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HAUSDORFF'S MOMENT PROBLEM AND EXPANSIONS IN LEGENDRE POLYNOMIA--ETC(U)
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HAUSDORFF'S MOMENT PROBLEM AND EXPANSIONS IN LEGENDRE POLYNOMIALS

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ABSTRACT

A new proof is given for Hausdorff's condition on a set of moments which determines when the function generating these moments is in L^2. The proof uses Legendre polynomials and their discrete extensions found by Tchebychef. Then an extension is given to a weighted L^2 space using Jacobi polynomials and their discrete extensions.

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SIGNIFICANCE AND EXPLANATION

The paper describes a method of obtaining in terms of the moments approximations to the solutions of the finite moment problem

$$(1) \qquad \int_0^1 f(x)x^\nu dx = \mu_\gamma, \qquad (\nu = 0,1,2,\ldots) .$$

In his paper [2] Hausdorff gave conditions on the moments μ_ν for the problems (1) to have a solution $f(x)$ which is squares integrable. However, our approximations are constructed in terms of the coefficients c_ν of the Legendre series expansion

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu P_\nu(2x - 1), \qquad (0 \leqq x \leqq 1) ,$$

where $P_\nu(x)$ are the Legendre polynomials. The main result is that Hausdorff's condition for a square integrable $f(x)$ are here expressed in terms of the c_ν. This transition from the μ_ν to the c_ν is done by using a set of orthogonal polynomials on the discrete set $x = 0, x = 1,\ldots,x = n$ due to Tchebychef.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

HAUSDORFF'S MOMENT PROBLEM AND EXPANSIONS IN LEGENDRE POLYNOMIALS

R. Askey, I. J. Schoenberg, and A. Sharma

1. Introduction. We refer to [3] for a description of the problem of Bellman, Kalaba and Lockett [1] of obtaining approximation to the inverse Laplace transform. They reduce the problem to the solution of the finite moment problem

$$\int_0^1 f(x)x^{\nu}dx = \mu_{\nu}, \qquad (\nu = 0,1,\ldots,n-1) \tag{1.1}$$

and obtain approximations for $f(x)$ by applying Gauss' n-point quadrature formula to the integrals (1) and use numerical approximations to the inverse of the matrix of the system so obtained.

In [3] it is shown that the inverse of the Gauss matrix is not needed. Rather approximations to $f(x)$ are obtained if we determine the polynomial

$$R_{n-1}(x) = \sum_{0}^{n-1} c_{\nu}P_{\nu}(1-2x) \tag{1.2}$$

of degree $n-1$ which is the least square approximation to $f(x)$ in $[0,1]$ having moments $\mu_0,\mu_1,\ldots,\mu_{n-1}$. The coefficients c_{ν} in (1.2) are given by the lower triangular transformation

$$c_{\nu} = (2\nu+1)\sum_{i=0}^{\nu}(-1)^{i}\binom{\nu+i}{\nu}\binom{\nu}{i}\mu_i, \qquad (\nu = 0,1,\ldots,n-1) . \tag{1.3}$$

The numerical problem of Bellman, Kalaba and Lockett is thereby solved. However, this approach shows that the infinite problem in $[0,1]$

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(1.4) $$\int_0^1 f(x)x^{\nu}dx = \mu_{\nu}, \qquad (\nu = 0,1,2 \ldots \text{ to infinity})$$

might be attacked in terms of the Legendre series expansion

(1.5) $$f(x) \sim \sum_{\nu=0}^{\infty} c_{\nu}P_{\nu}(1 - 2x) .$$

Hausdorff devoted to the problem (1.4) his famous paper [2] in which he showed the following:

A. The system

(1.6) $$\int_0^1 x^{\nu}d\psi(x) = \mu_{\nu}, \qquad (\nu = 0,1,2,\ldots)$$

has a non-decreasing solution $\psi(x)$ if and only if

$$\Delta^n\mu_m = \mu_m - \binom{n}{1}\mu_{m+1} + \cdots + (-1)^n\mu_{m+n} \geq 0$$

for $m,n \geq 0$.

B. The system (1.6) has a solution $\psi(x)$ of bounded variations in $[0,1]$ if and only if

$$\sum_{\nu=0}^{n} \binom{n}{\nu}|\Delta^{n-\nu}\mu_{\nu}| = O(1) \quad \text{as} \quad n \to \infty .$$

For a direct derivation of Hausdorff's conditions for A and B see [4].

