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(6) # Approximations to the Cumulative Distribution Function of the Magnitude-Squared Coherence Estimate

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Surface Ship Sonar Department

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Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, *Applications of Statistical Communication Theory to Acoustic Signal Processing*, Principal Investigator Dr. A. H. Nuttall (Code 33), Program Manager J. H. Probus, Naval Material Command (MAT 08T1); and under Project No. A12611, *Interarray Processing*, Principal Investigator, Dr. G. C. Carter (Code 313), Program Manager, J. Neely, Naval Sea Systems Command (SEA 63D).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An approximate method of evaluating the cumulative distribution function of the magnitude-squared coherence estimate is provided. The approximation is good when the number of averaged pieces in the coherence estimate is greater than approximately 16, when the true coherence values are in the range (.01,.99) and for cumulative probabilities in the range (.01, .99). A simple formula for obtaining confidence limits in the same range of parameters is also presented. An extension to a more accurate approximation is also given; it involves the

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inversion of a nonlinearity that requires computer aid.

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Approximations to the Cumulative Distribution Function of The Magnitude-Squared Coherence Estimate

Introduction*

The cumulative distribution function of the estimate of magnitude-squared coherence, obtained by averaging over N statistically independent pieces of Gaussian data, is available in reference 1 as a sum of N−1 hypergeometric functions. Direct evaluation of this quantity has recently been simplified in reference 2; however, it is still a tedious and time-consuming calculation for large N and suffers from overflow and underflow unless special care is taken in programming. Furthermore, no simple result for obtaining confidence limits is available.

In this report, we investigate the suggestion of Fisher (reference 3) that the nonlinearity arc tanh ($\sqrt{x}$) converts the magnitude-squared coherence estimate to a near-Gaussian random variable. In particular, we present simple approximations for the mean and variance of this nonlinearly distorted magnitude-squared coherence estimate and fit a Gaussian cumulative distribution function over a wide range of: N, the number of pieces averaged in the magnitude-squared coherence estimate; C, the true magnitude-squared coherence; and P, the values of the cumulative distribution function. Inversion of the Gaussian cumulative distribution function affords a simple way of getting confidence limits for specified probabilities of threshold crossings.

Since the arc tanh ($\sqrt{x}$) nonlinear distortion takes no account of the known number, N, of pieces entering the magnitude-squared coherence estimate, an improved nonlinear distortion that utilizes this information is presented; it converts the magnitude-squared coherence estimates to more nearly a Gaussian random variable over a wider range of parameters N, C, and P. Evaluation of confidence limits requires the inversion of this nonlinearity; analytic inversion is not possible, but a numerical procedure converges rapidly.

Investigation of Arc tanh ($\sqrt{x}$) Nonlinearity

Moments

The probability density function of the magnitude-squared coherence estimate, $\hat{C}$, as given in reference 1, is

$$p_1(x) = (N-1)\ \frac{(1-C)^N (1-x)^{N-2}}{(1-Cx)^{2N-1}}\ F(1-N, 1-N;\ 1;\ Cx) \text{ for } 0 < x < 1 \text{ and } N \geq 2, \quad (1)$$

where N is the number of pieces averaged; C is the true magnitude-squared coherence; and F is a Gaussian hypergeometric function.

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* Portions of this work, which were done jointly with G. C. Carter (NUSC), are being prepared for journal publication.

The nonlinearly distorted version of random variable $\hat{C}$, which we are interested in, is

$$D = \operatorname{arc\,tanh}(\sqrt{\hat{C}}) = \frac{1}{2}\, \ln\left(\frac{1+\sqrt{\hat{C}}}{1-\sqrt{\hat{C}}}\right). \tag{2}$$

If the nonlinear operation in (2) results in a Gaussian random variable for D, then we will be interested in the mean and variance of D. This problem is considered analytically in appendix A; the only closed-form results that we can obtain are listed below. Let the m-th moment of D be denoted by

$$\mu_m(N,C) = \overline{D^m} = \int_0^1 dx\, [\operatorname{arc\,tanh}(\sqrt{x})]^m\, p_1(x)\,, \tag{3}$$

where the overbar denotes a statistical average. Then for $C = 0$, the mean of D is

$$\mu_1(N,0) = \frac{\sqrt{\pi}}{2}\, \frac{\Gamma(N-1)}{\Gamma(N-1/2)} \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as } N\to\infty\,; \tag{4}$$

the m-th moment is

$$\mu_m(N,0) \sim \frac{\Gamma(1+m/2)}{(N-1)^{m/2}} \quad \text{as } N\to\infty\,; \tag{5}$$

and the variance is

$$\sigma^2(N,0) = \mu_2(N,0) - \mu_1^2(N,0) \sim \frac{1-\pi/4}{N-1} \quad \text{as } N\to\infty\,. \tag{6}$$

This last expression is the asymptotic variance of D for $C = 0$.

In order to deduce the fundamental behavior of the mean and variance of D for $C \neq 0$, it was necessary to evaluate (3) numerically; this problem is considered in appendix B. The main results of the numerical investigation are listed below. We find mean

$$\mu_1(N,C) \cong \operatorname{arc\,tanh}(\sqrt{C+B}) \text{ for } C > 0\,, \tag{7}$$

where

$$B = \frac{1-C^2}{2(N-1)}\,, \tag{8}$$

and variance

$$\sigma^2(N,C) \cong \frac{1}{2(N-1)} \quad \text{for } C > 0\,. \tag{9}$$

The imprecise qualifier $C > 0$ reflects the fact that the probabilistic behavior of $\hat{C}$ and D is distinctly different for $C = 0$ versus $C > 0$; for example, compare (6) and (9). The precise region where (7), (8), and (9) are valid will become clear in later plots. The result (9) is a slight modification of Fisher that better fits the calculated values of variance in appendix B and the slope of the cumulative distribution function plots. Its independence of true magnitude-squared coherence value C is striking and convenient.

Probabilistic Statements

If D is nearly a Gaussian random variable with mean $\mu_1(N,C)$ and variance $\sigma^2(N,C)$, then the cumulative distribution function of D, i.e., the probability that D is less than some threshold T, is given by the approximation

$$P_2(T) \equiv \text{Prob}\{D < T\} \cong \int_{-\infty}^{T} \frac{dx}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right) = \Phi\left(\frac{T-\mu_1}{\sigma}\right), \tag{10}$$

where

$$\Phi(y) = \int_{-\infty}^{y} \frac{du}{\sqrt{2\pi}} \exp(-u^2/2) \tag{11}$$

is the cumulative distribution function for a zero-mean unit-variance Gaussian random variable. A plot of (10) on normal probability paper, with T as the linear abscissa, yields a straight line with abscissa-intercept μ_1 and slope $1/\sigma$, since the nonlinear transformation to plot ordinates in this case is the inverse function* $\Phi^{I}(\ )$.

The exact cumulative distribution function of D is given by

$$P_2(T) = \text{Prob}\{D < T\} = \text{Prob}\{\text{arc tanh}\sqrt{\hat{C}} < T\}$$

$$= \text{Prob}\{\hat{C} < \tanh^2(T)\} = P_1(\tanh^2(T)), \tag{12}$$

where we employed (1) and (2) and defined P_1 as the cumulative distribution function of $\hat{C}$:

$$P_1(A) = \text{Prob}\{\hat{C} < A\} = \int_0^A dx\, p_1(x). \tag{13}$$

Thus, we can relate cumulative distribution function P_2 to the known cumulative distribution function P_1 given in reference 1. Programs for the exact evaluation of (13) and (12) are given in appendices C and D, respectively.

The availability of approximation (10) for the cumulative distribution function of D enables us to give a simple approximation to the cumulative distribution function P_1 of $\hat{C}$; namely

$$P_1(A) = \text{Prob}\{\hat{C} < A\} = \text{Prob}\{D < \text{arc tanh}(\sqrt{A})\}$$

$$= P_2(\text{arc tanh}(\sqrt{A})) \cong \Phi\left(\frac{\text{arc tanh}\sqrt{A} - \mu_1}{\sigma}\right), \tag{14}$$

where μ_1 and σ are given by (7), (8), and (9).

*A superscript I (^{I}) will be used to denote the inverse of a function.

Furthermore, we can now solve the inverse problem of determining a threshold A for specified values of cumulative distribution function P_1 and for given parameter values C and N. From (14), we have

$$A \cong \tanh^2(\mu_1 + \sigma \Phi^{-1}(P_1)) , \tag{15}$$

where μ_1 and σ are in (7), (8), and (9). Simplification of (15) results in

$$A \cong \left(\frac{\alpha + \beta}{1 + \alpha\beta}\right)^2 , \tag{16}$$

where

$$\alpha = \left[C + \frac{1 - C^2}{2(N-1)}\right]^{1/2} , \tag{17}$$

$$\beta = \tanh\left[\frac{\Phi^{-1}(P_1)}{\sqrt{2(N-1)}}\right] . \tag{18}$$

A program for the evaluation of (16), (17), and (18) is given in appendix E.

Plots

Before embarking on the plots of exact cumulative distribution function P_2 of distorted random variable D, as given by (12), we plot the exact cumulative distribution function P_1 in (13) of the original random variable $\hat{C}$, as given by reference 1. Plots on normal probability paper for N = 8, 16, 32, 64, 128 are given in figures 1-5*, respectively. In these figures, each dotted straight line corresponds to a Gaussian random variable with mean and variance as derived for magnitude-squared coherence estimate $\hat{C}$ in reference 4, page 20. The discrepancy with the exact cumulative distribution function (in solid lines) indicates that $\hat{C}$ is not well-approximated by a Gaussian random variable, especially for the small and large values of C and the extreme values of probability near .01 and .99.

Plots of the exact cumulative distribution function P_2 of D, given by (12), are presented in figures 6-10 for N = 8, 16, 32, 64, 128. Dotted straight lines corresponding to a Gaussian random variable satisfying (7)-(10) have been drawn for every C and N value being considered; however, they have been overdrawn by the exact cumulative distribution function (12) in some cases and are not visible. The agreement between exact and Gaussian cumulative distribution functions is extremely good except for very low values of C and N. Notice also that the curves are approximately parallel straight lines over a wide range and can, therefore, be interpolated more easily. Thus, approximate probabilistic relation (10) and its inverse (16) are very useful and accurate for a wide range of C and N, encompassing most of the useful values of these parameters.

* All figures are grouped at the end of the main text.

An Improved Nonlinear Transformation

Desired Transformation

The arc tanh ($\sqrt{x}$) nonlinear transformation in (2) is independent of the known quantity N, the number of pieces used in getting magnitude-squared coherence estimate $\hat{C}$. An improved transformation that utilizes N should, therefore, be possible and should be investigated to see if a more-nearly Gaussian random variable can result.

