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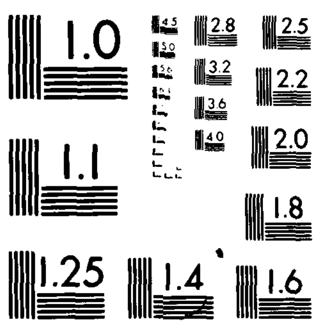

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REPORT SIT-DL-2118

November 1980

A PARTIALLY CAVITATING HYDROFOIL IN A GUST

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by

A. S. Peters

T. R. Goodman

J. P. Breslin

Final Report

November 1978 - November 1980

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ABSTRACT (Continued)

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A PARTIALLY CAVITATING HYDROFOIL IN A GUST

by

A. S. Peters
T. R. Goodman
J. P. Breslin

Performed Under
Office of Naval Research Contract N00014-77-C-0122
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Approved: J. P. Breslin
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ABSTRACT

An analysis is made of a hydrofoil with partial cavitation in a traveling gust. The problem is formulated in the time domain with pressure (acceleration potential) as the fundamental dependent variable. It is shown that, in the limit, as the aspect ratio of the foil becomes large, the second time derivative of the cross-sectional area of the cavity vanishes regardless of the shape of the gust. Integral equations are presented through which the length of the cavity and the unsteady pressure distribution on the foil can be determined. These equations are highly nonlinear and require formidable numerical procedures to solve; no attempt is made to solve them here. It is shown in an appendix that the earliest possible appearance of a nonvanishing second time derivative of the cross-sectional area of the cavity is to the order of the square of the reciprocal of the aspect ratio and that to this order the pressures at distances large in comparison with the chord of the foil (but not the span) grow logarithmically. It may be inferred from this result that the pressures in the far field can be reduced by increasing the aspect ratio of the foil.

NOMENCLATURE

A	nondimensional cross-sectional area of the cavity
B	denotes wetted portion of foil
C	denotes cavity
c	nondimensional local chord of the cavity
$\bar{c}$	c/ε
F	complex function whose real part is P
f	in the appendix ℓ/ε (the same as h in the text)
h	ℓ/ε
i	$\sqrt{-1}$
$\underline{i}_1$	unit vector in free stream direction
$\underline{i}$	unit vector along the chord of the foil
$\underline{j}_1$	unit vector normal to free stream direction
$\underline{j}$	unit vector normal to $\underline{i}$
K_o	a constant
K	a constant
ℓ	nondimensional local chord of foil
n	ordinate of cavity surface
P	acceleration potential
P_c	nondimensional cavity pressure
p	nondimensional pressure
S	semi-span
s	nondimensional distance along span
$\hat{t}$	time
t	nondimensional time
$\hat{\underline{u}}$	perturbation velocity due to foil
U	free stream velocity
$\underline{u}$	nondimensional perturbation velocity
$\hat{\underline{v}}_\infty$	vector sum of free stream velocity and gust velocity
$\hat{\underline{v}}$	unsteady velocity of fluid
$\hat{v}_g$	gust velocity
v_g	nondimensional gust velocity
W	denotes wake
$\hat{x}_1$	distance in free stream direction
$\hat{x}$	distance along chord of foil
x	nondimensional distance along chord of foil
$\hat{y}$	distance normal to chord of foil

y	nondimensional distance normal to chord of foil
z	$x + iy$
α	angle of attack
β	$\sqrt{(\ell-c)/c}$
γ	x/ε
ε	small quantity inversely proportional to aspect ratio
ζ	mapped complex plane = $\xi + i\eta$
η	imaginary coordinate in mapped plane
κ_1, κ_2	unknown functions of time
λ	dummy spacelike variable
μ	nondimensional difference between ambient pressure and cavity pressure
ν	ξ/ε
ξ	nondimensional dummy chordwise coordinate in first section; also in second section real coordinate in mapped plane
ρ	fluid density
σ	nondimensional dummy spanwise coordinate; also in third section denotes dummy time
τ	t/ε ; also in third section denotes $t-x$
Φ	transformed acceleration potential
ϕ	nondimensional velocity potential
ω	y/ε
$(\ )_1$	except where specifically defined, denotes first order in ε; similarly for $(\ )_2$, $(\ )_3$, $(\ )_4$

INTRODUCTION

Ship propeller blades are known to cavitate intermittently during their traverse of regions of large hull wake fractions or low inflow velocity. Such cavities are responsible for large pressure changes on the hull with concurrent intense hull and sub-structural vibration. It is known that the vibratory forces generated by cavitating propellers are much larger than the forces generated by noncavitating propellers. The reason for this is that the cavity behaves in the far field like a source whose strength is proportional to the second time derivative of the cavity volume[1]. It is also significant that cavity-induced forces arising at the first few integer multiples of blade frequency are of the same order as those at blade frequency. This is in sharp distinction to the excitations induced by non-cavitating blade loadings which are invariably small at all higher multiples of blade frequency, as has been pointed out by Breslin[1]. Thus, it is clear that the dynamics of unsteady cavities on propeller blades must be understood and reduced to a calculation yielding their time-varying geometry before the pressure field (which induces the hull vibration) can be predicted.

The complete description of intermittent cavitation on propeller blades requires a three-dimensional representation accounting for the mutual interaction of cavitating and non-cavitating blade elements as well as the mutual interaction of the blades. This is clearly a formidable task. As a first step, we consider the blades to be hydrofoils acting independently and we examine the limiting case of large aspect ratio. The problem is thereby reduced to a single two-dimensional hydrofoil subjected to a periodic gust pattern which is spatially stationary in hull-fixed coordinates and, therefore, travels with the negative speed of the section in foil-fixed coordinates.

In the past, there was, for an extensive period of time, a continuing controversy about the possibility of solving unsteady cavity problems in two dimensions because the existence of time-varying cavities of finite length implies a time-varying source flow whose potential and non-convective pressure fields do not vanish at large distance, but grow as log r. Birkhoff[2], Leehey[3] and others sought to invoke compressibility as a means for resolving this paradox. Benjamin[4] put the question to rest by pointing out that such two-dimensional problems are artifacts of attempts to reduce three-dimensional problems to manageable problems in two dimensions and that they really represent the inner behavior of three-dimensional flows with potentials and pressures vanishing as a three-dimensional source, i.e., as $(r^2 + s^2)^{-1/2}$, where s is the spanwise coordinate. Thus, the logarithmic character of the mathematical representation of two-dimensional time-varying cavity flows must be included in order to match the inner expansion of the outer field. Benjamin went on to assert that the rate of expansion of the cavity section should be regarded as a parameter of the inner field and could only be determined by resorting to the three-dimensional outer field. With these concepts in mind, therefore, the problem will be set up as a three-dimensional one and the aspect ratio will then be taken to be large.

Several attempts to solve unsteady, partially cavitating hydrofoil problems have been made, but all have fallen short of the goal set here, principally because they considered only small disturbances on an established steady cavity.* In contrast, we must address flow patterns with cavities whose size can grow and diminish over several orders of magnitude during one cycle. In one of the earliest assaults on the non-steadily cavitating hydrofoil, Steinberg and Karp[6] studied the small disturbance case, but they imposed a dubious condition, requiring that the velocity be continuous across the foil-cavity wake while ignoring the cavity closure condition altogether. Wang and Wu[7] studied only small perturbations on an established steady cavity. Unruh and Bass[8] presume that the cavity leading edge can be assigned arbitrarily downstream of the foil leading edge, whereas, many observations in water tunnels and a vacuum tank reveal that, on propeller blades, cavitation almost invariably initiates at the leading edge. Furthermore, they also imposed the, by now familiar, restriction to perturbations of an already existing steady cavity.

The three-dimensional problem posed by intermittent cavitation on ship propellers has been addressed by Johnsson and Søntvedt[9], Noordzij[10] and, more recently, by Kaplan, Bentson and Breslin[11]. All of these have employed quasi-steady strip theory utilizing the steady, partially cavitating, section theory of Guerst[12] with no consideration of unsteady flow mechanics.

An exception to this restriction is presented by Van Houten[5] using a numerical procedure.

Despite this restriction, surprisingly good correlation with measurements of hull pressures on models and on ships are reported in Reference 11.

The problem is posed on the pressure as the primary dependent variable. Because the length and cross-sectional area of the cavity (both unknown *a priori*) may undergo large excursions during a cycle, it is not possible to solve the problem in the frequency domain. Instead, it must be set up as an initial value problem in the time domain. As will be seen, this results in a pair of nonlinear integro-differential equations (with time as independent variable) which must be solved numerically. Inasmuch as the input is periodic, the output must also be periodic and, even though an initial value problem is posed, periodicity will be achieved by running through several cycles of input allowing the transients introduced by the starting conditions to decay. No numerical results are presented here, however, and the primary result of the analysis is to show that, in two dimensions, the second time derivative of the cross-sectional area of the cavity vanishes identically regardless of the shape of the gust. As a consequence, neither the potential nor the pressure grow logarithmically at large distance. In an appendix, it is then shown that only to the order of the square of the reciprocal of the aspect ratio is there a possibility that the second time derivative of the cross-sectional area does not vanish. The solution at this order allows for the possibility that the three-dimensional flow behaves like a three-dimensional source in the far field. The practical inference to be drawn from this result is that the vibratory pressures in the far field will be reduced as the aspect ratio is increased. Propeller blades of high aspect ratio are known to be more prone to cavitate than propeller blades of low aspect ratio. So a compromise must be drawn between low aspect ratio blades which cavitate less and high aspect ratio blades which induce lower vibratory pressures in the far field even though they are more prone to cavitate.

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A PARTIALLY CAVITATING FOIL

Our present interest is in the linear theory for a partially cavitating planar foil which is immersed in an inviscid, incompressible, fluid of constant density ρ. We suppose that the unsteady velocity of the fluid is given by

$$\underline{\hat{v}} = \underline{\hat{v}}_\infty + \underline{\hat{u}}$$

where $\underline{\hat{u}}$ denotes a perturbation velocity while

$$\underline{\hat{v}}_\infty = U\,\underline{i}_1 + \hat{v}_g(U\hat{t} - \hat{x}_1)\underline{j}_1$$

$$\underline{i}_1 = \underline{i}\cos\alpha + \underline{j}\sin\alpha$$

$$\underline{j}_1 = -\underline{i}\sin\alpha + \underline{j}\cos\alpha$$

$$\hat{x}_1 = \hat{x}\cos\alpha + \hat{y}\sin\alpha$$

represents a uniform flow upon which is superposed a traveling cross gust $\hat{v}_g$. We will take $\hat{v}/U$ and the angle of attack α to be of the same order of small magnitude.

Hereafter, all terms will be rendered in dimensionless form by using U; the semi-span S; and the constant density ρ as unit quantities. The linearized velocity vector is then

$$\underline{v} = \underline{i} + [\alpha + v_g(t-x)]j + \underline{u}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors along the x and y axes respectively. We also suppose that $\underline{u}$ is source free in the domain of the fluid and that $\underline{u}$ is irrotational in the fluid domain minus the wake so that in the latter domain there exists the velocity potential function $\phi(x,s,y,t)$.

