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ON THE INITIAL VALUE PROBLEM FOR A NONLINEAR SCHRÖDINGER EQUATION

Guang-Chang Dong and Shujie Li

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

In this paper we study conditions that ensure existence and uniqueness of solution to the initial value problem for the nonlinear Schrödinger equation. Many known results in the one dimensional space are extended to the higher dimensional cases. We also establish bounds and investigate decay properties for the solution.

AMS (MOS) Subject Classification: 35G25

Key Words: non-linear Schrödinger equation, a-priori estimation, Riesz-Thorin interpolation theorem

Work Unit Number 2 (Physical Mathematics)

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*Permanent address: Department of Mathematics, Zhejiang University, Hangzhou, China.

**Permanent address: Institute of Mathematics, Academia Sinica, Peking, China.

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SIGNIFICANCE AND EXPLANATION

The initial value problem for the nonlinear Schrödinger equation has important applications in physics and engineering. In this paper we study sufficient conditions for this problem to have a unique solution. We further establish bounds for the solution and also investigate its decay properties.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE INITIAL VALUE PROBLEM FOR A NONLINEAR SCHRÖDINGER EQUATION

Guang-Chang Dong* and Shujie Li**

1. Introduction

Consider the following initial value problem for the nonlinear Schrödinger equation

$$iu_t + \Delta u = F(u) , \tag{1}$$

$$u|_{t=0} = \phi(x) \tag{2}$$

where $x = (x_1,x_2,\ldots,x_n)$, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $F(u)$ is a complex-valued function of two real variables Reu, Imu.

For the case $n = 1$ and $u \in R$, it is shown in [1] that if for sufficiently small u and for $p > 4$, the function F satisfies the condition

$$F(u) = O(|u|^p) \tag{3}$$

and if ϕ and its derivatives up to some order are also sufficiently small then the problem has a unique solution. Furthermore, the solution u is as smooth as ϕ and F allow and is bounded in the L^2 and L^∞ norms. More specifically, we have

$$\|u(t)\|_\infty \leqq K(1 + t)^{-1/2} \tag{4}$$

where K is a constant.

It is also shown by Klainerman [2] that if F satisfies (3) with $p > 2 + \sqrt{3}$, a solution to Equations (1) and (2) still exists and is unique. But, in this case the order of $(1 + t)$ in (4) is a small negative number.

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*Permanent address: Department of Mathematics, Zhejiang University, Hangzhou, China.

**Permanent address: Institute of Mathematics, Academia Sinica, Peking, China.

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In this paper we show that the same results established in [1,2] are still true when the number p in (3) satisfies $p > (3 + \sqrt{17})/2$. Furthermore, we show that if

$$\frac{1}{2} + \frac{1}{n} + \sqrt{\frac{3}{4} + \frac{3}{n} + \frac{1}{n^2}} < p < \frac{n+2}{n-2} \tag{5}$$

the above results hold also for the higher dimensional cases $(n = 2,3,\ldots)$.

We note here that the problem has been widely studied for the special case that

$$F(u) = f(|u|)u \tag{6}$$

where f is a real-valued function, [see, for example, 3, 4]. Condition (5) is used in [4] to ensure a decay property for the solution.

In the sequel we use $\|\cdot\|_p$ to denote L_p-norm and use $K_1, K_2, \ldots,$ to denote positive constants.

2. Results

Theorem 1. Suppose that p satisfies (5), $F(u) \in C^1$ satisfies (3) and

$$F'(u) = O(|u|^{p-1}) . \tag{7}$$

If there exists an $\eta > 0$ sufficiently small such that for $\frac{n}{2} < k \leq \frac{n}{2} + 1$,

$$\|\phi\|_1 , \quad \|D^k \phi\|_2 < \eta < 1 \tag{8}$$

then (1) and (2) has a unique weak solution $u(x,t) \in C(L^2(\mathbb{R}), \mathbb{R}^+)$ satisfying (2) and for each v with

$$v, v_t, D_x^2 v \in C(L^2(\mathbb{R}^n), \mathbb{R}^+) , \tag{9}$$

the solution u satisfies

$$i(u(s),v(s))\Big|_{s=0}^{s=t} + \int_0^t (u(s), iv_s + \Delta v)ds = \int_0^t (F(u), v(s))ds . \tag{10}$$

