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Recent results on the stability of equilibrium solutions of a parabolic equation are given with indications of the proofs. Particular attention is devoted to dependence of the stability properties on the shape of the domain and the manner in which nonhomogeneous stable equilibria can occur through a bifurcation induced by varying the domain.

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STABILITY AND BIFURCATION IN A PARABOLIC EQUATION.*

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STABILITY AND BIFURCATION IN A PARABOLIC EQUATION

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Abstract

Recent results on the stability of equilibrium solutions
of a parabolic equation are given with indications of the
proofs. Particular attention is devoted to dependence of
the stability properties on the shape of the domain and
the manner in which nonhomogeneous stable equilibria can
occur through a bifurcation induced by varying the domain.

In this paper, we present a few recent results on the asymptotic behavior of the solutions of a parabolic equation of the form

$$u_t = \Delta u + f(u) \quad \text{in} \quad \Omega$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary. We also discuss how the qualitative behavior of the stable equilibria depend upon the shape of Ω and the nonlinear function f. The function f is supposed to satisfy conditions which ensure that Eq. (1) defines a strongly continuous semigroup $T_f(t)$ on $H^1(\Omega)$.

We remark that the boundary conditions in (1) are not important as far as the spirit of the problems to be discussed. Of course, the details will depend generally in a very significant manner upon the boundary conditions.

Let $E_f(\Omega)$ be the set of equilibrium solutions of (1); that is, the set of solutions of the equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \ . \tag{2}$$

If $\Omega = (0,\lambda) \subseteq \mathbb{R}$, then the set $E_f(0,\lambda)$ is given by

$$E_f(0,\lambda) = \{\text{periodic solutions of period } 2\lambda \text{ of the} \tag{3}$$
$$\text{equation } u_{xx} + f(u) = 0\}$$

In the following, we let $W^u(\phi)$, $W^s(\phi)$ denote, respectively, the unstable and stable manifolds for an equilibrium point ϕ of (1). The following result for $n = 1$ is due to a number of authors. The references are in the proof.

<u>Theorem 1.</u> (n=1) If $\Omega = (0,\lambda) \subset \mathbb{R}$, then

(i) the ω-limit set of any bounded solution of (1) is in $E_f(0,\lambda)$.

(ii) the ω-limit set of any bounded solution of (1) is a single point in $E_f(0,\lambda)$.

(iii) the only stable equilibrium points of (1) are constants.

(iv) If $\int_0^u f(s)ds \to -\infty$ as $u \to \pm\infty$, then every solution of (1) is bounded. If, in addition, $E_f(0,\lambda)$ is a bounded set, then there is a maximal compact invariant set $A_f(0,\lambda)$ of (1),

$$A_f(0,\lambda) = \bigcup_{\phi \in E_f(0,\lambda)} W^u(\phi),$$

$A_f(0,\lambda)$ is uniformly asymptotically stable and, for any bounded set B in $H^1(\Omega)$, $\text{dist}(T_f(t)B, A_f(0,\lambda)) \to 0$ as $t \to \infty$.

(v)   If, in addition to the hypothesis in (iv), all

$\phi \in E_f(0,\lambda)$ are hyperbolic, then, for any bounded

set B in $H^1(\Omega)$, the set

$$(\cup_{\phi \, stable} W^s(\phi)) \cap B$$

is open and dense in B.

Proof: (i)   This is due to Chafee [4] and is independent of the
boundary conditions. The idea is very simple.

If

$$V(u) = \int_0^\lambda (u_x^2 - \int_0^u f(s)ds)dx$$

then the derivative of V along the solutions of (1) satisfies

$$\dot{V}(u) = -\int_0^\lambda u_t^2 \, dx \leq 0.$$

Since every bounded orbit is precompact, a simple application
of the invariance principle implies the result.

(ii)   This result is due to Matano [13] and is independent
of the boundary   conditions. He used a rather sophisticated
application of the invariance principle. We sketch a proof based
on the theory of dynamical systems. The details will appear in
Hale and Massatt [6]. The idea is very simple and can be traced
to Malkin [12], Hale and Stokes [7] and perhaps even further.

