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BOUNDS FOR THE T-TAIL AREA FOR ARBITRARY DEGREES OF FREEDOM.(U)
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BOUNDS FOR THE t-TAIL AREA FOR ARBITRARY DEGREES OF FREEDOM

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ABSTRACT

The bounds of Birnbaum (1942) and Sampford (1953) for the upper tail area of the normal distribution are extended to the upper tail of the t-distribution for arbitrary real degrees of freedom. This generalizes the results of Soms (1977a) for integral degrees of freedom. Numerical and theoretical comparisons are made with the bounds of Peizer and Pratt (1968), Wallace (1959) and Soms (1977b).

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SIGNIFICANCE AND EXPLANATION

Real degrees of freedom arise in the modification of the two-sample t-test when the variances cannot be assumed to be equal. For small degrees of freedom both linear interpolation and standard computer routines may be unsatisfactory. The present paper provides simple techniques for satisfactory estimates of small tail probabilities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BOUNDS FOR THE t-TAIL AREA FOR ARBITRARY DEGREES OF FREEDOM

Andrew P. Soms

1. Introduction

The purpose of this paper is to extend to the tail area of the t-distribution bounds given by Birnbaum (1942) and Sampford (1953) for the tail area of the normal distribution based on Mills' ratio. This extension of Mills' ratio type bounds is completed in Soms (1977b) and (1979) and the basis for these extensions is the asymptotic expansion obtained in Soms (1976). An excellent summary of the normal results can be found in Chapter 33 of Johnson and Kotz (1970). The organization of this paper is as follows. In Section 2, we introduce the notation and give needed previous results. The proofs are given in Section 3, in Section 4 theoretical comparisons between the bounds are made, and Section 5 consists of numerical comparisons.

2. Statement of Results

Let $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$, $\bar{F}_k(x) = \int_x^\infty \phi(t)dt$, and $R_x = \bar{F}_k(x)/\phi(x)$, $x > 0$. Then Birnbaum (1942) and Sampford (1953) gave the following lower and upper bounds, respectively, for R_x:

$$(x/2 + (1+x^2/4)^{1/2})^{-1} < R_x < (3x + (1/2+x^2/16)^{1/2})^{-1} \quad . \tag{2.1}$$

Now let $f_k(t) = c_k(1+t^2/k)^{-(k+1)/2}$, $c_k = \dfrac{\Gamma((k+1)/2)}{\Gamma(k/2)(\pi k)^{1/2}}$, $\bar{F}_k(x) = 1 - F_k(x) = \int_x^\infty f_k(t)dt$, and $R_x = \bar{F}_k(x)/((1+x^2/k)f_k(x))$, for real $k > 0$ and real $x > 0$ here and throughout the paper. Then we will show here that

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$$\left[\frac{(k-1)x}{2k} + \left[1+(\frac{(k+1)x}{2k})^2\right]^{\frac{1}{2}}\right]^{-1} < R_x < \left[\frac{(3k-1)x}{4k} + \left[\frac{k-1}{2k} + (\frac{(k+1)x}{4k})^2\right]^{\frac{1}{2}}\right]^{-1} , \tag{2.2}$$

where the lower bound is valid for $k \geq 1$ and the upper for $k \geq 2$. Note that if for fixed x, $k \to \infty$ in (2.2) we obtain (2.1) with "$\leq$" replacing "$<$". A result from Soms (1976) which will be used subsequently is that

$$1/x - k/((k+2)x^3) < R_x < 1/x , \tag{2.3}$$

for real $k > 0$ (actually the proof assumed k a positive integer but this fact was nowhere used).

3. Proofs

Let $\nu(x) = 1/R_x$. Then

$$\lambda(x) = \nu'(x) = \nu(x)(\nu(x) - \frac{k-1}{k} x)/(1+x^2/k) \tag{3.1}$$

and the first result is that $0 < \lambda(x) < 1$ for $k \geq 1$ (here and throughout, " ' " means the derivative).

<u>Theorem 3.1</u>: Let $\lambda(x)$ be given by (3.1). Then $0 < \lambda(x) < 1$ for $k \geq 1$.