Moreover, there exist positive constants K_1, K_2, K_3 and K_4 such that

$$\|u(t)\|_2 \leq K_1 , \tag{11}$$

$$\|u(t)\|_{p+1} \leq K_2(1 + t)^{-\frac{n(p-1)}{2(p+1)}} , \tag{12}$$

for $n = 1$

$$\|u(t)\|_\infty \leq K_3(1 + t)^{-\min\left(\frac{1}{2}, \frac{p-3}{2}\right)^*} , \tag{13}$$

for $n \geq 2$, $0 < \varepsilon < 1 - \frac{n(p - 1)}{2(p + 1)}$,

$$\|u(t)\|_{\frac{2n}{n-2+2\varepsilon}} \leq K_4(1 + t)^{\varepsilon-\min\left(1, n\left(\frac{pn}{n+2-2\varepsilon} - 1\right)\right)^*} . \tag{14}$$

---

*When the two terms of the minimum expression are equal, we must multiply by the factor $\log(1 + t)$ in the right hand side. We meet the similar situation in the following formulas (32), (33), (61), and (62).

Proof: Find a solution w of

$$iw_t + \Delta w = 0 \tag{15}$$

with the initial condition

$$w(0) = \phi . \tag{16}$$

Let $u^{(0)} = w$. Construct a sequence of functions $u^{(m)}$, $(m = 0,1,2,\ldots)$

$$\begin{cases} iu_t^{(m)} + \Delta u^{(m)} = F(u^{(m-1)}) \\ u^{(m)}(0) = \phi \end{cases} . \tag{17}$$

If we can prove that $u^{(m)}$ converges, say $u^{(m)} \to u$ $(m \to \infty)$, then u is a solution of (1) and (2). Write V, W to denote $u^{(m-1)}$, $u^{(m)} - w$, then (17) is reduced to

$$\begin{cases} iW_t + \Delta W = F(V) \\ W(0) = 0 \end{cases} . \tag{18}$$

We proceed to derive some estimates for the solution of (15), (16) and then for the solution of (17). Then we will prove that $u^{(m)}$ is convergent.

When $\phi \in L^1(R^n)$ and $D^k\phi \in L^2(R^n)$, we write the solution of (15), (16) in the form

$$w(x,t) = (4\pi it)^{-\frac{n}{2}} \int_{R^n} \exp\left(\frac{\sum_{j=1}^{n} (x_j - y_j)^2}{4it}\right)\phi(y)dy \equiv R(t)\phi . \tag{19}$$

Hence

$$\|w(t)\|_2 \equiv \|R(t)\phi)\|_2 = \|\phi\|_2 , \tag{20}$$

$$\|w(t)\|_\infty \equiv \|R(t)\phi\|_\infty < (4\pi t)^{-\frac{n}{2}} \|\phi\|_1 . \tag{21}$$

By the Riesz-Thorin interpolation theorem [5] we have

$$\|w(t)\|_q \equiv \|R(t)\phi\|_q \le (4\pi t)^{\frac{n}{q}-\frac{n}{2}} \|\phi\|_{\bar{q}} \le K_5 t^{\frac{n}{q}-\frac{n}{2}} (\|\phi\|_1 + \|\phi\|_2) \tag{22}$$

where $q \ge 2$, $\frac{1}{q} + \frac{1}{\bar{q}} = 1$.