If ϕ is an element of an ω-limit of an orbit which is not a single point, then ϕ belongs to a continuum in $E_f(0,\lambda)$ and ϕ is not hyperbolic. The linear variational equation about ϕ has the simple eigenvalue zero with all other eigenvalues being nonzero. Thus, ϕ belongs to a smooth one dimensional submanifold M of E_f. In modern terminology, ϕ is normally hyperbolic. One now can show that any solution of (1) which remains in a sufficiently small neighborhood of M for t sufficiently large must be of $W^s(M)$, the stable manifold of M. Finally, one shows that each orbit in $W^s(M)$ approaches a single point.

(iii) This result is due to Chafee [4]. The following proof is taken from a preprint of Bardos, Matano and Smoller [1]. Suppose ϕ is a nonconstant equilibrium solution of (1). Then $v = d\phi/dx \not\equiv 0$, $v = 0$ at $x = 0, x = \lambda$, and v satisfies the equation

$$v_{xx} + f'(\phi)v = 0.$$

Let σ_N, σ_D be the spectrum of this differential operator with, respectively, homogeneous Neumann and Dirichlet boundary conditions. Since $\inf \sigma_N < \inf \sigma_D$ and $0 \in \sigma_D$, the result is proved.

(iv) Using the function $V(u)$ in part (i), one easily shows that every solution is bounded. Thus, the ω-limit set of every solution of (1) belongs to $E_f(0,\lambda)$. Since $E_f(0,\lambda)$ is

bounded, there is a bounded set B such that every solution of
(1) eventually enters B; that is, Eq. (1) is point dissipative.
Since the semigroup $T_f(t)$ is compact for $t > 0$, the results
follow from the general theory of dissipative processes (see, for
example, Hale [9,10]).

(v) This result is due to Henry [11]. The idea of the
proof is to observe first that the solution operator for the
linear variational equation of (1) about any point is one-to-one.
This can then be used to show that, for any $\phi \in E_f(0,\lambda)$ which
is not stable, the set $W^s(\phi) \cap B$ is nowhere dense in B for
any bounded set B. As remarked by Mañé, the one-to-oneness of
this solution operator implies that the stable manifold $W^s(\phi)$
can actually be given globally a manifold structure. This also
gives a proof of the assertion in (v).

For $\Omega = (0,\lambda)$, we have remarked that the set $E_f(0,\lambda)$
coincides with the set of 2λ-periodic solutions of $u_{xx} + f(u) = 0$.
For any $a \in \mathbb{R}$, let $u(x,a)$ be the solution of this equation with
$u(0,a) = a, u_x(0,a) = 0$. If a is such that $u(x,a)$ is periodic
in x, let $2\lambda_f(a)$ be the period. The function $u(\cdot,a) \in E_f(0,\lambda_f(a))$.
For $f(u)$ an arbitrary cubic polynomial in u, Smoller and
Wasserman [15] have shown that the function $\lambda_f(a)$ has a finite
number (≤ 2) of maxima and minima and the second derivative of λ
at these points is different from zero; that is, $\lambda_f(a)$ is a
Morse function.

The above result has important implications for the applications. In fact, for f any cubic polynomial in u, and for λ fixed and different from the maxima and minima of the function $\lambda_f(a)$, the set $E_f(0,\lambda)$ consists only of hyperbolic points. For λ equal to one of the extreme values of $\lambda_f(a)$, there is a bifurcation of the saddle-node type.

The following qualitative result of Brunovsky and Chow [2] has recently been proved.

<u>Theorem 2</u>. There is a residual set $\mathscr{F} \in \mathscr{C}^2(\mathbb{R})$ with the Whitney topology such that, for any $f \in \mathscr{F}$, the function $\lambda_f(a)$ above is a Morse function.