<u>Proof</u>: For convenience, $\lambda(x)$ and $\nu(x)$ will be denoted by λ and ν. From (2.3) $\nu > x$, and hence $\lambda > 0$. Suppose $\lambda \geq 1$ for some x. Then there exists an x such that $\lambda \geq 1$ and $\lambda' = 0$, since from (2.3) $\lim_{x\to\infty} \lambda = 1$ and it will be shown in Lemma 3.1 that $\lambda(0) = 4c_k^2 < 1$. Now

$$\lambda' = \frac{\nu}{(1+x^2/k)^2}\,(\nu - \frac{k+1}{k}\,x)(\nu - \frac{k-1}{k}\,x) + \frac{\nu}{1+x^2/k}\left[\frac{\nu}{1+x^2/k}\,(\nu - \frac{k-1}{k}\,x) - \frac{k-1}{k}\right]$$

$$= \frac{\nu}{1+x^2/k}\left[\frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k} - \frac{k-1}{k}\right]$$

$$= \frac{\nu}{1+x^2/k}\left[\lambda(2 - \frac{k+1}{k}\,\frac{x}{\nu}) - \frac{k-1}{k}\right] > \frac{\nu}{1+x^2/k}\,(\frac{k-1}{k})(\lambda-1) \geq 0 \ ,$$

and hence $\lambda' > 0$, a contradiction, giving the conclusion.

From Theorem 3.1,

$$\nu(\nu - (k-1)x/k) < 1 + x^2/k \ . \tag{3.2}$$

Completing the square in (3.2) gives

$$(\nu - (k-1)x/2k)^2 < 1 + ((k+1)x/2k)^2 \ ,$$

or since $\nu - (k-1)x/2k > 0$ from (2.3),

$$\left[\frac{(k-1)x}{2k} + \left(1 + (\frac{(k+1)x}{2k})^2\right)^{\frac{1}{2}}\right]^{-1} < R_x , \tag{3.3}$$

the desired result.

Recall that

$$\lambda' = \frac{\nu}{1+x^2/k}\left[\frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k} - \frac{k-1}{k}\right] =$$

$$= \frac{\nu}{1+x^2/k}\,(\phi - \frac{k-1}{k}),\ \phi = \frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k} \ . \tag{3.4}$$

Then the second result which will give the upper bound in (2.2) is

<u>Theorem 3.2</u>: Let λ' and ϕ be as in (3.4). Then, for $k \geq 2$, $\lambda' > 0$ or, equivalently, $\phi > (k-1)/k$.

<u>Proof</u>: Suppose $\lambda' \leq 0$ for some x. Then there exists an x such that $\phi \leq (k-1)/k$ and $\phi' = 0$, since $\phi(0) = 8c_k^2 > (k-1)/k$ by Lemma 3.1 and $\lim_{x\to\infty} \phi(x) = (k-1)/k$, since $\lim_{x\to\infty} \nu/x = 1$ by (2.3). Now using (3.1),

$$\phi' = \frac{[(\lambda-(k-1)/k)(2\nu-(k+1)x/k) + (\nu-(k-1)x/k)(2\lambda-(k+1)/k)](1+x^2/k)-2(1+x^2/k)x\phi/k}{(1+x^2/k)^2}$$

$$= \frac{(\nu-2x/k)\phi-((k-1)/k)(2\nu-(k+1)x/k) + (\nu-(k-1)x/k)(2\lambda-(k+1)/k)}{(1+x^2/k)} \quad . \tag{3.5}$$

Suppose $\nu \leq 2x/k$. Then $x < \nu \leq 2x/k$, since $\nu > x$ by (2.3), and so $kx < 2x$, which is impossible since $k \geq 2$, and so $\nu > 2x/k$. Adding and subtracting $(\nu-2x/k)(k-1)/k$ to the numerator of (3.5), gives, after some simple algebra, $\phi' = [(\nu-2x/k)(\phi-(k-1)/k) + 2(\nu-(k-1)x/k)(\lambda-1)]/(1+x^2/k) < 0$, since $\phi \leq (k-1)/k$ by assumption and $\lambda < 1$ by Theorem 3.1, which is a contradiction and completes the proof.

From Theorem 3.2,

$$(\nu-(k-1)x/k)(2\nu-(k+1)x/k) > (k-1)(1+x^2/k)/k \ , \tag{3.6}$$

and completing the square in (3.6) gives

$$(\nu-(3k-1)x/4k)^2 > (8(k-1)/k + ((k+1)x/k)^2)/16 \ , \tag{3.7}$$

and since $\nu > (3k-1)x/4k$, (3.7) gives, after some algebra,

$$R_x < \left[\frac{(3k-1)x}{4k} + \left(\frac{k-1}{2k} + (\frac{(k+1)x}{4k})^2\right)^{\frac{1}{2}}\right]^{-1} \ , \tag{3.8}$$

which is the remaining part of (2.2).