We also have

$$\|D_x^k w\|_2 = \|D^k\phi\|_2 . \tag{23}$$

From (20), (22), (23) and elementary inequality

$$\|w\|_q \le \|w\|_\infty^{\frac{q-2}{q}} \|w\|_2^{\frac{2}{q}} \le (K_6\|w\|_2 + K_7\|D_x^k w\|_2)^{\frac{q-2}{q}} \|w\|_2^{\frac{2}{q}} ,$$

we have

$$|||w|||_{q,t} \le K_8(\|\phi\|_1 + \|\phi\|_2 + \|D^k\phi\|_2) \le K_9(\|\phi\|_1 + \|D^k\phi\|_2) \tag{24}$$

where $|||\ |||_{q,t}$ denotes the norm

$$|||w|||_{q,t} = \sup_{t>0} \left[(1+t)^{\frac{n}{2}-\frac{n}{q}} \|w\|_q\right] .$$

Similarly, we have that

$$|||D_x w|||_{q,t} \le K_{10}(\|D\phi\|_1 + \|D^{k+1}\phi\|_2) , \tag{25}$$

$$|||D_x^2 w|||_{q,t} \le K_{11}(\|D^2\phi\|_1 + \|D^{k+2}\phi\|_2) . \tag{26}$$

Now we estimate the solution of (18). The solution W can be written in the form

$$W(t) = \int_0^t R(t-s)F(V(s))ds . \tag{27}$$

It follows from (3), (22), (27) that

$$\|W(t)\|_{p+1} \le K_{10} \int_0^t (t-s)^{\frac{n}{p+1} - \frac{n}{2}} \| |V(s)|^p \|_{1+\frac{1}{p}} ds$$

$$= K_{12} \int_0^t (t-s)^{\frac{n}{p+1} - \frac{n}{2}} \|V(s)\|_{p+1}^p ds$$

$$\le K_{13} |||V|||_{p+1,t}^p \int_0^t (t-s)^{-\frac{n(p-1)}{2(p+1)}} (1+s)^{-\frac{np(p-1)}{2(p+1)}} ds \ .$$

Since $\frac{np(p-1)}{2(p+1)} > 1$ and $\frac{n(p-1)}{2(p+1)} < 1$ by (5), hence we have

$$|||W|||_{p+1,t} \le K_{14} |||V|||_{p+1,t}^p \ . \tag{28}$$

Multiplying (18) by $2\bar{W}$ and integrating, we get

$$\|W\|_2^2 = 2\mathrm{Im} \int_0^t ds \int_{\mathbb{R}^n} \bar{W} F(V) dx \le 2 \int_0^t \|W(s)\|_{p+1} \|F(V(s))\|_{1+\frac{1}{p}} ds$$

$$\le K_{15} |||V|||_{p+1,t}^{2p} \int_0^t (1+s)^{\frac{n(1-p)}{2}} ds \ .$$

Hence

$$|||W|||_{2,t} \le K_{16} |||V|||_{p+1}^p \ . \tag{29}$$

Similarly, we have

$$\begin{aligned} |||W(t+\delta t) - W(t)|||_{p+1,t} &= O(|||V|||_{p+1,t}^{p-1}) |||V(t+\delta t) - V(t)|||_{p+1,t} \\ &\quad + O(\delta t) |||V|||_{p+1,t}^p \ , \end{aligned} \tag{30}$$

$$\begin{aligned} |||W(t+\delta t) - W(t)|||_{2,t}^2 &= O(|||V|||_{p+1,t}^{2p-2}) |||V(t+\delta) - V(t)|||_{p+1,t}^2 \\ &\quad + O(\delta t) |||V|||_{p+1,t}^{2p} \ . \end{aligned} \tag{31}$$

When $n = 1$, (27) implies that

$$
\begin{aligned}
\|W(t)\|_\infty &\le K_{17}\int_0^t (t-s)^{-\frac{1}{2}}\|F(V(s)\|_1 ds \\
&\le K_{18}\int_0^t (t-s)^{-\frac{1}{2}}\|V(s)\|_2^{\frac{1}{p-1}}\|V(s)\|_{p+1}^{\frac{(p-2)(p+1)}{p-1}} ds \\
&\le K_{19}|||V|||_{2,t}^{\frac{1}{p-1}}\ |||V|||_{p+1,t}^{\frac{(p-2)(p+1)}{p-1}} \int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{1-\frac{p}{2}}ds \\
&\le K_{20}|||V|||_{2,t}^{\frac{2}{p-1}}\ |||V|||_{p+1,t}^{\frac{(p-2)(p+1)}{p-1}}(1+t)^{-\min\left(\frac{1}{2},\ \frac{p-3}{2}\right)} .
\end{aligned}
\tag{32}
$$

