show that f is W - lineal if and only if it has the form

$$f(z) = e^{z \cdot \zeta^0} \prod_{j=1}^{m} (z^j \cdot z) \prod_{j=1}^{\infty} (1 + \zeta^j \cdot z)$$

where ζ^j, z^j are in W^0 and $\sum_j \|\zeta^j\| < \infty$. Let us denote this class by $E_s(W)$.

Theorem 2. Let $f_n(z) = \prod_1^n (1 + \zeta^{n,j} \cdot z)$, $\zeta^{n,j} \in W^0$ *and suppose* W *is an admissible wedge. If*

$$\lim_{n \to \infty} f_n(ix) = f(ix), \qquad x \in R^s$$

uniformly in $\|x\| < a$ *for some* a. *Then* f *has an analytic extension to* $\mathbb{C}^s$ *which is in* $E_s(W)$ *and* $f_n(z)$ *converges uniformly to* $f(z)$ *on compact subsets of* $\mathbb{C}^s$.

PROOF. Choose any $y \in W$ with $\|y\| > 0$ then $\lim_{n \to \infty} f_n(ity) = f(ity)$ uniformly in $|t| < a' = a/\|y\|$. Since $f_n(ty)$ is a polynomial with only negative zeros we know that (c.f. Karlin [5]) that

$$\lim_{n\to\infty} f_n(zy) = f(zy) ,$$

uniformly on compact subsets of $\mathbb{C}$ and $f(zy)$ has the form

$$f(zy) = e^{a(y)z} \prod_{j=1}^{\infty} (1 + b_j(y)z)$$

for some constants satisfying $a(y) \geq 0$, $b_j(y) \geq 0$ and $\Sigma b_j(y) < \infty$.

Just as in the proof of Theorem 1, we may choose $b_j(y) = \zeta^j \cdot y$ where ζ^j are the cluster values of $\{\zeta^{n,j}\}$. Moreover, since $\zeta^j \cdot y \geq 0$ for all $y \in W$ we have $\zeta^j \in W^0$. Also,

(1) $$\sum_j |\zeta^j \cdot y| < \infty$$

for $y \in W$ implies that $\sum_j |\zeta^j \cdot y| < \infty$ for any y in the linear span of W. Thus (1) holds for all $y \in R^s$ and it follows that

$$\sum_j \|\zeta^j\| < \infty .$$

Also, since

$$\lim_{n\to\infty} \nabla f_n(0) \cdot y = \nabla f(0) \cdot y$$

we obtain for every $y \in W$

$$\lim_{n\to\infty} \sum_{j=1}^{n} \zeta^{n,j} \cdot y = \sum_{j=1}^{\infty} \zeta^j \cdot y + a(y) .$$

Since W is an admissible this equation extends to all $y \in R^s$ and so we conclude that $a(y) = \zeta^0 \cdot y$ for some $\zeta^0 \in W^0$, $\sum_j \|\zeta^j\| < \infty$, and

$$f(z) = e^{z \cdot \zeta^0} \prod_{j=1}^{\infty} (1 + \zeta^j \cdot z) .$$

It remains to show that $f_n \to f$ uniformly on compact subsets of $\mathbb{C}^s$. In view of property c) in Definition 2 it is sufficient to prove f_n is bounded on compact subsets of $\mathbb{C}^s$. To this end, observe that for any $y \in W$ there is a constant $M(y)$ such that

$$\sum_1^n |\zeta^{n,j} \cdot y| = \sum_1^n \zeta^{n,j} \cdot y \le M(y)$$

for all n. Since W is an admissible wedge it follows that

$$\sum_{j=1}^n \|\zeta^{n,j}\| \le M$$

for some constant M and consequently

$$|f_n(z)| = \left|\prod_{j=1}^n (1 + \zeta^{n,j} \cdot z)\right| \le \prod_{j=1}^n (1 + \|\zeta^{n,j}\| \|z\|) \le e^{\sum_{j=1}^n \|\zeta^{n,j}\| \|z\|} \le e^{M\|z\|} .$$

Since we know that f_n is uniformly bounded on compact subsets of $\mathbb{C}^s$ and $f_n(z) \to f(z)$ pointwise on $\{zy : |z| \le 1,\ z \in \mathbb{C},\ y \in W\}$ the convergence is uniform on all compact subsets of $\mathbb{C}^s$. This completes the proof.

An obvious corollary of this result is the following characterization of the class $E_s(W)$.

Corollary 1. *An entire function f is W - lineal if and only if it is in $E_s(W)$.*

An essential point of view in our previous presentation was the analysis of an entire function on $\mathbb{C}^s$ along an arbitrary ray. The form of the resulting univariate function allowed us to make conclusions about the multivariate entire function. On this basis, one might be tempted to conclude that a function f is in E_s if and only if $f(zy)$ is in E_1 for all $y \in R^s$. This is unfortunately false as can easily be seen by the following simple example.

Example 2. Let $f(z_1,z_2) = z_1^2 + z_2^2 - 1$. Then along every ray (ty_1,ty_2), f is the quadratic polynomial

$$f(ty_1,ty_2) = t^2(y_1^2 + y_2^2) - 1$$

which has two real zeros and so is in E_1. However, f is not in E_2 because its zero set is the unit circle (elements in E_2 vanish only on lines).

This example can be substantially generalized. Let $g(-z)$ be any (univariate) one-sided Pólya frequency function given by

$$g(z) = e^{-\gamma z} \prod_1^{\infty} (1 - a_j z), \quad \gamma, a_j \geq 0, \quad \sum_{j=1}^{\infty} a_j < \infty$$

and define

$$f(z) = g(z_1^2+\cdots+z_s^2), \quad z = (z_1,\ldots,z_s) . \tag{2}$$

Then it is clear that $f(zy)$ is in E_1 for all y. However, the only radial function in E_s is $e^{-z_1^2-\cdots-z_s^2}$ while f defined by (2) is always radial. We will come back to this class of functions in the last section of the paper.

Let us also point out that there are functions other than those given by (2) which are in E_1 along every ray. For instance,

$$f(z) = z_1^2 + \cdots + z_{s-1}^2 - z_s$$

has this property. The analytic form of all such functions is unknown. However, when rays are replaced by lines the problem has a satisfactory answer.

Corollary 2. A function f defined on $\mathbb{C}^s$ is in E_s if and only if $f(x+zy)$ is in E_1 for all $x,y \in R^s$.