To complete the proofs of Theorems 3.1 and 3.2 we need

Lemma 3.1: $1-4c_k^2 > 0$ for $k > 0$ and $8c_k^2 > (k-1)/k$ for $k \geq 2$.

Proof: The proof is an immediate consequence of the fact shown in Soms (1979) that for $k > 0$ the c_k are a strictly increasing function of k with limit $1/\sqrt{2\pi}$. For the sake of completeness we provide a brief outline of that proof. Let $g(k) = \frac{d}{dk} \ln c_k = \frac{1}{2}\Psi(\frac{k+1}{2}) - \frac{1}{2}\Psi(\frac{k}{2}) - \frac{1}{2k}$, where $\Psi(x)$ is the digamma function (see, e.g., Abramowitz and Stegun, 1965, p. 258). From Artin (1964,

p. 17), $h(k) = \frac{1}{2}\,\Psi(\frac{k+1}{2}) - \frac{1}{2}\,\Psi(\frac{k}{2}) = \sum_{i=0}^{\infty} \frac{1}{(k+2i)(k+2i+1)}$. By the Euler-MacLaurin formula,

$$\frac{1}{2}\,\ln\frac{k+1}{k} + \frac{1}{2}\,\frac{1}{k(k+1)} \le h(k) \ ,$$

and so $h(k) > \frac{1}{2k}$, giving the conclusion.

It is reasonable to ask whether consideration of derivatives of R_x would also lead to bounds. In fact it does lead to analogues of the bounds of Gordon (1941), but these bounds, just as in the normal case, are inferior to (2.2). For the details of these and some other second-best bounds the reader is referred to Soms (1977a).

4. Theoretical Comparisons

We give here the three types of other bounds with which either numerical, theoretical, or both comparisons will be made. The bounds of Peizer and Pratt (1968) are

$$\frac{1}{x}\,\frac{x^2(k+2)}{x^2(k+2)+k} < R_x < \frac{1}{x}\,\frac{(k+2)(k+4)x^2+2(k+1)k}{(k+2)(k+4)x^2+2(k+1)k+k(k+4)} \ , \tag{4.1}$$

the bounds of Wallace (1959)

$$\frac{1-\Phi[(k\,\ln(1+x^2/k))^{\frac{1}{2}}]}{(1+x^2/k)f_k(x)} < R_x < \frac{1-\Phi[(1-1/2k)^{\frac{1}{2}}(k\,\ln(1+x^2/k))^{\frac{1}{2}}]}{(1+x^2/k)f_k(x)}, \tag{4.2}$$

$k > \frac{1}{2}$ for the upper bound, and of Soms (1977b)

$$p(x,\gamma_{min}) < R_x < p(x,\gamma_{max}) \ , \tag{4.3}$$

where $p(x,\gamma) = \frac{1+\gamma}{(x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}}+\gamma x}$, for $k > 2$, $\gamma_{min} = \frac{k}{2(k+2)c_k{}^2} - 1$, $\gamma_{max} = 4c_k{}^2/(1-4c_k{}^2)$, for $k < 2$ the definitions of γ_{min} and γ_{max} are interchanged and for $k = 2$, $\gamma_{min} = \gamma_{max}$ and $R_x = p(x,\gamma_{max})$.

Let us denote the lower bounds of (2.2), (4.1), (4.2), and (4.3) by SL, PL, WL, and BL, respectively, and the upper bounds by SU, PU, WU, and BU.

Then, after some algebra, comparison of (2.2) with (4.1) yields the following results. SL is better (bigger) than PL for $x < k/(k+2)^{\frac{1}{2}}$. SU is better (smaller) than PU for $x < -\frac{b}{2a} + (\frac{b^2-4ac}{4a^2})^{\frac{1}{2}}$, where

$$a = -2(k+2)(k+4)^2 ,$$

$$b = -2k(k+4)(2k^2+3k+4) ,$$

and $$c = 4(k-1)k^2(k+1)^2 .$$

Analytical comparisons with Wallace (1959) appear impossible.

BL is better than SL for $k > k_o$, where $1 < k_o < 2$, and for sufficiently large x and $1 \leq k \leq k_o$, SL is better. For sufficiently large x, SU is better than BU. Further analytical comparisons here are prohibitively difficult.

