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APPLICATION OF INTERVAL INTEGRATION TO THE SOLUTION OF INTEGRAL--ETC(U)
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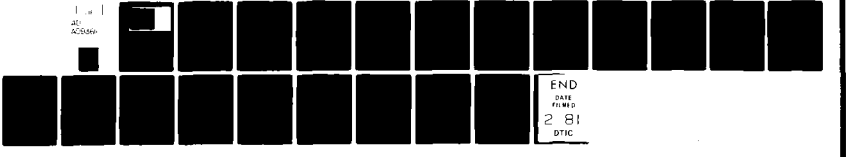

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APPLICATION OF INTERVAL INTEGRATION TO THE SOLUTION OF INTEGRAL EQUATIONS

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LEVEL II

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ABSTRACT

A method of numerical solution is presented for a class of integral equations which includes linear and nonlinear equations encountered in applications. A brief sketch is given of methods from classical analysis (inversion of power series) and functional analysis (functional and monotone iteration), together with some of their shortcomings (difficulty of implementation, error estimation, and special conditions on operators and initial approximations). Interval functions, which may be considered to be sets consisting of all functions bounded above and below by given endpoint functions, are defined, and the recently developed theory of interval integration is used to construct interval extensions of real integral operators. These interval operators are used to define an interval iteration process which converges if the initial interval contains a solution of the integral equation. Furthermore, the endpoint functions of the iterated interval functions provide upper and lower bounds for the solution at each stage of the iteration, and the interval iteration operator can be constructed so that the results of each transformation can be represented exactly. A numerical example is given of an interval iteration which gives a numerical solution of a nonlinear integral equation outside the limits of convergence of classical and functional methods. Some problems connected with the use of interval iteration are also discussed.

AMS (MOS) Subject Classifications: 65R20, 65G10, 45L10, 45L05, 45G10

Key Words: Numerical solution of integral equations, Nonlinear integral equations, Interval analysis, Interval integration, Iteration, Upper and Lower bounds, Error estimation

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SIGNIFICANCE AND EXPLANATION

Integral equations, particularly nonlinear integral equations, appear as mathematical models in many applications. In order to extract information from such models, these integral equations have to be solved, at least approximately. A method for doing this is presented in this paper and compared to some previous methods. For example, the methods of classical analysis for the solution of integral equations require more or less complicated manipulations with infinite series. There are problems of convergence and error estimation, as the integrals involved usually have to be calculated numerically. Functional analysis views the integral equation as a fixed point problem $y = \Phi y$ for an integral operator Φ, and suggests solution by functional iteration $y_{n+1} = \Phi y_n$, the convergence of which depends on the choice of the initial approximation y_0 and properties of the operator Φ. Although implementation of functional iteration is simpler in concept than the classical method, the transformation Φ usually cannot be applied exactly, which leads to an error estimation problem for the approximate sequence actually computed. Monotone iteration avoids some of these difficulties by providing the possibility of obtaining computable upper and lower bounds for the fixed point y; however, implementation is possible only if the operator Φ has special monotonicity properties, and the initial lower and upper bounds $\underline{y}_0$, $\overline{y}_0$ satisfy special conditions.

Some of the shortcomings of classical and functional analysis can be avoided by the methods of interval analysis. An interval function Y can be considered to be the set of all real functions y such that $\underline{y}(x) \le y(x) \le \overline{y}(x)$ on the interval $a \le x \le b$. However, even though the endpoint functions $\underline{y}$, $\overline{y}$ are smooth, the set Y may contain real functions y which are nasty little devils with no continuity, differentiability, or integrability properties. This creates a difficulty in the definition of integral transforms of interval functions. This difficulty has been resolved by the recent theory of interval integration (see MRC TSR #2087), which assigns an interval value to the integral of each real function. This permits construction of an interval extension T of the integral transform Φ with endpoint functions which can be integrated exactly. The interval iteration process $Y_{n+1} = Y_n \cap TY_n$ defined using T will converge if Y_0 contains a solution of the integral equation; furthermore, each step of the iteration provides upper and lower bounds for the solution. An example is given where this method works when classical and functional analytic techniques fail. Problems with interval iteration, namely, excess width (inaccuracy), stalling ($Y_1 = Y_0$), and verification of existence of solutions are discussed frankly, and some remedies are suggested.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