The method to determine this nonlinear transformation is as follows: suppose a random variable with cumulative distribution function $P_1(X)$ is given, and a nonlinear monotonically increasing transformation,

$$y = g(x) , \tag{19}$$

that will result in a specified cumulative distribution function $P_2(Y)$, is desired. We have

$$\begin{aligned} P_1(X) &= \mathrm{Prob}\{x < X\} = \mathrm{Prob}\{g(x) < g(X)\} \\ &= \mathrm{Prob}\{y < g(X)\} = P_2(g(X)) . \end{aligned} \tag{20}$$

Therefore, inverting this equation, the desired nonlinearity is

$$g(x) = P_2^{I}(P_1(x)) . \tag{21}$$

Thus, we need to know the inverse of the desired cumulative distribution function P_2.

In particular, if y is Gaussian with mean μ_1 and variance σ^2, then

$$P_2(Y) = \Phi\left(\frac{Y-\mu_1}{\sigma}\right),$$

$$P_2^{I}(t) = \mu_1 + \sigma\Phi^{I}(t) , \tag{22}$$

yielding

$$g(x) = \mu_1 + \sigma\Phi^{I}(P_1(x)) . \tag{23}$$

This nonlinear transformation (23) will result in a Gaussian random variable with mean $\mu_1$and σ^2. However, the quantities $\mu_1$and P_1 depend on true coherence C as well as N; see (7) and (1). Since C is unknown (we are using $\hat{C}$ to estimate C), (23) is useless unless it turns out that (23) is substantially independent of C; it can, of course, depend on N. To ascertain this possibility, we have plotted, in figures 11, 12, and 13, the nonlinear distortion (23) for N = 8, 16, and 32, respectively; the program is given in appendix F. Within each figure corresponding to a fixed value of N, the five distortions (23) for C = .01, .05, .1, .5, and .99 have been plotted as solid lines; in addition, the arc tanh ($\sqrt{x}$) nonlinearity is superimposed in dotted lines. Furthermore, (23) has only been plotted in the range of probabilities [.01, .99] for P_1, since this is the range we are concentrating on.

The ideal situation would occur if, within each figure, the same identical nonlinear function occurred for all five cases of C = .01, .05, .1, .5, and .99; of course, the function can change with N. Although this fortuitous situation does not occur precisely, inspection of each of the five cases within each figure reveals a marked independence of C. Thus, the desired goal of converting magnitude-squared coherence estimate $\hat{C}$ to a nearly Gaussian random variable, by means of a nonlinear transformation that depends only on N, is achievable.

Now, the problem remains to find as simple an analytic transformation as possible to approximate the solid curves in figures 11, 12, and 13. A great deal of trial-and-error has resulted in the following candidate. The distorted version of $\hat{C}$ we adopt is not (2), but rather

$$D_N = \text{arc tanh}\left(g_N(\hat{C})\right) , \tag{24}$$

where

$$g_N(x) = \sqrt{x} + \frac{(1-x)^N \, \ln\left(\frac{N-1}{2} x\right)}{2(N+2)} ; \; g_\infty(x) = \sqrt{x} . \tag{25}$$

The nonlinear transformation in (24) and (25) depends on N as well as x, but is independent of the unknown parameter C. The leading term of (25) is, of course, the earlier result. The correction term goes quickly to zero as $x \rightarrow 1$, and as N increases, as figures 11, 12, and 13 indicated desirable.

Probabilistic Statements

In order to measure the success of transformation (24) and (25), we need to evaluate the exact cumulative distribution function of random variable D_N. We have the exact relation

$$\begin{aligned}
\text{Prob}\{D_N < T\} &= \text{Prob}\{\text{arc tanh}(g_N(\hat{C})) < T\} \\
&= \text{Prob}\{g_N(\hat{C}) < \tanh(T)\} \\
&= \text{Prob}\{\hat{C} < g_N^I(\tanh(T))\} = P_1\left(g_N^I(\tanh(T))\right) ,
\end{aligned} \tag{26}$$

where g_N^I is the inverse function of g_N, and P_1 is the cumulative distribution function of $\hat{C}$. The program for evaluation of (26) is given in appendix G.

If we have succeeded in realizing a Gaussian random variable for D_N, then an approximation to (26) is given by

$$\text{Prob}\{D_N < T\} \cong \Phi\left(\frac{T-\mu_1}{\sigma}\right), \tag{27}$$

where we have chosen to use (7), (8), and (9) here also. Therefore, an approximation to cumulative distribution function P_1 follows according to

$$\begin{aligned} P_1(A) &= \mathrm{Prob}\{\hat{C} < A\} \\ &= \mathrm{Prob}\{\mathrm{arc\,tanh}(g_N(\hat{C})) < \mathrm{arc\,tanh}(g_N(A))\} \\ &= \mathrm{Prob}\{D_N < \mathrm{arc\,tanh}(g_N(A))\} \\ &\cong \Phi\left(\frac{\mathrm{arc\,tanh}(g_N(A)) - \mu_1}{\sigma}\right) . \end{aligned} \tag{28}$$

Furthermore, (28) affords an approximate expression for the required threshold value, A, to use for a specified probability value P_1; it is

$$A \cong g_N^{I}\left(\tanh(\mu_1 + \sigma\Phi^{I}(P_1))\right) . \tag{29}$$

This relation generalizes (15), which applied to the arc tanh ($\sqrt{x}$) transformation. The main drawback to (26) and (29) is that we need the inverse function g_N^I to (25). (This inverse is not needed for (27) or (28).) When the correction term in (25) is zero (as for $N = \infty$), the inverse relation is simply $g_\infty^I(y) = y^2$; however, no such simple relation exists for (25) in general, and g_N^I must be found numerically. It would be worthwhile to uncover a simpler nonlinearity than (25) that still converts $\hat{C}$ to a nearly Gaussian random variable, and, at the same time, is easily invertible; whether this is possible is unknown.

Plots

The final measure of success of transformation (24) and (25) is afforded by figures 14-18 for the cumulative distribution function of D_N for N = 8, 16, 32, 64, and 128, respectively. The discrepancy from a Gaussian random variable (dotted lines) is far less than for the corresponding cases in figures 6-10 for the arc tanh ($\sqrt{x}$) nonlinearity. Even for as low a value as N = 8, a fair approximation is given for C = .01.

Summary

Two nonlinear transformations have been presented that convert magnitude-squared coherence estimate $\hat{C}$ to nearly a Gaussian random variable. They afford quick approximate evaluation of the cumulative distribution function of $\hat{C}$ and can be used for arbitrarily large N; see (14) and (28). They also allow for rapid calculation of confidence limits since these relations can be inverted to find the threshold required for a specified cumulative distribution function value; see (15) and (29). A drawback of the latter more-accurate result is that no simple analytical inversion to nonlinear transformation (25) exists; however, since computer aid would likely be used to evaluate the expressions given here, inversion of (25) is easily achieved. A simple recursive scheme was employed by the author; see appendix G.