When $n \ge 2$ and $0 < \varepsilon < 1 - \frac{n(p-1)}{2(p+1)}$, it also follows from (27) that

$$
\begin{aligned}
\|W(t)\|_{\frac{2n}{n-2+2\varepsilon}} &\le K_{21}\int_0^t (t-s)^{\varepsilon-1}\|F(V(s))\|_{\frac{2n}{n+2-2\varepsilon}} ds \\
&\le K_{22}\int_0^t (t-s)^{\varepsilon-1}\|V(s)\|_2^{\frac{1}{p-1}\left(2p+2-\frac{pn}{n+2-2\varepsilon}\right)}\|V(s)\|_{p+1}^{2\frac{p+1}{p-1}\left(\frac{pn}{n+2-2\varepsilon}-1\right)} ds \\
&\le K_{23}|||V|||_{2,t}^{\frac{1}{p-1}\left(2p+2-\frac{pn}{n+2-2\varepsilon}\right)}|||V(s)|||_{p+1,t}^{2\frac{p+1}{p-1}\left(\frac{pn}{n+2-2\varepsilon}-1\right)} \cdot \\
&\qquad \cdot \int_0^t (t-s)^{\varepsilon-1}(1+s)^{n\left(\frac{pn}{n+2-2\varepsilon}-1\right)} ds \\
&\le K_{24}|||V(s)|||_{2,t}^{\frac{1}{p-1}\left(2p+2-\frac{pn}{n+2-2\varepsilon}\right)}|||V(s)|||_{p+1,t}^{2\frac{p+1}{p-1}\left(\frac{pn}{n+2-2\varepsilon}-1\right)} \cdot \\
&\qquad \cdot (1+t)^{\varepsilon-\min\left(1,n\left(\frac{pn}{n+2-2\varepsilon}-1\right)\right)} .
\end{aligned}
\tag{33}
$$

Changing V and W to their original notation $u^{(m-1)}$, $u^{(m)} - w$ in (28) respectively, we have

$$
|||u^{(m)} - w|||_{p+1,t} \le K_{14}|||u^{(m-1)}|||_{p+1,t}^{p} . \tag{34}
$$

Hence it follows that

$$|||u^{(m)}|||_{p+1,t} \leq |||w|||_{p+1,t} + K_{14}|||u^{(m-1)}|||^{p}_{p+1,t} . \tag{35}$$

If $|||w|||_{p+1,t}$ is so small that the following inequality holds

$$K_{14}(2|||w|||_{p+1,t})^{p-1} \leq \frac{1}{2} , \tag{36}$$

then we prove by induction that

$$|||u^{(m)}|||_{p+1,t} \leq \left(2 - \frac{1}{2^m}\right)|||w|||_{p+1,t}, \qquad (n = 0,1,2,\ldots) . \tag{37}$$

The inequality (37) is true for $m = 0$ because $u^{(0)} = w$. Assume that (37) is true for m, by (35), (36), (37) we have

$$|||u^{(m+1)}|||_{p+1,t} \leq |||w|||_{p+1,t} + \frac{1}{2} |||u^{(m)}|||_{p+1,t} \leq \left(2 - \frac{1}{2^{m+1}}\right)|||w|||_{p+1,t} .$$

Hence (37) is true for $m = 0,1,2,\ldots$. Therefore we have

$$|||u^{(m)}|||_{p+1,t} \leq 2|||w|||_{p+1,t}, \qquad (n = 0,1,2,\ldots) . \tag{38}$$