PROOF. If $f \in E_s$ then it easily follows from Definition 1 that $f(x+zy) \in E_1$. To prove the converse, let us first note that since for all $x,y \in R^s$, $f(x+zy) \in E_1$ it follows that f is a real analytic function on R^s.

In the case that $f(x) \neq 0$, $x \in R^s$, then

$$f(x+zy) = e^{-\alpha z^2 + cz},$$

for some real constants $\alpha = \alpha(x,y) \geq 0$, $\beta = \beta(x,y)$. Since f is entire we may differentiate this equation to see that

$$f(z) = e^{-Az\cdot z+b\cdot z}$$

where A an $s \times s$ positive semi-definite matrix and $b \in R^s$. In the general case, we know according to [3] that the zero set of f consists of real hyperplanes $\zeta^j \cdot z = t_j$ where $\zeta^j \in R^s$, $\|\zeta^j\| = 1$ and $t_j \in R^s$. Thus we may express f as

$$f(z) = e^{-Az\cdot z+b\cdot z} \prod_{j=1}^{m} (z^j \cdot z) \lim_{n\to\infty} \prod_{j=1}^{n} E\left(\frac{\zeta^j \cdot z}{t_j}; \alpha_j - 1\right),$$

where $E(z;p)$ are the Weierstrass polynomials given by

$$E(z;0) = 1 - z$$

$$E(z;p) = (1 - z)e^{z+\frac{z^2}{z}+\cdots+\frac{z^p}{p}}, \quad p = 1,2,\ldots,$$

and α_n are any integers such that $\Sigma\left(\frac{r}{|t_n|}\right)^{\alpha_n} < \infty$ for all finite r, [3,7]. Therefore $f(zy)$ vanishes only at $z = \zeta^j \cdot y/t_j$ and since $f(zy) \in E_1$ it follows that

$$\sum \left(\frac{\zeta^j \cdot y}{t_j}\right)^2 < \infty .$$

Thus, $\Sigma t_j^{-2} < \infty$ and so we may choose $\alpha_n = 2$. Consequently, $f \in E_s$ which completes the proof of the theorem.

There is a similar theorem for the class $E_s(W)$ where W is an admissible wedge which we state below.

<u>Corollary 3</u>. <u>A function</u> $f \in E_s(W)$ <u>if and only if</u>

1) $f(x+zy) \in E_1$ <u>for all</u> $x,y \in R^s$

2) $f(zy)$ <u>is a one-sided Pólya frequency function for all</u> $y \in W$.

<u>PROOF</u>. Clearly, according to Definition 2, properties 1) and 2) hold whenever $f \in E_s(W)$. Conversely, according to the Corollary 2, property 1) above implies that

$$f(z) = e^{-Az\cdot z + \zeta^0 \cdot z} \prod_{j=1}^{m} (\zeta^j \cdot z) \prod_{j=1}^{\infty} (1+\zeta^j \cdot z) e^{-z\cdot\zeta^j}$$

where $\Sigma\|\zeta^j\|^2 < \infty$, ζ^0, $\zeta^j \in \mathbb{C}^s$ and A is an $s \times s$ positive semi-definite matrix. However, we also know that for $y \in W$

$$f(zy) = a(y)z^m e^{b(y)z} \prod_1^{\infty} (1+c_j(y)z)$$

where $c_j(y)$, $b(y) \geq 0$, $\Sigma c_j(y) < \infty$. Thus it follows that $A = 0$, $\Sigma\|\zeta^j\| < \infty$ and $\zeta^j \in W^0$ which implies that $f \in E_s(W)$.

Section 3. LIMITS OF MULTIVARIATE B-SPLINES

In this section we turn to the principal application of the previous results on entire functions. We begin by recalling the definition of the multivariate B-spline.

Definition 4. Let $x^0,\ldots,x^n \in R^s$ and suppose $[x^0,\ldots,x^n]$ = convex hull of $x^0,\ldots,x^n$. If $[x^0,\ldots,x^n]$ has dimension s then the linear functional

$$\int_{[x^0,\ldots,x^n]} f = \int_{S^n} f(v_0x^0 + \cdots + v_nx^n)dv_1 \cdots dv_n$$

where

$$S^n = \{(v_0,\ldots,v_n) : v_i \geq 0, \sum_0^n v_i = 1\}$$

has for $n > s$ a (unique) continuous density $M(x|x^0,x^1,\ldots,x^n)$

$$\int_{S^n} f(v_0x^0 + \cdots + v_nx^n)dv_1 \cdots dv_n = \int_{R^s} M(x|x^0,\ldots,x^n)f(x)dx ,$$

$f \in L^1(R^s)$, which we call the (multivariate) B-spline of degree $k = n - s$.

A great deal is known about this function and certain linear combinations of such functions. It suffices for our purpose to mention here that when $s = 1$ this function corresponds to the well-known univariate B-spline. Thus, it is a piecewise polynomial of degree $n - 1$ with $n - 2$ continuous derivatives when $x^0,\ldots,x^n$ are distinct (these are real numbers in this case). In general, it has been shown that $M(x|x^0,\ldots,x^n)$ is a piecewise polynomial of total degree $\leq k$ with $k - 1$ continuous derivatives when the convex hull of every subset of $s + 1$ points of $\{x^0,\ldots,x^n\}$ forms an s-dimensional simplex.

In this section, we will identify all distribution functions which are limits of distributions of the form

$$F_n(x) = n! \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_s} M(y|x^{0,n},\ldots,x^{n,n})\,dy$$

$x = (x_1,\ldots,x_s)$ (we will sometimes refer to F_n as a B-spline distribution). Our basic tool in what follows is the Radon transform of a distribution. We review below the elementary facts which we require.

Let μ stand for a probability measure on R^s. Sometimes we will deal simultaneously with probability measures on R^1 and R^s. However, the context should make the domain of definition of the measures apparent.

Definition 5. The Radon transform of a probability distribution μ *on* R^s *is a map from* $\Omega^s = \{\lambda : \lambda \in R^s, \|\lambda\| = 1\}$ *into probability distributions on* R^1 *defined by*

$$(R\mu)(I,\lambda) = \mu(\{y : y \in R^s, \lambda\cdot y \in I\})$$

for any interval $I \subseteq R^1$.