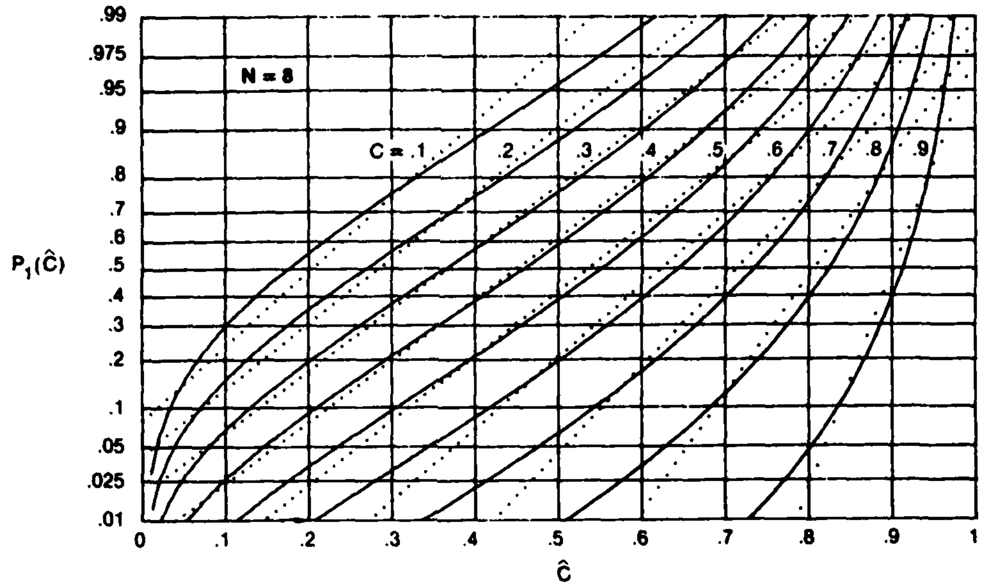


Figure 1. Cumulative Distribution Function of $\hat{C}$ for N = 8

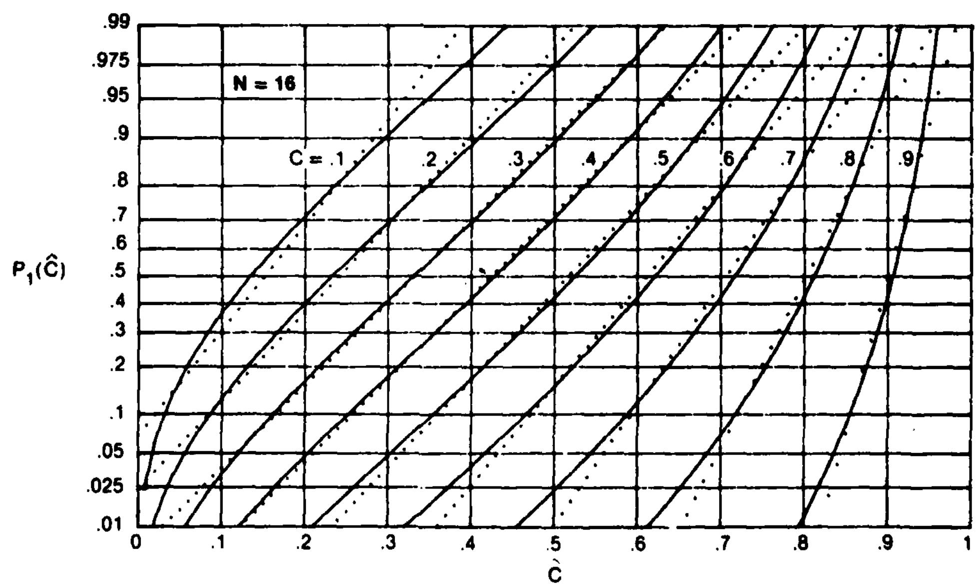


Figure 2. Cumulative Distribution Function of $\hat{C}$ for N = 16

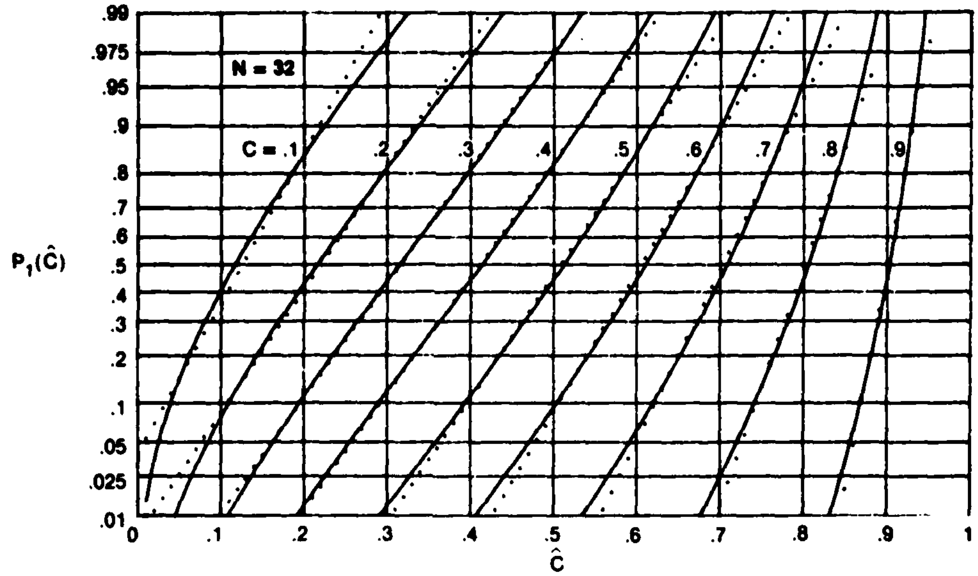


Figure 3. Cumulative Distribution Function of $\hat{C}$ for N = 32

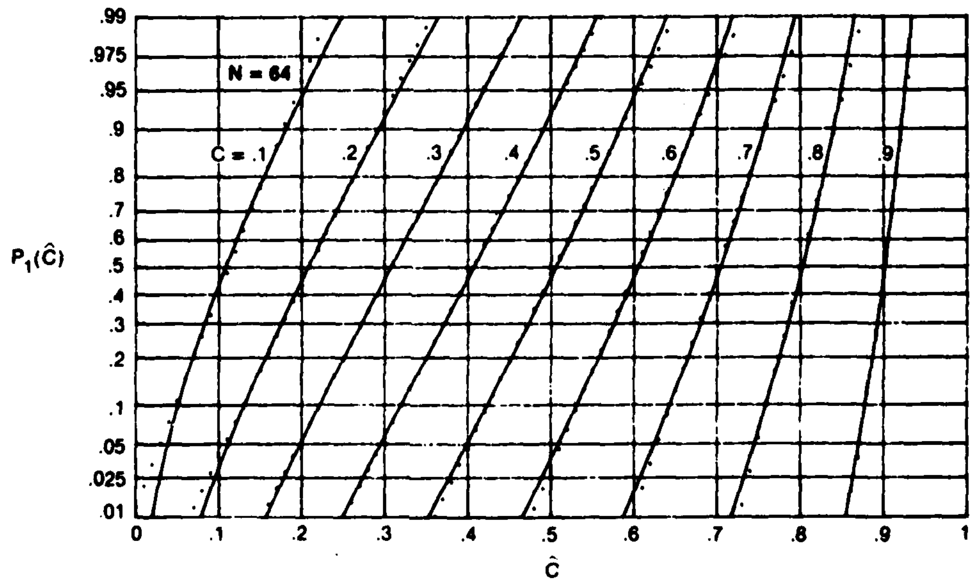


Figure 4. Cumulative Distribution Function of $\hat{C}$ for N = 64

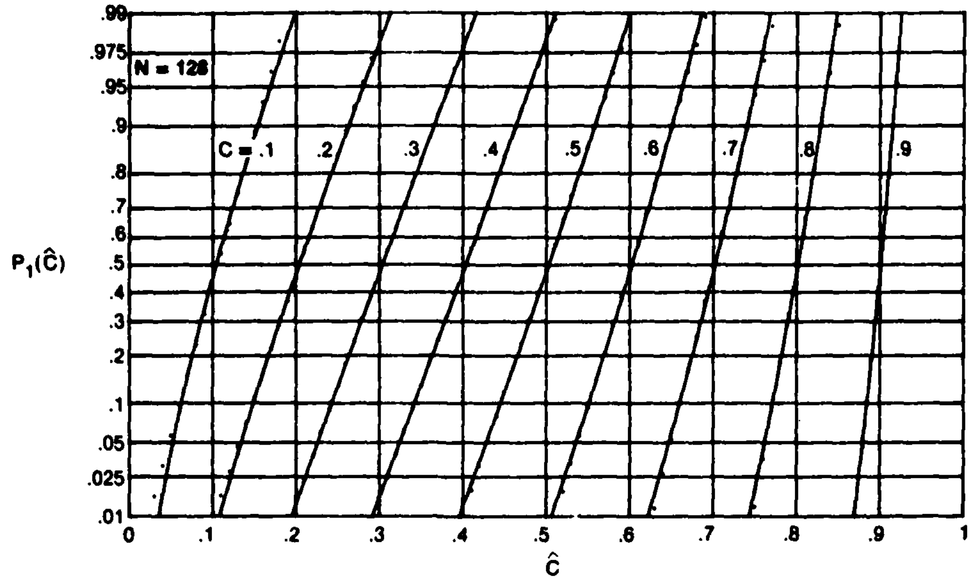


Figure 5. Cumulative Distribution Function of $\hat{C}$ for N = 128