Then

$$|||u^{(m+1)}-u^{(m)}|||_{p+1,t} = \sup_{t\geq 0}\left\{(1+t)^{\frac{n}{2} - \frac{n}{p+1}} \left\|\int_0^t R(t-s)[F(u^{(m)})-F(u^{(m-1)})]ds\right\|_{p+1}\right\}$$

$$\leq K_{25} \sup_{t\geq 0}\left\{(1+t)^{\frac{n}{2} - \frac{n}{p+1}} \int_0^t (t-s)^{\frac{n}{p+1} - \frac{n}{2}} \|(u^{(m)} - u^{(m-1)})F'(w + \theta u^{(m)}\right.$$

$$+ (1-\theta)u^{(m-1)})\|_{1+\frac{1}{p}} ds, \qquad (0 < \theta < 1)$$

$$\leq K_{26}|||w|||^{p-1}_{p+1,t}|||u^{(m)} - u^{(m-1)}|||_{p+1,t} . \tag{39}$$

Let $|||w|||_{p+1,t}$ be sufficiently small such that

$$K_{26}|||w|||^{p-1}_{p+1,t} < 1 . \tag{40}$$

From (24) we see that both (36) and (40) can be satisfied if $\|\phi\|_1$, $\|D^k\phi\|_2$ are small. Then from (39) we have that $u^{(m)}$ is convergent in the norm $|||\cdot|||_{p+1,t}$, $u^{(m)} \to u$ $(m \to \infty)$. Therefore (12), (11), (13), (14) follow from (38), (29), (32), (33) and (24).

In (30) and (31), letting $m \to \infty$, we then have that

$$|||(u-w)(t+\delta t) - (u-w)(t)|||_{p+1,t} = O(|||u|||^{p-1}_{p+1,t})|||u(t+\delta t) - u(t)|||_{p+1,t}$$

$$+ O((\delta t)|||u|||^{p}_{p+1,t} , \tag{41}$$

$$|||(u-w)(t+\delta t) - (u-w)(t)|||_{2,t} = O(|||u|||^{2p-2}_{p+1,t})|||u(t+\delta t) - u(t)|||^2_{p+1,t}$$

$$+ O((\delta t)^2)|||u|||^{2p}_{p+1,t} . \tag{42}$$

And we have

$$w(t + \delta t) - w(t) = \int_t^{t+\delta t} w_s(s)ds = -i \int_t^{t+\delta t} \Delta w(s)ds .$$

Combining the above equation with (26) we have

$$|||w(t + \delta t) - w(t)|||_q = O(\delta t), \qquad (q \geq 2) . \tag{43}$$

It follows from (41), (42) and (43) that $|||u(t + \delta t) - u(t)|||_{p+1,t} = O(\delta t)$, $|||u(t + \delta t) - u(t)|||_{2,t} = O((\delta t)^{1/2})$.

If v satisfies (9), then

$$i(u^{(m)},v)|^{t=T}_{t=0} + \int_0^T (u^{(m)},iv_t + \Delta v)dt = (F(u^{(m-1)}),v) . \tag{44}$$

Equation (10) follows from (44) by letting $m \to \infty$.

Consider now the uniqueness of the weak solution. Let u, $u + U$ be two solutions $C(L^2(\mathbb{R}^n),\mathbb{R}^+)$ satisfying (2), (10). We have

$$U|_{t=0} = 0 , \tag{45}$$

$$i(U(T),v(T)) + \int_0^T (U,iv_t + \Delta v)dt = \int_0^T (F(u + U) - F(u),v)dt . \tag{46}$$

Taking $v = U$ in (46) (see appendix) and then taking the imaginary part, we have

$$\frac{1}{2}\|U(T)\|^2 = \operatorname{Im}\int_0^T (F(u+U) - F(u), U)\,dx$$

$$\leq \max_{0<\theta<1,u,U} |F'(u+\theta U)| \int_0^T \|U(t)\|^2 dt = K_{27}\int_0^T \|U(t)\|^2 dt .$$

Hence $U(t) = 0$.

Thus, the proof of Theorem 1 is completed.

Theorem 2 (decay property). Let $u(x,t)$ be the weak solution of Theorem 1. There is a unique solution u_+ of the linearized equation corresponding to (1), i.e. (15), such that

$$\|u(x,t) - u_+(x,t)\|_2 \to 0. \tag{47}$$

as $t \to +\infty$.