Figure 1 (see following page) shows the area B of the foil and the cavity surface whose height n is assumed to be small. The trailing edge of the cavity is assumed to lie always in B; here $x = c(s,t)$ and the foil leading edge bound the area C under the cavity. The quantity ε which appears in the equation for the trailing edge of the foil is supposed to be small so that the foil is one of large aspect ratio. This foil with the straight leading edge suffices for the exposition of the basic ideas. Its wake W is defined by

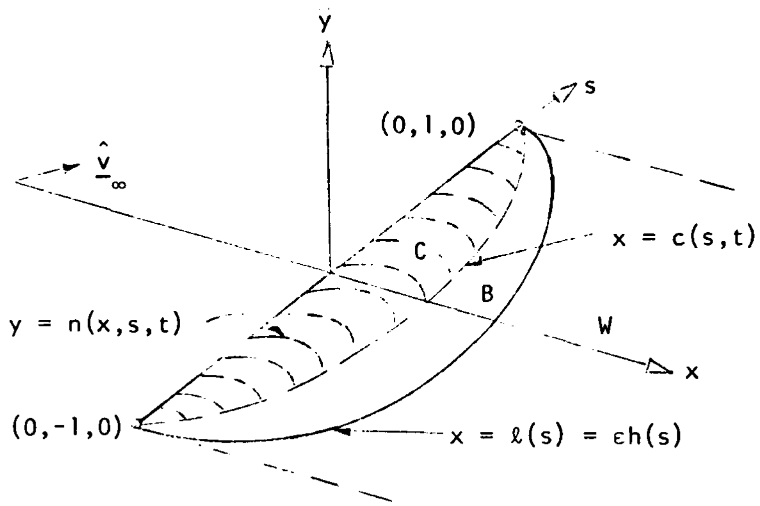

FIGURE 1. SKETCH OF CAVITATING FOIL

W:- $y = 0; \quad \ell(s) < x \leq \infty; \quad -1 < s < 1.$

The velocity potential ϕ (x,s,y,t) must satisfy

$$\phi_{xx} + \phi_{ss} + \phi_{yy} = 0$$

in the domain of the fluid minus the wake and be such that

$$\phi_y(x,s,0+,t) = -[\alpha + v_g(t-x)] \quad c(s,t) < x < \ell(s);\ -1 < s < 1$$

$$\phi_y(x,s,0-,t) = -[\alpha + v_g(t-x)] \quad 0 < x < \ell(s);\ -1 < s < 1.$$

For the same domain, a linearization of the Euler momentum equations provides the first order acceleration potential

$$P(x,s,y,t) = \phi_x + \phi_t = -p + |\phi_x + \phi_t + p|_\infty$$

where p is the pressure. The pressure $p[x,s,n(x,s,t),t]$ at the cavity wall is required to be a function of t only. Since n is small, the linearized version of this condition is

$$p(x,s,0+,t) = p_c(t)$$

which leads to the condition

$$\phi_x(x,s,0+,t) + \phi_t(x,s,0+,t) = -p_c(t) + |\phi_x + \phi_t + p|_\infty = \mu(t)$$

for (x,s) in c, that is

$$0 < x < c(s,t)$$

$$-1 < s < 1 \quad .$$

The cavity function $n(x,s,t)$ is determined by using the linearized kinematic boundary condition for the surface

$$y = n(x,s,t).$$

This linearized condition is

$$n_x(x,s,t) + n_t = \alpha + v_g(t-x) + \phi_y(x,s,0+,t), 0 < x < c(s,t); \ -1 < s < 1$$

subject to

$$n(0,s,t) = 0 \quad -1 < s < 1$$

and the closure condition

$$n[c(s,t),s,t] = 0 \quad -1 < s < 1 \quad .$$

Let us begin the determination of $\phi(x,s,y,t)$ with the representation

$$\phi(x,s,y,t) = \frac{1}{4\pi} \iint_{B+W} \frac{y}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot [\phi] d\xi d\sigma$$

$$- \frac{1}{4\pi} \iint_{B+W} \frac{1}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot [\phi_y] d\xi d\sigma$$

$$= I_1 + I_2$$

$$[\phi] = \phi(\xi,\sigma,0+,t) - \phi(\xi,\sigma,0-,t)$$

$$[\phi_y] = \phi_y(\xi,\sigma,0+,t) - \phi_y(\xi,\sigma,0-,t) \quad .$$

This implies a representation for the acceleration potential $P(x,s,y,t)$. We have

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) I_1 = - \frac{1}{4\pi} \iint_{B+W} \frac{\partial}{\partial \xi} \frac{y}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot [\phi] d\xi d\sigma$$

$$+ \frac{1}{4\pi} \iint_{B+W} \frac{y}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot [\phi_t] d\xi d\sigma \quad .$$

At the leading edge $[\phi] = 0$; therefore, since $y/[\quad]^{3/2} \to 0$ as $\xi \to \infty$, an integration by parts yields

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) I_1 = \frac{1}{4\pi} \iint_{B+W} \frac{y}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \{[\phi_\xi] + [\phi_t]\} d\xi d\sigma \quad .$$

However, we require the acceleration potential

$$P(x,s,y,t) = \phi_x + \phi_t$$

to be continuous in the wake and so

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) I_1 = \frac{1}{4\pi} \iint_B \frac{y}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot [P] d\xi d\sigma \quad .$$

Since we also require the normal velocity ϕ_y to be continuous in the wake, we have $[\phi_y] = 0$ there, from which

$$I_2 = -\frac{1}{4\pi} \iint_B \frac{1}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot [\phi_y] d\xi d\sigma$$

$$= \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{\ell(\sigma)} \frac{1}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot \frac{\partial}{\partial \xi} \int_{\xi}^{\ell(\sigma)} [\phi_y] d\lambda \, d\xi d\sigma$$

$$= -\frac{1}{4\pi} \int_{-1}^{1} \frac{1}{[x^2 + (\sigma-s)^2 + y^2]^{1/2}} \int_{o}^{\ell(\sigma)} [\phi_y] d\xi \, d\sigma$$

$$+ \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{\ell(\sigma)} \frac{(\xi-x)}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot \int_{\xi}^{\ell(\sigma)} [\phi_y] d\lambda \, d\xi d\sigma \quad .$$

Then

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) I_2 = -\frac{1}{4\pi} (\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) \int_{-1}^{1} \frac{1}{[x^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot \int_{o}^{\ell(\sigma)} [\phi_y] d\xi \, d\sigma$$

$$- \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{\ell(\sigma)} \frac{\partial}{\partial \xi} \left\{ \frac{(\xi-x)}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \right\} \cdot \int_{\xi}^{\ell(\sigma)} [\phi_y] d\lambda \, d\xi d\sigma$$

$$+ \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{\ell(\sigma)} \frac{(\xi-x)}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \cdot \frac{\partial}{\partial t} \int_{\xi}^{\ell(\sigma)} [\phi_y] d\lambda \, d\xi d\sigma$$

and an integration by parts of the second term on the right-hand side of the last equation shows

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) I_2 = - \frac{1}{4\pi} \frac{\partial}{\partial t} \int_{-1}^{1} \frac{1}{[x^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot \int_{o}^{\ell(\sigma)} [\phi_y] d\xi \, d\sigma$$

$$+ \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{\ell(\sigma)} \frac{(\xi-x)}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \left\{ \frac{\partial}{\partial t} \int_{\xi}^{\ell(\sigma)} [\phi_y] d\lambda - [\phi_y] \right\} d\xi d\sigma \; .$$

Downstream of the partial cavity and on B, we have $[\phi_y] = 0$ by virtue of the boundary conditions. Therefore

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) I_2 = - \frac{1}{4\pi} \frac{\partial}{\partial t} \int_{-1}^{1} \frac{1}{[x^2 + (\sigma-s)^2 + y^2]^{1/2}} \cdot \int_{o}^{c(\sigma,t)} [\phi_y] d\xi \cdot d\sigma$$

$$+ \frac{1}{4\pi} \int_{-1}^{1} \int_{o}^{c(\sigma,t)} \frac{(\xi-x)}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}} \left\{ \frac{\partial}{\partial t} \int_{\xi}^{c(\sigma,t)} [\phi_y] d\lambda - [\phi_y] \right\} d\xi d\sigma \; .$$

This can be expressed in terms of $n(x,s,t)$ if we use the condition $n_x(x,s,t) + n_t = \phi_y(x,s,0+,t) + \alpha + v_g(t-x) = [\phi_y]$ which holds on C; and the closure condition

$$n[c(s,t),s,t] = 0 \quad ;$$

with

$$n(0,s,t) = 0 \quad .$$

We find

$$\int_{o}^{c(\sigma,t)} [\phi_y] d\xi = \int_{o}^{c} n_t d\xi = \frac{\partial}{\partial t} \int_{o}^{c} n(\xi,\sigma,t) d\xi = A_t(\sigma,t)$$

where A is the cross-sectional area of the cavity. Also

$$M(\xi,\sigma,t) \equiv \frac{\partial}{\partial t} \int_{\xi}^{c(\sigma,t)} [\phi_y] d\lambda - [\phi_y]$$

$$= \frac{\partial}{\partial t} \int_{\xi}^{c} \{n_\lambda(\lambda,\sigma,t) + n_t\} d\lambda - n_\xi(\xi,\sigma,t) - n_t$$

$$M(\xi,\sigma,t) = \frac{\partial}{\partial t}\int_{\xi}^{c(\sigma,t)} n_t(\lambda,\sigma,t)d\lambda - 2n_t(\xi,\sigma,t) - n_\xi(\xi,\sigma,t) \quad .$$

Our development shows that the acceleration potential can be expressed as

$$P(x,s,y,t) = \phi_x(x,s,y,t) + \phi_t(x,s,y,t)$$

$$= -\frac{1}{4\pi}\int_{-1}^{1}\frac{1}{[x^2+(\sigma-s)^2+y^2]^{1/2}}\cdot A_{tt}(\sigma,t)d\sigma$$

$$+\frac{1}{4\pi}\int_{-1}^{1}\int_{o}^{c(\sigma,t)}\frac{(\xi-x)\ M(\xi,\sigma,t)d\xi d\sigma}{[(\xi-x)^2+(\sigma-s)^2+y^2]^{3/2}}$$

$$+\frac{1}{4\pi}\int_{-1}^{1}\int_{o}^{\ell(\sigma)}\frac{y[P]d\xi d\sigma}{[(\xi-x)^2+(\sigma-s)^2+y^2]^{3/2}}$$

which is the same as

$$P(x,s,y,t) = \frac{1}{4\pi}A_{tt}(s,t)\ \ell n(x^2+y^2)$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\frac{|\sigma-s|}{\sigma-s}\ \ell n[|\sigma-s|+\sqrt{(\sigma-s)^2+x^2+y^2}]A_{tt}(\sigma,t)d\sigma$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{o}^{c(\sigma,t)}\frac{(\xi-x)}{(\xi-x)^2+y^2}\ \frac{(s-\sigma)\ M(\xi,\sigma,t)d\xi d\sigma}{[(s-\sigma)^2+(\xi-x)^2+y^2]^{1/2}}$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{o}^{\ell(\sigma)}\frac{y}{(\xi-x)^2+y^2}\cdot\frac{(s-\sigma)\ [P]d\xi d\sigma}{[(s-\sigma)^2+(\xi-x)^2+y^2]^{1/2}}$$

We introduce the quantities:

$$x = \varepsilon\gamma \qquad \xi = \varepsilon\nu \quad ; \quad c(\sigma,t) = \varepsilon\bar{c}(\sigma,t) \quad ; \quad A = \varepsilon^2\bar{A}$$
$$t = \varepsilon\tau \qquad y = \varepsilon\omega \quad , \quad \ell(\sigma) = \varepsilon h(\sigma)$$

which implies a high aspect ratio wing and, because of the scaling of the time, it also implies oscillations whose wave length is of the order of the chord.