APPLICATION OF INTERVAL INTEGRATION TO THE SOLUTION OF INTEGRAL EQUATIONS

L. B. Rall

Dedicated to Professor Lothar Collatz on his 70th Birthday

1. Integral equations. Integral equations of the form

$$y(x) = \int_a^b \phi(x,t,y(x),y(t))dt, \qquad a \le x \le b, \tag{1.1}$$

are mathematical models of widespread utility in applications. Equation (1.1) is an integral equation of second kind, characterized as a biargument equation by Collatz [9]. This formulation is quite general; it includes the nonlinear integral equations of Hammerstein type for

$$\phi(x,t,y(x),y(t)) = K(x,t)g(t,y(t)), \tag{1.2}$$

and Uhrysohn type if

$$\phi(x,t,y(x),y(t)) = f(x,t,y(t)). \tag{1.3}$$

Furthermore, (1.1) is a Volterra equation in the case that

$$\phi(x,t,y(x),y(t)) \equiv 0, \qquad t > x. \tag{1.4}$$

For simplicity of discussion, it will be assumed that the unknown function y is sought as a real-valued function of the single real variable x. However, many of the considerations introduced below extend to more general cases.

In actual practice, numerical rather than exact solutions are obtained for integral equations (1.1). Such a numerical solution may take the form of an approximate solution y_n and an associated error bound ε_n, or as upper and lower bounds for the solution y, that is, functions $\bar{y}$, $\underline{y}$ such that

$$\underline{y}(x) \le y(x) \le \bar{y}(x), \qquad a \le x \le b. \tag{1.5}$$

Given upper and lower bounds (1.5), it is not difficult to derive an approximate solution and an associated error bound. This approach will be followed in this paper, using the methods of interval analysis [15], [16] and the recently developed theory of interval integration [4]. First, some numerical

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methods from classical and functional analysis will be sketched briefly for comparison. The discussion will be limited to direct, as opposed to variational, methods.

2. Classical analysis. Classical methods apply to equations (1.1) in which $\phi(x,t,y(x),y(t))$ is a polynomial or has a convergent power series expansion in $y(x)$ and $y(t)$. Such equations may be written

$$(2.1) \qquad y(x) = f(x) + \lambda\int_a^b g(x,t,y(x),y(t))dt, \quad \text{where } g(x,t,0,0) \equiv 0.$$

A simple example is the linear equation of Fredholm [11],

$$(2.2) \qquad y(x) = f(x) + \lambda\int_a^b K(x,t)y(t)dt,$$

for which classical analysis provides the solution

$$(2.3) \qquad y(x) = f(x) + \lambda\int_a^b \Gamma(x,t;\lambda)f(t)dt,$$

where $\Gamma(x,t;\lambda) = N(x,t;\lambda)/\Delta(\lambda)$ is the ratio of convergent power series in λ. The solution (2.3) of (2.2) exists if $\Delta(\lambda)$ (the Fredholm determinant) does not vanish, and is unique; nonexistence or multiplicity in the case $\Delta(\lambda) = 0$ are also understood [11].

An example of a nonlinear polynomial integral equation is the quadratic equation of Chandrasekhar [5] from the theory of radiative transfer:

$$(2.4) \qquad y(x) = 1 + \int_0^1 \frac{x}{x + t}\,\Psi(t)y(x)y(t)dt,$$

which has been investigated numerically extensively for the case $\Psi(t) = \lambda/2$, $0 \le \lambda \le 1$ [17], [18], [19] , by a number of techniques. Other examples of polynomial integral equations are the algebraic integral equations defined by Schmeidler [22], [23].

Nonlinear boundary value problems also provide a source of classical integral equations [12]. For example, the equation of the forced pendulum [10] is

$$(2.5) \qquad y(x) = f(x) + \lambda\int_0^1 K(x,t)\sin(y(t))dt,$$

where

$$(2.6) \qquad K(x,t) = \begin{cases} x(1 - t), & x \le t, \\ t(1 - x), & x \ge t, \end{cases}$$

and

$$(2.7) \qquad f(x) = -\int_0^1 K(x,t)F(t)dt, \quad F(-x) = -F(x),\ F(x + 2) = F(x),$$

F(x) being the driving function applied to the pendulum. Other examples of classical nonlinear integral equations can be found in the works of Hammerstein [12] and Lichtenstein [13].

The basic solution technique, developed by E. Schmidt [24], consists of a combination of the solution (2.3) of Fredholm with the classical method of inversion of power series. The solution of (2.1) is obtained as

$$y(x) = f(x) + \lambda \int_a^b \Gamma(x,t,f(x),f(t);\lambda)\,dt, \tag{2.8}$$

where $\Gamma(x,t,f(x),f(t);\lambda)$ is an infinite power series in λ with coefficients which are integral power forms (polynomials) in f(x) and f(t). The following comments may be made about the classical method:

(i) Convergence of the inverse power series may be difficult to establish, and may be limited. For example, Rall [17] shows that the inverse power series solution of the Chandrasekhar equation

$$y(x) = 1 + \frac{\lambda}{2} \int_0^1 \frac{x}{x+t}\, y(x)y(t)\,dt \tag{2.9}$$

converges for $0 \le \lambda < 1/(2 \ln 2) \cong 0.72$, omitting the values $1/(2 \ln 2) \le \lambda \le 1$.