Sometimes we will denote the Radon transform of μ by $\mu(\cdot,\lambda)$.

Lemma 1. For all $t \in R^1$, $\lambda \in \Omega^s$

$$\int_{R^1} e^{-it\sigma}\,d\mu(\sigma,\lambda) = \int_{R^s} e^{-it\lambda\cdot y}\,d\mu(y) .$$

PROOF. Let $g(t) = \sum_{i=1}^{m} c_i \chi_{I_i}(t)$ be any simple function on R^1, with $I_j \cap I_\ell = \phi$, $j \neq \ell$. Then

$$\int_{R^1} g(t)\,d\mu(t,\lambda) = \sum_{j=1}^{n} [illegible] = \sum_{j=1}^{n} [illegible]$$

$$= \int_{R^s} g(x\cdot y)\,d\mu(y) .$$

Hence

$$\int_{R^1} g(t)\,d\mu(t,\lambda) = \int_{R^s} g(\lambda\cdot y)\,d\mu(y)$$

for all bounded continuous functions on R^1 and, in particular, for $g(t) = e^{-it}$.

Definition 6. Let μ_n, μ be probability distributions on R^s. We will say μ_n converges to μ, provided that $\mu_n(I) \to \mu(I)$ for all rectangles $I \subseteq R^s$.

We will use the notation

$$\hat{\mu}(y) = \int_{R^s} e^{-iy\cdot x}\,d\mu(x)$$

for the characteristic function of μ.

Proposition 1.

a) Let μ_n, μ be probability measures on R^s. If $\mu_n \to \mu$ then $(R\mu_n)(\cdot,\lambda) \to (R\mu)(\cdot,\lambda)$, uniformly in λ.

b) Suppose μ_n is a sequence of probability measures on R^s such that $(R\mu_n)(\cdot,\lambda)$ converges to a probability measure $\mu(\cdot,\lambda)$ whose characteristic function is uniformly continuous at the origin in R^1. Then $\mu(\cdot,\lambda)$ is the Radon transform of some probability measure μ on R^s and $\mu_n \to \mu$.

PROOF. a) If $\mu_n \to \mu$ then it is well-known that $\hat{\mu}_n(y) \to \hat{\mu}(y)$ uniformly on any rectangle on R^s. Thus according to Lemma 1, $\hat{\mu}_n(t\lambda) \to \hat{\mu}(t\lambda)$ uniformly for t in any interval of R^1 and $\lambda \in \Sigma^s$. Thus, it follows that $\mu_n(\cdot,\lambda) \to \mu(\cdot,\lambda)$ for each $\lambda \in \Sigma^s$. The convergence is uniform in $\lambda \in \Sigma^s$ because if $\lambda_n \to \lambda$ then the characteristic function of $\mu_n(\cdot,\lambda_n)$ is $\hat{\mu}_n(t\lambda_n)$ which converges to the characteristic function of $\mu(\cdot,\lambda)$ and so $\mu_n(\cdot,\lambda_n) \to \mu(\cdot,\lambda)$.

b) If $\mu_n(\cdot,\lambda) \to \mu(\cdot,\lambda)$ then clearly $\phi_n(y) \to \Psi(y)$ where $\Psi(y) = \phi(t,\lambda)$, the characteristic of $\mu(\cdot,\lambda)$ and $y = t\lambda$, $\lambda \in \Omega^s$. Our hypothesis implies that $\Psi(y)$ is continuous which insures that it too is the characteristic function of some probability measure μ on R^s. (c.f. H. R. Pitt, Integration, Measure and Probability, Oliver and Boyd, London 1963). Thus by Lemma 1 $(R\mu)(\cdot,\lambda) = \mu(\cdot,\lambda)$ and $\mu_n \to \mu$.

We will say that a probability measure μ on R^s is a Pólya distribution if the reciprocal of its Laplace transform is in the (normalized) Pólya-Laguerre class on R^s, that is,

$$\int_{R^s} e^{-z\cdot x} d\mu(x) = \frac{1}{f(z)}$$

where

$$f(z) = e^{-Az\cdot z+\zeta^0\cdot z} \prod_{j=1}^{\infty} (1 + \zeta^j\cdot z)e^{-\zeta^j\cdot z} .$$

(It should be understood, if not otherwise explicitly stated, that the range of validity of the above expression is the hyperstrip $\mathrm{Re}(\zeta^j\cdot z) > -1$ in $\mathbb{C}^s$.)

A B-spline distribution μ has the form

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\ldots,x^{n,n})dx$$

for some $x^{0,n},\ldots,x^{n,n} \in R^s$.

<u>Proposition 2. If μ is a Pólya distribution then it is a limit of B-spline distributions.</u>

<u>PROOF</u>. Since μ is a Pólya distribution we have

(3)
$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)}$$

where f is a normalized Pólya frequency function. Thus according to Motzkin and Schoenberg (see the remark following Theorem 1) there is a set of vectors $\{x^{0,n},\ldots,x^{n,n}\}$ in R^s such that

$$\lim_{n\to\infty} \prod_{j=0}^{n} (1 + z\cdot x^{j,n}) = f(z)$$

uniformly on compact subsets of R^s. Thus for every $\lambda \in R^s$

$$f(z\lambda) = \lim_{n\to\infty} \prod_{j=0}^{n} (1 + z\lambda\cdot x^{j,n})$$

uniformly on compact subsets of R^1. Hence by the Curry-Schoenberg theorem we know that the sequence of distributions

(4) $$\mu_n(I,\lambda) = n! \int_I M(\sigma|\lambda\cdot x^{0,n},\ldots,\lambda\cdot x^{n,n})d\sigma, \quad I \subseteq R^1$$

converges to a distribution $\mu(t,\lambda)$ which satisfies

(5) $$\int_{R^1} e^{-z\sigma} d\mu(\sigma,\lambda) = \frac{1}{f(z\lambda)} .$$

According to (3), $(R\mu)(\cdot,\lambda) = \mu(\cdot,\lambda)$. Moreover, if we define

(6) $$\mu_n(I) = \int_I M(x|x^{0,n},\ldots,x^{n,n})dx, \quad I \subseteq R^s$$

then we can see that the Radon transform of $\mu_n(I)$ is $\mu_n(I,\lambda)$, because by Definition 4 and Lemma 1, if $I \subseteq R^1$

$$(R\mu_n)(I,\lambda) = \int_{R^s} \chi_I(\lambda\cdot x)M(x|x^{0,n},\ldots,x^{n,n})dx = \int_{S^n} \chi_I(\Sigma v_j\lambda\cdot x^{j,n})dv_1\cdots dv_n$$

$$= n! \int_I M(t|\lambda\cdot x^{0,n},\ldots,\lambda\cdot x^{n,n})dt = \mu_n(I,\lambda) .$$

Thus we have shown that $(R\mu_n)(\cdot,\lambda) \to (R\mu)(\cdot,\lambda)$. According to Proposition 1, $\mu_n \to \mu$ because the characteristic function of $(R\mu)(\cdot,\lambda)$ is obviously uniformly continuous in a neighborhood of zero. This completes the proof.