We now seek an asymptotic expansion for $\phi(x,s,y,t)$ such that

$$\phi(x,s,y,t) = \phi(\varepsilon\gamma,s,\varepsilon\omega,\varepsilon\tau) = \sum_{k=1}^{\infty} \varepsilon\nu_k(\varepsilon)\phi_k(\gamma,s,\omega,\tau)$$

where

$$L_{\varepsilon\to o} \frac{\nu_{k+1}(\varepsilon)}{\nu_k(\varepsilon)} = 0 \quad .$$

Each ϕ_k is to be determined by equating terms of like order after the expansion has been substituted for ϕ in the representation for P. However, for the time being, let us forego a detailed development of the expansion and be concerned in this paper with only the lowest order term $\phi_1(x,s,y,t)$. In the appendix, however, the higher order terms are examined.

Apart from the term

$$\frac{1}{4\pi} A_{tt}(s,t)\ \ell n(x^2+y^2) \quad , \ (\equiv \frac{1}{4\pi} \bar{A}_{\tau\tau}\ \ell n(x^2+y^2)),$$

the lowest order approximation to P can be obtained by neglecting the combinations x^2+y^2 and $(\xi-x)^2+y^2$ which appear in the radicals in the last expression for P. With respect to the non-integrated logarithmic term, we need to consider

$$\frac{1}{4\pi} A_{tt}\ \ell n\ \varepsilon^2(\gamma^2+\omega^2)$$

and hence

$$\frac{1}{2\pi} A_{1tt}(s,t)\ \ell n\ \varepsilon = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^c [\phi_{1y}]d\xi \cdot \ell n\ \varepsilon \quad .$$

Without loss of generality, we can take $\nu_1(\varepsilon) = 1$ so that the order of

$$\frac{1}{2\pi} A_{1tt}(s,t)\ \ell n\ \varepsilon$$

is given by $\ell n\ \varepsilon$. Now this order cannot be matched with that of any other term which appears in the expansion of P. We, therefore, conclude that we are forced to set

$$A_{1tt}(s,t) = 0 \qquad -1 < s < 1 \quad .$$

With this, it follows that the first order approximation to P is given by

$$P_1(x,s,y,t) = \phi_{1x}(x,s,y,t) + \phi_{1t}(x,s,y,t)$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{0}^{c(\sigma,t)} \frac{s-\sigma}{|s-\sigma|}\,\frac{(\xi-x)M_1(\xi,\sigma,t)\,d\xi\,d\sigma}{(\xi-x)^2+y^2}$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{0}^{\ell(\sigma)} \frac{s-\sigma}{|s-\sigma|}\cdot\frac{y[P_1]\,d\xi\,d\sigma}{(\xi-x)^2+y^2}$$

or

$$P\;(x,s,y,t) = \frac{1}{2\pi}\int_{0}^{c(s,t)} \frac{(\xi-x)M_1(\xi,s,t)\,d\xi}{(\xi-x)^2+y^2}$$

$$+ \frac{1}{2\pi}\int_{0}^{\ell(s)} \frac{y\;[P_1]\,d\xi}{(\xi-x)^2+y^2} \quad .$$

We see from the last result that, with s as a parameter, $P_1(x,s,y,t)$ is a two-dimensional harmonic function in the exterior of the cut along the x-axis from 0 to $\ell(s)$; and we also see that P_1 vanishes as $\sqrt{x^2+y^2} \to \infty$.

Analytic Determination of ϕ_1

Let us turn to the determination of $P_1 = \phi_{1x}(x,s,y,t) + \phi_{1t}$ and $\phi_1(x,s,y,t)$. For typographical convenience, we will drop the subscript and often suppress the parameter s. With this understanding, the first problem we have to solve is this: Find $P(x,y,t)$ such that

$$P(x,y,t) = \phi_x(x,y,t) + \phi_t(x,y,t)$$

is harmonic in the exterior of the slit along the x-axis from 0 to $\ell(s)$, while

$$P(x,0+,t) = \phi_x(x,0+,t) + \phi_t(x,0+,t) = \mu(t) \qquad 0 < x < c(s,t);$$

$$P_y(x,0+,t) = \phi_{yx}(x,0+,t) + \phi_{yt}(x,0+,t) = 0 \qquad c(s,t) < x < \ell(s);$$

$$P_y(x,0-,t) = \phi_{yx}(x,0-,t) + \phi_{yt}(x,0-,t) = 0 \qquad 0 < x < \ell(s) \quad .$$

We assume that, if $P(x,y,t)$ possesses any singularities, then they exist at only one or both of the points

$$(0,0) \quad ; \quad (c,0+) \quad ;$$

where (0,0)denotes the leading edge of the foil and (c,0+) denotes the moving trailing edge of the cavity. Let us suppose that the physically acceptable solution for $P(x,y,t)$ is the one with the mildest singularities at $(0,0)$ and $(c,0+)$. With respect to the behavior in the neighborhood of infinity, we impose the condition dictated by the three-dimensional analysis, namely

$$P \to 0$$

as $\sqrt{x^2 + y^2} \to \infty$

If $z = x + iy$, the function defined by

$$\zeta = \xi + i\eta = \sqrt{\sqrt{\frac{\ell - z}{z}} - \sqrt{\frac{\ell - c}{c}}}$$

$$= \sqrt{\sqrt{\frac{\ell - z}{z}} - \beta}$$

$$\beta = \beta(t) = \sqrt{\frac{\ell - c(t)}{c(t)}}$$

maps the exterior of the slit along the x-axis from 0 to ℓ into the quarter plane $\mathrm{Im}\,\zeta > 0$, $\mathrm{Re}\,\zeta > 0$ where Im denotes the imaginary part and Re denotes the real part. For this mapping, the branch $\sqrt{(\ell - z)/z}$ is defined as shown:

$$\sqrt{(\ell - z)/z} = \left|\sqrt{(\ell - z)/z}\right| \exp i(\theta_1 - \theta)/2$$

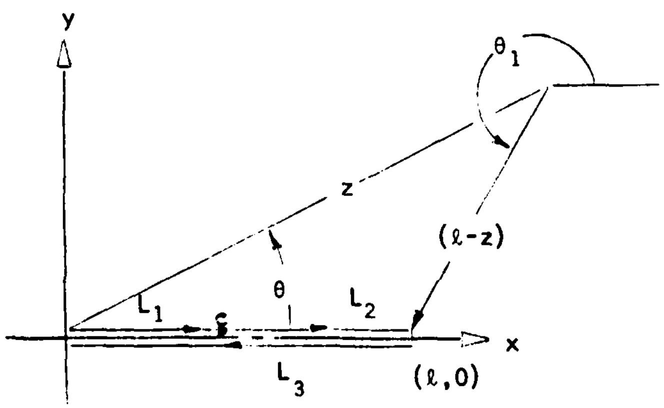

FIGURE 2. PHYSICAL PLANE

(The quantity $\beta = \sqrt{(\ell-c)/c}$, for example, is negative.) Then L_1 is *mapped*

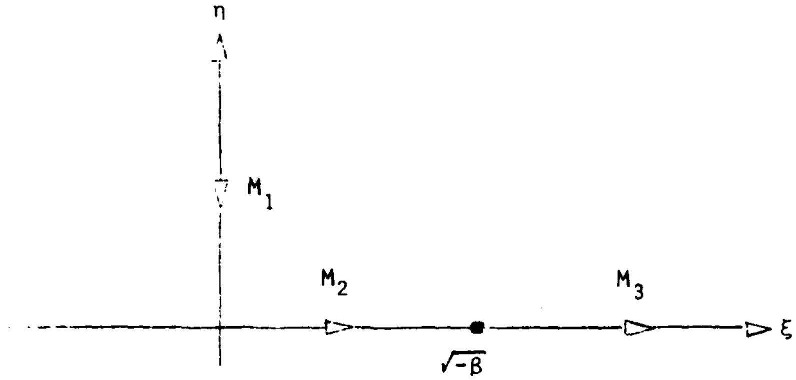

FIGURE 3. MAPPED PLANE

into M_1 defined by

$$\xi = 0 \quad ; \quad \infty > \eta > 0 \quad ;$$

L_2 is mapped into M_2 defined by

$$\eta = 0 \quad ; \quad 0 < \xi < \sqrt{-\beta} \quad ;$$

and L_3 is mapped into M_3 defined by

$$\eta = 0 \quad ; \quad \sqrt{-\beta} < \xi < \infty \quad .$$

The point at infinity in the z-plane is mapped into

$$\sqrt{\sqrt{\frac{\ell-\infty}{\infty}} - \beta} = \sqrt{i-\beta}$$

Under this mapping P(x,y,t) is transformed into a function $\Phi(\xi,\eta)$ which is harmonic in $\xi > 0$; $\eta > 0$ with singularities at $\zeta = 0$ and $\zeta = \infty$. The function

$$\Phi - \mu(t)$$

vanishes along M_1 while

$$\frac{\partial}{\partial\eta}\{\Phi - \mu(t)\}$$

vanishes along M_2 and M_3 . By using the theory of analytic continuation, it follows that Φ is the real part of a function of ζ which, except for poles, is analytic in the ζ-plane. (The continuation shows that we cannot admit branch point singularities.) If we admit only the mildest singularities at $\zeta = 0$ and $\zeta = \infty$, we have

$$\Phi = \mu(t) + \kappa_1(t)\mathrm{Re}\,\frac{1}{\zeta} + \kappa_2(t)\mathrm{Re}\,\zeta$$

or in terms of z

$$P(x,y,t) = \mu(t) + \kappa_1(t)\mathrm{Re}\,\frac{1}{\sqrt{\sqrt{\frac{\ell-z}{z}}-\beta}} + \kappa_2(t)\mathrm{Re}\sqrt{\sqrt{\frac{\ell-z}{z}}-\beta} \quad .$$

In order to avoid excessive writing in the sequel, let us express the last result as

$$P(x,y,t) = \mathrm{Re}\,F(z,t) \quad .$$

We now have the problem of finding the velocity potential $\phi(x,y,t)$ which is harmonic in the xy-plane minus the positive x-axis; exhibits a *wake downstream along the x-axis from* $x = \ell$ *to* $x = \infty$; *and which satisfies*

$$\phi_x + \phi_t = \mathrm{Re}\,F(z,t)$$

The general solution of this equation can be expressed as

$$\phi(x,y,t) = \int_{a-t}^{0} \mathrm{Re}\,F(\sigma+z,\ \sigma+t)d\sigma + \phi(x-t+a,y,a)$$

where a is an arbitrary constant. We may suppose that $\phi(x,y,0) = 0$ for all (x,y). Then, if we take a = 0, we have

$$\phi(x,y,t) = \int_{-t}^{0} \mathrm{Re}\,F(\sigma+z,\sigma+t)d\sigma \quad .$$

This function is harmonic for $0 < \mathrm{ang}\ z < 2\pi$; and, if we write it in the form

$$\phi(x,y,t) = \int_{x-t}^{x} \mathrm{Re}\,F(\lambda+iy,\lambda+t-x)d\lambda \quad ,$$

it is not difficult to verify that it shows a downstream wake in the sense that ϕ and ϕ_x suffer a jump, provided t is sufficiently large, as z crosses the downstream x-axis.

