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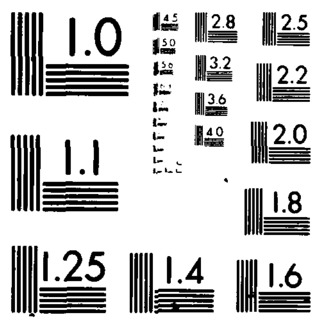
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ADAPTIVE DESIGNS IN ATTRIBUTE LIFE TESTING

By

Mohammad Kazim Khan

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The study is concerned with the problem of adaptively determining the (survillance) observation epochs for the purpose of estimating the parameter(s) of a life distribution under attribute life testing. A discrete adaptive procedure is developed and proved to be ϵ-optimal under the criterion of maximizing the Fisher information function.

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Adaptive Determination of Designs in Attribute Life Testing (*)

by

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1) Introduction

Consider a system of N components (N is fixed positive integer) working independently and having identical cumulative distribution functions (c.d.f.) of the time till failure $F(t;\theta)$. F is a known function and θ is an unknown parameter, belonging to a parameter space Θ. The components fail randomly at unobservable times. The system is inspected after X units of time and the number of failed components is observed. The replacement of the components could be according to the following two policies:

(A) Only failed components are replaced at each inspection.

(B) All items in the system are replaced at each inspection (frequent replacement policy, or block replacement policy).

The determination of the appropriate replacement policy depends upon the type of system under consideration. For example, policy (B) is preferred over policy (A) when it costs more to inspect and replace only failed components as compared to changing the whole system. Moreover, sometimes it is practically impossible to change only failed components without effecting the whole system. For examples see Barlow and Proschan (1967). In quantal response bioassay studies policy B is followed, where, after experimentation, the whole batch of experimental units (mice, fish, etc.) is replaced by a

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(*) Part of the Ph.D. dissertation written under the guidance of Professor S. Zacks for the Department of Mathematics and Statistics, Case Western Reserve University, Cleveland Ohio.

new one. Finney (1978) provides an exhaustive reference list for such bioassay designs.

Let $J(x_1), J(x_2), \ldots, J(x_n), \ldots$ denote the number of components failing during the intervals $(0, x_1), (x_1, x_1, +x_2), \ldots, (\sum_{i=1}^{n-1} x_i, \sum_{i=1}^{n} x_i), \ldots$. Intuitively, we would like to use the information $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$ to determine x_{n+1} such that $J(x_{n+1})$ will provide as much information on θ as possible. To define the best or optimal interinspection time at the (n+1)st stage, we shall use the criterion of maximizing the conditional Fisher information about θ given $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$. More specifically, let F_n denote the sigma algebra generated by $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$ and let $I(\theta; x_{n+1} | F_n)$ denote the conditional Fisher information of θ at the (n+1) at stage given F_n. Generally, $I(\theta; x_{n+1} | F_n)$ depends on θ and on F_n. Hence, the optimal value of x_{n+1} is a function of the unknown parameter θ and of the past history of the system. Since θ is unknown one has to change the criterion of optimality in a suitable manner.

It is readily seen (Khan 1980) that, if $I(F(x_1;\theta))$ and $I(\theta;x_1)$ represent the Fisher information function of $F(x_1;\theta)$ and θ given x_1 respectively, then

$$I(F(x_1;\theta)) = N/\{F(x_1;\theta)(1-F(x_1;\theta))\} \qquad (1.1)$$

and

$$I(\theta;x_1) = I(F(x_1;\theta))\left(\frac{\partial}{\partial\theta} F(x_1;\theta)\right)^2 \qquad (1.2)$$

One can also easily show (Khan 1980) that under replacement policy B, the conditional Fisher Information function, given F_{n-1}, is

$$I(\theta;x_n | F_{n-1}) = I(F(x_n;\theta))\left(\frac{\partial}{\partial\theta} F(x_n;\theta)\right)^2 \qquad (1.3)$$

In particular, for the negative exponential failure distribution, (1.3) holds under both replacement policies.

The terms 'design levels', 'dose levels', 'interinspection times' will be used interchangeably, depending upon the special application of the methods.

2) Adaptive Designs

As discussed in the previous section, the tolerance or failure time distribution yields for a design level x the probability of response $F(x;\theta)$. Let $J(x)$ be the total number of responses at a design level x, among N identical and independent units. Given x, $J(x)$ has a binomial distribution, provided all units started to function together. That is,

$$J(x) \sim B(N, F(x,\theta)) \ .$$

From the point of view of maximizing the Fisher information function θ, we may proceed to define the optimality criterion as follows.

Considering $I(\theta,x)$ as a function of x, we would like to find the design level x_0 such that

$$I(\theta;x) \leq I(\theta;x_0) \tag{2.1}$$

for all $x \in \mathcal{X}$ where $\mathcal{X}$ is some appropriate design space. If $F(x,\theta)$ is a continuous function of x such that $\frac{\partial}{\partial\theta} F(x,\theta)$ is also a continuous function of x, then there exists a unique x_0 such that (2.1) holds provided $\mathcal{X}$ is a compact subset of $\mathbb{R}$. Usually, x_0 will depend on θ by some (known) functional relation, say, g i.e.

$$\begin{aligned} g &: \Theta \to \mathcal{X} \\ \theta &\to g(\theta) = x_0(\theta) \end{aligned} \tag{2.2}$$

Hence, if θ is unknown, x_0 cannot be determined. The following are some typical examples of g encountered in application

(i) $F(x,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}(t-\theta)^2)\, dt \ ; \ g(\theta) = \theta$.