Proposition 3. Suppose μ is a probability measure on R^s and

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\dots,x^{n,n})dx$$

converges to μ. Then μ is a Pólya distribution with Laplace transform

$$\int_{R^s} e^{-z\cdot x} d\mu(x) = \frac{1}{f(z)}$$

where

$$\lim_{n\to\infty} \prod_{j=1}^{n} \left(1 + \frac{x^{j,n}\cdot z}{n}\right) = f(z) .$$

PROOF. As before we have

$$(R\mu_n)(I,\lambda) = n! \int_I M(t|\lambda\cdot x^{0,n},\dots,\lambda\cdot x^{n,n})dt$$

and since $(R\mu_n)(I,\lambda) \to (R\mu)(I,\lambda)$ for each λ, the Curry-Schoenberg theorem implies that

$$\lim_{n\to\infty} \prod_{j=0}^{n} \left(1 + z\frac{x^{j,n}\cdot\lambda}{n}\right) = g(z,\lambda)$$

converges uniformly on compact sets for every fixed λ to some $g(\cdot,\lambda) \in E_1$. Now just as in the proof of Theorem 1 we conclude that

$$g(z,\lambda) = f(z\lambda)$$

where f is given by

$$\lim_{n\to\infty} \prod_{j=0}^{n} \left(1 + \frac{x^{j,n}\cdot z}{n}\right) = f(z) . \tag{7}$$

This convergence is uniform on any compact part of any ray through the origin. The Curry-Schoenberg result tells us that

$$\int_{R^1} e^{-it\sigma} d\mu(\sigma,\lambda) = \frac{1}{g(t,\lambda)} = \frac{1}{f(t\lambda)}$$

and so invoking Lemma 1 again we obtain the desired result, namely

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)} .$$

To complete the proof we only have to show that the convergence in (7) is uniform on compact sets of $\mathbb{C}^s$.

In (7) the derivatives of the product converge to the respective derivatives of f. Thus we see, as in Theorem 1, that these products are uniformly bounded on any compact set and so the convergence is uniform.

Propositions 2 and 3 combine to give the main theorem of this section.

Theorem 3. A probability distribution μ is the limit of B-spline distributions if and only if

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)}$$

where $f(0) = 1$ and $f \in E_s$.

Corollary 4. A necessary and sufficient condition for

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\dots,x^{n,n})dx$$

to converge to the distribution μ defined by

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = e^{\lambda z\cdot z - \xi^0\cdot z}$$

is that

a) $$\lim_{n\to\infty} \frac{1}{n} \max_{0\le j\le n} \|x^{j,n}\| = 0$$

b)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x^{j,n} = \zeta^0$$

c)

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} (x^{j,n} \cdot y)^2 = Az \cdot z \ .$$

PROOF. The proof of this corollary follows immediately from Theorem 4. For instance, a) is equivalent to the fact that the limit of $\prod_{j=0}^{n} \left(1 + \frac{x^{j,n} \cdot z}{n}\right)$ has no hyperplane in its zero set. The remaining conditions then characterize the parameters which appear in the Gaussian distribution above.

Example 3. If $x^{j,n}$ is any bounded sequence in R^s then

$$\mu_n(I) = \int_I M(x|x^{0,n},\ldots,x^{n,n})dx$$

converges to the measure supported at y if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x^{j,n} = y \ .$$

Example 4. Let $x_{r,m}$, $0 \le r \le m$, be any scalar sequence such that the corresponding univariate B-spline distribution converges to the Gaussian distribution $\pi^{-1/2} e^{-t^2} dt$. (The conditions on $x_{r,m}$ needed to insure this is given by Corollary 4.) Now, form a vector sequence $x^{j,n}$, $0 \le j \le n = ms$, by "placing" $x_{r,m}$ on each coordinate direction. Then we may again use Corollary 4 to conclude that the corresponding multivariate B-spline converges to the multivariate Gaussian distribution $\pi^{-s/2} e^{-\|x\|^2}$.

We will say that a B-spline distribution

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\ldots,x^{n,n})dx$$

is W-restricted if $x^{j,n} \in W^0$.

Theorem 4. *A probability distribution* μ *is the limit of W-restricted B-spline distributions* μ_n *if and only if*

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)} \tag{8}$$

where $f(0) = 1$ *and* $f \in E_s(W)$.

We will prove this theorem in two propositions.

Proposition 4. *If* μ *is a distribution satisfying* (8) *then it is a limit of W-restricted B-spline distributions and* $\mathrm{supp}(\mu) \subseteq \mathrm{cohull}(W^0)$.

PROOF. According to Corollary 1, there is a sequence $\{x^{0,n},\dots,x^{n,n}\} \subseteq W^0$ such that

$$\lim_{n\to\infty} \prod_{j=0}^{n} (1 + z\cdot x^{j,n}) = f(z) \ .$$

Thus, as in Proposition 2, $\mu_n(I) = n! \int_I M(x|x^{0,n},\dots,x^{n,n})dx$ converges to μ, since $f \in E_s$. Moreover, because $\mathrm{supp}\ \mu_n \subseteq [x^{0,n},\dots,x^{n,n}]$ it follows that the support of $\mu \subseteq \mathrm{cohull}(W^0)$.

Proposition 5. *Suppose* μ *is a probability measure on* R^s *and*

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\dots,x^{n,n})dx \ ,$$

where $x^{\nu,n} \in W^0$. *If* μ_n *converges to* μ *then* μ *is a W-restricted Pólya distribution with Laplace transform*

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)} \ ,$$

$$f(z) = \lim_{n\to\infty} \prod_{j=0}^{n} \left(1 + \frac{x^{j,n}\cdot z}{n}\right) \ ,$$

and $\mathrm{supp}(\mu) \subseteq \mathrm{cohull}(W^0)$.