Determination of the Coefficients

The next step is to enforce the two boundary conditions which ϕ must satisfy: the closure condition, and the condition on the regular part of P at infinity.

The component ϕ_y is

$$\phi_y(x,y,t) = \int_{-t}^{o} \frac{\partial}{\partial y} \text{Re } F(\sigma+z,\sigma+t)d\sigma$$

$$= \frac{\partial}{\partial x} \int_{-t}^{o} \text{Re } i\ F(\sigma+z,\sigma+t)d\sigma$$

$$= \frac{\partial}{\partial x} \int_{o}^{t} \text{Re } i\ F(\sigma+z-t,\sigma)d\sigma \quad .$$

For $0 < x - 0i < \ell$, we must have

$$-[\alpha+v_g(t-x)] = \phi_y(x,0-,t) = \frac{\partial}{\partial x} \int_{o}^{t} \text{Re } i\,F(\sigma-t+x-0i,\sigma)d\sigma$$

$$= \frac{\partial}{\partial x} \{ \int_{o}^{t-x} + \int_{t-x}^{t} \text{Re } i\,F(\sigma-t+x-0i,\sigma)d\sigma\} \quad .$$

However, $\text{Re } i\,F(\sigma-t+x-0i,\sigma) = 0$ for $0 < \sigma - t+x - 0i < \ell$ or $t-x < \sigma < t + \ell-x$ and, consequently, we must have

$$-[\alpha+v_g(t-x)] = \frac{\partial}{\partial x} \int_{o}^{t-x} \text{Re } i\,F(\sigma-t+x-0i,\sigma)d\sigma \quad .$$

Then, if we set $t-x = \tau$, we see that for $\tau > 0$

$$\alpha + v_g(\tau) = \frac{\partial}{\partial \tau} \int_{o}^{\tau} \text{Re } i\,F(\sigma-\tau,\sigma)d\sigma$$

or

$$\int_{o}^{\tau} [\alpha + v_g(\sigma)]d\sigma = \int_{o}^{\tau} \text{Re } i\,F(\sigma-\tau,\sigma)d\sigma \quad . \tag{1}$$

This is the first equation which relates the unknowns κ_1, κ_2 and β.

For $c(t) < x + 0i < \ell$, we must have

$$-[\alpha+v_g(t-x)] = \frac{\partial}{\partial x}\int_0^t \mathrm{Re}\, i\, F(\sigma-t+x+0i,\sigma)d\sigma$$

$$= \frac{\partial}{\partial x}\left\{\int_0^{t-x} + \int_{t-x}^{t} \mathrm{Re}\, i\, F(\sigma-t+x+0i,\sigma)d\sigma\right\} \quad .$$

We already know that

$$-[\alpha+v_g(t-x)] = \frac{\partial}{\partial x}\int_0^{t-x} \mathrm{Re}\, i\, F(\sigma-t+x-0i,\sigma)d\sigma$$

$$= \frac{\partial}{\partial x}\int_0^{t-x} \mathrm{Re}\, i\, F(\sigma-t+x+0i,\sigma)d\sigma$$

for $t-x \geq 0$ because the space variable $\sigma-t+x$, in the range

$$-(t-x) < \sigma - t + x < 0$$

required in the last integrals lies ahead of the foil and F is continuous there. Therefore

$$0 = \frac{\partial}{\partial x}\int_{t-x}^{t} \mathrm{Re}\, i\, F(\sigma-t+x+0i,\sigma)d\sigma \tag{2}$$

$$c(t) < x < \ell$$

is the second equation which relates the unknowns.

Equation (2) can be simplified. A change in the dummy variable of integration, $\sigma-t+x=\lambda$, gives

$$0 = \frac{\partial}{\partial x}\int_0^{x} \mathrm{Re}\, i\, F(\lambda+0i,\lambda+t-x)d\lambda \tag{2a}$$

$$c < x < \ell \quad .$$

Note that

$$\mathrm{Re}\, i\, F(\lambda+0i,\lambda+t-x) = 0$$

when

$$c(\lambda+t-x) < \lambda < \ell \quad .$$

This inequality defines an upper limit for the integral in (2a). For simplicity, suppose that

$$\lambda = c(\lambda+t-x)$$

has but one root λ_1 for each t-x. Then

$$\lambda_1 \equiv c(\lambda_1+t-x)$$

and this root is a function of t-x so that we can write

$$\lambda_1 = \lambda_1(t-x) \quad .$$

(The root λ_1 is given by the ordinate of the point of intersection of

$$\lambda = c(\sigma)$$

and the line

$$\lambda = \sigma - t + x$$

which passes through the point (t,x) with slope equal to 1.) Consequently, equation (2a) becomes

$$\frac{\partial}{\partial x} \int_0^{\lambda_1(t-x)} \mathrm{Re}\, i\, F(\lambda+0i, \lambda+t-x)\, d\lambda = 0$$

or

$$\frac{\partial}{\partial x} \int_0^{c(\lambda_1+t-x)} \mathrm{Re}\, i\, F(\lambda+0i, \lambda+t-x)\, d\lambda = 0 \quad ;$$

and, if we introduce $t - x = \tau$, we have

$$\frac{d}{d\tau} \int_0^{c(\lambda_1+\tau)} \mathrm{Re}\, i\, F(\lambda+0i, \lambda+\tau)\, d\lambda = 0 \quad .$$

Integration gives

$$\int_0^{c(\lambda_1+\tau)} \mathrm{Re}\, i\, F(\lambda+0i, \lambda+\tau)\, d\lambda$$

$$= \int_0^{\lambda_1(0)} \mathrm{Re}\, i\, F(\lambda+0i, \lambda)\, d\lambda = \mathrm{cons't.} = K_0$$

This can be written as

$$\int_{o}^{c(\lambda_1+\tau)} \text{Re } i\, F[\lambda+0i,\lambda+\tau+\lambda_1-c(\lambda_1+\tau)]d\lambda = K_o$$

and if we set

$$t^* = \lambda_1 + \tau$$

we have

$$\int_{o}^{c(t^*)} \text{Re } i\, F[\lambda+0i,\lambda+t^*-c(t^*)]d\lambda = K_o \tag{2b}$$

where t^* is arbitrary.

We now turn our attention to the cavity surface, the equation for which is

$$y = n(x,s,t) \qquad 0 \leq x \leq c(s,t)$$

or, if we suppress s,

$$y = n(x,t) \qquad 0 \leq x \leq c(t)$$

and, as we have seen in the three-dimensional analysis, $n(x,t)$ can be determined by using the linearized kinematic condition

$$n_x(x,t) + n_t(x,t) = \phi_y(x,0+,t) + [\alpha+v_g(t-x)] \qquad 0 \leq x \leq c(t)$$

In terms of F this is

$$n_x(x,t) + n_t(x,t) = \frac{\partial}{\partial x}\int_{o}^{t} \text{Re } i\, F(\sigma-t+x+0i,\sigma)d\sigma + [\alpha+v_g(t-x)]$$

$$= \frac{\partial}{\partial x}\left\{\int_{o}^{t-x} + \int_{t-x}^{t} \text{Re } i\, F(\sigma-t+x+0i,\sigma)d\sigma\right\} + [\alpha+v_g(t-x)]$$

which, by virtue of the first condition (1), reduces to

$$n_x(x,t) + n_t(x,t) = \frac{\partial}{\partial x}\int_{t-x}^{t} \text{Re } i\, F(\sigma-t+x+0i,\sigma)d\sigma \qquad 0 \leq x \leq c(t)$$

for $t - x > 0$. Now $n(x,t)$ must vanish at the leading edge, i.e.,

$$n(0,t) = 0$$

and the only solution of the differential equation for $n(x,t)$ which satisfies this condition is

$$n(x,t) = \frac{\partial}{\partial x}\int_0^x (x-\lambda)\,\mathrm{Re}\; i\,F(\lambda+0i,\lambda+t-x)\,d\lambda \quad .$$

The formula for the cross-sectional area A(s,t) determined by the cavity surface and the foil has already been used in the three-dimensional analysis. Here it is:

$$A(t) = \int_0^{c(t)} n(x,t)\,dx$$

$$= \int_0^{c(t)} [c(t) - \lambda]\,\mathrm{Re}\; i\,F[\lambda+0i,\lambda+t-c(t)]\,d\lambda \quad .$$

The closure condition to be imposed is

$$n[c(t),t] = 0 \quad .$$

This can be enforced by integrating

$$n_x(x,t) + n_t(\lambda,t) = \frac{\partial}{\partial x}\int_{t-x}^{t} \mathrm{Re}\; i\,F(\sigma-t+x+0i,\sigma)\,d\sigma$$

$$= \frac{\partial}{\partial x}\int_0^x \mathrm{Re}\; i\,F(\lambda+0i,\lambda+t-x)\,d\lambda$$

$$0 \leq x \leq c(t)$$

Since $n(0,t) = 0$, we have

$$n(x,t) + \int_0^x n_t(\xi,t)\,d\xi = \int_0^x \mathrm{Re}\; i\,F(\lambda+0i,\lambda+t-x)\,d\lambda$$

or, using $n[c(t),t] = 0$,

$$\int_0^{c(t)} \frac{\partial}{\partial t}\, n(\xi,t)\,d\xi = \frac{d}{dt}\int_0^{c(t)} n(\xi,t)\,d\xi = \int_0^{c(t)} \mathrm{Re}\; i\,F(\lambda+0i,\lambda+t-c)\,d\lambda$$

which is the same as

$$A_t(t) = \int_0^{c(t)} \mathrm{Re}\; i\,F[\lambda+0i,\lambda+t-c(t)]\,d\lambda \quad . \qquad (3)$$

This equation relates the rate of change of the cross-sectional area of the cavity to the unknown parameters κ_1, κ_2 and β. However, Eqs. (2b) and (3) imply that

$$A_t = K_o$$

from which

$$A_{tt} = 0$$

(as we already know is required) and

$$A = K_o t + K \quad .$$

If a monotonic growth of the cavity cross-sectional area is precluded, it follows that we must take

$$K_o = 0 \quad .$$

It appears then that the first order approximation produces a constant value for the cross-sectional area of the partial cavity. Equation (3) then becomes

$$\int_o^{c(t)} \mathrm{Re}\, i\, F[\lambda+0i, \lambda+t-c(t)]\, d\lambda = 0 \tag{3a}$$

and we now have equations (1) and (3a) for the determination of κ_1, κ_2, and β.