(ii) Implementation requires somewhat complicated algebraic manipulations to obtain the terms of the inverse power series; furthermore, approximations to the solution resulting from numerical integration and truncation of the inverse series lead to a problem in

(iii) Error estimation which may be difficult, or at least tedious.

Some of these difficulties may be overcome by the use of the methods of functional analysis.

3. Functional analysis. Another approach to the solution of nonlinear integral equations is through the methods of functional analysis, which is analysis in normed linear spaces (see, for example, the books by Collatz [7], [8] for basic definitions and examples). The fundamental idea is to formulate (or reformulate) the integral equation (1.1) as a fixed point problem

$$y = \Phi y \tag{3.1}$$

in a complete normed linear (Banach) space S[a,b] of functions having some distinguishing property (continuity, square-integrability, etc.). For example, in a direct formulation of (1.1) as a fixed point problem (3.1), the operator Φ would be defined by

$$(\Phi z)(x) = \int_a^b \phi(x,t,z(x),z(t))\,dt, \quad z \in S[a,b]. \tag{3.2}$$

Appropriate conditions are imposed on the integrand so that the transformed

function Φz also belongs to the space S[a,b]. There is a vast literature on the solution of (3.1) by functional analytic methods. The techniques which will be mentioned here are functional (or simple) iteration, and monotone iteration.

4. Functional iteration. The form of the fixed point problem (3.1) suggests solution by iteration, that is, the formation of the sequence $\{y_n\}$ defined by

$$y_{n+1} = \Phi y_n, \qquad n = 0,1,2,\ldots . \tag{4.1}$$

This method will be called functional iteration. If Φ is continuous, and the sequence $\{y_n\}$ converges to a point y, then y will satisfy (3.1). A constructive fixed point theorem will guarantee the convergence of the sequence (4.1) for some y_0 to a fixed point y of Φ in some region Y_0 containing y_0. Such a theorem will be useful numerically if it also provides an error bound

$$\varepsilon_n \geq \|y - y_n\|, \qquad n = 0,1,2,\ldots, \tag{4.2}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$, in terms of the norm $\|\cdot\|$ for S[a,b]. Examples of constructive fixed point theorems which provide error bounds are the contraction mapping theorem and the theorem of Kantorovich on the convergence of Newton's method [19]. The following comments may be made about functional iteration:

(i) Convergence of the sequence $\{y_n\}$ depends strongly, in most cases, on the choice of the initial point y_0, for which there is no general recipe. For example, with the initial point $y_0 \equiv 0$ in the space C[0,1] of continuous functions, the contraction mapping theorem guarantees convergence to the solution of the Chandrasekhar equation (2.9) only for $0 \leq \lambda < 1/(4 \ln 2) \cong 0.36$, while the Kantorovich theorem gives $0 \leq \lambda \leq 1/(2 \ln 2) \cong 0.72$ as the corresponding range, which is essentially the same result given by classical analysis [19]. In some cases, this situation can be remedied by a better choice of initial points y_0, but there are also problems with fixed points which repel the sequence $\{y_n\}$ even for some y_0 with $\|y - y_0\| > 0$ arbitrarily small.

(ii) Implementation of functional iteration is conceptually much simpler than inversion of power series; however, it is usually not possible to apply the transformation Φ exactly. This means that some approximations, such as numerical integration, must be used, and instead of $\{y_n\}$, an approximate sequence $\{z_n\}$ is actually computed.

(iii) Error estimation thus will require bounds for $\|z_n - y_n\|$ in addition to the bounds (4.2) given by the theory. Although it is a challenge to find such error bounds, the necessity to do so is hardly an advantage of the use of functional iteration.

In some problems, more favorable results can be obtained by the use of partial ordering of the space S[a,b].

5. Monotone iteration. A natural way to introduce a partial ordering into a space S[a,b] of functions is by means of the definition that $y \le z$ if and only if

$$y(x) \le z(x), \qquad a \le x \le b. \tag{5.1}$$

A monotone iteration T is a process which generates a sequence of lower bounds $\{\ell_n\}$ and a sequence of upper bounds $\{u_n\}$ for a fixed point y of Φ:

$$\ell_0 \le \ell_1 \le \ldots \le \ell_n \le \ldots \le y \le \ldots \le u_n \le \ldots \le u_1 \le u_0. \tag{5.2}$$

The iteration (5.2) is symbolized by

$$\langle \ell_{n+1}, u_{n+1} \rangle = T\langle \ell_n, u_n \rangle, \qquad n = 0,1,2,\ldots, \tag{5.3}$$

where $\langle\ ,\ \rangle$ denotes an interval in the partially ordered space S[a,b], that is,

$$\langle v, w \rangle = \{z \mid v \le z \le w, \quad v,w,z \in S[a,b]\}. \tag{5.4}$$

The study of monotone iteration owes much to the work of Collatz [6], [7], [8] and Schröder [25], [26].