This result is proved by arguments similar to those used to prove Proposition 4. We omit the details.

The analog of Corollary 4 is

Corollary 5. Let W be an admissible cone. Then a necessary and sufficient condition for

$$\mu_n(I) = n! \int_I M(x|x^{0,n},\ldots,x^{n,n})dx, \qquad x^{j,n} \in W^0$$

to converge to the Dirac measure at $\zeta^0 \in W^0$ is that

a) $$\lim_{n\to\infty} \frac{1}{n} \max_{0\le j\le n} \|x^{j,n}\| = 0 ,$$

b) $$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x^{j,n} = \zeta^0 .$$

Section 4. INVERSION FORMULA AND SMOOTHNESS PROPERTIES OF POLYA DISTRIBUTIONS

In this section we investigate some further properties of Polya distributions. Their characteristic property, as defined earlier, is that

$$\int_{R^s} e^{-z\cdot y} d\mu(y) = \frac{1}{f(z)}, \qquad \mathrm{Re}(\zeta^j\cdot z) > -1 ,$$

where

$$f(z) = e^{-Az\cdot z+\zeta^0\cdot z} \prod_{j=1}^{\infty} (1+\zeta^j\cdot z)e^{-\zeta^j\cdot z} ,$$

$\sum_{j=1}^{\infty} \|\zeta^j\|^2 < \infty$, A is a positive semi-definite $s \times s$ matrix, and $\zeta^j \in P^s$, $j = 0,1,2,\ldots$.

We begin with the finite kernel case, i.e.,

$$(9) \qquad \int_{P^s} e^{-z\cdot y} d\mu(y) = \frac{1}{\prod_{j=1}^{n} (1+\zeta^j\cdot z)} , \qquad \mathrm{Re}(\zeta^j\cdot z) > -1, \qquad \zeta^1,\ldots,\zeta^n \in R^s .$$

In general, we can always represent this measure as an <u>n-dimensional</u> integral in the following way. We define the linear functional

$$T_{\zeta^1,\ldots,\zeta^n}(\phi) = \int_0^\infty \cdots \int_0^\infty e^{-(t_1+\cdots+t_n)} \phi(t_1\zeta^1+\cdots+t_n\zeta^n)dt_1\cdots dt_n$$

and observe that

$$T_{\zeta^1,\ldots,\zeta^n}(e^{-z\cdot x}) = \int_0^\infty \cdots \int_0^\infty e^{-(t_1+\cdots+t_n)} e^{-z\cdot \sum_{j=1}^n t_j\zeta^j} dt_1\cdots dt_n$$

$$= \frac{1}{\prod_{j=1}^n (1 + z\cdot\zeta^j)} ,$$

where $\mathrm{Re}(z\cdot\zeta^j) > -1$, $j = 1,\ldots,n$.

Thus, for μ defined by (9), we obtain

$$T(f) = \int_{R^s} f(x)d\mu(x)$$

at least for functions which are bounded and continuous on R^s. Perhaps, it should be pointed out that this "inversion formula" for the measure μ in the univariate case is not the representation given by Hirshman-Widder, [4]. A "conventional" representation would express μ as an s-dimensional integral. To this end, let us observe that if $m = \dim \mathrm{span}\{\zeta^1,\ldots,\zeta^n\} < s$ then T does not come from an L^1-density, i.e. there is no L^1-density $G(x)$ such that

$$T(f) = \int_{R^s} f(x)G(x)dx .$$

Otherwise, there would be a $\lambda \in R^s - \{0\}$ such that $\lambda\cdot\zeta^j = 0$, $j = 1,\ldots,n$. Then

$$1 = T(e^{-it\lambda\cdot x}) = \int_{R^s} e^{-it\lambda\cdot x}G(x)dx .$$

However, because $G \in L^1(R^s)$ the limit of the right side of this equation is zero for $t \to \infty$.

When $m = s$ we consider two cases. For $n = s$ we write $\zeta^j = Ae^j$, $j = 1,\ldots,n$ where $(e^j)_k = \delta_{jk}$, $j,k = 1,\ldots,n$ and A is the nonsingular $s \times s$ matrix whose j-th columns is the vector ζ^j. Thus, in this case,

$$G(x) = \begin{cases} \dfrac{1}{|\det A|} e^{-1\cdot(A^{-1})^T x}, & x \in [\zeta^1,\ldots,\zeta^s]_c \\ 0 & , \text{otherwise} \end{cases} \tag{10}$$

where $[\zeta^1,\ldots,\zeta^s]_c$ is the cone spanned by $\zeta^1,\ldots,\zeta^s$.

The last and principal case occurs when $m = s < n$. In this case we define the density corresponding to T recursively. We suppose for ease of notation that $\zeta^1,\ldots,\zeta^s$ are linearly independent and define

$$G(x|\zeta^1,\ldots,\zeta^{\ell+1}) = \int_0^\infty e^{-t} G(x - t\zeta^{\ell+1}|\zeta^1,\ldots,\zeta^\ell)dt \tag{11}$$

initialized by

$$G(x|\zeta^1,\ldots,\zeta^s) = G(x) \tag{12}$$

where $G(x)$ is given by (10). It is an easy matter to prove that

$$T_{\zeta^1,\ldots,\zeta^n}(f) = \int f(x)G(x|\zeta^1,\ldots,\zeta^n)dx \tag{13}$$

(induction will do). Since $T_{\zeta^1,\ldots,\zeta^n}$ is invariant under a permutation of the vectors $\zeta^1,\ldots,\zeta^n$ so too is $G(x|\zeta^1,\ldots,\zeta^n)$. Consequently, the recursion above can begin at any set of linearly independent vectors and proceed through any ordering of the remaining elements of $\{\zeta^1,\ldots,\zeta^n\}$. We might also mention that this fact can be used to bound $G(x|\zeta^1,\ldots,\zeta^n)$. In particular,

$$\max_x |G(x|\zeta^1,\dots,\zeta^n)| \le \min_K \mathrm{vol}_s^{-1} K$$

where the minimum is taken over all s-dimensional simplicies $K = [0,\zeta^{i_1},\dots,\zeta^{i_s}]$ formed by any s vectors from $\{\zeta^1,\dots,\zeta^s\}$ and the origin in R^s.