This model is known as the Probit model in bioassay studies.

(ii) $F(x,\theta) = \{1 + \exp(-(x-\theta))\}^{-1} \ ; \ g(\theta) = \theta$.

This model is known as the Logit model in bioassay literature.

(iii) $F(x,\theta) = (e-1)^{-1} [e \ \exp\{-e^{-\theta x}\} - 1] \ , \ x \geq 0 \ ;$

$g(\theta) = 1.9366/\theta$.

(iv) $F(x,\theta) = 1 - \exp(-x/\theta)$, $x \geq 0$; $g(\theta) = 1.5936\ \theta$.

To overcome this difficulty of the dependence of x_0 on the unknown parameter θ, we apply an adaptive techniques for estimating θ by using a sequence of design levels $x_1, x_2, \ldots, x_n, \ldots$ and corresponding to each design level we define an estimate of θ, θ_n. The double sequence $\{x_n, \theta_n; n \geq 1\}$ is called an adaptive design.

Definition (2.1)

An adaptive design $\{x_n, \theta_n\}$ is asymptotically optimal if for each fixed $\theta \in \Theta$,

(i) $\theta_n \xrightarrow{a.s.} \theta$ as $n \to +\infty$,

(ii) $x_n \xrightarrow{a.s.} x_0(\theta)$ as $n \to +\infty$

(iii) $I(\theta_n; x_n) \xrightarrow{a.s.} I(\theta; x_0(\theta))$.

Note that if $I(\theta,x)$ is a continuous function of θ and x then (iii) follows from (i) and (ii).

A slightly relaxed (weaker) definition of optimality is as follows:

Definition (2.2)

An adaptive design $\{x_n, \theta_n\}$ is asymptotically ε-optimal if for each fixed $\varepsilon > 0$ and each fixed $\theta \in \Theta$,

(i) $\theta_n \xrightarrow{a.s.} \theta$ as $n \to +\infty$

(ii) $\lim_n \sup |x_n - x_0(\theta)| < \varepsilon$ a.s.

(iii) $\limsup_n |I(\theta_n; x_n) - I(\theta, x_0(\theta))| < \varepsilon$ a.s.

Note that if the adaptive design $\{x_n, \theta_n\}$ is independent (functionally) of ε, then definitions (2.1) and (2.2) are equivalent.

Venter and Gastwirth (1964) defined the optimality criterion (in the negative exponential case) to be that a sequence $\{x_n, \theta_n\}$ is asymptotically optimal if

$$\liminf n^{-1} \sum_{i=1}^{n} E\{I(\theta_i; x_i)\} = I(\theta; x_0(\theta)) , \qquad (2.3)$$

for each $\theta \in \Theta$.

Note that if an adaptive design is optimal in the sense of definition (2.1) then by the Lebesque dominated convergence theorem and the regularity property of the Césaro means (Powell and Shah (1972), Knopp (1956)) it is asymptotically optimal in Venter's sense. However the converse may not hold. Therefore, definition (2.1) is a stronger optimality criterion.

One of the shortcomings of definition (2.1) is that it does not take into account the rate of convergence of the adaptive design. This factor could be very important for experiments of moderate sample size. Therefore, it is preferable to find some bound to the asymptotic variance of the adaptive design.

3) <u>Discrete Adaptive Designs</u>

In this section we shall assume, unless otherwise stated, that

(i) X, Θ are compact subsets of $\mathbb{R}$.

(ii) $F(x,\theta)$ is a differentiable function of θ for each fixed x so that the Fisher information function exists.

(iii) $F(x,\theta)$ has a unique continuous inverse for each fixed x.

(iv) $\frac{\partial}{\partial\theta} F(x,\theta)$ is a continuous function of x and θ.

(v) $g(\theta)$ is a continuous function of θ, where g is as defined in (2.4).

(vi) $I(\theta,x)$ satisfies the Lipschitz condition.

Condition (i) will not have any effect in applications of the results. Conditions (ii) - (vi) will be satisfied in almost all the models encountered in bioassay and reliability studies. For example for the Probit and Logit models the above conditions will be satisfied when θ is the shift parameter.

Since Θ is a compact set, we can, without loss of generality, assume that

$$\Theta = [a,b] \; ; \; -\infty < a < b < +\infty \, . \tag{3.1}$$

Furthermore, since g is a continuous function, $g(\Theta)$ is also a compact set. Therefore it is reasonable to define the design space

$$X = g(\Theta) = [a', b'] \tag{3.2}$$

where $$a' = \inf_{\theta\epsilon\Theta} g(\theta) \tag{3.3}$$

and $$b' = \sup_{\theta\epsilon\Theta} g(\theta) \tag{3.4}$$

For any given $\varepsilon > 0$, let K be an integer such that

$$K > (b' - a')/\varepsilon \tag{3.5}$$

and define

$$d_i = a' + K^{-1}(b' - a')\, i \ , \ i=0,\ldots,K \ . \tag{3.6}$$

All the experiments are performed at some of these design levels. Note that

$$d_{i+1} - d_i < \varepsilon \ ; \ i = 0,1,\ldots,K-1 \tag{3.7}$$

Define the set of discrete designs to be

$$X' = \{d_i \ ; \ 1 = 0,1,2,\ldots,K\} \tag{3.8}$$

Unless otherwise stated we shall assume that $g(\theta) \notin X'$. The first design level $X_{i,K}$ is a random variable taking values in X'. The maximum likelihood estimate of $F(d_i,\theta)$ given $X_{i,K} = d_i$ and N is

$$N^{-1} J_1(d_i) = \bar{J}_1(d_i) \ .$$

We define the estimate of θ as

$$\theta_{1,K} = [F^{-1}(d_i \ , \ \bar{J}_1(d_i))]_a^b \ . \tag{3.9}$$

where

$$[x]_a^b = \begin{cases} a, \text{ if} & x < a \\ x, \text{ if} & a \leq x \leq b \\ b, \text{ if} & x > b \end{cases}$$