The last equation for the determination of the unknowns is

$$L_{|z|\to\infty} \mathrm{Re}\, F(z,t) = 0$$

or

$$\mathrm{Re}\{\mu(t) + \frac{\kappa_1(t)}{\sqrt{i-\beta}} + \kappa_2(t)\sqrt{i-\beta}\} = 0 \quad . \tag{4}$$

The equations (1), (3a) and (4) form a set for the determination of κ_1, κ_2 and β. The equations are linear in κ_1 and κ_2, but disconcertingly nonlinear in β. Two of them are highly nonlinear integral equations in t and the parameter s. Their solution for given $\alpha, v_g(t)$ and $\mu(t)$ can only be accomplished by using formidable numerical procedures.

CONCLUDING REMARKS

The result $\phi_x + \phi_t = \mathrm{Re}\, F(z,t)$ presented on page 14, where $F(z,t)$ is also given there, can be shown to subsume some classical results. For example, if the dependence on time is eliminated, and κ_1 and κ_2 are evaluated from the kinematic condition that the vertical velocity is simply α, then, together with the (steady) closure condition, the result can be shown to reduce to that of Guerst[12].

As another example, take $\kappa_1 = 0$ and write

$$\phi_x + \phi_t = \mu(t) + \sqrt{-\beta}\,\kappa_2\, \mathrm{Re}\, \left(1 - \frac{\sqrt{(\ell - z)/z}}{\sqrt{-\beta}}\right)^{1/2}$$

As $\beta \to \infty$, take $\mu - \sqrt{-\beta}\,\kappa_2 = 0$ and we have

$$\phi_x + \phi_t = \frac{\kappa_2}{2}\, \mathrm{Re} \sqrt{(\ell - z)/z}$$

which is the case of the unsteady, completely wetted, foil.

The analysis presented thus far is in a somewhat unsatisfactory state inasmuch as it has been shown that $A_{tt} = 0$ so that no two-dimensional source-like flow at distances large in comparison with the chord of the foil is possible. It is clear that this result comes about because the aspect ratio has been taken to be infinite. It is desirable to determine to what order in aspect ratio a non-zero value of A_{tt} will occur. To this end, we are devising a procedure for the methodical determination of higher order asymptotic approximations. If ε is the reciprocal of the aspect ratio, and, if the orders of the asymptotic approximations are characterized by 1, ε, $\varepsilon^2 \ell n \varepsilon$, ε^2...., we find that there are strong indications that $A_{\kappa tt}$ remains equal to zero at least until κ corresponds to ε^2. Thus, it may be expected that, as the aspect ratio is increased, the time variations of the cavity cross-sectional area and, hence, the variation of the total volume of the cavity on the three-dimensional foil) will diminish.

Finally, comparison of the formalism of this report with the two-dimensional analysis of Van Houten shows that, whereas we have demonstrated the necessity for taking $A_{tt} = 0$, thereby eliminating the source-like behavior of the pressure at large distances, Van Houten has retained this term in his work. We believe this to be improper, especially in view of the requirement that the boundary condition at the cavity can only be a statement that the difference between the cavity pressure and the ambient pressure is given, whereas, if the pressure grows logarithmically at large distances, no ambient pressure can be defined.

The solution of the integral equations which Van Houten analyzes numerically is equivalent to finding the acceleration potential

$$P^*(x,s,y,t) = \phi_x^* + \phi_t^*$$

$$= \frac{1}{2\pi} A_{tt}^*(s,t)\ \ell n\sqrt{x^2+y^2}$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\frac{|\sigma-s|}{\sigma-s}\ \ell n 2|\sigma-s|\ A_{tt}^*(s,t)d\sigma$$

$$+ \frac{1}{2\pi}\int_0^{c(s,t)}\frac{(\xi-x)M^* d\xi}{(\xi-x)^2+y^2}$$

$$+ \frac{1}{2\pi}\int_0^{\ell(s)}\frac{y[P^*]d\xi}{(\xi-x)^2+y^2}$$

where

$$A_t^*(s,t) = \int_0^{c(s,t)} [\phi_y^*]d\xi$$

$$M^* = \frac{\partial}{\partial t}\int_{\xi}^{c(s,t)} [\phi_y^*]d\lambda - [\phi_y^*]$$

and P^* is subject to the cavity condition

$$P^*(x,s,0+,t) = \mu(t) \qquad 0 < x < c(s,t);\ -1 < s < 1;$$

and the condition

$$P_y^*(x,s,0^{\pm},t) = 0$$

along the wetted surface if we take

$$\phi_y^*(x,s,0^{\pm},t) = -\alpha - v_g(t-x)$$

there. The representation for P^* shows that it may possess a logarithmic singularity at infinity. However, if we admit such a singularity when P^* is determined by the function theory method used above, it turns out again that $A^*_{tt}(s,t)$ must be zero in order to satisfy the boundary condition on the upper side of the foil downstream of the cavity and the closure condition. Thus, it appears that the first order approximation $A^*_{tt} = 0$ can be regarded as a consequence of the imposed boundary conditions for the two-dimensional approximating problem rather than a consequence of matching in an asymptotic expansion.

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APPENDIX

HIGHER ORDER APPROXIMATIONS
FOR A CAVITATING FOIL

We proceed to give a method for the theoretical determination of higher order approximations for the cavitating foil. It is beyond the scope of the present report to supply all of the details of the development. It is expected that a fuller analysis will be produced later. Here, our object is to find the order of approximation which gives the first indication that $A_{tt} \neq 0$ where A is the cross-sectional area of the cavity.

We have found that the acceleration potential is

$$P(x,s,y,t) = \phi_x(x,s,y,t) + \phi_t(x,s,y,t)$$

$$= -\frac{1}{4\pi}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \int_{-1}^{1} \int_{0}^{c(\sigma,t)} \frac{[\phi_y] d\xi d\sigma}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{1/2}}$$

$$+ \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{\ell(\sigma)} \frac{y[P] d\xi d\sigma}{[(\xi-x)^2 + (\sigma-s)^2 + y^2]^{3/2}}$$

where

$$[\phi] = [\phi(\xi,\sigma,t)] = \phi(\xi,\sigma,0+,t) - \phi(\xi,\sigma,0-,t) \quad .$$

In order to satisfy the boundary condition for the cavity, we must satisfy

$$P(x,s,0+,t) - \frac{1}{2}[P(x,s,t)] = -\frac{1}{4\pi}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \int_{-1}^{1} \int_{0}^{c(\sigma,t)} \frac{[\phi_y] d\xi d\sigma}{[(\xi-x)^2 + (\sigma-s)^2]^{1/2}}$$

for $\quad 0 < x < c(s,t) \quad ; \quad -1 < s < 1 \quad .$

In order to satisy the boundary condition for the wetted part of the foil, we must satisfy

$$\phi_{xy}(x,s,0\pm,t) + \phi_{ty}(x,s,0\pm,t)$$

$$= \pm \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) [\phi_y(x,s,t)] - \frac{1}{4\pi} \frac{\partial^2}{\partial x \partial s} \int_{-1}^{1} \int_{0}^{\ell(\sigma)} \frac{\sqrt{(x-\xi)^2 + (s-\sigma)^2}\,[P] d\xi d\sigma}{(x-\xi)(s-\sigma)} \quad .$$

An integration gives

$$\frac{\partial}{\partial x} \int_{x}^{\infty} \phi_y(\xi,s,0\pm,t) d\xi + \frac{\partial}{\partial t} \int_{x}^{\infty} \phi_y(\xi,s,0\pm,t) d\xi$$

$$= \pm \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \int_x^\infty [\phi_y(\xi,s,t)]d\xi$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{0}^{\ell(\sigma)} \left\{\frac{\sqrt{(x-\xi)^2 + (s-\sigma)^2}}{(x-\xi)(s-\sigma)} - \frac{1}{s-\sigma}\right\} [P]d\xi d\sigma$$

and if we solve this for

$\int_x^\infty \phi_y(\xi,s,0\pm,t)d\xi$ we find

$$\int_x^\infty \phi_y(\xi,s,0\pm,t)d\xi = \pm \frac{1}{2}\int_x^\infty [\phi_y(\xi,s,t)]d\xi$$

$$+ \frac{1}{4\pi}\int_{x-t}^{x}\frac{\partial}{\partial s}\int_{-1}^{1}\int_{0}^{\ell(\sigma)} \left\{\frac{\sqrt{(\lambda-\xi)^2 + (s-\sigma)^2}}{(\lambda-\xi)(s-\sigma)} - \frac{1}{s-\sigma}\right\} [P(\xi,\sigma,\lambda+t-x)]d\xi d\sigma d\lambda \quad .$$

In dimensional terms, the normal velocity along the wetted part of the foil is required to be

$$-[U\alpha + \hat{v}_g(U\hat{t}-\hat{x})]$$

In dimensionless terms, this velocity is

$$-[\alpha + \frac{\hat{v}_g}{U}(St-Sx)]$$

where S is the semi-span. For a foil of large aspect ratio, we can take

$$S = \frac{\kappa}{\varepsilon}$$

where ε is small and κ is the chord length, so that the normal velocity can be expressed as

$$-\{\alpha + \frac{\hat{v}_g}{U}[\frac{\kappa}{\varepsilon}(t-x)]\}$$

or

$$-[\alpha + \bar{v}_g(\frac{t-x}{\varepsilon})]$$

In dimensional terms, the acceleration potential at the cavity wall is $\hat{p}(\hat{t})$.

In dimensionless terms, it is

$$\frac{\hat{p}(\frac{St}{U})}{\rho U^2} = \frac{\hat{p}(\frac{\kappa t}{\varepsilon U})}{\rho U^2} = \bar{p}(\frac{t}{\varepsilon}) \quad .$$

In dimensional terms, the trailing edge of the foil is

$$\hat{x} = S\ell(s) = S\varepsilon f(s)$$

and in dimensionless terms

$$x = \varepsilon f(s) \quad .$$

In dimensional terms, the trailing edge of the cavity is given by

$$\hat{x} = \hat{c}(s,\hat{t}) = S\varepsilon\bar{c}(s, \frac{St}{U}) = Sc(s,t)$$

and in dimensionless terms

$$x = \varepsilon\,\bar{c}(s, \frac{\kappa t}{U\varepsilon}) = \varepsilon\tilde{c}(s,\frac{t}{\varepsilon}) \quad .$$

Since the unknowns depend on the parameter ε, let us introduce

$$x = \varepsilon\gamma \quad ; \quad \xi = \varepsilon\omega \quad ; \quad y = \varepsilon\mu \quad ; \quad t = \varepsilon\tau$$

and assume the inner asymptotic expansions

$$\phi(x,s,y,t) = \phi(\varepsilon\gamma,s,\varepsilon\mu,\varepsilon\tau) = \varepsilon\tilde{\phi}(\nu,s,\mu,\tau;\varepsilon) = \sum_{\kappa=1}^{\infty} \varepsilon\nu_\kappa(\varepsilon)\ \phi_\kappa(\nu,s,\mu,\tau)$$

$$\tilde{c}(s,\frac{t}{\varepsilon}) = \sum_{\kappa=1}^{\infty} \nu_\kappa(\varepsilon)c_\kappa(s,\tau)$$

where

$$\mathop{L}_{\varepsilon\to o}\frac{\nu_{\kappa+1}(\varepsilon)}{\nu_\kappa(\varepsilon)} = 0 \quad .$$

In terms of the new variables, the boundary condition for the cavity is

$$\tilde{\mathcal{P}}(\nu,s,0+,\tau;\varepsilon) - \frac{1}{2}[\tilde{P}(\gamma,s,\tau;\varepsilon)]$$

$$= -\frac{1}{4\pi}(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}) \int_{-1}^{1} \int_{0}^{\tilde{c}(\sigma,\tau)} \frac{[\tilde{\phi}_\mu(\omega,\sigma,\tau;\varepsilon)]d\omega d\sigma}{[(s-\sigma)^2 + r^2]^{1/2}}$$

where $r^2 = \varepsilon^2(\omega-\gamma)^2$

and $0 < \gamma < \tilde{c}$.