Because of (11) it is to be expected that $G(x|\zeta^1,\dots,\zeta^n)$ has a number of continuous derivatives. To make this precise, we denote the directional derivative of f in the direction $y \in R^s$ by $D_y f$. Then (11) clearly implies that

(14) $$(D_{\zeta^\ell} + 1)G(x|\zeta^1,\dots,\zeta^{\ell+1}) = G(x|\zeta^1,\dots,\zeta^\ell)$$

at every point of continuity of $G(x|\zeta^1,\dots,\zeta^\ell)$. In general, we see that

(15) $$D_y G(x|\zeta^1,\dots,\zeta^n) = \sum_{j=1}^n \mu_j G(x|\zeta^1,\dots,\zeta^{j-1},\zeta^{j+1},\dots,\zeta^n)$$

where

$$y = \sum_{j=1}^n \mu_j \zeta^j, \qquad \sum_{j=1}^n \mu_j = 0 .$$

The proof of (15) uses (14) and the fact that $G(x|\zeta^1,\dots,\zeta^n)$ is invariant under any permutation of the vectors $\zeta^1,\dots,\zeta^n$. Returning to (11) we note that $G(x|\zeta^1,\dots,\zeta^{s+1})$ is continuous on R^s, if the vectors $\zeta^1,\dots,\zeta^{i-1},\zeta^{i+1},\dots,\zeta^s,\zeta^{s+1}$, $i = 1,\dots,s$, are linearly independent. Since in this case for every $x \in R^s$ the line $x - t\zeta^{s+1}$ intersects the cone $[\zeta^1,\dots,\zeta^s]_c$ at most once. Thus we may combine our previous remarks to conclude that whenever every subset of s vectors from $\{\zeta^1,\dots,\zeta^n\}$ is linearly independent $G(x|\zeta^1,\dots,\zeta^n)$ has $n - s - 1$ continuous derivatives.

In one dimension the precise form of $G(x|\zeta^1,\dots,\zeta^n)$ as a combination of <u>two</u> exponential polynomials is known, Hirshman and Widder (Theorem 8.2, page 31) [4]. For the purpose of presenting a similar result in R^s we prove the following lemma.

The points $\zeta^1,\dots,\zeta^n$ are said to be in <u>general position</u> if every subset of $s + 1$ points form an s-dimensional simplex.

Let $0, \zeta^1, \ldots, \zeta^n$ be in general position. Then for every set $I \subseteq \{1,\ldots,n\}$ containing $|I| = s$ elements there is a unique $x_I \in R^s$ such that $1 + \zeta^j \cdot x_I = 0$, $j \in I$. Moreover, x_I necessarily has the property that $1 + \zeta^j \cdot x_I \neq 0$, $j \notin I$.

Lemma 2. Let $\{0, \zeta^1, \ldots, \zeta^n\}$ be in general position. Then for every polynomial Q of total degree $\leq n - s$ we have

$$Q(x) = \sum_{|I|=s} Q(x_I) \frac{\prod_{j \notin I} (1 + \zeta^j \cdot x)}{\prod_{j \notin I} (1 + \zeta^j \cdot x_I)} .$$

PROOF. The space of polynomials of total degree $\leq n - s$ has dimension $N = \binom{n}{s}$. Thus it suffices to observe that the N polynomials (of degree $\leq n - s$)

$$Q_I(x) = \frac{\prod_{j \notin I} (1 + \zeta^j \cdot x)}{\prod_{j \notin I} (1 + \zeta^j \cdot x_I)}, \qquad I \subseteq \{1,\ldots,n\}, \qquad |I| = s$$

satisfy $Q_I(x_J) = 0$, $I \neq J$ for then they must be linearly independent.

We may now specialize Lemma 2 to the polynomial $Q(x) = 1$ to obtain the partial fraction decomposition

$$\frac{1}{\prod_{j=1}^{n} (1 + \zeta^j \cdot z)} = \sum_{|I|=s} a_I \frac{1}{\prod_{j \in I} (1 + \zeta^j \cdot z)}$$

where

$$a_I = \frac{1}{\prod_{j \notin I} (1 + \zeta^j \cdot x_I)} .$$

Hence we obtain the basic relation

$$(16) \qquad G(x|\zeta^1,\ldots,\zeta^n) = \sum_{|I|=s} a_I G(x|\zeta^{i_1},\ldots,\zeta^{i_s}), \qquad I = \{i_1,\ldots,i_s\} .$$

Using (10), we can readily see that

$$G(x|\zeta^{i_1},\dots,\zeta^{i_s}) = c_I \chi_I(x) e^{x_I \cdot x} \tag{17}$$

where $c_I = \mathrm{vol}_s[0,\zeta^{i_1},\dots,\zeta^{i_s}]$ and χ_I is the characteristic function of $[\zeta^{i_1},\dots,\zeta^{i_s}]_c$. Setting $b_I = a_I c_I$ and combining (16) and (17) gives

$$G(x|\zeta^1,\dots,\zeta^n) = \sum_{|I|=s} b_I \chi_I(x) e^{x_I \cdot x} \tag{18}$$

which reveals the precise piecewise exponential form of $G(x|\zeta^1,\dots,\zeta^n)$. We might add that since

$$G(x|\zeta^1,\dots,\zeta^n) = \sum_{j=1}^{n} \lambda_j G(x|\zeta^1,\dots,\zeta^{i-1},\zeta^{i+1},\dots,\zeta^n) \tag{19}$$

whenever $\sum_{j=1}^{n} \lambda_j = 1$ and $\sum_{j=1}^{n} \lambda_j \zeta^j = 0$ we could generate this representation recursively.