Adaptively at the n^{th} stage ($n \geq 2$) define the design level $X_{n,K} \in X'$ such that

$$X_{n,K} = d_i$$

$$\text{if } g(\theta_{n-1,K}) \in [d_i \ , \ d_{i+1}) \ . \tag{3.10}$$

This implies that for all $n \geq 2$.

$$|X_{n,K} - g(\theta_{n-1,K})| < \varepsilon \text{ almost surely} . \tag{3.11}$$

Let $J_n(d_i)$ denote the total number of responses among N units at the nth stage given $X_{n,K} = d_i$. Also let F_n be the sigma algebra generated by $(X_{1,K},\ldots,X_{n,K},J_1,\ldots,J_{n-1})$, then due to the block replacement policy, (and the memory less property of the exponential distribution in policy A).

$$J_n(d_i)|F_n \sim B(N, F(d_i,\theta)) . \tag{3.12}$$

(For more details see Khan (1980) and Zacks (1973).)

At the nth stage we define an estimate of θ as

$$\theta_{n,K} = \sum_{i=0}^{K} n^{-1} M_{i,n}[F^{-1}(d_i,\bar{J}_{M_{i,n}})]_a^b . \tag{3.13}$$

Where $M_{i,n}$ is the total number of times in n stages the design level d_i was repeated and $J_{M_{i,n}}$ denotes the average number of responses during these $M_{i,n}$ repititions of d_i.

Hence, for each fixed $\varepsilon > 0$, we define the discrete adaptive design $\{X_{n,K} , \theta_{n,K}\}$ for $\{g(\theta),\theta\}$. Note that K depends on ε, for simplicity of notation, however, we will write K instead of $K(\varepsilon)$.

The following lemma and a theorem of Anscombe are needed for further developments.

Lemma (3.1)

If $\{Y_n\}$ is any sequence of random variables such that

$$Y_n \xrightarrow{a.s.} \theta \text{ as } n \to +\infty$$

and $\{N_r\}$ is any sequence of non-negative integer valued random variables such that

$$N_r \xrightarrow{a.s.} +\infty \text{ as } r \to +\infty$$

then,

$$Y_{N_r} \xrightarrow{a.s.} \theta \text{ as } r \to +\infty .$$

Proof: See M.K. Khan (1980) or R.A. Khan (1975).

Theorem (3.1) (F.R. Anscombe)

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are independent and identically distributed random variables with mean zero and variance one. Let

$$S_n = X_1 + X_2 + \ldots + X_n \;;\; n \geq 1 \quad .$$

Furthermore, let $\nu(t)$ denote a positive integer-valued random variable for any $t > 0$ such that

$$\frac{\nu(t)}{t} \xrightarrow{P} c \quad \text{as} \quad t \to +\infty \quad ,$$

where $c > 0$ is a constant. Then for any $x \in \mathbb{R}$,

$$\lim_{t \to +\infty} P\left\{ \frac{S_{\nu(t)}}{\sqrt{\nu(t)}} < x \right\} = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-u^2/2)\, du \quad .$$

Proof.

See Anscombe (1952). For a simpler proof of the theorem see Renyi (1957).

Lemma (3.2)

For the discrete adaptive design, let d_i be the maximal design level smaller than $g(\theta)$ and $M_{i,n}$ is the associated number of repetitions of d_i, then

$$M_{i,n} \xrightarrow{a.s.} +\infty \text{ as } n \to +\infty \; [P_\theta]$$

Proof.

By negation, if the lemma is false, there exists a set $A \subset \Omega$ such that

$$P(A) > 0 \qquad (3.14)$$

and for each $\omega \in A$, as $n \to +\infty$

$$M_{i,n}(\omega) \to M_i(\omega) < +\infty \quad .$$

(Note that $\lim_{n \to +\infty} M_{i,n}(\omega)$ always exists, being a non-decreasing sequence of integers.)

Since,

$$\sum_{j=0}^{K} M_{j,n}(\omega) = n, \text{ for all } n \text{ and for each } \omega \in \Omega \;,$$

as $n \to +\infty$, there exist some design levels, say, $d_{j_1}, d_{j_2}, \ldots, d_{j_q}$, $q \geq 1$, such that as $n \to +\infty$,

$$M_{j_\ell, n}(\omega) \to +\infty \quad , \quad \ell = 1,2,\ldots,q \quad ,$$

and for each $j \neq j_\ell$, $\ell = 1,2,\ldots,q$, there exist constants $M_j(\omega)$ such that as $n \to +\infty$

$$M_{j,n}(\omega) \to M_j(\omega) < +\infty \quad .$$

By the strong law of large numbers

$$|\bar{J}_{n_j} - F(d_j,\theta)| \overset{a.s.}{\to} 0 \quad \text{as} \quad n_j \to +\infty \quad .$$