C. The system (1.6) has a solution

$$\psi(x) = \int_0^x \varphi(x)dx$$

where $\varphi(x) \in L^p(0,1)$ with $1 < p < \infty$, if and only if

$$(n+1)^{p-1} \sum_{\nu=0}^{n} \{\binom{n}{\nu}|\Delta^{n-\nu}\mu_\nu|\}^p = O(1) .$$

A particular case of C is this (p = 2):

<u>The moment problem</u>

(1.7) $$\int_0^1 f(x)x^\nu dx = \mu_\nu, \qquad (\nu = 0,1,2,\ldots)$$

<u>has a solution</u> $f(x) \in L^2(0,1)$ <u>if and only if</u>

(1.8) $$S_n = (n+1) \sum_{\nu=0}^{n} \binom{n}{\nu}^2 (\Delta^{n-\nu}\mu_\nu)^2 = O(1) .$$

While Hausdorff's results A and B are most apt, it seems that the result (1.8) might be profitably deduced from the expansion (1.5). Since $\sqrt{2n+1}\, P_n(1-2x)$ are orthonormal, we derive from the Riesz-Fisher theorem and

$$f(x) \sim \sum_0^\infty \frac{c_\nu}{\sqrt{2\nu+1}} \sqrt{2\nu+1}\, P_\nu(1-2x)$$

the following: The moment problem (1.7) has a solution $f(x) \in L^2(0,1)$ if and only if

(1.9) $$\sum_{\nu=0}^{n} \frac{c_\nu^2}{2\nu+1} = O(1) \quad \text{for all } n .$$

Formula (1.3) can be inverted and assumes the form

(1.10) $$\mu_\nu = \frac{1}{(\nu+1)\binom{2\nu+1}{\nu}} \{\binom{2\nu+1}{\nu}c_0 - \binom{2\nu+1}{\nu-1}c_1 + \cdots + (-1)^\nu \binom{2\nu+1}{0}c_\nu\} .$$

Substituting (1.10) in (1.8) leads to

$$S_1 = c_0^2 + \frac{1}{9} c_1^2$$

$$S_2 = c_0^2 + \frac{1}{6} c_1^2 + \frac{1}{50} c_2^2$$

$$S_3 = c_0^2 + \frac{1}{5} c_1^2 + \frac{1}{25} c_2^2 + \frac{1}{245} c_3^2 .$$

These simple expressions were a surprise and suggested that

$$S_n = \sum_{\nu=0}^{n} a_{n,\nu} c_\nu^2$$

with $a_{n,\nu}$ given by a reasonably simple expression. This will be shown in the next section

2. Hausdorff Theorem via Orthogonal Polynomials

Using the notation introduced in the first section, we have

$$\Delta^{n-\nu}\mu_\nu = \int_0^1 x^\nu (1 - x)^{n-\nu} f(x)dx$$

$$= \sum_{k=0}^{n} c_k \int_0^1 P_k(1 - 2x)x^\nu (1 - x)^{n-\nu} dx$$

$$= \sum_{k=0}^{n} c_k \sum_{j=0}^{k} \frac{(-k)_j (k + 1)_j}{(1)_j j!} \int_0^1 x^{\nu+j} (1 - x)^{n-\nu} dx$$

(2.1) $$= \frac{\Gamma(\nu + 1)\Gamma(n - \nu + 1)}{\Gamma(n + 2)} \sum_{k=0}^{n} c_k \cdot {}_3F_2\left(\begin{matrix} -k, & k+1, & \nu+1 \\ & 1, & n+2 \end{matrix}; 1\right) .$$

The shifted factorial $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(n + a)}{\Gamma(a)} .$$

Murphy's formula for Legendre polynomials was used

(2.2) $$P_n(x) = {}_2F_1\left(\begin{matrix} -n, & n+1 \\ & 1 \end{matrix}; \frac{1 - x}{2}\right)$$

and the generalized hypergeometric function is defined by

(2.3)
$$ {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}; t\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{t^n}{n!} . $$

Using (2.1) in S_n gives

(2.4)
$$ S_n = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{\ell=0}^{n} c_k c_\ell \sum_{\nu=0}^{n} {}_3F_2\left(\begin{matrix} -k, & k+1, & \nu+1 \\ & 1, & n+2 \end{matrix}; 1\right) {}_3F_2\left(\begin{matrix} -\ell, & \ell+1, & \nu+1 \\ & 1, & n+2 \end{matrix}; 1\right) $$

If this quadratic form is to be diagonal, then the following orthogonality relation must hold:

$$ \sum_{x=0}^{n} {}_3F_2\left(\begin{matrix} -k, & k+1, & x+1 \\ & 1, & n+2 \end{matrix}; 1\right) {}_3F_2\left(\begin{matrix} -\ell, & \ell+1, & x+1 \\ & 1, & n+2 \end{matrix}; 1\right) = 0, \quad (0 \leq k \neq \ell \leq n) . $$