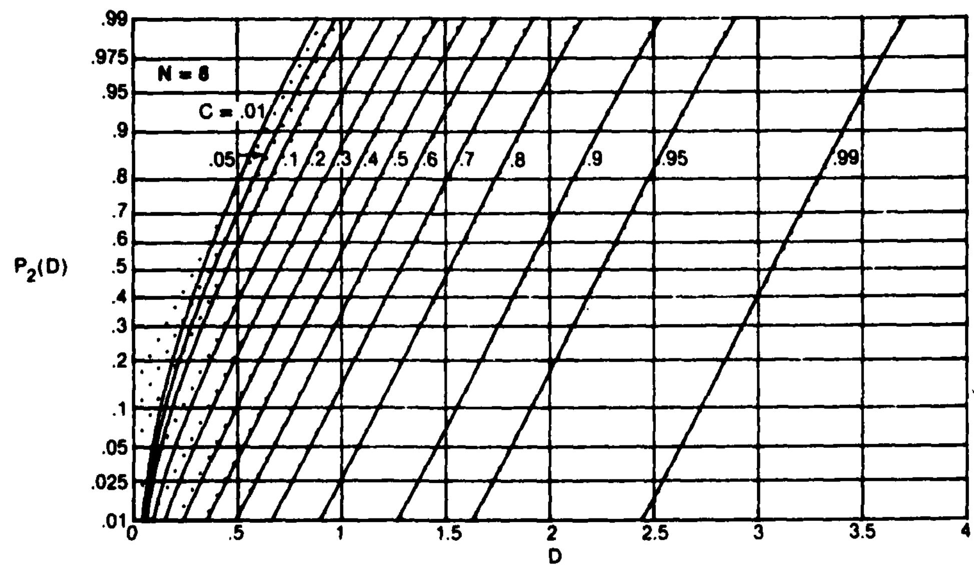


Figure 6. Cumulative Distribution Function of D for N = 8

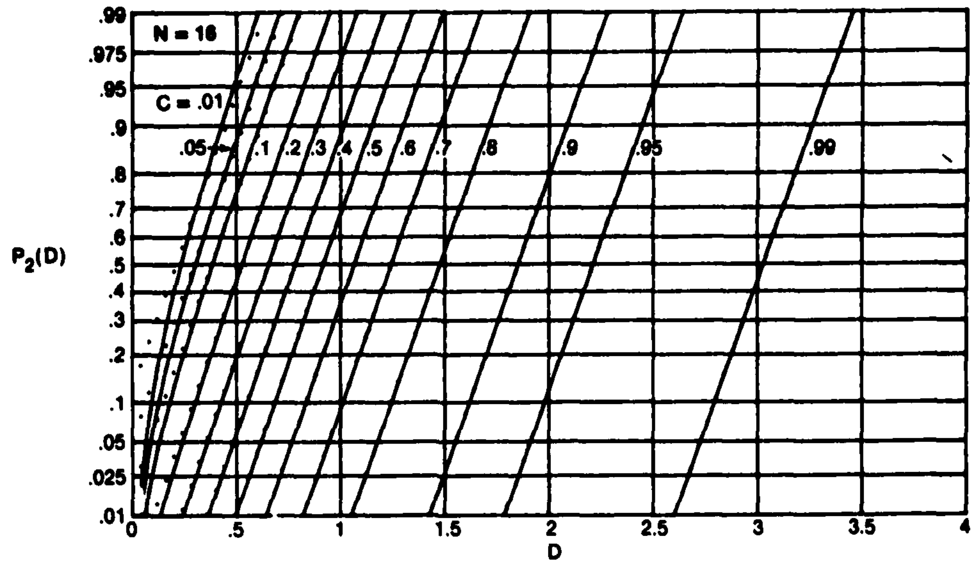


Figure 7. Cumulative Distribution Function of D for N = 16

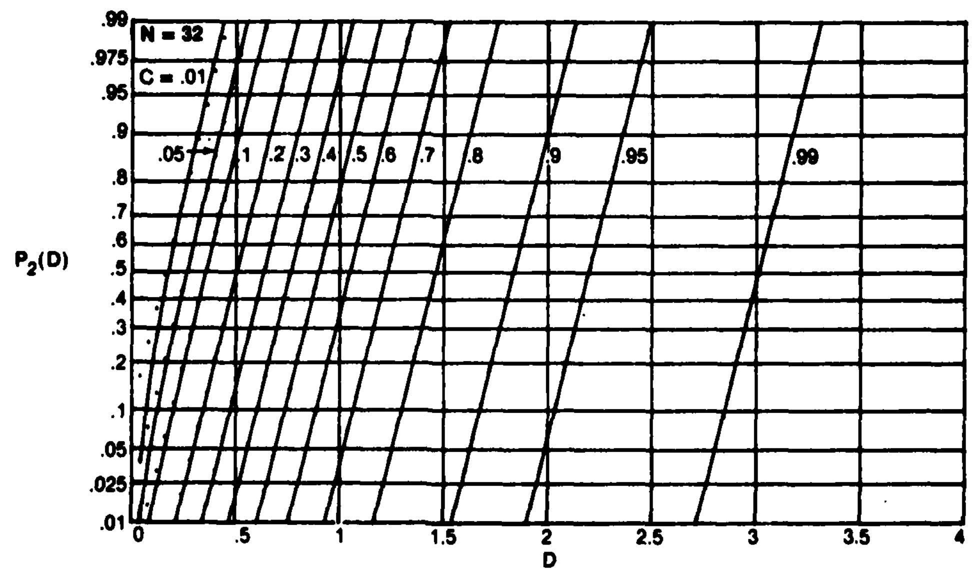


Figure 8. Cumulative Distribution Function of D for N = 32

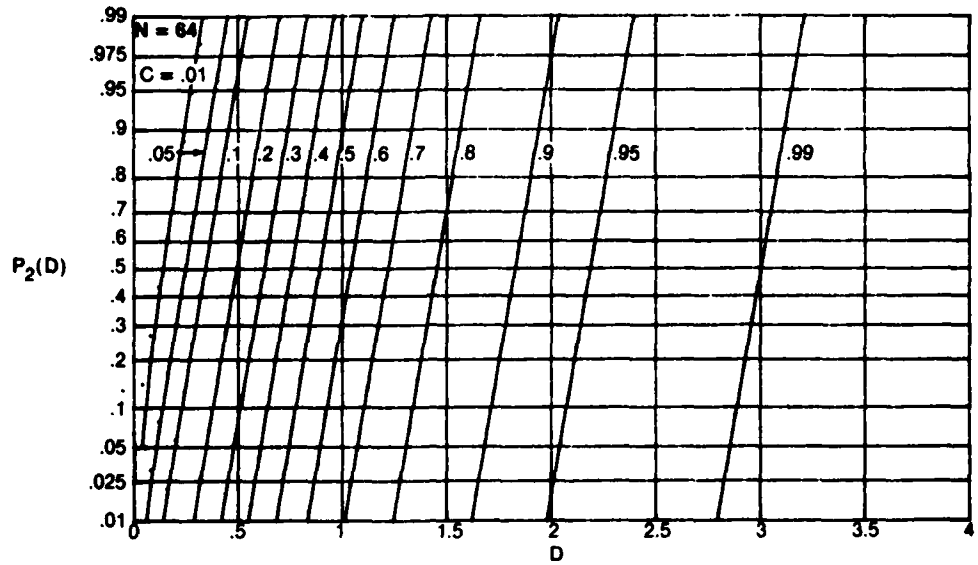


Figure 9. Cumulative Distribution Function of D for N = 64

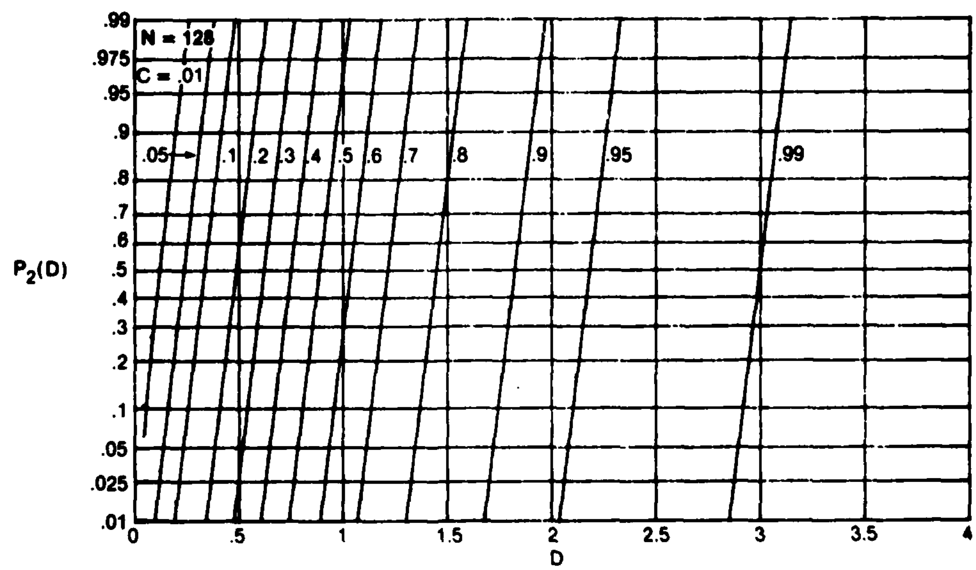
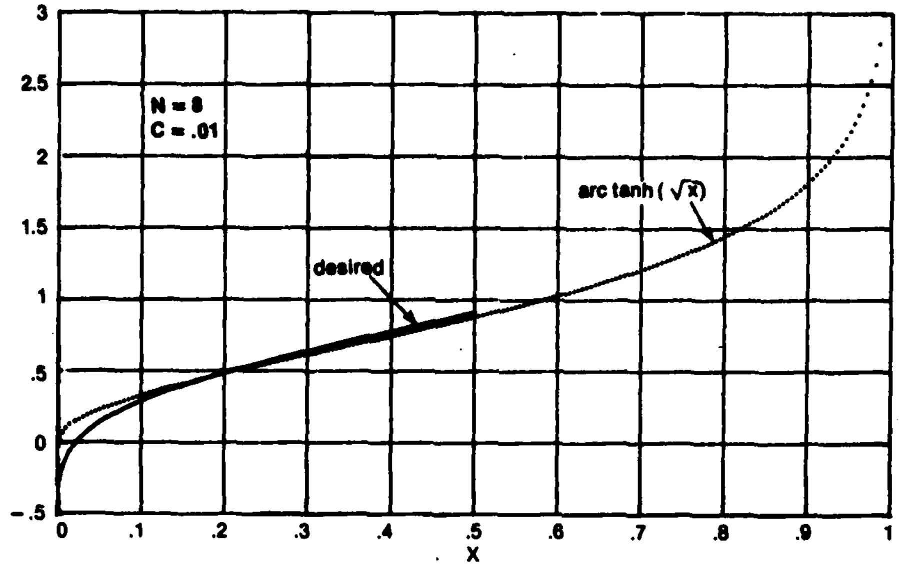
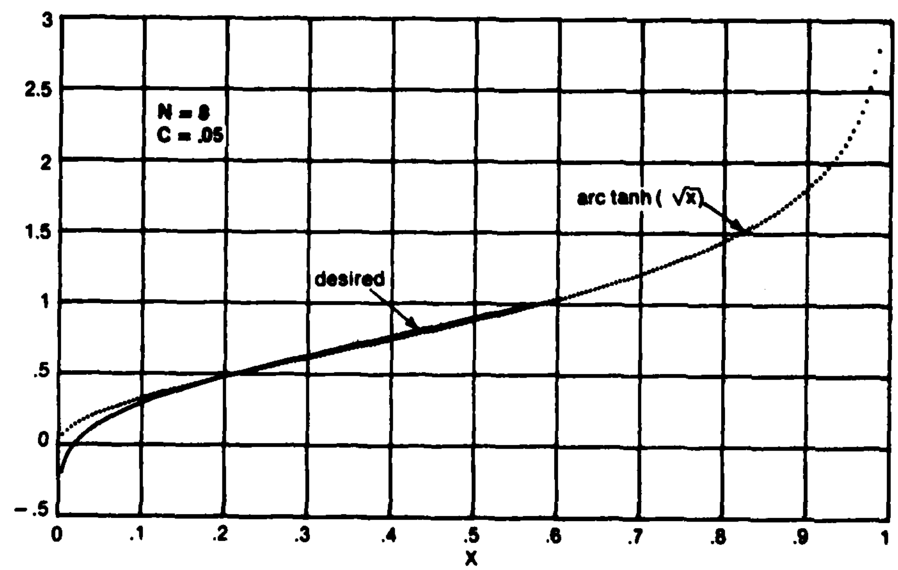