Therefore, for almost all $\omega \in A$, the subsequence

$$|\bar{J}_{M_{j_\ell},n}(\omega) - F(d_j,\theta)| \to 0$$

$$\text{as} \quad n \to +\infty \; , \; \ell = 1,2,\ldots,q \quad .$$

By the continuity of F^{-1} and the fact that $\theta \in [a,b]$,

$$[F^{-1}(d_{j_\ell}, J_{M_{j_\ell},n})]_a^b = Y_{M_{j_\ell},n}(\omega) \to \theta$$

$$\text{as} \quad n \to +\infty \quad , \quad \text{for each } \ell = 1,2,\ldots,q \quad .$$

Now

$$|\theta_{n,K} - \theta| \leq \sum_{j=0}^{K} n^{-1} M_{j,n} |Y_{M_{j,n}} - \theta|$$

$$\leq \sum_{j=0}^{K} n^{-1} M_{j,n}(b-a) + \sum_{\ell=1}^{q} n^{-1} M_{j_\ell,n} |Y_{M_{j_\ell},n} - \theta|$$

$$(j \neq j_\ell \; ; \; \ell = 1,2,\ldots,q).$$

Therefore, for almost all $\omega \in A$, we have

$$|\theta_{n,K}(\omega) - \theta| \leq o(1) + o(1) \sum_{\ell=1}^{q} n^{-1} M_{j_\ell,n}$$

$$= o(1) \quad . \tag{3.15}$$

By the definition of $X_{n,K}$ we have

$$g(\theta_{n-1,K}) - (b-a)/K \leq X_{n,K} \leq g(\theta_{n-1,K})$$

This implies that for almost all $\omega \in A$

$$\limsup_n |X_{n,K}(\omega) - g(\theta)| \leq (b-a)/K \quad . \tag{3.16}$$

It follows that

$$X_{n,K}(\omega) = d_i \quad \text{infinitely often}, \tag{3.17}$$

for almost all $\omega \in A$.

Hence, for almost all $\omega \in A$,

$$M_{i,n}(\omega) \to +\infty \quad \text{as} \quad n \to +\infty \quad .$$

This contradicts the definition (3.14) of the set A. Hence,

$$P(A) = 0$$

i.e.,

$$M_{i,n} \xrightarrow{a.s.} +\infty \quad \text{as} \quad n \to +\infty \quad . \qquad \text{(Q.E.D.)}$$

Lemma (3.3)

For the discrete adaptive design

$$P_\theta(M_{j,n} \to +\infty) = 0 \ ,$$

for all $j \neq i$, where d_i is the maximal design level less than $g(\theta)$.

Furthermore,

$$n^{-1} M_{i,n} \xrightarrow{a.s.} 1 \quad .$$

Proof.

If the Lemma is wrong, there exists a $j \neq i$ and a set $B \subset \Omega$ such that

$$B = \{\omega: \ M_{j,n}(\omega) \to +\infty\}$$

and

$$P(B) > 0 \quad . \tag{3.18}$$

Let $\omega \in B$, since by the negation hypothesis

$$M_{j,n}(\omega) \to +\infty \quad ,$$

$$X_{n,K}(\omega) = d_j \quad \text{infinitely often}. \tag{3.19}$$

By lemma (3.2),

$$M_{i,n}(\omega) \to +\infty \quad , \quad \text{as} \quad n \to +\infty \quad ,$$

for almost all $\omega \in \Omega$.

Therefore, without loss of generality,

$$X_{n,K}(\omega) = d_i \quad \text{infinitely often} \tag{3.20}$$
$$\text{for almost all } \omega \in B \quad .$$

Since $j \neq i$, we have

$$|d_i - d_j| \geq (b-a)/K \ .$$

Hence, for almost all $\omega \in B$, we have

$$\limsup_n |X_{n,K}(\omega) - g(\theta)| > (b-a)/K \text{ if } \quad j \neq i+1$$

and (3.21)

$$\limsup_n X_{n,K}(\omega) \geq d_{i+1} \qquad \text{if} \quad j = i+1$$

By similar arguments as used in lemma (3.2), for almost all $\omega \in B$, we have

$$\theta_{n,K}(\omega) \to \theta \text{ as } \quad n \to +\infty \quad .$$
$$g(\theta_{n,K}(\omega)) \to g(\theta) \quad \text{as} \quad n \to +\infty \quad .$$

Since,

$$g(\theta_{n-1,K}) - (b-a)/K \leq X_{n,K} \leq g(\theta_{n-1,K}), \text{ for all } n,$$

for almost all $\omega \in B$, we get

$$\left.\begin{aligned} &\limsup_n |X_{n,K} \ (\omega) - g(\theta)| \leq (b-a)/K \\ &\text{and} \\ &\limsup_n X_{n,K} \ (\omega) \leq g(\theta) < d_{i+1} \end{aligned}\right\} \tag{3.22}$$

which contradicts (3.21)

Hence,

$$P(B) = 0$$

i.e.,

$$P(M_{j,n} \to +\infty) = 0$$
$$\text{for all } \ j \neq i \quad .$$

(Q.E.D.)

Theorem (3.2)

The discrete adaptive design $\{X_{n,K}, \theta_{n,K}\}$ is asymptotically ε - optimal for $\{g(\theta),\theta\}$.

Proof.

Without loss of generality let $g(\theta) \in (d_i, d_{i+1})$.