Now

$$ R_k(x) = {}_3F_2\left(\begin{matrix} -k, & k+1, & x+1 \\ & 1, & n+2 \end{matrix}; 1\right) $$

is a polynomial of degree n in x and it is relatively well-known that Tchebychef found a set of polynomials which are orthogonal on $x = 0,1,\ldots,n$ with respect to the uniform distribution (see [5], §2.8). This is what we want, but at first glance, it seems we do not have it, since Tchebychef's polynomials are usually given as

$$ Q_k(x,n) = {}_3F_2\left(\begin{matrix} -k, & k+1, & -x \\ & 1, & -n \end{matrix}; 1\right), \qquad x,k = 0,1,\ldots,n $$

and this does not seem to be the same as $R_k(x)$. However, there is a transformation formula which reconciles this difference,

(2.5)
$$ {}_3F_2\left(\begin{matrix} -k, & a, & b \\ & c, & d \end{matrix}; 1\right) = \frac{(c-a)_k}{(c)_k} \cdot {}_3F_2\left(\begin{matrix} -k, & a, & d-b \\ d, & a+1-k-c \end{matrix}; 1\right) . $$

To obtain (2.5), write ([4], (4.1.3)) as an identity between hypergeometric series, that is

$$ {}_2F_1\left({-k,\ a \atop c};\ x\right) = \frac{(c-a)_k}{(c)_k}\ {}_2F_1\left({-k,\ a \atop a+1-k-c};\ 1-x\right) $$

and integrate with respect to a beta distribution. Take $a = k + 1$, $b = x + 1$, $c = n + 2$ and $d = 1$ in (2.5) to get

$$ R_k(x) = \frac{(n+1-k)_k}{(n+2)_k} \cdot {}_3F_2\left({-k,\ k+1,\ -x \atop 1,\ -n};\ 1\right) . \tag{2.6} $$

Using (2.6) above gives

$$ S_n = \frac{1}{n+1} \sum_{k=0}^{n} c_k^2 \frac{(n+1-k)_k^2}{(n+2)_k^2} \cdot \sum_{x=0}^{n} [Q_k(x)]^2 . $$

The orthogonality relation for $Q_k(x)$ is

$$ \sum_{x=0}^{n} Q_k(x,n)Q_\ell(x,n) = \delta_{k\ell} \frac{(n+1)(n+2)_k}{(n+1-k)_k \cdot (2k+1)} $$

so

$$ S_n = \sum_{k=0}^{n} \frac{c_k^2}{2k+1} \cdot \frac{(n+1-k)_k}{(n+2)_k} . $$

Since $(n+1-k)_k/(n+2)_k \leq 1$, $S_n \leq \sum_{k=0}^{n} c_k^2/(2k+1)$, which proves one of the required inequalities.

Conversely, if $S_n = O(1)$, then

$$\sum_{k=0}^{n} \frac{c_k^2}{2k+1} = \sum_{k=0}^{n} \frac{c_k^2}{(2k+1)} \cdot \frac{(n^2+1-k)_k}{(n^2+2)_k} \cdot \frac{(n^2+2)_k}{(n^2+1-k)_k}$$

$$\leq \frac{(n^2+2)_n}{(n^2+1-n)_n} \cdot \sum_{k=0}^{n} \frac{c_k^2}{2k+1} \cdot \frac{(n^2+1-k)_k}{(n^2+2)_k}$$

$$\leq \frac{\left(1 + \frac{n}{n^2+1}\right)^n}{\left(1 - \frac{n}{n^2+1}\right)^n} S_{n^2} = O(1) .$$

3. A Weighted Hausdorff Moment Problem

Extensions of Legendre polynomials and the discrete Tchebychef polynomials exist, so it is natural to see if they can be used to obtain an extension of Hausdorff's theorem. To this end, set

$$(3.1) \qquad \mu_\nu = \int_0^1 f(x) x^{\nu+\alpha}(1-x)^\beta dx, \quad \nu = 0,1,2,\ldots \qquad (\alpha;\beta > -1) .$$

Polynomials orthogonal with respect to $x^\alpha(1-x)^\beta$ on $[0,1]$ are known. They are called Jacobi polynomials and are given by

$$(3.2) \qquad P_n^{(\alpha,\beta)}(1-2x) = \frac{(\alpha+1)_n}{n!} \, {}_2F_1\left(\begin{matrix}-n, & n+\alpha+\beta+1 \\ & \alpha+1\end{matrix}; x\right) .$$