The use of monotone iteration to solve the fixed point problem (3.1) depends on a monotone decomposition of the operator Φ. An operator Φ is said to be isotone if $v \le w \Rightarrow \Phi v \le \Phi w$, and antitone if $v \le w \Rightarrow \Phi v \ge \Phi w$. Φ has a monotone decomposition if one can write

$$\Phi = \Phi_1 + \Phi_2, \tag{5.5}$$

with Φ_1 isotone and Φ_2 antitone. The iteration operator T defined by

$$T\langle \ell, u \rangle = \langle \Phi_1 \ell + \Phi_2 u, \Phi_1 u + \Phi_2 \ell \rangle \tag{5.6}$$

will generate a monotone iteration if $T\langle \ell_0, u_0 \rangle \subset \langle \ell_0, u_0 \rangle$ ([8], §21.1). Furthermore, the Schauder fixed point theorem ([8], p. 358) states that a fixed point y of Φ satisfying (5.2) will exist if $\langle \ell_N, u_N \rangle$ is compact for some N.

Monotone iteration lends itself to numerical approximation somewhat more readily than functional iteration, for if a numerical iteration operator $\overline{T}$ can be found such that $\overline{T}\langle \ell, u \rangle \supset T\langle \ell, u \rangle$, then the satisfaction of the condition $T\langle \ell_0, u_0 \rangle \subset \langle \ell_0, u_0 \rangle$ guarantees that the sequences obtained from

$$\langle \ell_n, u_n \rangle = \overline{T}^n \langle \ell_0, u_0 \rangle, \qquad n = 0,1,2,\ldots, \tag{5.7}$$

will satisfy (5.2). Thus, the need for a separate error estimation can be avoided. For example, Rall [18] gives an example of monotone iteration which bounds the solution of the Chandrasekhar equation (2.9) above and below by step functions for $\lambda = 0.5$. The success of monotone iteration, however, depends upon:

(i) Monotone decomposition of the operator Φ, which may not be easy (or possible), and

(ii) Convergence of the iteration depends on the choice of the initial interval, as the condition

$$T(\ell_0, u_0) \subset \langle \ell_0, u_0 \rangle \tag{5.8}$$

must be satisfied. For example, the operator Φ defined by (2.9) is isotone for $y \geq 0$, so one may take $T = \Phi$ ($\Phi_2 = 0$). Condition (5.8) is satisfied for intervals of the form $\langle 1, M \rangle$ only if $0 \leq \lambda \leq 1/(2 \ln 2)$ and

$$\frac{1}{\lambda \ln 2}\left(1 - \sqrt{1 - 2\lambda \ln 2}\right) \leq M \leq \frac{1}{\lambda \ln 2}\left(1 + \sqrt{1 - 2\lambda \ln 2}\right) \tag{5.9}$$

[18], so that limitations again apply in this simple problem.

(iii) Compactness of the interval $\langle \ell_N, u_N \rangle$ must be verified for some N in order to establish existence of the fixed point y. Collatz ([8], §5.4) gives some useful criteria for compactness of sets of functions defined by integral operators of the types occurring in practice. However, it should be pointed out that if the fixed point y is known to lie in the initial interval on the basis of some independent argument, then the monotone iteration will provide upper and lower bounds without the necessity of verifying compactness.

Interval iteration, which will be introduced below, is similar to monotone iteration in that it provides upper and lower bounds for fixed points which belong to the initial interval. However, the convergence of interval iteration will be seen not to require monotone decomposition of the operator nor satisfaction of an inclusion condition of the type (5.8).

6. Interval analysis. The basic units of interval analysis [15], [16] are the closed, nonempty real intervals

$$I = [a,b] = \{x \mid a \leq x \leq b\}. \tag{6.1}$$

The set of real numbers R is identified as a subset of the set IR of intervals (6.1), with each real number r corresponding to the degenerate interval $[r,r]$ with equal lower and upper endpoints. One writes

$$r = [r, r] \tag{6.2}$$

for brevity.

In the same way that real analysis is concerned with transformations f of one real number into another, symbolized by $y = f(x)$, interval analysis is the branch of mathematics which deals with transformations F of one interval into another, for which the notation $J = F(I)$ is used. A crucial property of interval transforms is inclusion monotonicity, which will now be defined.