Equations (15) and (19) are similar to formulas satisfied by the multivariate B-*spline* and truncated power, *[26]*. This suggests that these other functions are closely *related* to $G(x)$. To present this relationship we define

$$\int_0^\infty \cdots \int_0^\infty \omega\Big(\sum_{j=1}^{n} v_j\Big) f\Big(\sum_{j=1}^{n} v_j \zeta^j\Big) dv_1 \cdots dv_n = \int_{R^s} G_\omega(x|\zeta^1,\dots,\zeta^n) f(x)\,dx$$

for a given univariate function $\omega(t)$. The choice $\omega(t) = e^{-t}$ corresponds to $G(x|\zeta^1,\dots,\zeta^n)$ while $\omega(t) = \chi_{[0,1]}(t)$ and $\omega(t) = 1$ correspond to the multivariate B-spline $M(x|0,\zeta^1,\dots,\zeta^n)$ (see Definition 4) and the multivariate truncated power $H(x|\zeta^1,\dots,\zeta^n)$ respectively. In general, we have

$$\int_{R^s} G_\omega(x|\zeta^1,\ldots,\zeta^n)dx$$

$$= \lim_{h\to 0^+} \int_{\sum_{j=1}^n v_j < h^{-1}} f(\sum_{j=1}^n v_j\zeta^j)\omega(\sum_{j=1}^n v_j)dv_1\cdots dv_n$$

$$= \lim_{h\to 0} h^{-n} \int_{\sum_{j=1}^n v_j<1} f\Big(h^{-1}\sum_{j=1}^n v_j\zeta^j\Big)\omega\Big(h^{-1}\sum_{j=1}^n v_j\Big)dv_1\cdots dv_n$$

$$= \lim_{h\to 0} h^{-n}\int_0^1 \Big(\int_{\sum_{j=1}^n v_j=\tau} f\Big(h^{-1}\sum_{j=1}^n v_j\zeta^j\Big)\omega(h^{-1}\tau)dv_1\cdots dv_{n-1}\Big)d\tau$$

$$= \lim_{h\to 0^+} h^{-n}\int_0^1 \tau^{n-1}\omega(h^{-1}\tau)\Big(\int_{S^{n-1}} f(h^{-1}\tau\sum_{j=1}^n v_j\zeta^j)dv_1\cdots dv_{n-1}\Big)d\tau$$

$$= \lim_{h\to 0^+}\int_{R^s}\int_0^{h^{-1}} \omega(h)h^{n-s-1}f(x)M(h^{-1}x|\zeta^1,\ldots,\zeta^n)dx\ .$$

Thus we obtain

$$G_\omega(x|\zeta^1,\ldots,\zeta^n) = \int_0^\infty \omega(h)h^{n-s-1}M(\tau^{-1}x|\zeta^1,\ldots,\zeta^n)dx\ .$$

Besides the case $\omega(h) = e^{-h}$, which gives the formula

$$G(x\zeta^1,\ldots,\zeta^n) = \int_0^\infty e^{-h}h^{n-s-1}M(\tau^{-1}\chi|\zeta^1,\ldots,\zeta^n)d\tau$$

the choices $\omega(t) = \chi_{[0,1]}(t)$ and $\omega(t) = 1$ yield the known formulas

$$M(x|0,\zeta^1,\ldots,\zeta^n) = \int_1^\infty \tau^{-n+s-1}M(\tau x|\zeta^1,\ldots,\zeta^n)d\tau$$

and

$$H(x|\zeta^1,\ldots,\zeta^n) = \int_0^\infty \tau^{-n+p-1} M(\tau x|\zeta^1,\ldots,\zeta^n)\,d\tau ,$$

respectively, [6]. (This last identity is true when $0 \notin [\zeta^1,\ldots,\zeta^n]$.)

Next, we turn to the infinite kernel case. For that purpose we introduce

Definition 7. *A function f in E_s given by*

$$f(z) = e^{-Az\cdot z+\zeta^0\cdot z} \prod_{j=1}^{\infty} (1 + \zeta^j\cdot z)e^{-\zeta^j\cdot z}$$

is called degenerate if and only if

$$S = \text{span}\{a_1,\ldots,a_s,\zeta^1,\zeta^2,\ldots,\zeta^j,\ldots,\} \neq R^s$$

where a_j are the rows of the matrix A.

We will also say $\Lambda(x)$, $x \in P^s$, is a Pólya frequency function provided that $d\mu(x) = \Lambda(x)dx$ is a Pólya distribution.

Theorem 5. *Let $f \in E_s$ and suppose $f(0) = 1$. Then $1/f$ is the Laplace transform of a Pólya frequency function if and only if f is nondegenerate.*

PROOF. Suppose f E_s, $f(0) = 1$, is degenerate and

$$\frac{1}{f(z)} = \int_{R^s} e^{-z\cdot x}\Lambda(x)dx, \quad \text{Re}(\zeta^j\cdot z) > -1, \quad j = 1,2,\ldots$$

for some density function $\Lambda(x)$.

Since f is degenerate there is a $\lambda \in R^s$, $\lambda \neq 0$, and $\lambda \perp S$. Hence $f(\lambda z) = e^{z\zeta^0\cdot\lambda}$ and so

$$e^{-it\zeta^0\cdot\lambda} = \int_{P^s} e^{-it\lambda\cdot x}\Lambda(x)dx$$

for all $t \in R^1$. This is obviously a contradiction because the right hand side of the equation goes to zero as $t \to \infty$, by the Riemann-Lebesgue theorem.

Conversely, suppose f is nondegenerate. We will now show how to construct a density whose Laplace transform is $1/f$. First let us observe that if f_1, f_2 are in E_s and $1/f_1$ is the Laplace transform of a density $\Lambda(x)$. Then $1/f$, for $f = f_1 f_2$, is the Laplace transform of the density

$$\int_{R^s} \Lambda(x - y)\, d\mu(y)$$

where $d\mu$ is the Pólya distribution corresponding to f_2. Thus to prove the theorem it suffices to find a factor of f which corresponds to a Pólya frequency function.

When A is of full rank, $r = s$ then the well-known formula

$$e^{Az\cdot z} = \frac{1}{(2\sqrt{\pi})^s \sqrt{\det A}} \int_{R^s} e^{-z\cdot x} e^{-(A^{-1}x,x)} dx$$

already provides the density. In the other extreme case, when $A = 0$, there are s-linearly independent vectors from $\{\zeta^j : j = 1,\ldots\}$ say $\zeta_1,\ldots,\zeta_s$. Then the formula

$$\frac{1}{\prod_{j=1}^{s} (1 + \zeta^j\cdot z)} = \int_{R^s} e^{-z\cdot x} G(x|\zeta_1,\ldots,\zeta_s) dx$$

which we used earlier in the section gives us a density. Finally if $0 < r < s$ then there are $\ell = s - r$ vectors say $\zeta^1,\ldots,\zeta^\ell$ such that $\{a_1,\ldots,a_r,\zeta^1,\ldots,\zeta^\ell\}$ spans R^s. In this case, the function

$$F(z) = e^{-Az\cdot z} \prod_{j=1}^{\ell} (1 + \zeta^j\cdot z)$$

is a factor of f. To see that it corresponds to a density we may as well make a linear change of variables and assume it has the form

$$e^{-z_1^2 \cdots - z_r^2} \prod_{j=r+1}^{s} (1 + z_j) \,.$$

Now, it is clear that

$$\frac{1}{F(z)} = \frac{1}{(2\sqrt{\pi})^z} \int_{R^s} e^{-z\cdot x} \Lambda(x) dx$$

where Λ is the density

$$\Lambda(x) = e^{-x_1^2-\cdots-x_r^2-(x_{r+1})_+^0-\cdots-(x_s)_+^0} .$$

This last case proves the theorem.