The proof is not trivial because the function f depends only on u and not on (x,u). The proof is based on a detailed analysis of an analytic expression of $\lambda_f(a)$ as a function of f,a. It is not a trivial exercise because there is no simple way to determine the qualitative properties of the derivatives of this function in a from the derivatives of f. In fact, there are nonlinear functions f for which $\lambda_f(a)$ is constant (see, for example, Urabe [16]).

Theorem 2 can be appropriately generalized to other boundary conditions (see Brunovsky and Chow [2]). Smoller and Wasserman [15] have also considered other boundary conditions.

Our next objective is to discuss the extent to which the above results are valid for a bounded set Ω in $\mathbb{R}^n$.

Theorem 3. If Ω is a bounded set in $\mathbb{R}^n$ with smooth boundary,
then

(i) the ω-limit set of a bounded orbit is in $E_f(\Omega)$.

(ii) the ω-limit set of a bounded orbit is a single point
if the following condition is satisfied:

If $\phi \in E_f(\Omega)$ is not hyperbolic and k is the
dimension of the null space of the operator $\Delta + f'(\phi)$ in Ω
with homogeneous Neumann conditions, then ϕ belongs to a smooth
submanifold of dimension k.

(iii) Ω convex implies the only stable equilibrium points
are constants.

(iv) Same statement as (iv) in Theorem 1 holds.

(v) Same statement as (v) in Theorem 1 holds.

Proof: The proof of (i), (ii), (iv) and (v) are essentially the
same as the corresponding assertions in Theorem 1.

(iii) This result was independently discovered by Casten
and Holland [3], Matano [14]. The proof exploits
special properties of the Laplacian on convex regions
to prove that the linear variational equation has a
negative eigenvalue for any nonconstant equilibrium.

An analogue of Theorem 2, as far as hyperbolicity of
equilibrium and saddle node bifurcations, is not known for Ω in
$\mathbb{R}^n$ and seems to be rather difficult.

We remark that part (iii) of Theorem 3 is also valid for Ω convex and the equations

$$u_t = \Delta u + f(u), \quad \text{in} \quad \Omega$$

$$v_t = -g(u,v)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

provided that the spectrum of the operator $\partial g(\xi,\eta)/\partial v$ belongs to the set $\{z : \text{Re}\, z > 0\}$ uniformly in ξ, η (see Bardos, Matano and Smoller [1]).

The remainder of the discussion centers around the case when Ω *is not convex* and the objective is to understand more about the set of stable equilibrium. Before doing this, we make the important remark that, when Ω is convex, the qualitative structure of the stable equilibria is independent of the non-linearity f. When Ω is not convex, this will no longer be the case.

The following result is due to Matano [14].

Theorem 4. There is a nonlinear function f and $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, such that (1) has a stable nonconstant equilibrium.

Idea of proof: Suppose f has zeros only at $a < 0 < b$, they are simple and $\int_0^u f(s)ds \to -\infty$ as $u \to \pm\infty$. Suppose Ω has the

shape shown in Figure 1 and let λ_2 be the minima of the
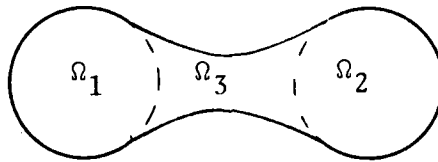

Figure 1

second eigenvalues of the Laplacian on Ω_1 and Ω_2. Matano [ ]
gives a specific continuous function $G: \mathbb{R}^2 \to \mathbb{R}$ such that the
set where $G < 0$ is nonempty with the property that, if λ_2, Ω_3
are such that $G(\lambda_2, \text{meas } \Omega_3) < 0$, then there is a nonconstant
stable equilibrium of (1). For fixed Ω_1, Ω_2, he shows there is
an Ω_3 such that the above inequality is satisfied.