Proof: We define

$$u_+ = u(t) + \int_t^\infty R(t-s)F(u(s))ds . \tag{48}$$

We wish to prove that the above integral is absolutely convergent in $\| \ \|_2$.

When $n = 1$, by (12), (13) we have

$$\|F(u(s))\|_2^2 \leq K_{28}\||u(s)|^{2p}\|_1 \leq K_{29}\|u(s)\|_{p+1}^{p+1}\|u(s)\|_\infty^{p-1} \leq K_{30}S^{-\frac{p-1}{2}-(p-1)\min(\frac{1}{2},\frac{p-3}{2})} \tag{49}$$

When $p > 3$, we have $p-1 > 2$ and when $p \leq 3$, we have $\frac{(p-1)(p-2)}{2} > 4$ by (5), hence in both cases we have

$$\|F(u(s))\|_2 = O(S^{-1-\eta}) \quad (\eta > 0) \tag{50}$$

When $n \geq 2$, by (12), (14) we have

$$\|F(u(s))\|_2^2 \leq K_{28}\||u(s)|^{2p}\|_1 \leq K_{31}\|u(s)\|_{p+1}^{\alpha}\|u(s)\|_{\frac{2n}{n-2+2\varepsilon}}^{2p-\alpha} \leq K_{32}S^{-\beta} \tag{51}$$

where

$$\alpha = \frac{1-\frac{n-2+2\varepsilon}{n}p}{\frac{1}{p+1}-\frac{n-2+2\varepsilon}{2n}} = \frac{2p(1-\varepsilon) - n(p-1)}{1-\varepsilon-\frac{n(p-1)}{2(p+1)}} \tag{52}$$

$$\beta = \frac{n(p-1)}{2(p+1)} + (2p-\alpha)[-\varepsilon + \min(1, n(\frac{pn}{n+2-2\varepsilon} - 1))] \ . \tag{53}$$

When $n(\frac{pn}{n+2-2\varepsilon} - 1) > 1$ or equivalently,

$$p > (1 + \frac{1}{n})(1 + \frac{2-2\varepsilon}{n}) \ , \tag{54}$$

it follows from (5) that

$$\begin{aligned}\beta = 2p(1-\varepsilon) - \alpha[1-\varepsilon\frac{n(p-1)}{2(p+1)}] &= 2p(1-\varepsilon) - [2p(1-\varepsilon) - n(p-1)] \\ &= n(p-1) > 2(1 + \frac{1}{p}) > 2\end{aligned} \tag{55}$$

when

$$p < (1 + \frac{1}{n})(1 + \frac{2-2\varepsilon}{n}) \tag{56}$$

we have

$$\begin{aligned}&[2(p+1)(1-\varepsilon)-n(p-1)](\beta-2) = np(p-1)\,\frac{n^2+4}{n+2} + n^2-2n-4-(n^2-2n+4)p + O(\varepsilon) \\ &= [np(p-1)-2(p+1)]\,\frac{n^2+4}{n+2} + \frac{n^3-2n^2}{n+2}\,[(1 + \frac{1}{n})[1 + \frac{2-2\varepsilon}{n}) - p] + \frac{(n-2)^2}{n+2} + O(\varepsilon) \ .\end{aligned} \tag{57}$$

The right hand side of (57) is positive when ε is small. Using (5), (56) for $n \geq 2$, we see that $\beta > 2$. Combining this with (55), (51), we have (50) for the case $n \geq 2$.

Therefore

$$\|u_+(t) - u(t)\|_2 \leq \int_t^\infty \|F(u(s))\|_2 ds \to 0 \qquad (t \to \infty)$$

i.e. (47) is satisfied.

The solution of (1), (2) can be written in the form

$$u(t) = R(t)\phi + \int_0^t R(t-s)F(u(s))ds \ .$$

Therefore, we have

$$u_+(t) = R(t)\phi + \int_0^\infty R(t-s)F(u(s))ds \ . \tag{58}$$

The right hand side of (58) is a solution of (15).

Theorem 2 is thus proved.

We now study the regularity of the solution of (1), (2). The result is given in the following theorem.