This can be expanded into

$$\tilde{P}(\gamma,s,0+,\tau;\varepsilon) - \frac{1}{2}[\tilde{P}(\gamma,s,\tau;\varepsilon)]$$

$$= \left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\left\{\begin{array}{l} \frac{1}{2\pi}\int_0^{\tilde{c}(s,\tau)} \ln\varepsilon|\omega-\gamma|\cdot[\tilde{\phi}_\mu]d\omega \\ + \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\int_0^{\tilde{c}(\sigma,\tau)} \frac{|\sigma-s|}{\sigma-s}\ln 2|\sigma-s| \cdot[\tilde{\phi}_\mu]d\omega d\sigma \\ - \frac{1}{8\pi}\frac{\partial^2}{\partial s^2}\int_0^{\tilde{c}} \varepsilon^2(\omega-\gamma)^2\ln\varepsilon|\omega-\gamma| \cdot [\phi_\mu]d\omega \\ + \frac{1}{4\pi}\frac{\partial^3}{\partial s^3}\int_{-1}^{1}\int_0^{c(\sigma,\tau)}\left\{\begin{array}{l} -\frac{|\sigma-s|}{\sigma-s}\cdot\frac{\varepsilon^2}{4}(\omega-\gamma)^2 \\ -\frac{|\sigma-s|}{\sigma-s}\cdot\frac{\varepsilon^2}{4}(\omega-\gamma)^2\ln 2|\sigma-s| \\ + o(\varepsilon^2;\sigma,s,\omega,\gamma)\end{array}\right\}|[\tilde{\phi}_\mu]d\omega d\sigma \end{array}\right. \tag{A-1}$$

In terms of the new variables the boundary condition for the wetted surface is

$$\int_\gamma^\infty \tilde{\phi}_\mu(\omega,s,0\pm,\tau;\varepsilon)d\omega = \pm\frac{1}{2}\int_\gamma^{\tilde{c}}[\tilde{\phi}_\mu(\omega,s,\tau;\varepsilon)]d\omega$$

$$+ \frac{1}{4\pi}\int_{\gamma-\tau}^{\gamma}\frac{\partial}{\partial s}\int_{-1}^{1}\int_0^{f(\sigma)}\left\{\frac{\varepsilon\sqrt{(s-\sigma)^2+q^2}}{q(s-\sigma)} - \frac{\varepsilon}{s-\sigma}\right\}[\tilde{P}(\omega,\sigma,\lambda+\tau-\gamma;\varepsilon)]d\omega d\sigma d\lambda$$

where

$q = \varepsilon(\lambda-\omega)$.

An expansion of this yields

$$\int_\gamma^\infty \tilde{\phi}_\mu(\omega,s,0\pm,\tau;\varepsilon)d\omega \mp \frac{1}{2}\int_\gamma^{\tilde{c}}[\tilde{\phi}_\mu(\omega,s,\tau;\varepsilon)]d\omega$$

$$+ \frac{\varepsilon}{4\pi}\int_{\gamma-\tau}^{\gamma}\frac{\partial}{\partial s}\int_{-1}^{1}\int_0^{f(\sigma)}\frac{[\tilde{P}(\omega,\sigma,\lambda+\tau-\gamma;\varepsilon)]}{s-\sigma}d\omega d\sigma d\lambda$$

$$= \frac{1}{2\pi} \int_{\gamma-\tau}^{\gamma} \int_{0}^{f(\sigma)} \frac{[\tilde{P}(\omega,s,\lambda+\tau-\gamma;\varepsilon)]}{\lambda-\omega} d\omega d\lambda$$

$$+ \frac{1}{4\pi} \int_{\gamma-\tau}^{\gamma} \frac{\partial^3}{\partial s^3} \int_{-1}^{1} \int_{0}^{f(\sigma)} \left\{ \begin{array}{l} -\frac{3}{4}\varepsilon^2 \frac{|s-\sigma|}{s-\sigma} \\ \\ -\frac{\varepsilon^2}{2} \frac{|s-\sigma|}{s-\sigma} \ell n 2|s-\sigma| \\ \\ -\varepsilon^2 \; h(\sigma-s)\ell n\varepsilon|\lambda-\omega| \end{array} \right\} (\lambda-\omega)[\hat{P}(\omega,s,\lambda+\tau-\gamma;\varepsilon)] d\omega d\sigma d\lambda \qquad \text{(A-2)}$$

$$+ \frac{1}{4\pi} \int_{\gamma-\tau}^{\gamma} \frac{\partial^3}{\partial s^3} \int_{-1}^{1} \int_{0}^{f(\sigma)} o(\varepsilon^2;\sigma,s,\lambda,\omega) \cdot [\tilde{P}(\omega,\sigma,\lambda+\tau-\gamma;\varepsilon)] d\omega d\sigma d\lambda$$

where

$$h(\sigma-s) = \begin{cases} 1 & \sigma > s \\ 0 & \sigma < s \end{cases}$$

The expression (A-2) shows that the inner asymptotic expansion of $\phi(x,s,y,t)$ should have the form $\phi(x,s,y,t) = \phi(\varepsilon\gamma,s,\varepsilon\mu,\varepsilon\tau) = \varepsilon\tilde{\phi}(\gamma,s,\mu,\tau;\varepsilon)$ where

$$\tilde{\phi}(\gamma,s,\mu,\tau,\varepsilon) = \phi_1(\gamma,s,\mu,\tau) + \varepsilon\,\phi_2 + \varepsilon^2\ell n\varepsilon \cdot \phi_3 + \varepsilon^2\phi_4 + \ldots$$

Then for $P(x,s,y,t)$, we have

$$P(x,s,y,t) = \phi_x(x,s,y,t) + \phi_t(x,s,y,t) \text{ or}$$

$$\tilde{P} = \tilde{\phi}_\gamma(\gamma,s,\mu,\tau;\varepsilon) + \tilde{\phi}_\tau(\gamma,s,\mu,\tau;\varepsilon)$$

$$= P_1(\gamma,s,\mu,\tau) + \varepsilon P_2$$

$$+ \varepsilon^2\ell n\varepsilon \cdot P_3 + \varepsilon^2 P_4 + \ldots$$

The function $P(x,s,y,t)$ is a three-dimension potential function in the exterior of the foil. If we apply the operator $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial y^2})$ to the assumed asymptotic expansion, we find

$$\sum_{\kappa=1}^{\infty} \nu_\kappa(\varepsilon)\ [P_{\kappa\gamma\gamma} + P_{\kappa\mu\mu} + \varepsilon^2 P_{\kappa ss}] = 0$$

and, hence, each $P_\kappa(\gamma,s,\mu,\tau)$ must satisfy

$$P_{\kappa\gamma\gamma}(\gamma,s,\mu,\tau) + P_{\kappa\mu\mu} = H_\kappa(\gamma,s,\mu,\tau)$$

in the exterior of the slit defined by

$$0 < \gamma < f(s) \quad ; \quad \mu = 0 \quad .$$

For example, we have

$$P_{\kappa\gamma\gamma} + P_{\kappa\mu\mu} = 0 \qquad \kappa = 1,2,3$$

while

$$P_{4\gamma\gamma} + P_{4\mu\mu} = -P_{1ss} \quad .$$

The equations for the ϕ_κ's can now be found by equating terms of like order in (A-1) and (A-2).

Consider first ϕ_1: In (A-1) there is no match for the term

$$\frac{\ell n\varepsilon}{2\pi}\ \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right) \int_0^{c_1(s,\tau)} [\phi_{1\mu}]d\omega \quad ;$$

and, therefore, we must take

$$\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right) \int_0^{c_1} [\phi_{1\mu}]d\omega = \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right) A_{1\tau}(s,\tau) = A_{1\tau\tau}(s,\tau) = 0 \quad .$$

Here A_1 is the first order approximation to the cavity area.

For the cavity, ϕ_1 must then satisfy the integral equation

$$\bar{p}(\tau) - \frac{1}{2}\ [P_1(\gamma,s,\tau)] = \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right) \int_0^{c_1(s,\tau)} \ell n|\omega-\gamma|\ [\phi_{1\mu}]d\omega \quad ;$$

and, for the wetted surface, it must satisfy

$$\int_\gamma^\infty \phi_{1\mu}(\omega,s,0\pm,\tau)\,d\omega = \pm\frac{1}{2}\int_\gamma^{c_1} [\phi_{1\mu}(\omega,s,\tau)]d\omega + \frac{1}{2\pi}\int_0^\tau\int_0^{f(s)} \frac{[P_1(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

from which

$$\phi_{1\mu}(\gamma,s,0\pm,\tau) = \pm\frac{1}{2}[\phi_{1\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{\tau}\int_0^{f(s)}\frac{[P_1(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

and

$$\phi_{1\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{1\mu\tau}(\gamma,s,0\pm,\tau)$$

$$= \pm\frac{1}{2}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)[\phi_{1\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{f(s)}\frac{[P_1(\omega,s,\tau)]d\omega}{\gamma-\omega}$$

where, on the wetted surface,

$$\phi_{1\mu} = -[\alpha + \bar{v}_g(\tau-\gamma)] \quad .$$

As we have seen, these integral equations can be solved by first finding the analytic function $F(z,t)$ such that

$$\text{Re}\,F(z,t) = \phi_{1x}(x,s,y,t) + \phi_{1t}(x,s,y,t)$$

$$= \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1}\ell n\sqrt{(\omega-\gamma)^2+\mu^2}\;[\phi_{1_\mu}]d\omega + \frac{1}{2\pi}\int_0^{f(s)}\frac{\mu[P_1]d\omega}{(\omega-\gamma)^2+\mu^2}$$

$$= \frac{1}{2\pi}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\int_0^{c(s,t)}\ell n\sqrt{(\xi-x)^2+y^2}\cdot[\phi_{1\mu}]d\xi + \frac{1}{2\pi}\int_0^{\ell(s)}\frac{y[P_1]d\xi}{(\xi-x)^2+y^2}$$

$$= \frac{1}{2\pi}\int_0^{c(s,t)}\frac{(\xi-x)\;M_1(\xi,s,t)d\xi}{(\xi-x)^2+y^2} + \frac{1}{2\pi}\int_0^{\ell(s)}\frac{y[P_1]d\xi}{(\xi-x)^2+y^2}$$

The above representations and the integral equations imply that we must have

$$\text{Re}\,F\,(x+0i,t) = p(t)$$

for the cavity and

$$\left.\frac{\partial}{\partial y}\text{Re}\,F\,(z,t)\right|_{y=0\pm} = 0$$

$$\phi_{1y}(x,s,0\pm,t) = -[\alpha+v_g(t-x)]$$

along the wetted surface while $\text{Re}\,F\,(z,t)$ vanishes at infinity.