Figure 10. Cumulative Distribution Function of D for N = 128

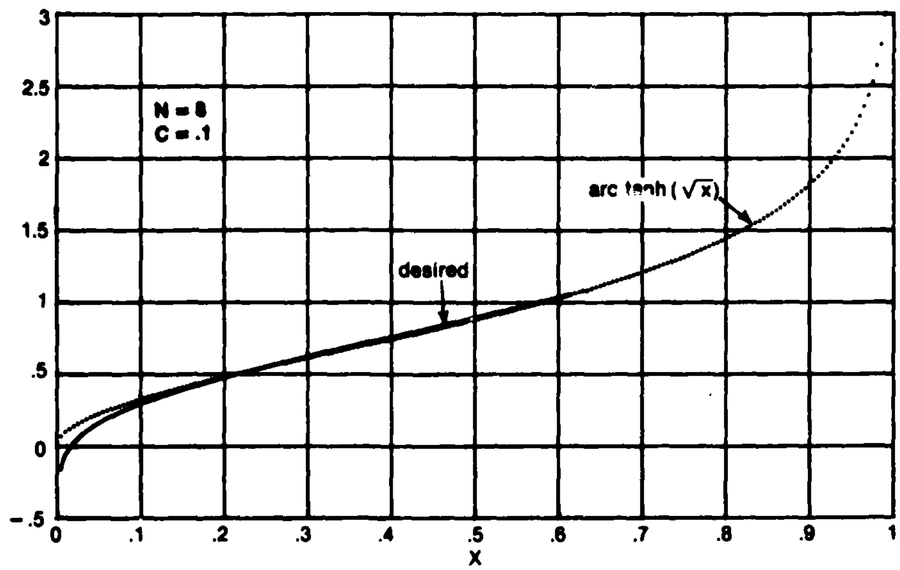


11a. C = 0.01

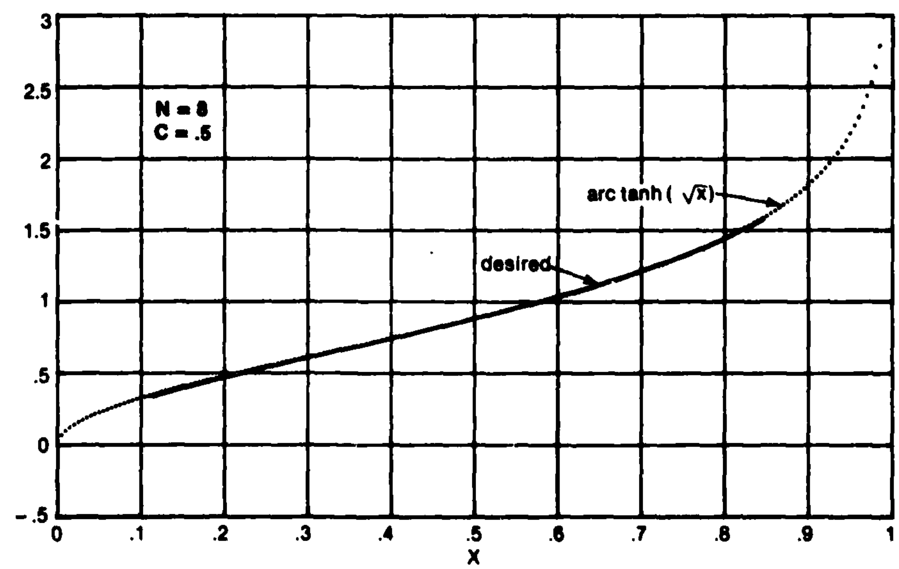


11b. C = 0.05

Figure 11. Desired Non-Linearity for N = 8

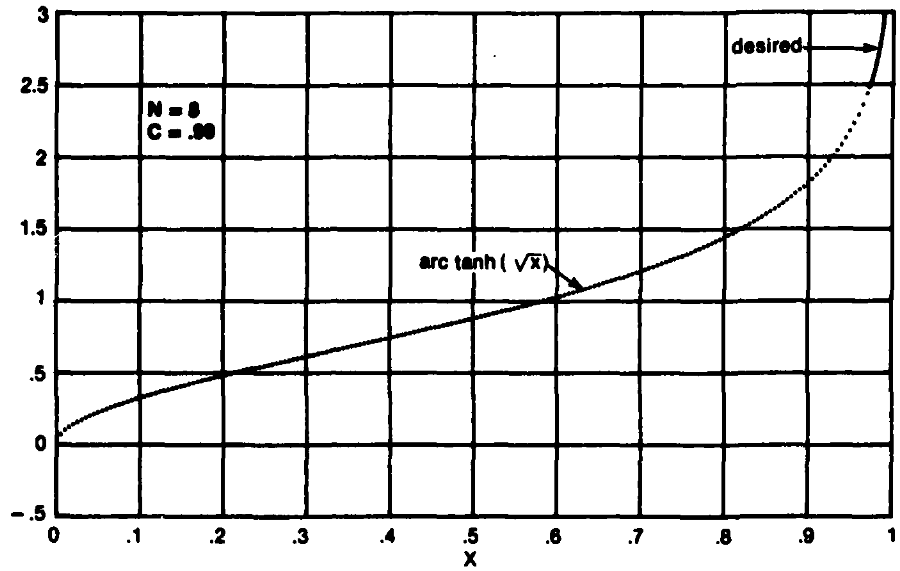


11c. C = 0.1



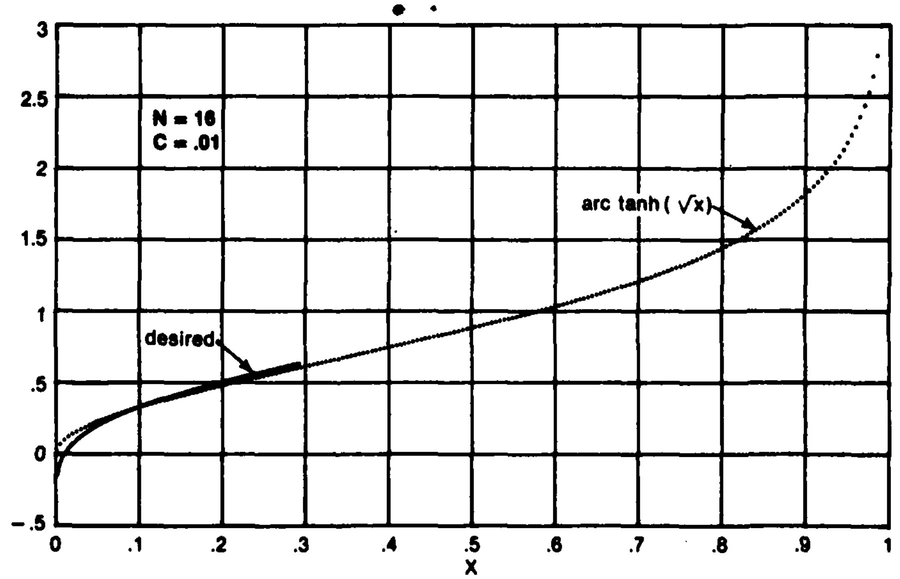
11d. C = 0.5

Figure 11. (Cont'd) Desired Non-Linearity for N = 8



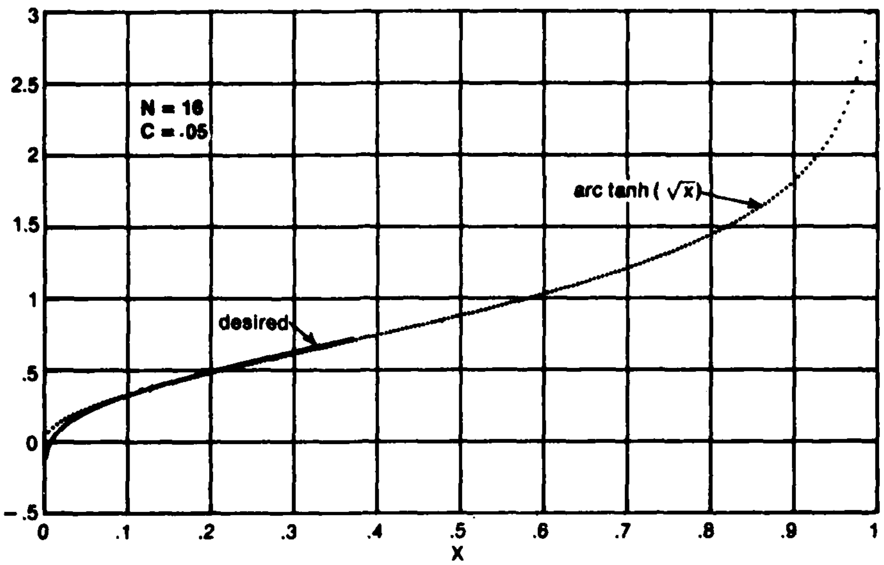
11e. C = 0.99

Figure 11. (Cont'd) Desired Non-Linearity for N = 8

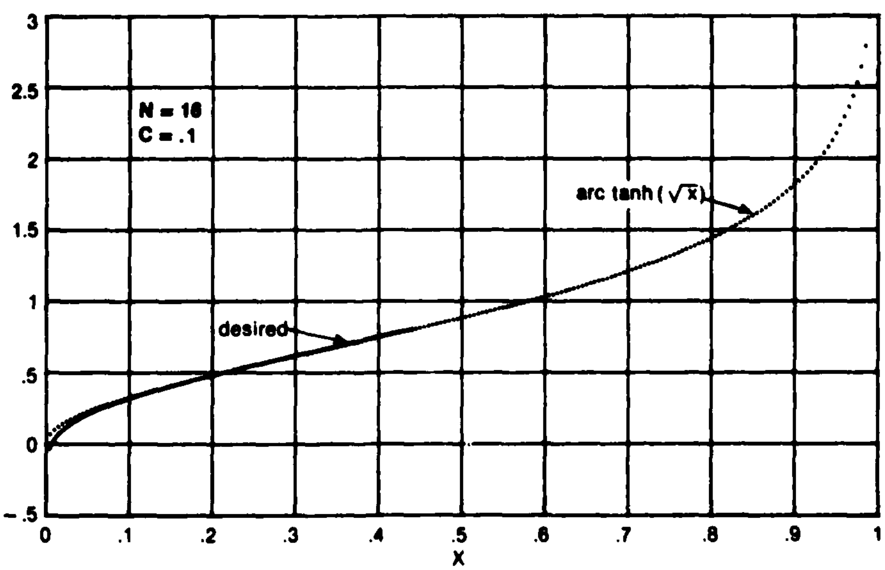


12a. C = 0.01

Figure 12. Desired Non-Linearity for N = 16

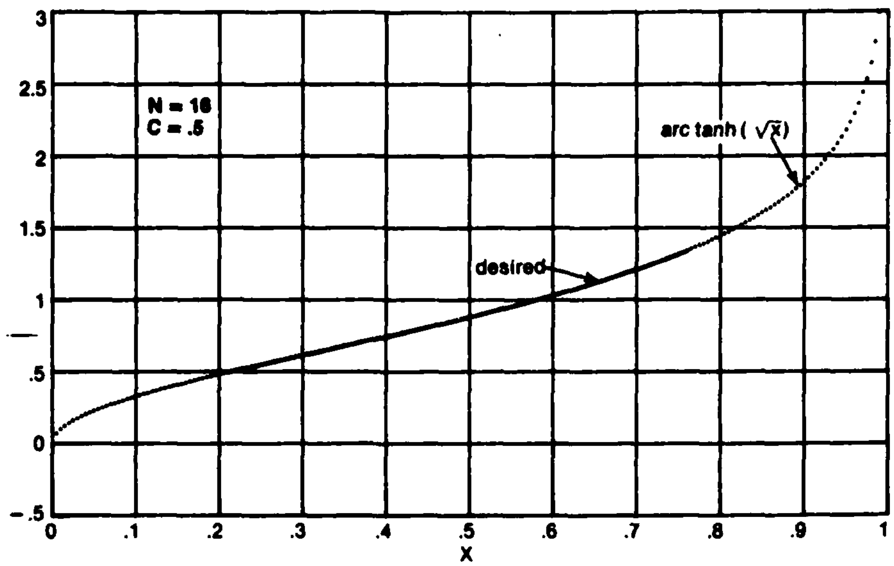


12b. C = 0.05

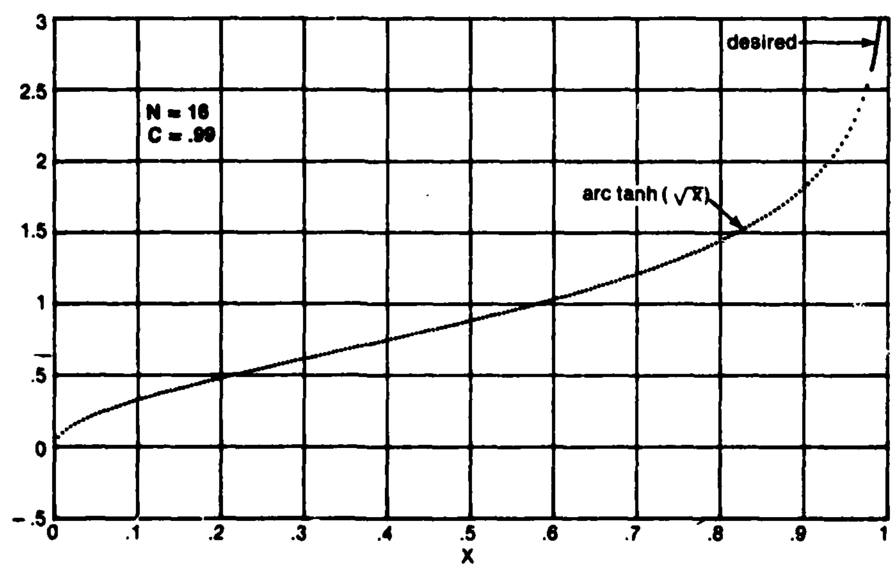


12c. C = 0.1

Figure 12. (Cont'd) Desired Non-Linearity for N = 16

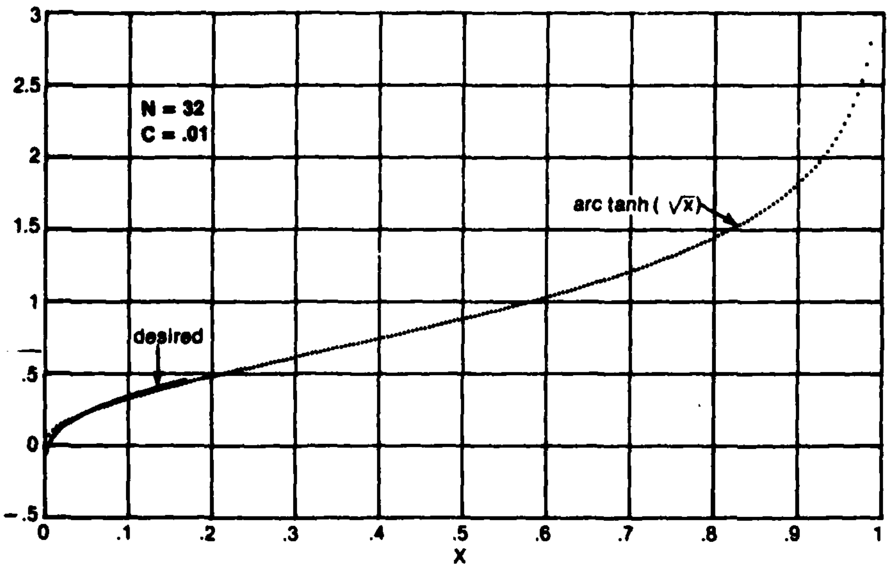


12d. C = 0.5

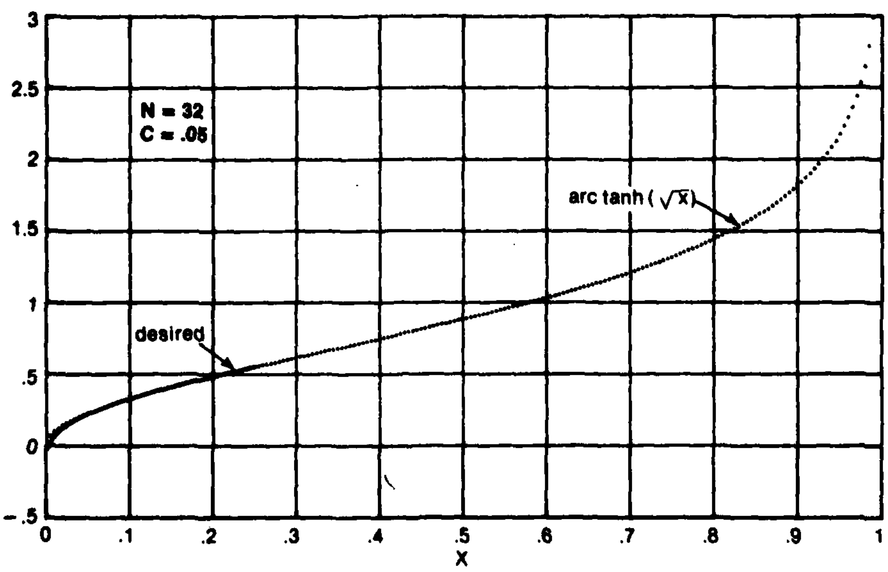


12e. C = 0.99

Figure 12. (Cont'd) Desired Non-Linearity for N = 16