Theorem 3. Suppose that $F^{(j)}(u) = O(|u|^{p-j})$ for each nonnegative integer $j \leq p$. *Then we have that*

(i) If $n = 1$, ϕ_{xx} exists or if $n = 2$, $p \geq 3$, $D_x^5\phi$ exists or if $n = 3$, $p \geq 4$, $D_x^6\phi$ exists, then the weak solution of Theorem 1 is a classical solution.

(ii) If $n = 2,3$ or if $n = 4,5$ and $p \geq 2$, then $D_x^2 u$, $u_t \in$ $L^2(\mathbf{R}^n) \cap L^{\frac{2n}{n-2+2\varepsilon}}(\mathbf{R}^n)$ for every $t \geq 0$, and u satisfies (1) almost everywhere, and $D_x^2 u$, u_t satisfy inequality (11)-(14) for some constants K_1, K_2, K_3 and K_4.

(iii) If $D_x u \in L^2(\mathbf{R}^n) \cap L^{\frac{2n}{n-2+2\varepsilon}}(\mathbf{R}^n)$ for every $t \geq 0$, then

$$i(u(s),v(s))\Big|_{s=0}^{s=t} + \int_0^t [(u(s),iv_s(s)) - (\mathrm{grad}\ u,\ \mathrm{grad}\ v)]\,ds = \int_0^t (F(u(s),v(s))ds$$

for every v with v, v_t, $D_x^2 v \in C(L^2(\mathbf{R}^n),\mathbf{R}^+)$. Moreover $D_x u$ satifies inequality (11)-(14) for some constants K_1, K_2, K_3 and K_4.

Proof: Differentiating (18) and estimating by the same method as in (28), (29), (32), (33), we have

$$|||D_x W|||_{p+1,t} \leq K_{33}|||V|||_{p+1,t}^{p-1}|||D_x V|||_{p+1,t} \tag{59}$$

$$|||D_x W|||_{2,t} \leq K_{34}|||V|||_{p+1,t}^{\ell-1}|||D_x V|||_{p+1,t} \tag{60}$$

$$|||D_x w|||_\infty <$$

$$K_{35}\left[|||v|||_{2,t}^{\frac{p-1}{p}}|||D_x v|||_{2,t}^{\frac{1}{p}}\right]^{\frac{1}{p-1}}\left[|||v|||_{p+1,t}^{\frac{p-1}{p}}|||D_x v|||_{p+1,t}^{\frac{1}{p}}\right]^{\frac{(p-2)(p+1)}{p-1}}$$

$$\cdot\ (1+t)^{-\min(\frac{1}{2},\ \frac{p-3}{2})} \quad (n = 1) \tag{61}$$

$$|||D_x w|||_{\frac{2n}{n-2+2\varepsilon}} < K_{36}\left[|||v|||_{2,t}^{\frac{p-1}{p}}|||D_x v|||_{2,t}^{\frac{1}{p}}\right]^{\frac{1}{p-1}(2p+2-\frac{pn}{n+2-2\varepsilon})}$$

$$\cdot\left[|||v|||_{p+1,t}^{\frac{p-1}{p}}|||D_x v|||_{p+1,t}^{\frac{1}{p}}\right]^{2\frac{p+1}{p-1}(\frac{pn}{n+2-2\varepsilon}-1)}(1+t)^{\varepsilon-\min(1,n(\frac{pn}{n+2-2\varepsilon}-1))}$$

$$(n \geq 2) \tag{62}$$

$$|||D_{xx}w|||_{p+1,t} < K_{37}\left[|||v|||_{p+1,t}^{p-1}|||D_{xx}v|||_{p+1,t}|||v|||_{p+1,t}^{p-2}|||D_x v|||_{p+1,t}^{2}\right]$$

$$(n \geq 1,\ p \geq 2). \tag{63}$$

And we have three additional similar inequalities.