The idea of the proof is the following. Let Y be the
subset of functions u in $H^1(\Omega)$ such that $\int_{\Omega_1} u > 0, \int_{\Omega_2} u < 0$.
If meas Ω_3 is small enough, it is then shown that the set Y has
a certain invariance property with respect to $T_f(t)$. A careful
application of the maximum principle gives a minimal equilibrium
solution v_m in Y stable from below and a maximal equilibrium
solution v_M in Y stable from above. If it were known that

there are only a finite number of equilibrium solutions, then the proof would be complete. Since this is not known, another argument must be used. Matano first proves that any solution unstable from above must be strongly unstable from above in the sense that it can be isolated from equilibrium solutions from above uniformly in Y. He then puts an ordering on the solutions from above, uses Zorn's lemma and the above property of solutions unstable from above.

In the proof of Theorem 4, the nonlinear function f has the three simple zeros at a < 0 < b. The equilibrium points a,b are stable and zero is a saddle point. A stable, nonconstant equilibrium solution was shown to exist. The argument of Matano can be used to show there must be another nonconstant equilibrium solution with $\int_{\Omega_1} u < 0$, $\int_{\Omega_2} u > 0$. Thus, there are at least four stable equilibrium solutions and one unstable equilibrium solution. This is impossible dynamically and there must be some other equilibrium solutions which are unstable. In fact, an index argument implies there must be at least three unstable equilibria. Using more of the detailed information from the paper of Matano, one can show there must be at least five unstable equilibrium solutions. Consequently, there are at least nine equilibrium solutions for this nonlinear function f and region Ω.

The basic problem is to understand in more detail how variations in the shape of the domain Ω cause these additional solutions to appear. We now summarize some work of Hale and Vegas [8] which give a possible explanation.

Let us begin with an intuitive discussion of how the
stable nonconstant equilibrium solutions could appear as
secondary bifurcations. Suppose $\mu \in [0,\infty)$, Ω_μ is a bounded
set in $\mathbb{R}^2$ with smooth boundary with the property that Ω_0 is
convex and the second eigenvalue $\lambda_2(\mu)$ of $-\Delta$ on
Ω_μ is a monotone decreasing function of μ, approaching zero
as $\mu \to \infty$. Also, suppose the third eigenvalue $\lambda_3(\mu)$ of
$-\Delta$ on Ω_μ satisfies $\lambda_3(\mu) \geq \delta > 0$ for all μ. Let
$f(\nu,u) = \nu^2 u - u^3$, $\nu > 0$, and let $A_{\nu,\mu}$ be the maximal compact
invariant set for the equation

$$u_t = \Delta u + f(\nu,u) \quad \text{in} \quad \Omega_\mu$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial\Omega_\mu.$$

Fix ν sufficiently small so that $\nu^2 < \delta$ and the only equilibrium
solutions in Ω_0 are the constant functions 0, $\pm\nu$. The set
$A_{\nu,0}$ is then the constant functions $\pm\nu$ and the unstable manifold
of 0, which is one dimensional. Let μ_0 be such that
$\lambda_2(\mu_0) = f'(0) = \nu^2$. At the point $\mu = \mu_0$, the zero solution
bifurcates creating two new equilibrium solutions which are unstable.
They are unstable because the unstable manifold of zero becomes two
dimensional - the direction of bifurcation is independent of the
direction of the original unstable manifold in $A_{\nu,\mu}$ for $\mu < \mu_0$.
The set $A_{\nu,\mu}$ for $\mu > \mu_0$ but close to μ_0 is then two dimensional
with three unstable and two stable equilibriums.

Now suppose that Ω_μ has the shape shown in Figure 1 and that meas $\Omega_\mu \to 0$ as $\mu \to \infty$. Then we can find a μ_1 such that the inequality in the proof of Theorem 4 is satisfied. Thus, there is a stable nonconstant equilibrium solution. It is conjectured that this occurs as a secondary bifurcation from the unstable nonconstant equilibria discussed above. In Figures 2 and 3 we have depicted, respectively, the set $A_{\nu,\mu}$ as a function of μ and the conjectured bifurcation diagram.