By lemma (3.3) we have,

$$n^{-1} M_{i,n} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to +\infty .$$

By the strong law of large numbers and lemma (3.1), we have

$$\bar{J}_{M_{i,n}} \overset{a.s.}{\to} F(d_i,\theta) \quad \text{as} \quad n \to +\infty .$$

By the continuity of $[F^{-1}(d_i,y)]_a^b$ with respect to y and the fact that $\theta \in [a,b]$, we deduce that

$$Y_{M_{i,n}} \overset{a.s.}{\to} \theta \text{ as } n \to +\infty . \tag{3.23}$$

Hence,

$$\begin{aligned}
|\theta_n - \theta| &= \Big| \sum_{j=0}^{K} n^{-1} M_{j,n} Y_{M_{j,n}} - \theta \Big| \\
&\le \sum_{j=0}^{K} n^{-1} M_{j,n} |Y_{M_{j,n}} - \theta| \\
&\le (b-a) \sum_{\substack{j=0 \\ j \ne i}}^{K} n^{-1} M_{j,n} + |Y_{M_{i,n}} - \theta| \\
&\overset{a.s.}{=} o(1) \quad \text{by (3.23) and lemma (3.3)}
\end{aligned}$$

which proves (i) of definition (2.2) .

Since,

$$\begin{aligned}
|X_{n,K} - g(\theta)| &\le |X_{n,K} - g(\theta_{n-1,K})| + |g(\theta_{n-1,K}) - g(\theta)| \\
&\le (b-a)/K + o(1) \quad ,
\end{aligned}$$

by taking $\limsup_n$, we get (ii) of definition (2.2) .

In fact, we have proved that with arbitrary large probability

$$X_{n,K} = d_i \tag{3.24}$$

for all, except finitely many, values of n.

Furthermore, $I(\theta,x)$ is a continuous function of (θ,x), satisfying the Lipschitz condition. Therefore, there exists a constant $A < +\infty$ such that

$$\limsup_n |I(\theta_{n,K},X_{n,K}) - I(\theta,g(\theta))| \leq A \frac{(b-a)}{K} . \qquad (3.25)$$

By taking K large enough we get (iii) of definition (2.2). (Q.E.D.)

<u>Theorem (3.3)</u>

For the discrete adaptive designs $\{X_{n,K}, \theta_{n,K}\}$

$$\sqrt{n}(\theta_{n,K} - \theta) \xrightarrow{\mathcal{D}} N(0,D) \quad \text{as} \quad n \to +\infty$$

for each $\theta \in (a,b)$ such that $g(\theta) \in (d_i,d_{i+1})$

where $D = N_i^{-1} F(d_i,\theta)(1-F(d_i,\theta)) \left. \frac{\partial}{\partial e} F^{-1}(d_i,e) \right|_{e=F(d_i,\theta)}^{2}$

<u>Proof.</u>

By the Central limit theorem

$$\sqrt{n_i}\,(\bar{J}_{n_i} - F(d_i,\theta)) \xrightarrow{\mathcal{D}} N(0,D_i)$$

$$\text{as} \quad n_i \to +\infty , \quad \text{where,}$$

$$D_i = N_i^{-1} F(d_i,\theta)(1 - F(d_i,\theta)) .$$

By lemma (3.3) and Anscombe's theorem (3.1),

$$\sqrt{M_{i,n}}\,(\bar{J}_{M_{i,n}} - F(d_i,\theta)) \xrightarrow{\mathcal{D}} N(0,D_i)$$

$$\text{as} \quad n \to +\infty$$

which implies,

$$\sqrt{n}\,(\bar{J}_{M_{i,n}} - F(d_i,\theta)) \xrightarrow{\mathcal{D}} N(0,D_i)$$

$$\text{as} \quad n \to +\infty .$$

Therefore by the well known property of continuous function of a sequence satisfying the central limit theorem (see Billingsley (1979) pp. 320, problem 27.10) we have

$$\sqrt{n}\,([F^{-1}(d_i,\bar{J}_{M_{i,n}})]_a^b - \theta)$$

$$= \sqrt{n}\,(Y_{M_{i,n}} - \theta) \xrightarrow{\mathcal{D}} N(0,D) \qquad (3.26)$$

for each $\theta \in (a,b)$, as $n \to +\infty$,

where

$$D = D_i \left\{\frac{\partial}{\partial z} F^{-1}(d_i,z) \Big| \, z = F(d_i,\theta)\right\}^2 .$$

Finally,

$$\sqrt{n}\,(\theta_{n,K}-\theta) = \sqrt{n}\,\Big(\sum_{j=0}^{K} n^{-1} M_{i,n} Y_{M_{i,n}} - \theta\Big)$$

$$= n^{-1} M_{i,n} \sqrt{n}\,(Y_{M_{i,n}} - \theta) + \sqrt{n} \sum_{\substack{j=0 \\ j \neq i}}^{K} n^{-1} M_{j,n} (Y_{M_{j,n}} - \theta)$$

By lemma (3.3)

$$n^{-1/2} M_{j,n} \xrightarrow{a.s.} 0$$

for each $j \neq i$.

Therefore,

$$\sqrt{n}\,(\theta_{n,K}-\theta) = n^{-1} M_{i,n} \sqrt{n}(Y_{M_{i,n}} - \theta) + o_p(1)$$

which implies by (3.26)

$$\sqrt{n}\,(\theta_{n,K}-\theta) \xrightarrow{\mathcal{D}} N(0,D)$$

as $n \to +\infty$.

(Q.E.D.)

<u>Remarks.</u>

(i) If $d_{i,n} = f(M_{i,n},n)$ where f is a measurable function of $M_{i,n}$, n such that

$$\sum_{i=0}^{K} d_{i,n} \overset{a.s.}{=} 1 \quad \forall\, n ,$$

$$0 \le d_{i,n} \le 1 \quad \text{a.s.} \quad \forall\, i \text{ and } \forall\, n$$

and define an estimate of θ by

$$\hat{\theta}_{n,K} = \sum_{i=0}^{K} d_{i,n} Y_{M_{i,n}} .$$

Then all of the above mentioned results for the discrete adaptive design will hold when $\theta_{n,K}$ is replaced by $\hat{\theta}_{n,K}$ under suitable conditions on f.