When $p \geq 2$, $|||w|||_{p+1,t}$ is small enough and $|||D_x w|||_{p+1,t}$, $|||D_x^2 w|||_{p+1,t}$ exist, from (59), (63) we deduce that $D_x u^{(m)}$, $D_x^2 u^{(m)}$ are uniformly bounded in the norm $\|\ \|_{p+1}$. Since they are weak convergent, by the Banach-Saks theorem, $D_x u$, $D_x^2 u$ exist for all t, or $u \in W_2^{p+1}(\mathbb{R}^n)$ for all $t > 0$, and

$$\|D_x u\|_{p+1} < K_{38}(1+t)^{-\frac{n(p-1)}{2(p+1)}}, \quad \|D_x^2 u\|_{p+1} < K_{39}(1+t)^{-\frac{n(p-1)}{2(p+1)}},$$

i.e. $D_x u$, $D_x^2 u$ satisfy inequality of type (12). It is also easy to deduce that they also satisfy inequalities of the type (11), (13), (14). Hence (ii) is proved.

By the same process and with the help of the Sobolev imbedding theorem, (i) and (iii) can also be proved straightforward.

APPENDIX

We cannot take $v = U$ in (45) directly because U is not smooth. However, we can proceed as follows.

Let $j(x)$ be a mollifier, i.e. satisfying the following conditions:

(i) $j(x) \in C_0^\infty \quad (x \in R^n)$; (ii) $\int_{R^n} j(x)dx = 1$;

(iii) $j(x) \equiv j(x,\lambda) \equiv j(\frac{x}{\lambda}) = 0 \quad (|x| \geq \lambda)$;

(iv) $j(x)$ is an even function of x ; (v) $j(x) = O(\lambda^{-n}), Dj(x) = O(\lambda^{-n-1})$.

Similary let $k(t)$ be a mollifier also, with (iii) changes to $k(t) \equiv k(\frac{t}{\mu}) = 0$ $(|t| \geq \mu)$. Define $U(x,t) = 0 \; (t < 0)$, then because of $U(x,0) = 0$, we have $U(x,t) \in C(L^2(\mathbf{R}^n),\mathbf{R}))$.

Take $v(x,t) = \int U(y,\tau))j(x-y)k(t-\tau)dyd\tau$ in (46), and then take imaginal part, let $\lambda, \mu \to 0$, we have (denote $\mathrm{Re}U = U_1$, $\mathrm{Im}\, U = U_2$)

$$\|U(T)\|^2 + \lim_{\lambda\to 0} I_1 + \lim_{\mu\to 0} I_2 = \mathrm{Im} \int_0^T (F(u+U) - F(u),U)dt$$

where

$$I_1 = \int_0^T dt \int_{\mathbf{R}^n\times\mathbf{R}^n} \Delta_x\, j(x-y)[U_1(y,t)U_2(x,t) - U_1(x,t)U_2(y,t)]\,dxdy = 0$$

(because of the integrand is an odd function of x,y)

$$I_2 = \int_{\mathbf{R}^n} dx \int_0^T dt \int_{t-\mu}^{t+\mu} [U_1(x,t)U_1(x,\tau) + U_2(x,t)U_2(x,\tau)]k_t(t-\tau)d\tau$$

$$= \int_{\mathbf{R}^n} dx(\int_{T-\mu}^T dt \int_T^{T+\mu} d\tau + \int_0^\mu dt \int_{-t}^0 d\tau)[U_1(x,t)U_1(x,\tau) + $$
$$+ U_2(x,t)U_2(x,\tau)]k_t(t-\tau)$$

(because of the integrand is an odd function of t,τ)

$$= \|U(T)\|^2 \int_{T-\mu}^T dt \int_T^{T+\mu} K_t(t-\tau)d\tau + o(1) \max_{0\leq t\leq T+\mu} \|U(t)\|^2 = -\frac{1}{2}\|U(t)\|^2 + o(1) .$$

Hence we have

$$\frac{1}{2}\|U(T)\|^2 = \mathrm{Im} \int_0^T (F(u+U) - F(u),U)dx .$$

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In this paper we study conditions that ensure existence and uniqueness of solution to the initial value problem for the nonlinear Schrödinger equation. Many known results in the one dimensional space are extended to the higher dimensional cases. We also establish bounds and investigate decay properties for the solution.

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