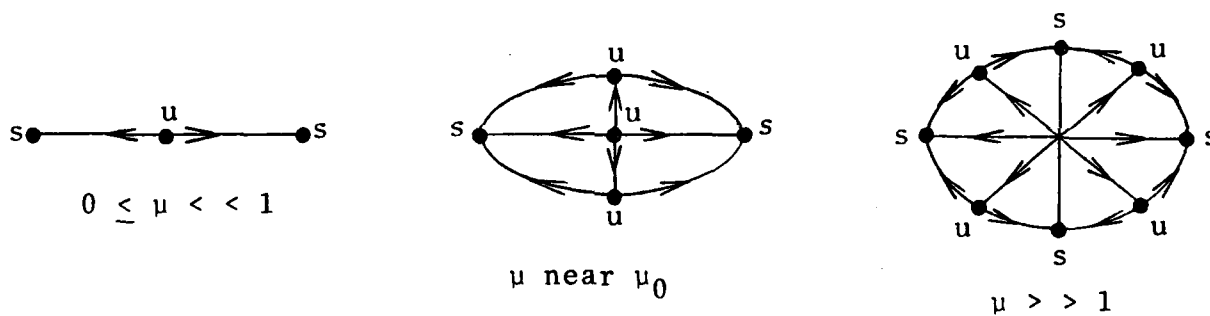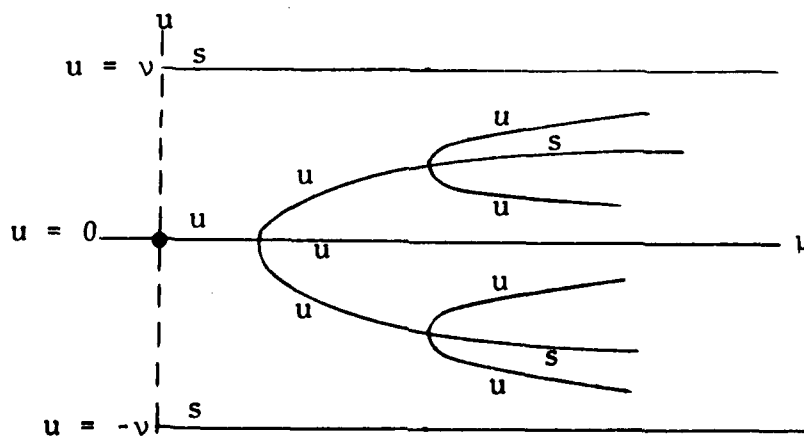$0 \le \mu << 1$

μ near μ_0

$\mu >> 1$

Figure 2

Figure 3

To do things more analytically, we have tried to discuss the neighborhood of $\mu = \infty$ treating it as a bifurcation problem from a double eigenvalue zero. More specifically, suppose $\mu = \varepsilon^{-1}$ and the region Ω_ε is shown in Figure 4, two circles Ω_1, Ω_2 and a small channel between them.
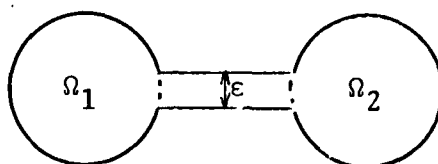

Figure 4

For $\varepsilon = 0$, it is clear there are nine solutions consisting of all combinations of 0, $\pm\nu$ on Ω_1 and Ω_2. Five of these are saddles and four are stable nodes. Let $\lambda_2(\varepsilon)$ be the second eigenvalue of $-\Delta$ on Ω_ε and let w_ε be a unit eigenvector corresponding to λ_ε. Let u_ε be the constant function (meas $\Omega_\varepsilon)^{-1}$; that is, u_ε is a unit eigenvector for the eigenvalue 0 of the Laplacian on Ω_ε.