One such function is

$$d_{i,n}(r) = r^{M_{i,n}} \left(\sum_{i=0}^{K} r^{M_{i,n}} \right)^{-1} \quad ; \quad r > 1 \quad .$$

Such a weighting smoothes out the effect of the first guess, $X_{1,K}$, very fast.

(ii) The discrete adaptive design may be used to prove the asymptotic optimality of certain continuous adaptive designs.

(iii) Asymptotic confidence band for θ are

$$L(\theta) = \theta_{n,K} - Z_{\alpha/2}\, D^{1/2}\, n^{-1/2}$$

and

$$U(\theta) = \theta_{n,K} + Z_{\alpha/2}\, D^{1/2}\, n^{-1/2}$$

where D is given in theorem (3.3.2) and $Z_{\alpha/2}$ represents the $(1-\alpha/2)$ inverse of the standard normal distribution function.

So far we have assumed that $g(\theta) \neq d_i$ for $i = 0,1,\ldots,K$. The following theorem considers the case when $g(\theta) = d_{i+1}$ for some i.

<u>Theorem (3.4)</u>

For the discrete adaptive design if there exists an i such that for some fixed K

$$g(\theta) = d_{i+1} \text{ , then}$$

(i) $\theta_{n,K} \xrightarrow{a.s.} \theta$ as $n \to +\infty$,

(ii) $\limsup\limits_{n} |X_{n,K} - g(\theta)| \leq (b-a)/K$,

(iii) $\limsup\limits_{n} |I(\theta_{n,K}, X_{n,K}) - I(\theta, g(\theta))| \leq AK^{-1}(b-a)$ where $A < +\infty$ is a constant

Proof.

By similar arguments as lemma(3.2) and lemma (3.3) one can easily show that

$$P(M_{j,n} \to +\infty) = 0$$

for all $j \neq i,\ i+1$.

Which implies that

$$n^{-1}(M_{i,n} + M_{i+1,n}) \overset{a.s.}{\to} 1 \qquad (3.27)$$
$$\text{as} \quad n \to +\infty \ .$$

By similar arguments as theorem (3.2) part (i) follows that

$$\theta_{n,K} \overset{a.s.}{\to} \theta \quad \text{as} \quad n \to +\infty \ .$$

Since,

$$g(\theta_{n-1,K}) - (b-a)/K \leq X_{n,K} \leq g(\theta_{n-1,K}),$$

by taking $\limsup_n$, we get (ii).

Since $I(\theta,x)$ satisfies the Lipschutz condition we get (iii) (Q.E.D.)

Remark.

Since $g(\theta) = d_{i+1}$, there may exist a $0 < p < 1$ such that

$$P(\theta_{n,K} < \theta \text{ i.o.}) = p$$

Thus, $X_{n,K} = d_i$ infinitely often with probability p

and $X_{n,K} = d_{i+1}$ infinitely often with probability 1-p.

Therefore the result on asymptotic normality may not hold.

Remark.

We can construct test of hypothesis concerning θ. Let $H:\theta = \theta_0$ vs. appropriate alternative, the quantity

$$N_i M_{i,n}(\bar{J}_{M_{i,n}} - F(d_i,\theta_0))^2 \{F(d_i,\theta_0)(1 - F(d_i,\theta_0))\}^{-1}$$

measures the discrepancy between the observed and the expected frequencies.

Let ν be the total number of design levels yielding (after pooling if necessary) the quantity

$$R_{i,n} = N_i M_{i,n} F(d_i,\theta_0)(1 - F(d_i,\theta_0))$$

greater than 9, then under H, and given

$$M_{i,n} \; ; \; i = 0,1,2,\ldots,K,$$

the statistics

$$\chi^2[\nu] = \sum_{i=0}^{K} (N_i M_{i,n})^2 R_{i,n}^{-1} \{\bar{J}_{M_{i,n}} - F(d_i,\theta_0)\}^2$$

has approximately a conditional chi-square distribution with ν degrees of freedom.

(4) Comparison of the d a d with Continuous Designs

There are many ways to define continuous adaptive designs $\{x_n,\theta_n\}$ such that

$$\theta_n \overset{a.s.}{\to} \theta \text{ as } n \to +\infty$$

$$x_n \overset{a.s.}{\to} g(\theta) \text{ as } n \to +\infty$$

$$\text{and } I(\theta_n,x_n) \overset{a.s.}{\to} I(\theta,g(\theta)) \; .$$

For example, consider the following stochastic approximation method. For the negative exponential model in which $F(x;\theta) = 1-e^{-x/\theta}$.

Let x_1 be any random variable taking values in $(0,+\infty)$. After defining $x_1,x_2,\ldots x_n$ and observing the responses $J(x_1),J(x_2),\ldots,J(x_n)$ we define the $(n+1)$th design level x_{n+1} as

$$x_{n+1} = x_n + a_n\{J(x_n)/N - 1 + \exp(-c)\} \tag{4.1}$$

where $J(x_n)$ is the total number of responses (failures) out of N items (components) and $c = 1.5936$.

And define the estimate of θ as

$$\theta_{n+1} = c^{-1} x_{n+1} \; . \tag{4.2}$$

Usually, the sequence a_n is taken to be d/n, where d is a constant.