Set

$$f(x) \sim \sum_{0}^{\infty} c_\nu P_\nu^{(\alpha,\beta)}(1-2x)$$

where c_ν is determined by

$$(3.3)\qquad c_\nu = \frac{1}{h_\nu^{(\alpha,\beta)}} \int_0^1 f(x) P_\nu^{(\alpha,\beta)}(1-2x)x^\alpha(1-x)^\beta dx$$

and

$$(3.4)\qquad \int_0^1 P_n^{(\alpha,\beta)}(1-2x)P_k^{(\alpha,\beta)}(1-2x)x^\alpha(1-x)^\beta dx = \delta_{kn} h_n^{(\alpha,\beta)}$$

with

$$h_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\cdot n!} .$$

As in the last section

$$\Delta^{n-\nu}\mu_\nu = \int_0^1 f(x)x^{\nu+\alpha}(1-x)^{n-\nu+\beta}dx$$

$$= \sum_{k=0}^{n} c_k \int_0^1 P_k^{(\alpha,\beta)}(1-2x)x^{\nu+\alpha}(1-x)^{n-\nu+\beta}dx$$

$$= \frac{\Gamma(\nu+\alpha+1)\Gamma(n-\nu+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{k=0}^{n} \frac{(\alpha+1)_k}{k!} c_k \cdot {}_3F_2\left(\begin{matrix}-k, & k+\alpha+\beta+1, & \nu+\alpha+1\\ \alpha+1, & n+\alpha+\beta+2\end{matrix}; 1\right) .$$

Using (2.5) gives

$$(3.5)\qquad \frac{\Gamma(n+\alpha+\beta+2)\Delta^{n-\nu}\mu_\nu}{\Gamma(\nu+\alpha+1)\Gamma(n-\nu+\beta+1)} = \sum_{k=0}^{n} \frac{(\alpha+1)_k}{k!} c_k \frac{(n+1-k)_k}{(n+\alpha+\beta+2)_k} \cdot {}_3F_2\left(\begin{matrix}-k, & k+\alpha+\beta+1\\ \alpha+1, & -n\end{matrix}; 1\right) .$$

The general discrete Tchebychef polynomials [6] (or to use their common name, the Hahn polynomials) are given by

$$Q_k(x;\alpha,\beta,n) = {}_3F_2\left(\begin{matrix}-k, & k+\alpha+\beta+1, & -x\\ \alpha+1, & -n\end{matrix}; 1\right), \quad (k,x = 0,1,\ldots,n) . \tag{3.6}$$

Their orthogonality relation is

$$\sum_{x=0}^{n} Q_k(x;\alpha,\beta,n)Q_j(x;\alpha,\beta,n)\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}$$

$$= \delta_{jk} \cdot \frac{(\alpha+\beta+2)_n \cdot k!(n+\alpha+\beta+2)_k(\beta+1)_k(\alpha+\beta+1)}{n!(n+1-k)_k(\alpha+1)_k(\alpha+\beta+1)_k(2k+\alpha+\beta+1)}, \tag{3.7}$$

$$0 \leq j,k \leq n .$$

Square (3.5), multiply by $\binom{\nu+\alpha}{\nu}\binom{n-\nu+\beta}{n-\nu}$ and sum. After simplification, the resulting identity is

$$\sum_{\nu=0}^{n} |\Delta^{n-\nu}\mu_\nu|^2\binom{n}{\nu}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\nu+\alpha+1)\Gamma(n-\nu+1)} \cdot \sum_{k=0}^{n} c_k^2 h_k^{(\alpha,\beta)} \cdot \frac{(n+1-k)_k}{(n+\alpha+\beta+2)_k} .$$

The Riesz-Fisher theorem for Jacobi series is

$$\int_0^1 (f(x))^2 x^\alpha (1-x)^\beta dx = \sum_{k=0}^{\infty} c_k^2 h_k^{(\alpha,\beta)}$$

so an argument similar to the one in §2 gives the following:

<u>Theorem 1</u>. <u>Define</u> μ_ν <u>by</u> (3.1). <u>Then for</u> $\alpha,\beta > -1$,

$$\int_0^1 |f(x)|^2 x^\alpha (1-x)^\beta dx < \infty$$

<u>if and only if</u>

$$\sum_{\nu=0}^{n} |\Delta^{n-\nu}\mu_\nu|^2\binom{n}{\nu}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\nu+\alpha+1)\Gamma(n-\nu+\beta+1)} = O(1) .$$

This can be rephrased as

$$\int_0^1 |f(x)x^{\alpha}(1 - x)^{\beta}|^2 \frac{dx}{x^{\alpha}(1 - x)^{\beta}} < \infty$$

if and only if

$$(n + 1) \sum_{\nu=0}^{n} |\Delta^{n-\nu}\mu_{\nu}|^2 \binom{n}{\nu}^2 \frac{1}{\left(\frac{\nu + \frac{1}{2}}{n + 1}\right)^{\alpha} \left(1 - \frac{\nu + \frac{1}{2}}{n + 1}\right)^{\beta}} = O(1) ,$$

when $\alpha,\beta > -1$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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