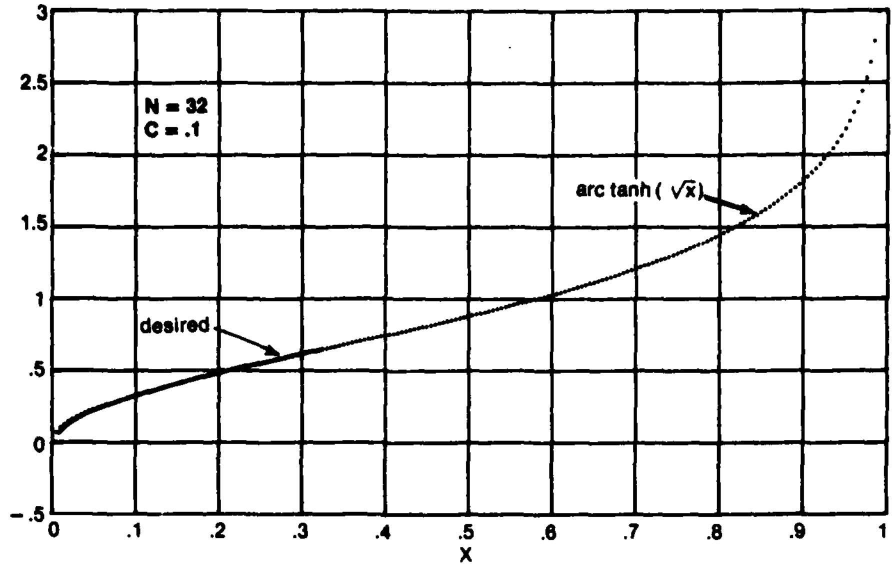


13a. C = 0.01

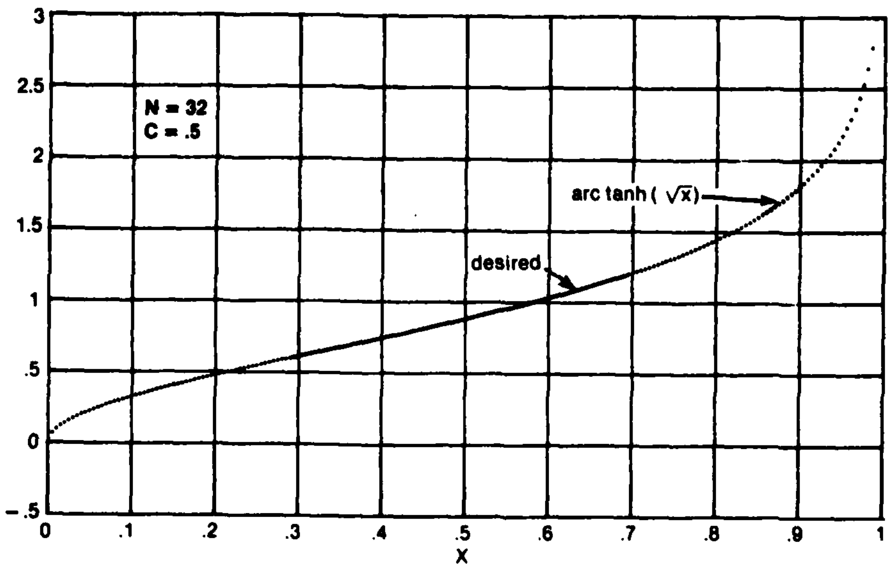


13b. C = 0.05

Figure 13. Desired Non-Linearity for N = 32

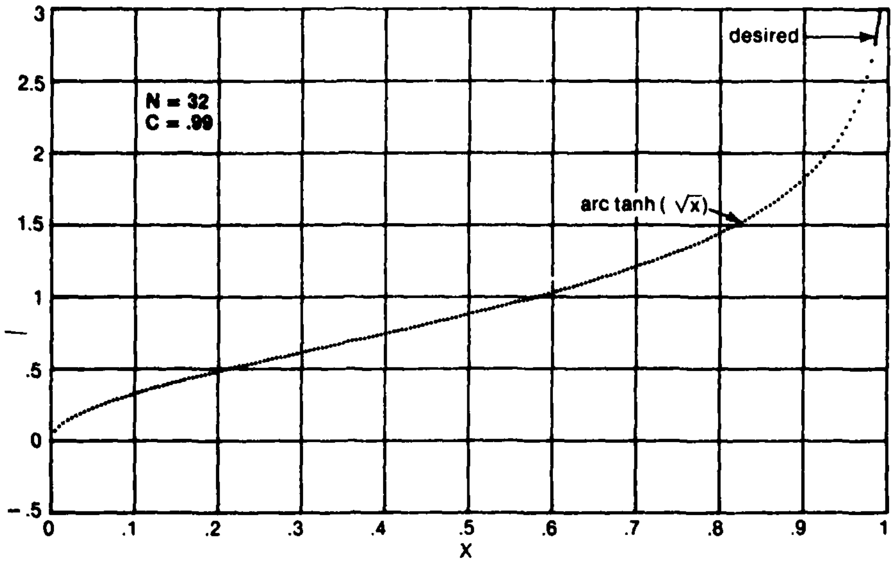


13c. C = 0.1



13d. C = 0.5

Figure 13. (Cont'd) Desired Non-Linearity for N = 32



13e. C = 0.99

Figure 13. (Cont'd) Desired Non-Linearity for N = 32

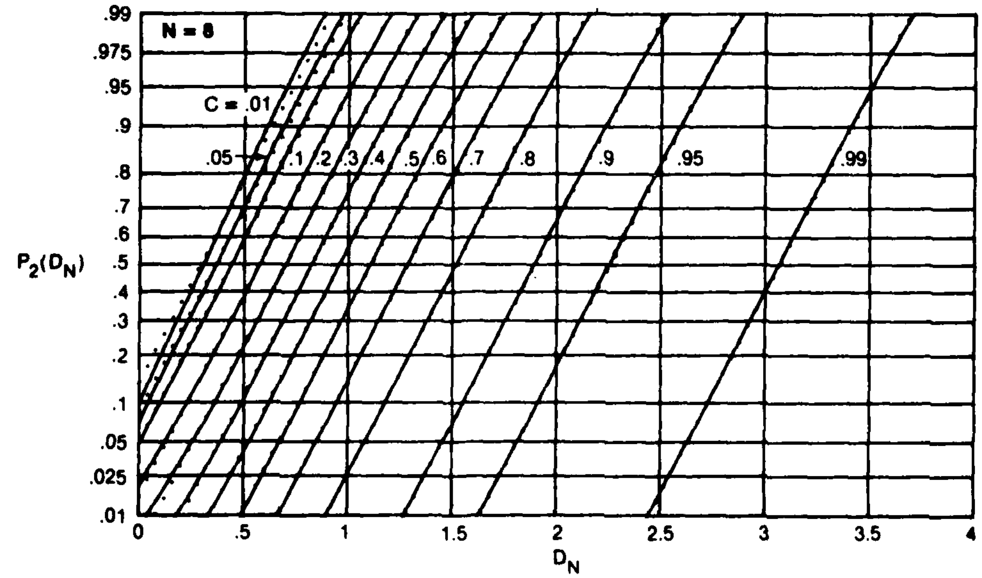


Figure 14. Cumulative Distribution Function of D_N for N = 8

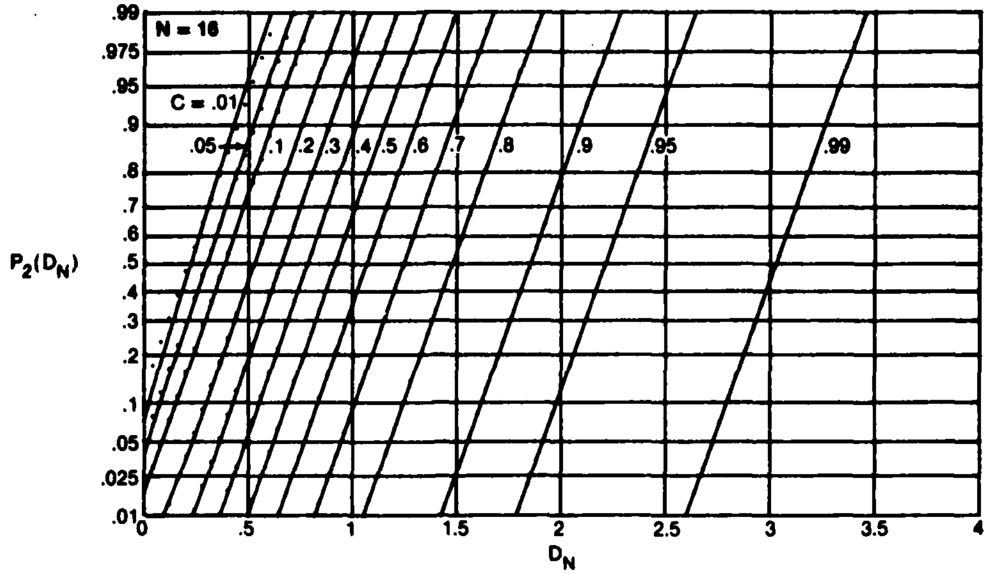


Figure 15. Cumulative Distribution Function of D_N for N = 16

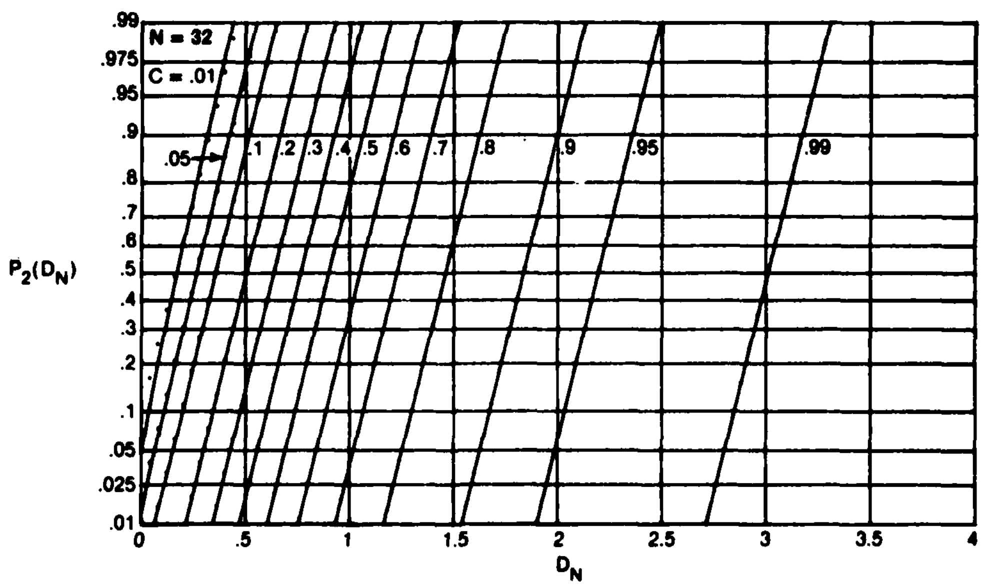


Figure 16. Cumulative Distribution Function of D_N for N = 32

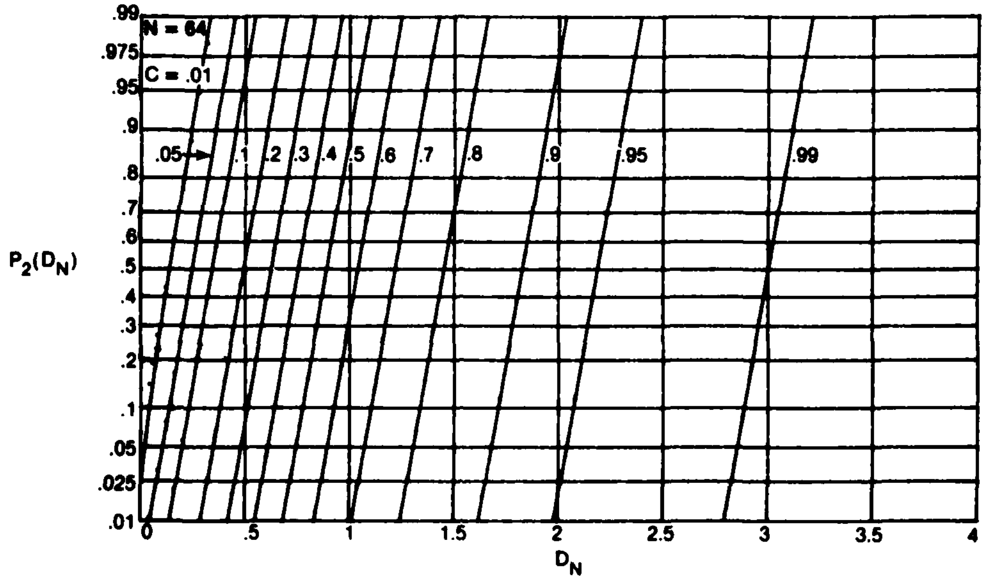


Figure 17. Cumulative Distribution Function of D_N for N = 64

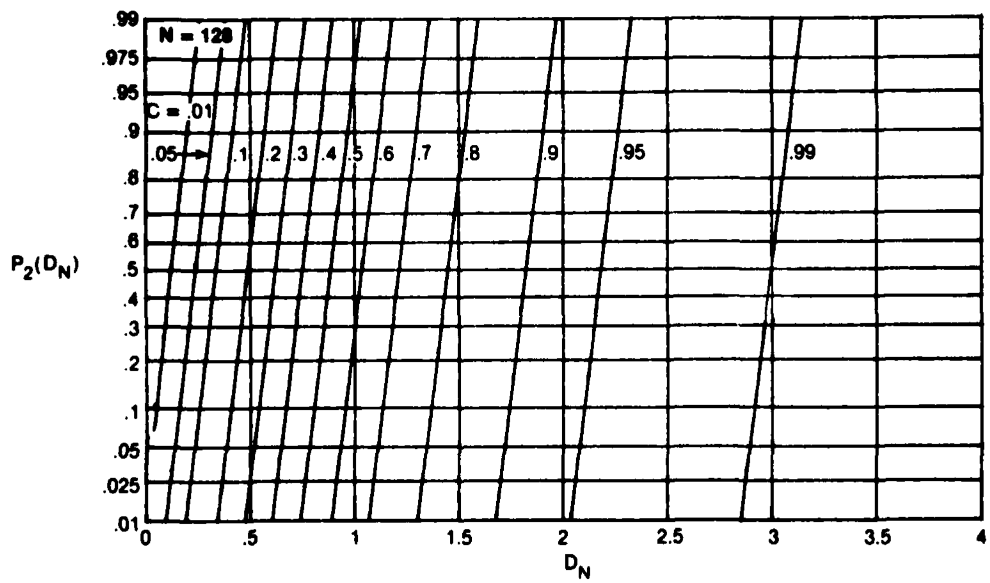


Figure 18. Cumulative Distribution Function of D_N for N = 128

Appendix A
Moments of D for C = 0

Exact Results