We note here that for $k = 1$, we have the Cauchy distribution and (2.2) and (2.3) combine for the interesting simple and sharp result

$$\frac{1}{\pi(1+x^2)^{\frac{1}{2}}} < \frac{1}{\pi} \int_x^\infty \frac{1}{1+t^2} dt < \frac{1}{\pi x} .$$

5. Numerical Comparisons

Let $g(x) = (1+x^2/k)f_k(x)$. Then (2.2) gives

$$g(x)SL < \bar{F}_k(x) < g(x)SU , \tag{5.1}$$

and we can approximate $\bar{F}_k(x)$ by $L = g(x)SL$ and $U = g(x)SU$. This is done in Table 1 for selected degrees of freedom and known t percentiles (Cramer, 1946, p. 560).

1. S-Approximations to $\bar{F}_k(x)$

	Degrees of Freedom								
$\bar{F}_k(x)$	6			20			120		
	x	L	U	x	L	U	x	L	U
.1	1.440	.0956	.102	1.325	.0959	.101	1.289	.0959	.101
.05	1.943	.0487	.0505	1.725	.0486	.0503	1.658	.0487	.0502
.025	2.447	.0246	.0252	2.086	.0245	.0251	1.980	.0245	.0251
.010	3.143	.00989	.0100	2.528	.00988	.0100	2.358	.00988	.0100
.005	3.707	.00496	.00502	2.845	.00496	.00501	2.617	.00496	.00501
.0005	5.959	.000498	.000500	3.850	.000497	.000500	3.373	.000498	.000501

From Table 1 is appears that the approximation using the SL and SU-bounds gives good results if $\bar{F}_k(x)$ is less than or equal to .1.

In Table 2, SL, PL, WL, and BL are given and in Table 3, SU, PU, WU, and BU, for selected x and k. In the calculation of the WL bounds for extreme tails the Boyd (1959) approximation for the tail area of the normal distribution was used. A "-" indicates the bound is inapplicable.

While the superiority of SL to BL for $k = 1$ is not apparent from the tables, it is seen by direct calculation that, for example, for $x = 50$ SL is bigger than BL. In general, for lower bounds BL is best while the picture is mixed for upper bounds. For x larger than in the tables the performance of SL and SU would be better than for BL and BU.

2. Lower Bounds to R_x

	Degrees of Freedom											
x	1				10				20			
	SL	PL	WL	BL	SL	PL	WL	BL	SL	PL	WL	BL
.5	.8944	.8571	1.0000	1.1046	.7923	.4615	.8892	.8949	.7866	.4314	.8828	.8807
1.0	.7071	.7500	.6363	.7834	.6284	.5455	.6490	.6687	.6233	.5238	.6521	.6591
2.0	.4472	.4615	.3213	.4630	.4190	.4138	.3890	.4277	.4167	.4074	.4031	.4238
4.0	.2425	.2449	.1450	.2449	.2372	.2376	.1888	.2384	.2366	.2366	.2049	.2375
10.0	.0995	.0997	.0498	.0997	.0991	.0992	.0609	.0992	.0991	.0991	.0674	.0991
20.0	.0499	.0500	.0225	.0500	.0499	.0499	.0257	.0499	.0499	.0499	.0279	.0499

3. Upper Bounds to R_x

x	Degrees of Freedom											
	1				10				20			
	SU	PU	WU	BU	SU	PU	WU	BU	SU	PU	WU	BU
.5	-	1.2156	1.1598	1.1092	.9549	1.3034	.9020	.9045	.9343	1.3388	.8891	.8921
1.0	-	.7917	.8735	.7860	.6897	.7349	.6735	.6766	.6780	.7403	.6644	.6688
2.0	-	.4638	.5807	.4637	.4315	.4322	.4310	.4312	.4271	.4301	.4258	.4282
4.0	-	.2450	.3675	.2450	.2388	.2385	.2450	.2392	.2378	.2377	.2400	.2385
10.0	-	.0997	.2022	.0997	.0992	.0992	.1135	.0992	.0991	.0991	.1067	.0992
20.0	-	.0500	.1310	.0500	.0499	.0499	.0665	.0499	.0499	.0499	.0605	.0499

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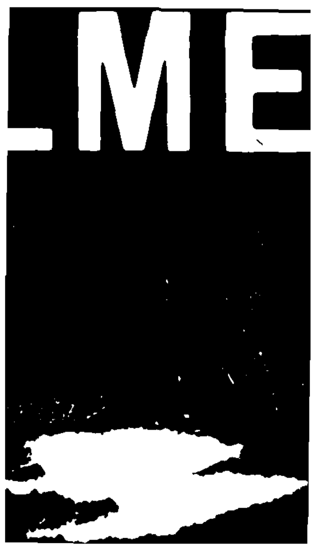
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