Definition 6.1. An interval transformation F is inclusion monotone if and only if

$$I \subset J \Rightarrow F(I) \subset F(J). \tag{6.3}$$

Real and interval transformations are related through the idea of an

interval extension of a real transformation.

Definition 6.2. An interval transformation F is called an extension of a real transformation f if

$$f(I) = \{y \mid y = f(x),\ x \in I\} \subset F(I) \tag{6.4}$$

for each interval I in the domain of definition D of f. In addition, if F is inclusion monotone, then it is called an interval extension of f.

What is called interval arithmetic is an interval extension of real arithmetic [15], [16]. The rules for the four basic operations are:

$$\begin{aligned} [a,b] + [c,d] &= [a+c\ ,\ b+d]; \\ [a,b] - [c,d] &= [a-d\ ,\ b-c]; \\ [a,b] \times [c,d] &= [\min\{ac,ad,bc,bd\}\ ,\ \max\{ac,ad,bc,bd\}]; \\ [a,b] \div [c,d] &= [a,b] \times [\tfrac{1}{d}\ ,\ \tfrac{1}{c}],\quad cd > 0. \end{aligned} \tag{6.5}$$

In actual computation, only a finite set of numbers are available, the so-called machine numbers. An interval I is said to be exactly representable, or representable for short, if its endpoints are machine numbers. The process of finding the smallest representable interval I which contains a given interval J is called directed rounding (lower endpoints are rounded down, and upper endpoints are rounded up). Directed rounding may be used to obtain interval extensions F of real transformations which are exactly representable, that is, F(I) is a representable interval for each representable interval I. The use of interval extensions of this type allows automatic estimation of the error of calculation of the corresponding real transformation [15], [16], as if the data x are known to lie in a representable interval I, then the results y = f(x) will be contained in the representable interval J = F(I). It is essential that the interval extension F of f be inclusion monotone in order to be able to perform directed rounding to obtain a representable interval extension. It will be assumed that the interval extensions cited in the following are representable.

7. Interval functions. The concept of an interval function, introduced in [4], is fundamental to the theory of interval integration.

Definition 7.1. An interval function Y assigns to each x in its interval of definition I = [a,b] the interval

$$Y(x) = [\underline{y}(x)\ ,\ \overline{y}(x)]. \tag{7.1}$$

One writes $Y = [\underline{y}\ ,\ \overline{y}]$, where the real functions $\underline{y}$, $\overline{y}$ are called the endpoint functions (or simply endpoints) of Y.

An interval function Y may be identified with its graph, which is the set of points

(7.2) $$Y = [a,b] \times Y(x) = \{(x,y) \mid \underline{y}(x) \le y \le \overline{y}(x),\ a \le x \le b\}$$

in the x,y-plane. For the present purposes, however, it will be more useful to consider an interval function Y as a set of functions, namely all real functions y such that $\underline{y} \le y \le \overline{y}$, that is

(7.3) $$Y = \{y \mid \underline{y}(x) \le y(x) \le \overline{y}(x),\ a \le x \le b\}.$$

In this light, a real function y is simply the degenerate interval function

(7.4) $$y = [y\ ,\ y]$$

with equal upper and lower endpoint functions.

An interval function Y, defined as the set (7.3), differs from an interval $\langle \underline{y}\ ,\ \overline{y} \rangle$ in a partially ordered space in the following way: In S[a,b], an interval consists only of elements of the space (continuous functions or whatnot), including the endpoint functions, while the endpoint functions $\underline{y} \le \overline{y}$ of an interval function Y are arbitrary, and the only property of a real function $y \in Y$ which is required is that (7.3) is satisfied. Thus, even if the endpoint functions $\underline{y}$, $\overline{y}$ of Y are smooth, Y may contain functions which have no continuity, differentiability, or integrability properties in the ordinary sense. The theory of interval integration developed in [4] resolves this difficulty.

8. Interval integration. As constructed in [4], the interval integral of an interval function Y over an interval I = [a,b] on which it is defined is the interval

(8.1) $$\int_a^b Y(x)\,dx = [\underline{\int}_I \underline{y}(x)\,dx\ ,\ \overline{\int}_I \overline{y}(x)\,dx],$$

where $\underline{\int}_I$ denotes the lower Darboux integral of a function over I (the supremum of the integrals of all step functions less than or equal to the function), and $\overline{\int}_I$ denotes the upper Darboux integral of a function over I (the infimum of the integrals of all step functions greater than or equal to the function) [14]. As these Darboux integrals always exist in the extended real number system [14], the theory of integration of interval (and hence real) functions is extremely simple in this sense [4].

Theorem 8.1. An interval function Y is integrable over each interval I on which it is defined.

