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A CHARACTERIZATION OF NORMAL OPERATORS

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Shmuel Friedland* and Luc C. Tartar**

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ABSTRACT

Let A be a bounded linear operator in a Hilbert space. If A is normal then $\log\|e^{At}u\|$ and $\log\|e^{A^*t}u\|$ are convex functions for all $u \neq 0$. In this paper we prove that these properties characterize normal operators.

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SIGNIFICANCE AND EXPLANATION

Consider the differential equation

$$\frac{dx}{dt} = Ax \qquad (1)$$

in a Hilbert space H. Assume that $A: H \to H$ is a bounded linear operator. Then any solution of (1) is of the form $x(t) = e^{At}u$. Suppose that A is a normal operator, i.e. $AA^* = A^*A$. Then one can show that the function $\log\|x(t)\|$ is a convex function on R. Here $\|x\|$ denotes the norm of x in H. The purpose of this paper is to study the converse of this statement. It turns out that there is a distinction between the finite and infinite dimensional case of H. In the first case the convexity of $\log\|x(t)\|$ for all non-trivial solutions $x(t)$ implies the normality of A. In the infinite dimensional case this result does not apply for a general A. We show, however, if we assume in addition that $\log\|y(t)\|$ is also convex for all non-trivial solutions of the system

$$\frac{dy}{dt} = A^*y \qquad (2)$$

then A must be a normal operator.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A CHARACTERIZATION OF NORMAL OPERATORS

Shmuel Friedland* and Luc C. Tartar**

1. Introduction.

Let H be a Hilbert space over the complex numbers C with an inner product (x,y). Assume that $A:H \to H$ is a bounded linear operator. A straightforward calculation shows (see the next section)

Lemma 1. Let $A:H \to H$ be a bounded linear operator. If $A^*A - AA^*$ is non-negative definite then $\log\|e^{At}u\|$ is convex on R for all $u \neq 0$.

Thus if A is normal then $\log\|e^{At}u\|$ and $\log\|e^{A^*t}u\|$ are convex. However, there are non-normal operators A such that $0 \leq A^*A - AA^*$. Here, as usual, for self-adjoint operators S,T the inequality $S \leq T$ denotes that $T-S$ is a non-negative definite operator. For example let $H = \ell_2$ and choose A to be the shift operator $A(x_1,x_2,\cdots) = (0,x_1,x_2,\cdots)$. In this case $\log\|e^{A^*t}u\|$ *is not convex for* $u = (0,1,0,\cdots)$. This situation can not hold in a finite dimensional H. More precisely we have

Theorem 1. Let $A = P + iQ$, where P and Q are bounded self-adjoint operators. Assume that P has only a point spectrum (i.e. H has an orthonormal basis consisting of eigen-elements of P). Then A is normal if and only if

$$(1) \qquad \frac{d^2}{dt^2}(\log\|e^{At}u\|)(0) \geq 0, \quad \text{for all } u \neq 0.$$

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Our main result is

Theorem 2. Let $A:H \to H$ be a bounded linear operator. Then A is normal if and only if (1) and

$$\frac{d^2}{dt^2}(\log\|e^{A^*t}u\|)(0) \geq 0 , \quad \text{for all } u \neq 0 , \tag{2}$$

hold.

We conjecture

Conjecture. Assume that (1) holds. Then $0 \leq A^*A - AA^*$.

2. Proofs.

Using the group properties of e^{At} we easily deduce

Lemma 2. *Let* $A : H \to H$ *be a bounded linear operator. Then* $\log \|e^{At}u\|$ *is convex on* R *for all* $u \neq 0$ *if and only if* (1) *holds.*

A straightforward calculation shows

$$\frac{d^2}{dt^2}(\log\|e^{At}u\|)(0) = \tfrac{1}{2}(u,u)^{-2}[((A^2 + A^{*2} + 2A^*A)u,u) - ((A + A^*)u,u)^2].$$

Thus (1) is equivalent to the inequality

$$(3) \qquad ((A + A^*)u,u)^2 \leq ((A^2 + A^{*2} + 2A^*A)u,u)(u,u).$$

The Cauchy-Schwarz inequality yields

$$((A + A^*)u,u)^2 \leq ((A + A^*)^2u,u)(u,u) .$$

As

$$(A + A^*)^2 = A^2 + A^{*2} + 2A^*A - (A^*A - AA^*)$$

the assumption that $A^*A - AA^* \geq 0$ implies the inequality (3). This establishes Lemma 1 .