It is readily seen that (4.1) is a special case of the Robbins-Monro (1951) stochastic approximation method. Therefore, by the well known results (Wetherill (1963), (1975)) on the properties of the Robbins-Monro procedure

$$x_n \xrightarrow{a.s.} c\theta \quad \text{as} \quad n \to +\infty \tag{4.3}$$

which implies that

$$\theta_n \xrightarrow{a.s.} \theta \quad \text{as } n \to +\infty \quad .$$

Since, $I(\theta,x) = Nx^2\theta^{-4}(\exp(x/\theta) -1)^{-1}$ is a continuous function of (θ,x) by (4.2) and (4.3) we have

$$I(\theta_n,x_n) \xrightarrow{a.s.} I(\theta,c\theta) \quad \text{as} \quad n \to +\infty \quad .$$

Also

$$\sqrt{n} \quad (\theta_n-\theta) \xrightarrow{\mathcal{D}} N(0,\sigma_d^2) \quad \text{as} \quad n \to +\infty \tag{4.4}$$

where

$$\sigma_d^2 = d^2(e^c - 1)/\{c^2N(2d\beta-1)\} \ ; \ d\beta > 1/2$$

$$\beta = \theta^{-1} \exp(-c) \quad .$$

One can easily check that σ_d^2 is minimized if

$$d = \beta^{-1}$$

and

$$\min_d \sigma_d^2 = (e^c-1)N^{-1}(\theta/c)^2 \quad . \tag{4.5}$$

Another solution to this problem was given by Venter and Gastwirth (1964) by proposing maximum likelihood (ML) and stochastic approximation (SA) methods. Both ML and SA procedures of Venter are strongly consistent and asymptotically normal, i.e., if $\hat{\theta}_n$ represents Venter's ML or SA estimate of θ at the nth stage, then

$$\sqrt{n} \ (\hat{\theta}_n-\theta) \to N(0,V)$$

where

$$V = (\exp(c)-1)(\theta/c)^2 \tag{4.6}$$

For the discrete adaptive design $\theta_{n,K}$, we have from theorem (3.2) that

$$\sqrt{n} \ (\theta_{n,K} - \theta) \xrightarrow{\mathcal{D}} N(0,D) \tag{4.7}$$

where

$$D = \exp(d_i/\theta) \ N_i^{-1} \ \theta^4 \ d_i^2$$

where $d_i \approx c\theta$ and N_i is the number of components used for design level d_i.

One can easily show that for the Venter's procedures, if the system is composed of N components (instead of one component)

$$\mathrm{Var}(\sqrt{n}\ \theta_n) \to V/N \quad . \tag{4.8}$$

Hence, the asymptotic variance of these three procedures have the following relationship

$$D \approx V = \min \sigma_d^2 \leq \sigma_d^2 \quad . \tag{4.9}$$

It can also be shown that $\min \sigma_d^2$ is in fact the least possible variance (Cramér-Rao lower bound) of designs which are strongly consistent. Therefore, in practical situations, especially for small sample sizes, the minimum asymptotic variance criterion may not help to choose the best adaptive design out of the two adaptive designs of Venter and the discrete adaptive design.

In terms of computational ease, note that for the discrete adaptive design

$$\bar{J}_{n+1}(x) = n\ (n+1)^{-1}\bar{J}_n(x) + (n+1)^{-1}\ J_{n+1}(x)$$

and

$$M_{i,n+1} = M_{i,n} + I\{X_{n+1} = d_i\} \quad .$$

By using these two equations, we see that the discrete adaptive design and the Venter's SA procedure have almost the same level of difficulty in computation. However, Venter's ML procedure is considerably more difficult to apply in real situations.

Furthermore, for the application of the discrete adaptive designs we need only finitely many levels which could be pre-defined and made ready before the actual experimentation, and hence, will be more suitable in those situations for which there is a natural discretization of the design space (e.g., experiments could be performed at certain hours only, or at certain fixed levels (which cannot be split into smaller fractions) of the process).

One should also note that none of these adaptive procedures yield unbiased estimate of θ for any fixed sample size.

5) Designs Having Minimal Cost

So far we have been dealing with maximization of the Fisher information function of θ. Now we shall discuss the construction of adaptive designs yielding minimum value of a specified cost function. One cost function is proposed as follows. Let us denote by T_i the (unobservable) failure time of component $i = 1,2,\ldots,N$. Let $[0,x)$ be an interinspection interval. Then, $(x-T_i)^+$ denotes the length of time the ith component has been in state of failure, where

$$(x)^+ = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} .$$

Accordingly, let $h((x-T_i)^+)$ denote the failure cost of the ith component. $h(\cdot)$ should be non-negative perhaps monotonically increasing function such that $h(0) = 0$. Let c_1 be the cost of replacing a failed component and c_2 be the cost of inspecting a component, then under the assumption of additive costs, the total cost associated with design level x is

$$B(x) = \sum_{i=1}^{N} h((x-T_i)^+) + c_1\, J(x) + Nc_2$$

where $J(x)$ denotes the total number of failures during the interval $(0,x)$. Venter and Gastwirth (1964), Zacks and Fenske (1973) and Anbar (1976) considered this cost function for (1.1.1), where $h(t)$ is a linear function of t. Let $C(x)$ denote the average cost per time unit associated with the design level x, i.e.,

$$C(x) = B(x)/x \quad .$$

The optimal design level, x^*, is defined as the value of x for which the expected average cost per time unit is minimized, i.e.,

$$E\{C(x^*)\} \leq E\{C(x)\} \qquad x \in X ,$$

provided such an x* exists.