Although simple in character, interval integrals have many of the properties of real integrals. Some of the most important are [4]:

Theorem 8.2. Interval integration is inclusion monotone with respect to integrands; that is,

$$(8.2) \qquad Y \subset Z \Rightarrow \int_a^b Y(x)dx \subset \int_a^b Z(x)dx.$$

In (8.2), inclusion of interval functions simply means set inclusion with respect to their graphs (7.2) or as sets of functions (7.3).

Theorem 8.3. Interval integration is an extension of real integration in the sense that if the real function y is Lebesgue (L) integrable, then

$$(8.3) \qquad (L)\int_a^b y(x)dx \in \int_a^b y(x)dx.$$

Interval integration is actually a Riemann-type integration. If the real function y is Riemann (R) integrable, then

$$(8.4) \qquad (R)\int_a^b y(x)dx = \int_a^b y(x)dx,$$

which follows directly from (7.4) and (8.1) as, by definition, a function is Riemann integrable if and only if its upper and lower Darboux integrals are equal [14].

From Definition 6.2 and Theorems 8.1 and 8.2, there follows

Theorem 8.4. Interval integration is an interval extension of real (Riemann and Lebesgue) integration.

Interval integrals also have a mean (interval) value property, and indefinite interval integrals have continuity and differentiability properties similar to real integrals [4]. For the purposes of this paper, only the following relationships between interval and Riemann integrals will be needed.

Theorem 8.5. If $\underline{y}$, $\overline{y}$ are Riemann (R) integrable, then

$$(8.5) \qquad \int_a^b Y(x)dx = [(R)\int_a^b \underline{y}(x)dx\ ,\ (R)\int_a^b \overline{y}(x)dx].$$

Equation (8.5) follows from (8.1) and the definition of the Riemann integral [14]. The importance of this result is that it allows the construction of representable interval extensions by choice of upper and lower endpoint functions which have known Riemann integrals (piecewise polynomials, etc.).

The following result [4] will also be useful, as it gives a necessary and sufficient condition for an interval integral to be degenerate (that is, a real number).

Theorem 8.6.

$$(8.6) \qquad \int_a^b Y(x)dx = [r\ ,\ r]$$

if and only if each real function $y \in Y$ is Riemann integrable, and

$$r = (R)\int_a^b y(x)\,dx, \quad y \in Y. \tag{8.7}$$

Before going on to an application of interval integration, it should be pointed out that intervals, and hence interval functions, do not form linear spaces. For example, from (6.5),

$$[0\ ,1] - [0\ ,1] = [-1\ ,1], \tag{8.8}$$

there being no inverse element for addition of nondegenerate intervals. Hence, interval analysis is a mathematical topic distinct from functional analysis.

9. Interval iteration. Using interval integrals, an interval extension T of the integral operator Φ defined by (3.2) can be constructed in the following way: Form a (representable) interval extension F of the function ϕ, and define T by

$$(TY)(x) = \int_a^b F(x,t,Y(x),Y(t))\,dt, \tag{9.1}$$

which will be a (representable) interval function for (representable) interval functions $Y(x)$.

Definition 9.1. The sequence $\{Y_n\}$ defined by

$$Y_{n+1} = Y_n \cap TY_n, \quad n = 0,1,2,\ldots, \tag{9.2}$$

is said to be generated by interval iteration, starting from the initial interval Y_0.

Convergence and divergence of the sequence $\{Y_n\}$ will now be defined.

Definition 9.2. The sequence defined by (9.2) is said to be divergent if

$$Y_N = \emptyset \text{ (empty)} \tag{9.3}$$

for some N.

Otherwise, the sequence is convergent, which means that

$$Y = \lim_{n\to\infty} Y_n = \bigcap_{n=0}^{\infty} Y_n \tag{9.4}$$

is nonempty, as the intervals Y_n are closed, and

$$Y_0 \supset Y_1 \supset \ldots \supset Y_n \supset \ldots \tag{9.5}$$

by (9.2).

Theorem 9.1. If the initial interval contains a fixed point y of Φ, then the interval iteration (9.2) will converge.

Proof: By construction, if $y \in Y_0$ is a fixed point of Φ, then $y \in TY_0$; thus $y \in Y_1$ and $y \in Y_2, Y_3, \ldots, Y_n, \ldots$ by mathematical induction. Thus $\{Y_n\}$ is a sequence of nonempty sets, which is convergent by definition. QED.

Interval iteration provides upper and lower bounds for fixed points y of Φ in Y_0; for $Y_n = [\underline{y}_n, \overline{y}_n]$, one has

$$\underline{y}_n(x) \le y(x) \le \overline{y}_n(x), \quad a \le x \le b. \tag{9.6}$$

Of course, (9.3) cannot happen if Y_0 contains a fixed point y of Φ. This gives

Theorem 9.2. If the interval iteration diverges, then there is no fixed point y of Φ in the initial interval Y_0.