To give an equivalent form of the inequality (3) we need the following lemma.

Lemma 3. *Let* $R,S,T:H \to H$ *be self-adjoint non-negative definite operators. Then*

$$(4) \qquad (Ru,u)^2 \leq (Su,u)(Tu,u), \quad \text{for all} \quad u \in H$$

if and only if

$$(5) \qquad 2R \leq \alpha^{-1}S + \alpha T$$

for all positive α .

Proof. The inequality (4) implies (5) in view of arithmetic-geometric inequality. Suppose that (5) holds. If $(Su,u) = 0$ then by letting α tend to zero we deduce that $(Ru,u) = 0$. Thus we may assume that $(Su,u)(Tu,u) > 0$. In that case choose $\alpha = [(Su,u)/(Tu,u)]^{1/2}$ to obtain (4). ∎

Lemma 4. Let $A = P + iQ$ *, where* P *and* Q *are self-adjoint. Then* (3) *is equivalent to the inequality*

$$\frac{i}{2}(QP - PQ) \leq (P - \alpha I)^2 \tag{6}$$

for all real α .

Proof. A straightforward computation shows that the inequality (3) is invariant under the transformation $A \to A + \omega I$. So we may assume that $P \geq 0$. Also in terms of P and Q (3) becomes

$$(Pu,u)^2 \leq ([P^2 + \frac{i}{2}(PQ - QP)]u,u)(u,u) .$$

In view of Lemma 3 the above inequality is equivalent to (6) for $\alpha > 0$. As $P \geq 0$ (6) trivially holds also for $\alpha \leq 0$. Again (6) is invariant under the transformation $A \to A + \omega I$. The proof of the lemma is completed. ∎

Lemma 5 Let $P,Q:H \to H$ *be bounded self-adjoint operators. Assume that* $Pu = \alpha u$, $u \neq 0$ *and suppose that* (6) *holds. Then*

$$P(Qu) = \alpha(Qu) . \tag{7}$$

Proof. Let $y = u + sx$, where $s \in C$ and $(u,x) = 0$. As

$$(Bu,u) = ((P - \alpha I)^2 u,u) = ((P - \alpha I)^2 u,x) = 0 , \quad B = \frac{i}{2}(QP - PQ),$$

(6) implies

$$2\mathrm{Re}\{\bar{s}(Bu,x)\} + |s|^2(Bx,x) \leq |s|^2((P - \alpha I)^2 x,x).$$

Since s is arbitrary we obtain that $(Bu,x) = 0$ if $(u,x) = 0$. So $Bu = \beta u$. Finally the equality $(Bu,u) = 0$ yields $\beta = 0$, i.e. $Bu = 0$. This proves (7). ∎

Proof of Theorem 1. As P has only a point spectrum H decomposes to a direct sum of invariant eigen-subspaces of P .

$$H = \sum_{\lambda \in \sigma(P)} \oplus H_\lambda , \qquad (P - \lambda I)H_\lambda = 0 .$$

Lemma 5 implies that $QH_\lambda \subset H_\lambda$. That is $PQ = QP$ which is equivalent to the normality of A . ∎

Assume now that $\log\|e^{At}u\|$ and $\log\|e^{A^*t}u\|$ are convex on R for all $u \neq 0$. According to Lemma 4 these conditions are equivalent to

(8) $$-(P - \alpha I)^2 \leq \frac{i}{2}(QP - PQ) \leq (P - \alpha I)^2$$

for all $\alpha \in \mathbf{R}$. Then Theorem 2 follows from our last theorem.

Theorem 3. Let $B, P: H \to H$ be bounded self-adjoint operators. Assume that

(9) $$-(P - \alpha I)^{\mu} \leq B \leq (P - \alpha I)^{\mu}, \quad \mu = 2m/(2\ell - 1)$$

for all real α, where $m \geq \ell \geq 1$ are integers. Then $B = 0$.