One can now show that it is possible to apply the method of Liapunov-Schmidt for the solutions of (1) near $u = 0$ for ε, ν near zero. More specifically, for real α, β sufficiently small and ε, ν sufficiently small, there is a function $u^*(\alpha, \beta, \varepsilon, \nu)$ continuously differentiable in α, β, ν and continuous in ε such that $u^*(0,0,0,0) = 0$, $\partial u^*(0,0,0,0)/\partial(\alpha,\beta) = 0$ and

$$\Delta u^* + f(\nu, \alpha u_\epsilon + \beta w_\epsilon + u^*) - \pi f(\nu, \alpha u_\epsilon + \beta w_\epsilon + u^*) = 0$$

where πu is the projection of u onto the span of the constant function u_ϵ and the function w_ϵ; that is,

$$\pi u = \int_{\Omega_\epsilon} u + w_\epsilon \int_{\Omega_\epsilon} w_\epsilon u.$$

If $u^*(\alpha, \beta, \epsilon, \nu)$ satisfies the above, then

$$u = \alpha u_\epsilon + \beta w_\epsilon + u^*(\alpha, \beta, \epsilon, \nu)$$

is a solution of (1) if and only if $(\alpha, \beta, \epsilon, \nu)$ satisfy the bifurcation equations

$$\int_{\Omega_\epsilon} f(\nu, \alpha u_\epsilon + \beta w_\epsilon + u^*(\alpha, \beta, \epsilon, \nu)) dx = 0$$

$$\int_{\Omega_\epsilon} w_\epsilon f(\nu, \alpha u_\epsilon + \beta w_\epsilon + u^*(\alpha, \beta, \epsilon, \nu)) dx = 0$$

are satisfied.

If we let $(\alpha, \beta) = \gamma$, then one can show that these equations have the form

$$c(\gamma) + \epsilon L_1 \gamma + \nu L_2 \gamma + h.o.t. = 0 \qquad (4)$$

where h.o.t. denotes higher order terms in γ, ϵ, ν, L_1, L_2 are

constant two by two matrices and $c(\gamma)$ is a homogeneous cubic
two vector in the components of γ. One can now apply the method
in, for example, Chow, Hale and Mallet-Paret [5] to obtain the
complete bifurcation diagram for the solutions of (4). These
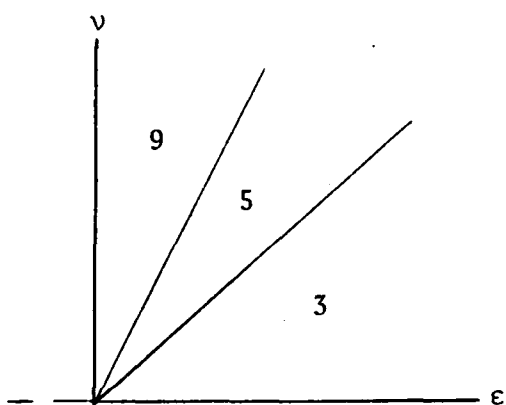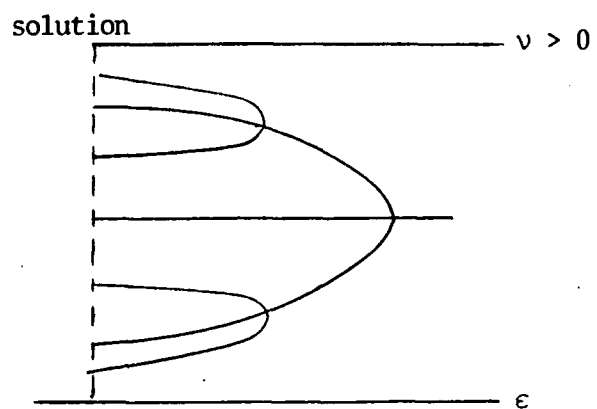are shown in Fiture 5. Figure 5a shows the number of solutions

Figure 5a

Figure 5b

for a fixed (ϵ, ν) and Figure 5b shows the way the solutions
bifurcation as a function of ϵ for a fixed ν.

In the verification of the previous results, it is crucial
to show that the third eigenvalue of the Laplacian on Ω_ϵ is
bounded away from zero. It would be very interesting to obtain
general geometric conditions on a region Ω_ϵ to have this latter
property satisfied.

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