For C = 0, the probability density function in (1), of magnitude-squared coherence estimate $\hat{C}$, simplifies to

$$p_1(x) = (N-1)(1-x)^{N-2} \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad N \geq 2. \tag{A-1}$$

The m-th moment (3) is then given by

$$\mu_m(N,0) = \int_0^1 dx \, p_1(x) \operatorname{arc\,tanh}^m(\sqrt{x}). \tag{A-2}$$

Let

$$x = \tanh^2(t), \quad dx = dt \; 2 \sinh(t)/\cosh^3(t), \tag{A-3}$$

to obtain

$$\mu_m(N,0) = 2\int_0^\infty dt \, \frac{\sinh(t)}{\cosh^3(t)} \, p_1\left(\tanh^2(t)\right) t^m$$

$$= 2(N-1)\int_0^\infty dt \, \frac{\sinh(t)}{\cosh^{2N-1}(t)} \, t^m. \tag{A-4}$$

For m = 0, the substitution u = cosh(t) immediately yields unit area. So, let m ≥ 1 in the following. Integrate by parts with

$$u = t^m, \quad dv = dt \, \frac{\sinh(t)}{\cosh^{2N-1}(t)}, \tag{A-5}$$

and obtain

$$\mu_m(N,0) = m\int_0^\infty dt \, \frac{t^{m-1}}{\cosh^{2N-2}(t)} \quad \text{for} \quad m \geq 1. \tag{A-6}$$