Proof. Suppose that $Pu = \alpha u$. Then (9) yields $(Bu,u) = 0$. Apply the arguments of the proof of Lemma 5 to deduce $Bu = 0$. Decompose $H = H_1 \oplus H_2$, $PH_i \subseteq H_i$ such that H_2 has an orthonormal basis consisting of eigen-elements of P and H_1 - the orthogonal complement of H_2 - does not contain any eigen-elements of P. Thus $BH_2 = 0$. Therefore it is enough to assume that P has only a continuous spectrum. Without restriction in generality we may asssume that the spectrum of P lies in $[0,1]$. Consider the spectral decomposition of P

$$P = \int_0^1 \lambda dE(\lambda).$$

Let

$$E_i = \int_{(i-1)/n}^{i/n} dE(\lambda), \quad i = 1,\cdots,n.$$

Thus

$$I = \sum_{i=1}^{n} E_i, \quad E_iE_j = \delta_{ij}E_j, \quad i,j = 1,\cdots,n.$$

Choose $\alpha = (2i - 1)/2n$. Then (9) yields

(10) $$-(2n)^{-\mu}E_i \leq E_iBE_i \leq (2n)^{-\mu}E_i.$$

Let $y = u + sy$, $u \in E_iH$, $y \in (I - E_i)H$. Then for the same choice of α (9) implies

$$|(Bu,u) + 2\mathrm{Re}\{s(By,u)\} + |s|^2(By,y)| \leq (2n)^{-\mu}(u,u) + |s|^2(v,y).$$

The same inequality applies if we replace s by $-s$. Combine these two inequalities to get

$$2|\mathrm{Re}\{s(By,u)\}| \leq (2n)^{-\mu}(u,u) + |s|^2(y,y) .$$

Choose $|s| = (2n)^{-\mu/2}$, $\arg s = -\arg(By,u)$ to deduce

(11) $|(By,u)| \leq (2n)^{-\mu/2}[(u,u) + (y,y)]/2, \quad u \in E_iH, \quad y \in (I - E_i)H .$

Let $\lambda \in \sigma(B)$. We claim that

(12) $$|\lambda| \leq 3(2n)^{-(\mu-1)/2} .$$

Indeed, there exists $x \in H$ such that

$$\|Bx - \lambda x\| \leq (2n)^{-\mu/2} , \qquad \|x\| = 1 .$$

As $\|x\|^2 = \sum_{i=1}^{n} \|E_ix\|^2 = 1$ we may assume that $\|E_jx\| \geq n^{-1/2}$ for some $1 \leq j \leq n$. So

$$\|E_jBx - \lambda E_jx\| \leq (2n)^{-\mu/2} .$$

Thus

$$|\lambda| \leq \sqrt{n}\ ((2n)^{-\mu/2} + \|E_jBx\|) .$$

We now estimate $\|E_jB\|$. Clearly

$$\|E_jB\| = \sup_{\|v\| = \|w\| = 1} \mathrm{Re}\{(E_jBv,w)\} = \sup_{\|v\| = \|E_jw\|=1} \mathrm{Re}\{(E_jBv,E_jw)\} \leq$$

$$\sup_{\|E_jv\| = \|E_jw\| = 1} \mathrm{Re}\{(E_jBE_jv,E_jw)\} +$$

$$+ \sup_{\|(I-E_j)v\| = \|E_jw\| = 1} \mathrm{Re}\{(E_jB(I-E_j)v,E_jw)\} .$$

In view of (10) and (11) we get

$$\sup_{\|E_jv\| = \|E_jw\| = 1} \mathrm{Re}\{(E_jBE_jv,E_jw)\} \leq (2n)^{-\mu} ,$$

$$\sup_{\|(I-E_j)v\| = \|E_jw\| = 1} \mathrm{Re}\{(E_jB(I-E_j)v,E_jw)\} \leq (2n)^{-\mu/2} .$$

Combine the above inequalities to deduce (12). As n is arbitrary and $\mu > 1$ (12) implies $\sigma(B) = \{0\}$.

As B is self-adjoint we conclude that $B = 0$.

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