It has been shown that the first approximation to the cavity surface $y = n\;(x,s,t)$ can be found from

$$n_{1x}(x,s,t) + n_{1t}(x,s,t) = [\phi_{1y}(x,s,0,t)]$$

$$= \frac{\partial}{\partial x}\int_0^x \operatorname{Re} i\, F(\lambda+0i,\ \lambda+t-x)d\lambda$$

$$= \operatorname{Re} i\, F(x+0i,t) + \int_0^x \frac{\partial}{\partial x} \operatorname{Re} i\, F(\lambda+0i,\lambda+t-x)d\lambda$$

for $0 < x < c$. From this

$$[\phi_{1y}(c-\varepsilon,s,0,t)] = \operatorname{Re} i\, F(c-\varepsilon+0i,t) - \int_0^{c-\varepsilon} \left|\frac{\partial}{\partial t} \operatorname{Re} i\, F(\lambda+0i,\lambda+t-x)\right|_{x=c-\varepsilon} d\lambda$$

$$= \frac{\operatorname{Re} i\, F(c-\varepsilon+0i,t)}{1-\dot{c}} - \frac{1}{1-\dot{c}}\frac{\partial}{\partial t}\int_0^{c-\varepsilon} \operatorname{Re} i\, F(\lambda+0i,\lambda+t-c+\varepsilon)d\lambda$$

if ε is a sufficiently small positive quantity.

The closure condition

$$n(c,s,t) = 0$$

gives

$$\int_0^c n_{1t}(\xi,s,t)d\xi = \int_0^c [\phi_{1y}(\xi,s,0,t)]d\xi$$

or

$$A_{1t} = \int_0^c \operatorname{Re} i\, F(\lambda+0i,\lambda+t-c)d\lambda$$

or

$$A_{1tt} = \frac{\partial}{\partial t}\int_0^c \operatorname{Re} i\, F(\lambda+0i,\lambda+t-c)d\lambda \quad ;$$

but we have found that A_{1tt} must vanish. Therefore

$$L_{\varepsilon\to 0}[\phi_{1y}(c-\varepsilon,s,0,t)] = L_{\varepsilon\to 0}\frac{\operatorname{Re} i\, F(c-\varepsilon+0i,t)}{1-\dot{c}}$$

which is infinite as can be seen from the explicit expression for F. On the other hand

$$L_{\varepsilon\to 0}\ [\phi_{1y}(c+\varepsilon,s,0,t] = 0 \quad .$$

Consider second ϕ_2: For the determination of ϕ_2, we need to consider the term

$$\frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial t}\right)\ln\varepsilon \cdot \varepsilon\int_0^{c_1}[\phi_{2\mu}]d\omega$$

We leave $\tilde{c}$ unexpanded and define it to be equal to c_1. Indeed, $[\phi_{1\mu}]$ vanishes beyond c_1 so that such terms can make no contribution to ϕ_2. A similar argument will hold for the higher orders. Since no other term of order $\varepsilon \ln\varepsilon$ appears in (A-1), we must take

$$A_{2\tau\tau} = \frac{\partial}{\partial\tau}\int_0^{c_1}[\phi_{2\mu}]d\omega = 0 \quad .$$

We see now that, even for the second order approximation, the second derivative of the cross-sectional cavity area is zero.

With this, ϕ_2 must satisfy

$$P_2(\gamma,s,0+,\tau) - \frac{1}{2}[P_2(\gamma,s,\tau)] = \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1(s,\ )}\ln|\omega-\gamma|[\phi_{2\mu}]d\omega$$

for the cavity.

For the wetted surface, ϕ_2 must satisfy

$$\int_\gamma^\infty \phi_{2\mu}(\omega,s,0\pm,\tau)d\omega \mp\frac{1}{2}\int_\gamma^{c_1}[\phi_{2\mu}(\omega,s,\tau)]d\omega$$

$$= -\frac{1}{4\pi}\int_0^\tau\frac{\partial}{\partial s}\int_{-1}^{1}\int_0^{f(\sigma)}\frac{[P_1(\omega,\sigma,\lambda)]d\omega d\sigma d\lambda}{s-\sigma} + \frac{1}{2\pi}\int_0^\tau\int_0^{f(s)}\frac{[P_2(\omega,s,\lambda))d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

from which

$$\phi_{2\mu}(\gamma,s,0\pm,\tau) \mp \frac{1}{2}[\phi_{2\mu}(\gamma,s,\tau)] = -\frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^\tau\int_0^{f(s)}\frac{[P_2(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

and hence

$$\phi_{2\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{2\mu\tau}(\gamma,s,0\pm,\tau)$$

$$= \pm\frac{1}{2}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)[\phi_{2\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{f(s)}\frac{[P_2(\omega,s,\tau)]d\omega}{\gamma-\omega}$$

These integral equations can be solved by first finding the potential function

$$\mathrm{Re}\, F_2(z,t) = \phi_{2x}(x,s,y,t) + \phi_{2t}(x,s,y,t)$$

$$= \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1} \ln\sqrt{(\omega-\gamma)^2+\mu^2}\cdot[\phi_{2\mu}]d\omega + \frac{1}{2\pi}\int_0^{f(s)} \frac{\mu[P_2]d\omega}{(\omega-\gamma)^2+\mu^2}$$

$$= \frac{1}{2\pi}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\int_0^{c} \ln\sqrt{(\xi-x)^2+y^2}\,[\phi_{2y}]d\xi + \frac{1}{2\pi}\int_0^{\ell(s)} \frac{y[P_2]d\xi}{(\xi-x)^2+y^2}$$

$$= \frac{1}{2\pi}\int_0^{c} \frac{(\xi-x)M_2\, d\xi}{(\xi-x)^2+y^2} + \frac{1}{2\pi}\int_0^{\ell(s)} \frac{y[P_2]d\xi}{(\xi-x)^2+y^2} \quad .$$

The function $\mathrm{Re}\,F_2(z,t)$ must vanish at infinity and the integral equations show that it must be such that

$$\mathrm{Re}\,F_2\,(x+0i,t) = P_2(x,s,0+,t)$$

for $0 < x < c$, while along the wetted surface

$$\frac{\partial}{\partial y}\,\mathrm{Re}\,F_2(z,t)\Big|_{y=0\pm} = \phi_{2xy}(x,s,0,t) + \phi_{2ty}(x,s,0,t)$$

where $\phi_{2y}(x,s,0,t)$ is an induced velocity on the wetted surface and $P_2(x,s,0+,t)$ is an induced acceleration potential at the cavity. The latter quantities are to be determined by matching inner and outer expansions.

Consider next ϕ_3: The component ϕ_3 is to be determined by equating terms of order $\varepsilon^2 \ln \varepsilon$ in (A-1) and (A-2); but first we need to consider the terms

$$\frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1} \varepsilon^2\ln\varepsilon\; [\phi_{3\mu}]d\omega$$

Because the term of order $\varepsilon^2(\ln\varepsilon)^2$ cannot be matched in (A-1), it follows that we must have

$$A_{3\tau\tau} = \frac{\partial}{\partial\tau}\int_0^{c_1}[\phi_{3\mu}]d\omega = 0 \quad .$$

We can now see that ϕ_3 must satisfy

$$P_3(\gamma,s,0+,\tau) - \frac{1}{2}[P_3(\gamma,s,\tau)]$$

$$= \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right)\left\{\begin{array}{l} \frac{1}{2\pi}\int\limits_{o}^{c_1} \ln|\omega-\gamma| \; [\phi_{3\mu}]d\omega \\ + \frac{1}{2\pi}\int\limits_{o}^{c_1} [\phi_{4\mu}]d\omega \\ - \frac{1}{8\pi}\frac{\partial^2}{\partial s^2}\int\limits_{o}^{c_1} (\omega-\gamma)^2 \; [\phi_{1\mu}]d\omega \end{array}\right.$$

for the cavity, i.e., $0 < \gamma < c_1$; $\mu = 0+$.

For the remainder of the foil ϕ_3 must satisfy

$$\int\limits_{\gamma}^{\infty} \phi_{3\mu}(\omega,s,0\pm,\tau)d\omega$$

$$= \pm\frac{1}{2}\int\limits_{\gamma}^{c_1} [\phi_{3\mu}(\omega,s,\tau)]d\omega + \frac{1}{2\pi}\int\limits_{o}^{\tau}\int\limits_{o}^{f(s)} [P_3(\omega,s,\lambda)]d\omega d\lambda$$

$$- \frac{1}{4\pi}\int\limits_{o}^{\tau}\frac{\partial^3}{\partial s^3}\int\limits_{-1}^{1}\int\limits_{o}^{f(\sigma)} h(\sigma-s)\cdot(\lambda+\gamma-\tau-\omega) \; [P_1(\omega,\sigma,\lambda)]d\omega d\sigma d\lambda$$

from which

$$\phi_{3\mu}(\gamma,s,0\pm,\tau) = \pm\frac{1}{2}[\phi_{3\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int\limits_{o}^{\tau}\int\limits_{o}^{f(s)} \frac{[P_3(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

$$+ \frac{1}{4\pi}\int\limits_{o}^{\tau}\frac{\partial^3}{\partial s^3}\int\limits_{-1}^{1}\int\limits_{o}^{f(\sigma)} h(\sigma-s)[P_1(\omega,\sigma;\lambda)]d\omega d\sigma d\lambda$$

and

$$\phi_{3\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{3\mu\tau}(\gamma,s,0\pm,\tau)$$

$$= \pm\frac{1}{2}\left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right)[\phi_{3\mu}(\gamma,s,\tau)]$$

$$- \frac{1}{2\pi} \frac{\partial}{\partial\gamma} \int_{o}^{f(s)} \frac{[P_3(\omega,s,\lambda)]d\omega}{\gamma-\omega}$$

$$+ \frac{1}{4\pi} \frac{\partial^3}{\partial s^3} \int_{-1}^{1} \int_{o}^{f(\sigma)} h(\sigma-s)\ [P_1(\omega,\sigma,\lambda)]d\omega d\sigma \quad .$$

These integral equations, which express the boundary conditions, can be solved by first finding the potential function

$$P_3(\gamma,s,\mu,\tau) = \mathrm{Re}\ F_3(z,\tau)$$

$$= \frac{1}{2\pi} \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right) \int_{o}^{c_1} \ell n\sqrt{(\omega-\gamma)^2+\mu^2}\ [\phi_{3\mu}]d\omega$$

$$+ \frac{1}{2\pi} \int_{o}^{f(s)} \frac{\mu[P_3]d\omega}{(\omega-\gamma)^2+\mu^2} + \frac{1}{2\pi} \frac{\partial}{\partial\tau} \int_{o}^{c_1} [\phi_{4\mu}]d\omega$$

$$- \frac{1}{8\pi} \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right) \frac{\partial^2}{\partial s^2} \int_{o}^{c_1} \{(\omega-\gamma)^2+\mu^2\}\ [\phi_{1\mu}]d\omega$$

$$- \frac{1}{4\pi} \frac{\partial^2}{\partial s^2} \int_{o}^{f(s)} \mu[P_1]d\omega$$

where $z = \gamma + i\mu$. From the integral equations, we find that Re $F_3(z,\tau)$ must satisfy

$\mathrm{Re}\ F_3(\gamma+0i,\tau) = P_3(\gamma,s,0+,\tau)$ for the cavity and

$$\frac{\partial}{\partial\mu} \mathrm{Re}\ F_3(z,\tau)\Big|_{\mu=0\pm} = \phi_{3\mu\gamma}(\gamma,s,0,\tau) + \phi_{3\mu\tau}(\gamma,s,0,\tau)$$

for the remainder of the foil; while $F_3(z,\tau)$ must have a pole at infinity.