Furthermore,

$$E\{C(x)\} = x^{-1}\{ \sum_{i=1}^{N} E[h((x-T_i)^+)] + c_1 NF(x,0) + c_2 N\} \quad , \qquad (5.1)$$

Since (5.1) depends on θ, the design level x^* yielding minimum value of (5.1) depends on θ. Therefore, we consider adaptive designs which converge to $x^*(\theta)$.

Approximate $h(\cdot)$ by the first two terms of its Taylor series expansion, i.e.,

$$h((x-T_i)^+) \approx h(0) + (x-T_i)^+ \, h'(0) \qquad (5.2)$$

since

$$h(0) = 0$$

$$h((x-T_i)^+ \approx h'(0) \; (x-T_i)^+ \quad .$$

Let $c_0 = h\ (0)$, which represents the cost of a unit of idle time of the failed component i. For (5.3) one can easily show that (see Anbar (1976) in the negative exponential model,

$$E\{C(x)\} = x^{-1} \; N\{c_0[x-\theta F(x,\theta)] + c_1 F(x,\theta) + c_2\} \qquad (5.4)$$

and the design level x^* minimizing (5.4) is the solution of

$$(1 + \theta^{-1/x}) \exp(-x/\theta) = 1 - c_2(c_0\theta - c_1)^{-1} \qquad (5.5)$$

provided

$$0 < c_2(c_0 \; - c_1)^{-1} < 1 \quad , \qquad (5.6)$$

otherwise we can reduce the cost by not inspecting the system.

By (5.5) we get

$$x^* = \theta \;\; {}^{-1}(1 - c_2(c_0 \; - c_1)^{-1}) \qquad (5.7)$$

where

$$(x) = (x+1) \exp(-x) \quad ; \quad x \geq 0 \quad .$$

This is, x^* is a continuous function of θ such that for $\theta \in [a,b]$, x^* is bounded away from zero, where

$$a > c_0^{-1}(c_1 + c_2) \quad . \tag{5.8}$$

Note that while $F(x^*,\theta)$ depends on θ, $F(c\theta,\theta) = 1-\exp(-c)$ is independent of θ. Due to this fact the usual Robbins-Monro type stochastic approximation methods (used for estimating y such that $F(y,\theta)$ is a known constant) cannot be applied.

To overcome this problem Venter and Gastwrith (1964) and Anbar (1976) proposed equivalent adaptive designs for the negative exponential model. We will not discuss these designs here. In the following we shall show that the discrete adaptive design can easily be modified in such situations.

Let $x^*(\theta)$ be any (known) continuous function of the parameter θ such that there exist two constants a', b' so that, for each $\theta \in [a,b] = \Theta$

$$0 < a' \leq x^*(\theta) \leq b' < +\infty \tag{5.9}$$

and

$$a' = \inf_{\theta\in\Theta} x^*(\theta)$$

$$b' = \sup_{\theta\in\Theta} x^*(\theta) \quad .$$

(Note that x^* defined by (5.7) satisfies these conditions).

Let $X^* = [a',b']$ and K be an integer.

Define

$$X' = \; d_i \; : d_i = a' + K^{-1}(b'-a') \; i=0,1,\ldots,K \tag{5.10}$$

Let $x^*_{1,K}$ be a random variable taking values in X'. After defining $x^*_{2,K},\ldots,x^*_{n-1,K}$ and $\theta^*_{1,K},\ldots,\theta^*_{n-1,K}$ define,

$$x^*_{n,K} = a_i \quad \text{if} \tag{5.11}$$

$$x^*(\theta_{n-1,K}) \in \; [d_i \; , \; d_{i+1})$$

and

$$\theta^*_{n,K} = \sum_{i=0}^{K} n^{-1} M_{i,n} [F^{-1}(d_i, J_{M_{i,n}})]_a^b \quad . \tag{5.12}$$

For the sake of completeness we state, without proof, the following theorems.

The proofs follow by simple reworking of the arguments in the theorems (3.2) and (3.3) after replacing $g(\theta)$ by $x^*(\theta)$ and $g(\theta_{n,K})$ by $x^*(\theta_{n,K})$.

Theorem (5.1)

For the modified discrete adaptive design defined by (3.5.11) and (3.5.12), if $d_j \neq x^*(\theta)$, $j = 0,1,2,\ldots,K$, then

(i) $\theta^*_{n,K} \overset{a.s.}{\to}$ as $n \to +\infty$,

(ii) $\limsup_n |X^*_{n,K} - x^*(\theta)| \overset{a.s.}{\leq} (b'-a')/K$,

(iii) $\sqrt{n}\,(\theta^*_{n,K} - \theta) \overset{\mathcal{D}}{\to} N(0,D^*)$ as $n \to +\infty$

where

$$D^* = N^{-1}\, F(d_i,\theta)(1 - F(d_i,\theta)) \left\{\frac{\partial}{\partial e} F^{-1}(d_i,e)/e = F(d_i,\theta)\right\}^2$$

and d_i is the maximal design level belonging to $\mathcal{X}'$ less that $x^*(\theta)$.

Theorem (5.2)

For the modified d.a.d. defined by (5.11) and (5.12) if $a_{i+1} = x^*(\theta)$ for some i, then

(i) $\theta^*_{n,K} \overset{a.s.}{\to} \theta$ as $n \to +\infty$,

(ii) $\lim\sup |X^*_{n,K} - x^*(\theta) \overset{a.s.}{\leq} (b'-a')/K$.

Remarks.

(i) All the remarks following theorems (3.2) and (3.3) trivially carry over to the modified d.a.d. procedure (5.11), (5.12).

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