This is simply the contrapositive of Theorem 9.1, and provides a conclusive demonstration of nonexistence in case (9.3) holds.

Interval iteration thus stands in a converse relationship to functional iteration. If the sequence $\{y_n\}$ defined by (4.1) converges to a point $y \in Y_0$, then this establishes existence of a fixed point y of Φ in Y_0, for continuous Φ. On the other hand, if Φ has no fixed point in Y_0, then the sequence $\{y_n\}$ cannot converge to a point of Y_0, it must diverge from Y_0. In this sense, the relationship between functional and interval iteration is shown in Table 9.1.

Functional Iteration $y_{n+1} = \Phi y_n$.	Interval Iteration $Y_{n+1} = Y_n \cap TY_n$.
Convergence ⇒ Existence	Existence ⇒ Convergence
Nonexistence ⇒ Divergence	Divergence ⇒ Nonexistence

Table 9.1. Functional and Interval Iteration

In actual computing practice, there are only finite sets of lower and upper bounds available. Thus, all convergent interval iterations will be observed to converge in a finite number of steps, that is

$$Y_{N+1} = Y_N \neq \emptyset \tag{9.7}$$

for some N. This is called <u>finite convergence</u>. In theory, of course, the limit (9.4) may require an infinite number of iterations, but the iteration (finite or infinite) can be stopped at any point a satisfactory answer is obtained from the standpoint of accuracy.

10. <u>Convergence in width</u>. This is related to the idea of an <u>interval contraction</u> [1].

Definition 10.1. The <u>width</u> of an interval function Y on an interval I = [a,b] is

$$w(Y) = \sup_{x\in I} w(Y(x)) = \sup_{x\in I}\{\overline{y}(x) - \underline{y}(x)\}. \tag{10.1}$$

It follows from this definition that $w(Y) = 0$ if and only if $Y = y$, a real (degenerate interval) function.

Definition 10.2. The interval iteration sequence $\{Y_n\}$ defined by (9.2) is said to converge in width if it is convergent and

$$\lim_{n\to\infty} w(TY_n) = 0. \tag{10.2}$$

Convergence in width obviously implies convergence to a real function, furthermore, the following result holds.

Theorem 10.1. If the sequence $\{Y_n\}$ defined by (9.2) converges in width, then

$$y = \lim_{n\to\infty} Y_n \tag{10.3}$$

satisfies the integral equation (1.1) in the sense of Riemann integration, that is

$$y(x) = (R)\int_a^b \phi(x,t,y(x),y(t))dt, \quad a \le x \le b. \tag{10.4}$$

Proof: The convergence (10.3) in width implies $y = Ty$, hence

$$y(x) = \int_a^b F(x,t,y(x),y(t))dt, \tag{10.5}$$

a degenerate interval integral for each x. As $\phi(x,t,y(x),y(t)) \in F(x,t,y(x),y(t))$, (10.4) follows from Theorem 8.6. QED.

11. A numerical example. As a simple example, a calculation was done of an interval iteration applied to the Chandrasekhar equation (2.9) for $\lambda = 1.0$, which is outside the range $0 \le \lambda < 2\ln 2$ in which the previously considered methods from classical and functional analysis could be applied. The initial interval is taken to be $Y_0 = [1\ ,\ 3]$, and the interval extension consists of bounding the integral above and below by step functions, making use of the fact that the integral transformation of the step function

$$\sigma(x) = M_i, \quad s_{i-1} < x < s_i, \quad i = 1,2,\ldots,n, \tag{11.1}$$

where $s_0 = 0$, $s_n = 1$, is

$$(\Phi\sigma)(x) = 1 + \frac{\lambda}{2} xM_i \cdot \sum_{j=1}^{n} \ln\{(x + s_j)/(x + s_{j-1})\}M_j, \tag{11.2}$$

$s_{i-1} < x < s_i$, $i = 1,2,\ldots,n$. The function (11.2) is monotone increasing in each subinterval, so upper and lower bounds were obtained by evaluating the transform of the upper step function at the right ends of the intervals, and the transform of the lower step function at the left endpoints. Directed rounding to two decimal places was employed, and the actual calculations were

done with an HP-33E pocket calculator with $s_i = i/8$, $i = 0(1)8$. The approximate solution y was taken to be the harmonic point

(11.3) $$y(x) = h[a,b] = 2ab/(a + b)$$

of the calculated interval $a = \underline{y}(x)$, $b = \overline{y}(x)$, which minimizes the maximum percentage error

(11.4) $$p = p[a,b] = 100 \cdot (b - a)/(a + b)$$

over the interval [21]. The results are summarized in Table 11.1.

x	$\underline{y}$	$\overline{y}$	y	Error
0.000	1.00	-	1.00	0.0%
0.125	1.22	1.45	1.33	8.7%
0.250	1.38	1.90	1.60	15.9%
0.375	1.52	2.41	1.86	22.7%
0.500	1.65	3.00	2.13	29.1%
0.625	1.76	3.00	2.22	26.1%
0.750	1.86	3.00	2.30	23.5%
0.875	1.95	3.00	2.36	21.3%
1.000	1.95	3.00	2.36	21.3%

Table 11.1. Interval Solution of (2.9) for $\lambda = 1.0$.