Let us end this section by noting that the density $\Lambda(x)$ will be $C^\infty(R^s)$ provided that either A is <u>positive definite</u> or that there is a set $\{\zeta^j: j \in J\}$, $|J| = \infty$, which are in general position with the origin in R^s. This latter fact follows from our earlier discussion of the finite kernel case.

Section 5. PÓLYA FREQUENCY FUNCTIONS AND VARIATION DIMINISHING TRANSFORMATION ON R^s

In this section we will indicate the extent to which we can generalize the variation diminishing property of univariate Pólya frequency functions to higher dimension. Our remarks are based on the following elementary observation.

A <u>ridge</u> function has the special form

$$f_\omega(x) = g(\omega\cdot x)$$

where g is a univariate function and $\omega \in \Omega^s$, the unit sphere in R^s. The convolution operator

$$(\Lambda * f)(x) = \int_{R^s} \Lambda(x - y)f(y)dy$$

maps the ridge function f_ω into the ridge function $h(\omega\cdot x)$ where

$$h(t) = \int_{R^s} \Lambda(y)g(t - \omega\cdot y)dy .$$

The induced map T_ω defined by $h = T_\omega g$ has the representation

$$T_\omega = (R_\omega\Lambda)*g$$

where

$$(R_\omega\Lambda)(t) = \int_{x\cdot\omega=t} \Lambda(x)dx$$

is the Radon transform of Λ.

Theorem 5. *If Λ is a Pólya Frequency function on R^s then for any $\omega \in \Omega^s$ the map T_ω defined above is variation diminishing.*

PROOF. To prove this result we must demonstrate that $(R_\omega \Lambda)(t)$ is a Pólya frequency function in t for every $\omega \in \Omega^s$. Since

$$\int_{R^s} e^{-z \cdot y} \Lambda(y)\,dy = \frac{1}{f(z)}$$

where $f \in E_s$ we see that the Laplace transform of $R_\omega \Lambda$ is $f^{-1}(z\omega \cdot y)$. But $f(z\omega \cdot y)$ is in the class E_1. This implies by one of Schoenberg's basic results [7,8] that $(R_\omega \Lambda)(t)$ is a Pólya frequency function.

The above result only gives a sufficient condition for Λ to be a Pólya frequency function on R^s. There is an extensive collection of radical functions satisfying the conclusion of Theorem 5 but which are not Pólya frequency functions. Specifically, we let

$$\Lambda(x) = h(\|x\|^2), \qquad \|x\|^2 = x_1^2 + \cdots + x_s^2,$$

where

$$h(t) = \frac{1}{(2\pi)^{s/2} t^{(s-2)/2}} \int_0^\infty \sigma^{s/2} \frac{J_{(s-2)/2}(t\sigma)}{g(-\sigma^2)}\,d\sigma\ ,$$

(the Bessel transform of $1/g(-\sigma^2)$) and g has the form

$$g(t) = e^{-\gamma t} \prod_{j=1}^\infty (1 + a_j t), \qquad \gamma, a_j > 0, \quad \sum_{j=1}^\infty a_j < \infty\ .$$

It is well-known that the Fourier transform of Λ is given by

$$\int_{R^s} e^{-ix \cdot y} \Lambda(y)\,dy = \frac{1}{g(-\|x\|^2)}$$

and thus

$$\int_{R^s} e^{-z\cdot y}\Lambda(y)\,dy = \frac{1}{g(z_1^2+\cdots+z_s^2)}, \qquad z = (z_1,\ldots,z_s)\ .$$

Now, it quickly follows as in the proof of Theorem 5 that $(R_\omega\Lambda)(t)$ is a Pólya frequency function in t for every $\omega \in \Omega^s$.

Postscript

In their paper Curry and Schoenberg showed how the Brunn-Minkowski Theorem concerning the volume of a convex body intersected with a hyperplane leads to a "convexity" inequality for the univariate B-spline. The same argument gives similar information about the multivariate B-spline. We record this result below.

Recall that a nonnegative function g defined on R^1 is a Pólya frequency function of order two, if

$$\begin{vmatrix} g(t_1-\sigma_1) & g(t_1-\sigma_2) \\ g(t_2-\sigma_1) & g(t_2-\sigma_2) \end{vmatrix} \geq 0$$

whenever $t_1 < t_2$, $\sigma_1 < \sigma_2$.

Theorem 7. *For any* $x,y \in R^s$ *the function* $g(t) = M(x + ty|x^0,\ldots,x^n)$ *is a Pólya frequency function of order two.*

PROOF. Using the geometric interpretation of the multivariate B-spline, [2,6], as the volume of a polyhedra and the Brünn-Minkowski theorem it follows just as in [1] for the univariate case that $(M(x + ty|x^0,\ldots,x^n))^{1/n-s}$ is a concave function. The theorem now follows by observing the simple fact that every positive power of a nonnegative concave function is a Pólya frequency function of order two.

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In a definitive series of papers I. J. Schoenberg with H. B. Curry clarified the relationship between several diverse properties of distribution functions on R^1. They showed that a distribution function is a limit of B-spline distributions if and only if the reciprocal of its Laplace transform is in the Pólya-Laguerre class. When the distribution function corresponds to a density $\Lambda(x)$

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20. ABSTRACT - Cont'd.

Schoenberg showed that these properties are equivalent to Λ being a Pólya frequency function or that the convolution transform $\Lambda * h$ is variation diminishing.

The purpose of this paper is to extend some of these properties to a multivariate setting. The major tool in this investigation is a notion of multivariate B-spline which we have both studied earlier.