If

$$\frac{1}{2\pi} \frac{\partial}{\partial\tau} \int_{o}^{c_1} [\phi_{4\mu}]d\omega$$

is not zero, ϕ_4 remains to be determined from the equations for order ϵ^2 and there is a coupling between the approximations of third and fourth order. In the solution for ReF_3, the induced quantities $\phi_{3\mu}(\gamma,s,0,\tau)$

and $P_3(\gamma,s,0+,\tau)$ are at first unknown. Ultimately they are to be found by matching inner and outer expansions.

Finally, let us consider ϕ_4: Along the cavity interval ϕ_4 must satisfy

$$P_4(\gamma,s,0+,\tau) - \frac{1}{2}\,[P_4(\gamma,s,\tau)]$$

$$= \left(\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\tau}\right)\left\{\begin{aligned} &\frac{1}{2\pi}\int_0^{c_1} \ln|\omega-\gamma|\;[\phi_{4\mu}(\omega,s,\tau)]d\omega \\ &+ \frac{1}{4\pi}\frac{\partial}{\partial s}\int_{-1}^{1}\frac{|\sigma-s|}{\sigma-s}\ln 2|\sigma-s|\int_0^{c_1}[\phi_{4\mu}]d\omega d\sigma \\ &- \frac{1}{8\pi}\frac{\partial^2}{\partial s^2}\int_0^{c_1}(\omega-\gamma)^2\ln|\omega-\gamma|\;[\phi_{1\mu}]d\omega \\ &- \frac{1}{16\pi}\frac{\partial^3}{\partial s^3}\int_{-1}^{1}\int_0^{c_1}\frac{|\sigma-s|}{\sigma-s}(\omega-\gamma)^2\{1 + \ln 2|\sigma-s|\}\;[\phi_{1\mu}]d\omega d\sigma \end{aligned}\right. .$$

Along the wetted surface, ϕ_4 must satisfy

$$\phi_{4\mu}(\gamma,s,0\pm,\tau)$$

$$= \pm\frac{1}{2}\,[\phi_{4\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{\tau}\int_0^{f(s)}\frac{[P_4(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

$$+ \frac{1}{4\pi}\int_0^{\tau}\frac{\partial^3}{\partial s^3}\int_{-1}^{1}\int_0^{f(\sigma)}\left\{\begin{aligned}&\frac{3}{4}\frac{|s-\sigma|}{s-\sigma} + \frac{1}{2}\frac{|s-\sigma|}{s-\sigma}\ln 2|s-\sigma| \\ &+ h(\sigma-s)\;\{\ln|\lambda+\gamma-\tau-\omega|+1\}\end{aligned}\right\}P_1(\omega,\sigma,\lambda)]d\omega d\sigma d\lambda$$

from which

$$\phi_{4\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{4\mu\tau}(\gamma,s,0\pm,\tau)$$

$$= \pm\frac{1}{2}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)[\phi_{4\mu}(\gamma,s,\tau)] - \frac{1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{f(s)}\frac{[P_4(\omega,s,\tau)]d\omega}{\gamma-\omega}$$

$$+ \frac{1}{4\pi}\frac{\partial^3}{\partial s^3}\int_{-1}^{1}\int_0^{f(\sigma)}\left\{\begin{aligned}&\frac{3}{4}\frac{|s-\sigma|}{s-\sigma} + \frac{1}{2}\frac{|s-\sigma|}{s-\sigma}\ln 2|s-\sigma| \\ &+h(\sigma-s)\;\{\ln|\gamma-\omega| + 1\}\end{aligned}\right\}[P_1(\omega,\sigma,\tau)]d\omega d\sigma .$$

In accordance with these integral equations, the function

$$\mathrm{Re}\, F_4(z,\tau) = \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1} \ln\sqrt{(\omega-\gamma)^2+\mu^2}\,[\phi_{4\mu}]d\omega + \frac{1}{2\pi}\int_0^{f(s)} \frac{\mu[P_4]d\omega}{(\omega-\gamma)^2+\mu^2}$$

$$= \frac{1}{2\pi}\ln\sqrt{\gamma^2+\mu^2}\,\frac{\partial}{\partial\tau}\int_0^{c_1}[\phi_{4\mu}]d\omega + \frac{1}{2\pi}\int_0^{f(s)}\frac{\mu[P_4]d\omega}{(\omega-\gamma)^2+\mu^2}$$

$$+ \frac{1}{2\pi}\int_0^{c_1}\frac{\omega-\gamma}{(\omega-\gamma)^2+\mu^2}\left\{\frac{\partial}{\partial\tau}\int_\omega^{c_1}[\phi_{4\mu}]d\omega - [\phi_{4\mu}]\right\}d\omega$$

must satisfy

$$\mathrm{Re}\, F_4(\gamma+0i,\tau) =$$

$$P_4(\gamma,s,0+,\tau)$$

$$+\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\begin{cases} -\dfrac{1}{4\pi}\dfrac{\partial}{\partial s}\displaystyle\int_{-1}^{1}\frac{|\sigma-s|}{\sigma-s}\ln 2|\sigma-s|\cdot\int_0^{c_1}[\phi_{4\mu}]d\omega d\sigma \\[2ex] +\dfrac{1}{8\pi}\dfrac{\partial^2}{\partial s^2}\displaystyle\int_0^{c_1}(\omega-\gamma)^2\ln|\omega-\gamma|\,[\phi_{1\mu}]d\omega \\[2ex] +\dfrac{1}{16\pi}\dfrac{\partial^3}{\partial s^3}\displaystyle\int_{-1}^{1}\int_0^{c_1}\frac{|\sigma-s|}{\sigma-s}(\omega-\gamma)^2\{1+\ln 2|\sigma-s|\}[\phi_{1\mu}]d\omega d\sigma \end{cases}$$

for the cavity, and

$$\frac{\partial}{\partial\mu}\mathrm{Re}\, F_4(z,\tau)\Big|_{\mu=0\pm}$$

$$=\phi_{4\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{4\mu\tau}(\gamma,s,0\pm,\tau)$$

$$-\frac{1}{4\pi}\frac{\partial^3}{\partial s^3}\int_{-1}^{1}\int_0^{f(\sigma)}\left\{\begin{matrix}\dfrac{3}{4}\dfrac{|s-\sigma|}{s-\sigma}+\dfrac{1}{2}\dfrac{|s-\sigma|}{s-\sigma}\ln 2|s-\sigma| \\[1ex] +h(\sigma-s)\quad\{\ln|\gamma-\omega|+1\}\end{matrix}\right\}[P_1]d\omega d\sigma$$

for the wetted part of the foil.

As we have seen, the function $P_4 = \phi_{4\gamma} + \phi_{4\tau}$ must satisfy the Poisson equation

$$P_{4\gamma\gamma} + P_{4\mu\mu} = - P_{1ss}$$

where

$$P_1(\gamma,s,\mu,\tau) = \frac{1}{2\pi}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\tau}\right)\int_0^{c_1} \ell n\sqrt{(\gamma-\omega)^2+\mu^2}\,[\phi_{1\mu}]d\omega$$

$$+ \frac{1}{2\pi}\int_0^{f(s)} \frac{\mu[P_1]d\omega}{(\gamma-\omega)^2+\mu^2}$$

It can be shown that if S_4 is an appropriate particular solution of the Poisson equation, then

$$P_4 = S_4 + \text{Re } F_4(z,\tau) \quad .$$

The potential function Re F_4 possesses a logarithmic singularity at infinity, namely,

$$\frac{1}{2\pi}\frac{\partial}{\partial\tau}\int_0^{c_1} [\phi_{4\mu}]d\omega \cdot \ell n\sqrt{\gamma^2+\mu^2}$$

provided

$$A_{4\tau\tau} = \frac{\partial}{\partial\tau}\int_0^{c_1} [\phi_{4\mu}]d\omega$$

does not vanish. So far there is no apparent reason for supposing that $A_{4\tau\tau} = 0$.

We conclude from our analysis that, if A(s,t) is the cross-sectional cavity area and if A_{tt} does not vanish, then it must be of order equal to or higher than ε^2.

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$$\int_{\gamma} \phi_{3\mu}(\omega,s,0\pm,\tau)d\omega$$

$$= \pm \frac{1}{2} \int_{\gamma}^{c_1} [\phi_{3\mu}(\omega,s,\tau)]d\omega + \frac{1}{2\pi} \int_{0}^{\tau} \int_{0}^{f(s)} [P_3(\omega,s,\lambda)]d\omega d\lambda$$

$$- \frac{1}{4\pi} \int_{0}^{\tau} \frac{\partial^3}{\partial s^3} \int_{-1}^{1} \int_{0}^{f(\sigma)} h(\sigma-s) \cdot (\lambda+\gamma-\tau-\omega) \; [P_1(\omega,\sigma,\lambda)]d\omega d\sigma d\lambda$$

om which

$$_{\mu}(\gamma,s,0\pm,\tau) = \pm \frac{1}{2} [\phi_{3\mu}(\gamma,s,\tau)] - \frac{1}{2\pi} \frac{\partial}{\partial \gamma} \int_{0}^{\tau} \int_{0}^{f(s)} \frac{[P_3(\omega,s,\lambda)]d\omega d\lambda}{\lambda+\gamma-\tau-\omega}$$

$$+ \frac{1}{4\pi} \int_{0}^{\tau} \frac{\partial^3}{\partial s^3} \int_{-1}^{1} \int_{0}^{f(\sigma)} h(\sigma-s)[P_1(\omega,\sigma;\lambda)]d\omega d\sigma d\lambda$$

id

$$\phi_{3\mu\gamma}(\gamma,s,0\pm,\tau) + \phi_{3\mu\tau}(\gamma,s,0\pm,\tau)$$

$$= \pm \frac{1}{2} \left(\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \tau}\right) [\phi_{3\mu}(\gamma,s,\tau)]$$

END

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