The results given are crude, but were obtained with very little effort, and can be used as initial intervals for a more refined calculation. Somewhat better results can be obtained with the choice $s_i = (i/8)^2$, $i = 0(1)8$, as used in [3], and the replacement of step functions by piecewise linear functions results in a large increase in accuracy at the expense of a little more computation, a main result of [2], [3]. The point of the above calculation is that it gives a qualitative idea of the behavior of y(x), and was obtained outside the range of straightforward application of classical and functional analysis, for example, the monotone iteration described in [18], which required an enormous amount of computer time to attain four decimal place accuracy for the case $\lambda = 0.5$.

12. Problems with interval iteration. Some features of interval iteration require further investigation, and should be considered in connection with its computational application.

(i) Existence of fixed points must be verified independently; convergence of the interval iteration is a necessary, not a sufficient condition.

This may not be much of a barrier in actual practice, as even nonconstructive fixed point theorems may be used in combination with interval iteration.

(ii) Excess width may be observed in the limit interval Y of a convergent interval iteration. This may be defined as follows: Suppose that S is the set of fixed points of Φ in the interval Y_0. S may be enclosed in an interval $Y^* = [\underline{y}^*, \overline{y}^*]$ with endpoint functions defined by

$$\begin{aligned} \underline{y}^*(x) &= \inf\{y(x) \mid y \in S\}, \quad a \le x \le b, \\ \overline{y}^*(x) &= \sup\{y(x) \mid y \in S\}, \quad a \le x \le b. \end{aligned} \tag{12.1}$$

It follows from $S \subset Y_0$ that $Y^* \subset Y_n$, $n = 0,1,2,\ldots$, and hence $Y^* \subset Y$. Thus, Y^* is the "best" result which can be obtained by interval iteration starting from Y_0, and the quantity

$$E(Y,Y^*) = \max\{ w[\underline{y}, \underline{y}^*], w[\overline{y}^*, \overline{y}] \} \tag{12.2}$$

is called the excess width of Y over Y^*.

One way to reduce excess width is by making the interval extension T of Φ a sufficiently accurate approximation. Caprani and Madsen [1] have shown that if Φ is a contraction, and T approximates Φ to $o(w(Y))$, then the interval iteration (9.2) will converge in width, an hence have excess width zero. In [2], methods for construction of such accurate approximations are shown, using mean value forms, with numerical examples given in [3].

(iii) Stalling of an interval iteration occurs if $Y_0 \subset TY_0$, which means that $Y_1 = Y_0$. Here, no information has been lost, but none has been gained, either. Remedies for stalling include more accurate approximation of Φ by the interval operator T, and "suitable choice" of the initial interval Y_0. As in the case of the choice of the initial point y_0 for functional iteration, no general rules can be prescribed for the latter task. However, it might be expected that the choice of a set containing a fixed point might be easier on the basis of outside information about the problem than the selection of a single point close to the solution.

In the application of interval iteration, the key step appears to be the construction of a sufficiently accurate interval extension in the initial phase. Once the interval iteration is started, it automatically yields lower and upper bounds for the solution of the integral equation, which can be used to construct approximate solutions with error bounds, as in the simple example in §11. Further implications and applications of interval iteration will be reported in subsequent papers.

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A method of numerical solution is presented for a class of integral equations which includes linear and nonlinear equations encountered in applications. A brief sketch is given of methods from classical analysis (inversion of power series) and functional analysis (functional and monotone iteration), together with some of their shortcomings (difficulty of implementation, error estimation, and special conditions on operators and

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20. Abstract, continued

initial approximations). Interval functions, which may be considered to be sets consisting of all functions bounded above and below by given endpoint functions, are defined, and the recently developed theory of interval integration is used to construct interval extensions of real integral operators. These interval operators are used to define an interval iteration process which converges if the initial interval contains a solution of the integral equation. Furthermore, the endpoint functions of the iterated interval functions provide upper and lower bounds for the solution at each stage of the iteration, and the interval iteration operator can be constructed so that the results of each transformation can be represented exactly. A numerical example is given of an interval iteration which gives a numerical solution of a nonlinear integral equation outside the limits of convergence of classical and function methods. Some problems connected with the use of interval iteration are also discussed.