We now specialize to m = 1; let $x = e^t$, dx = x dt, p = 2N-2, and obtain

$$\mu_1(N,0) = 2^{p-1}\int_0^\infty dx \, \frac{x^{p-1}}{(1+x^2)^p} = 2^{2N-4} \, \frac{\Gamma^2(N-1)}{\Gamma(2N-2)}$$

$$= \frac{\sqrt{\pi}}{2} \, \frac{\Gamma(N-1)}{\Gamma(N-\frac{1}{2})} = \mu_1(N-1,0)\,\frac{N-2}{N-1.5} \quad \text{for} \quad N \geq 3 \,; \quad \mu_1(2,0) = 1, \tag{A-7}$$

where we employed reference 5, 3.241 4, and reference 6, 6.1.18. Then using reference 6, 6.1.47, with the parameters selected to eliminate the first correction term, we find

$$\mu_1(N,0) \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as} \quad N \to \infty. \tag{A-8}$$

Asymptotic Expansion to One Term

We were unable to locate a closed-form expression for (A-6) when m ≥ 2. A recursion relation similar to (A-7) is derivable for m ≥ 2 and will be presented later. But, a useful approximation for any m and for large N follows quickly from (A-6):

$$\frac{1}{\cosh(t)} \sim \frac{1}{1+t^2/2} \sim \exp(-t^2/2) \quad \text{as} \quad t \to 0;$$

$$\frac{1}{\cosh^p(t)} \sim \exp(-p\,t^2/2) \quad \text{as} \quad t \to 0. \tag{A-9}$$

Therefore,

$$\mu_m(N,0) \sim m\int_0^\infty dt\, t^{m-1}\exp(-p\,t^2/2)$$

$$= \frac{\Gamma\left(\frac{m}{2}+1\right)}{(N-1)^{m/2}} \quad \text{as} \quad N \to \infty. \tag{A-10}$$

(The reason that (A-10) for m = 1 does not agree precisely with (A-8) is that (A-8) is exact to order $N^{-3/2}$, whereas (A-10) is correct only to order $N^{-1/2}$ for m = 1. This shortcoming is alleviated below.)

From (A-10), we can now evaluate the variance of D for C = 0; it is

$$\sigma^2 = \mu_2(N,0) - \mu_1^2(N,0) \sim \frac{1-\pi/4}{N-1} \quad \text{as} \quad N \to \infty. \tag{A-11}$$

Asymptotic Expansion to Two Terms

Our starting point is (A-6):

$$\mu_m(N,0) = m\int_0^\infty dx\, \frac{x^{m-1}}{\cosh^p(x)}. \tag{A-12}$$

Let

$$\cosh(x) = e^t, \quad t = \ln \cosh(x), \quad x = \text{arc} \cosh(e^t),$$

$$\frac{dx}{dt} = \frac{e^t}{\sqrt{e^{2t} - 1}} . \tag{A-13}$$

when we used reference 6, 4.6.38. Then

$$\mu_m(N, 0) = m \int_0^\infty dt \; f_1(t) \, f_2^{m-1}(t) \, e^{-pt} , \tag{A-14}$$

where

$$f_1(t) \equiv \frac{e^t}{\sqrt{e^{2t} - 1}} , \quad f_2(t) = \text{arc} \cosh(e^t), \; p = 2N-2. \tag{A-15}$$

We are now in a position to employ Watson's Lemma, reference 7, pp. 102-103. We need the expansions of $f_1(t)$ and $f_2(t)$ about $t = 0$. Consider

$$f_1(t) = \frac{1 + t + O(t^2)}{\sqrt{2t + 2t^2 + O(t^3)}} = \frac{1}{\sqrt{2t}} \; \frac{1 + t + O(t^2)}{1 + \frac{1}{2}t + O(t^2)}$$

$$= \frac{1}{\sqrt{2t}} \left[1 + \tfrac{1}{2} t + O(t^2) \right] . \tag{A-16}$$

Also via reference 6, 4.6.21 and 4.1.24,

$$f_2(t) = \ln \left[e^t + \sqrt{e^{2t} - 1} \right]$$

$$= \ln \left[1 + t + O(t^2) + \sqrt{2t + 2t^2 + O(t^3)} \right]$$

$$= \ln \left[1 + t + O(t^2) + \sqrt{2t} \left\{ 1 + \tfrac{1}{2} t + O(t^2) \right\} \right]$$

$$= \ln \left[1 + \sqrt{2t} + t + \frac{1}{\sqrt{2}} t^{3/2} + O(t^2) \right]$$

$$= \sqrt{2t} + t + \frac{1}{\sqrt{2}} t^{3/2} + O(t^2) - \tfrac{1}{2} \left[\sqrt{2t} + t + O(t^{3/2}) \right]^2 + \tfrac{1}{3} \left[\sqrt{2t} + O(t) \right]^3$$

$$= \sqrt{2t} + t + \frac{1}{\sqrt{2}} t^{3/2} + O(t^2) - \tfrac{1}{2} \left[2t + 2\sqrt{2} t^{3/2} + O(t^2) \right] + \tfrac{1}{3} \left[2\sqrt{2} t^{3/2} + O(t^2) \right]$$

$$= \sqrt{2t} \left[1 + \tfrac{1}{6} t + O(t^{3/2}) \right] . \tag{A-17}$$

Therefore,

$$f_2^{m-1}(t) = (2t)^{\frac{m-1}{2}}\left[1+\frac{m-1}{6}\,t + O(t^{3/2})\right], \tag{A-18}$$

and using (A-16)

$$f_1(t)\,f_2^{m-1}(t) = (2t)^{\frac{m}{2}-1}\left[1+\frac{m+2}{6}\,t + O(t^{3/2})\right]. \tag{A-19}$$

Term-by-term integration of (A-19) in integral (A-14) yields the desired result

$$\mu_m(N,0) \sim \frac{\Gamma\left(\frac{m}{2}+1\right)}{(N-1)^{m/2}}\left[1+\frac{m(m+2)}{24(N-1)}\right]$$

$$\sim \frac{\Gamma\left(\frac{m}{2}+1\right)}{\left(N-\frac{14+m}{12}\right)^{m/2}} \quad \text{as } N\to\infty. \tag{A-20}$$

Special cases are

$$\mu_1(N,0) \sim \frac{\sqrt{\pi}/2}{(N-1)^{1/2}}\left[1+\frac{1}{8(N-1)}\right] \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as } N\to\infty; \tag{A-21}$$

$$\mu_2(N,0) \sim \frac{1}{N-1}\left[1+\frac{1}{3(N-1)}\right] \sim \frac{1}{N-4/3} \quad \text{as } N\to\infty; \tag{A-22}$$

$$\sigma^2 = \mu_2 - \mu_1^2 \sim \frac{1-\pi/4}{N-1} + \left(\frac{1}{3}-\frac{\pi}{16}\right)\frac{1}{(N-1)^2}$$

$$\sim \left(1-\frac{\pi}{4}\right)\left(N-\frac{64-15\pi}{48-12\pi}\right)^{-1} \quad \text{as } N\to\infty. \tag{A-23}$$

Equation (A-21) agrees precisely with (A-8) when the latter is expanded to two terms in $(N-1)^{-1}$. More generally, (A-20) extends (A-10) to two terms in the asymptotic expansion, and (A-23) generalizes (A-11) to two terms. Comparison of (A-20)-(A-23) with results of the numerical procedure in appendix B reveals very close agreement.

Third Cumulant of Normalized D for C = 0

In order to determine whether random variable D, for C = 0, is approaching a Gaussian random variable as N increases, we evaluate the third cumulant of the normalized random variable

$$r = \frac{D - \mu_1}{\sigma}. \tag{A-24}$$

We have the usual first two cumulants of r:

$$\lambda_1^{(r)} = 0, \quad \lambda_2^{(r)} = 1. \tag{A-25}$$

Then, using (A-20)-(A-23),

$$\lambda_3^{(r)} = \overline{(r - \bar{r})^3} = \overline{r^3} = \frac{\overline{(D - \mu_1)^3}}{\sigma^3}$$

$$= \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{\sigma^3} \sim \frac{2\sqrt{\pi}(\pi - 3)}{(4 - \pi)^{3/2}} = .631 \quad \text{as } N \to \infty. \tag{A-26}$$

Since this quantity does not decrease to zero as N increases to ∞, the nonlinear *distortion D of magnitude-squared coherence estimate* $\hat{C}$ does not approach Gaussian, at least in the case when C = 0.

Recursion for $\mu_m(N,0)$

From (A-1) and (A-2),

$$\mu_m(N,0) = \int_0^1 dx\,(N-1)(1-x)^{N-2} A^m(\sqrt{x}), \tag{A-27}$$

where

$$A(t) = \operatorname{arctanh}(t), \quad A'(t) = \frac{1}{1-t^2}. \tag{A-28}$$

Let $t = \sqrt{x}$ in (A-27) to obtain

$$\mu_m(N,0) = \int_0^1 dt\; 2t(N-1)(1-t^2)^{N-2} A^m(t). \tag{A-29}$$

Let $m \geq 1$ in the following and integrate by parts with $u = A^m(t)$, $v = -(1-t^2)^{N-1}$, to obtain

$$\mu_m(N,0) = m\int_0^1 dt\,(1-t^2)^{N-2} A^{m-1}(t). \tag{A-30}$$

Now we assume that $N \geq 3$ and develop

$$\mu_m(N,0) = m\int_0^1 dt\,(1-t^2)^{N-3}(1-t^2)A^{m-1}(t)$$
$$= \mu_m(N-1,0) - m\int_0^1 dt\,(1-t^2)^{N-3}t^2A^{m-1}(t). \qquad \text{(A-31)}$$

Integrate by parts with $u = tA^{m-1}(t)$, $v = -(1-t^2)^{N-2}/[2(N-2)]$, to obtain

$$\mu_m(N,0) = \mu_m(N-1,0) - \frac{1}{2(N-2)}\mu_m(N,0) - \frac{m(m-1)}{2(N-2)}\int_0^1 dt\,(1-t^2)^{N-3}t\,A^{m-2}(t). \qquad \text{(A-32)}$$

For $m = 1$, this immediately reduces to the recursion in (A-7). Furthermore, for $m = 2$, the last integral can be evaluated immediately, and we find the convenient recursion for the second moment:

$$\mu_2(N,0) = \frac{2N-4}{2N-3}\mu_2(N-1,0) - \frac{1}{(N-2)(2N-3)} \text{ for } N \geq 3;\ \mu_2(2,0) = 2\ln 2. \qquad \text{(A-33)}$$

The starting value for $\mu_2(2,0)$ follows from (A-30) and (A-28):

$$\mu_2(2,0) = 2\int_0^1 dt\,A(t) = 2\int_0^1 dt\int_0^t \frac{dx}{1-x^2} = 2\int_0^1 \frac{dx}{1-x^2}\int_x^1 dt = 2\int_0^1 \frac{dx}{1+x} = 2\ln 2. \qquad \text{(A-34)}$$

We now assume that $m \geq 3$ and conclude the recursion derivation in (A-32). Integrate by parts with $u = A^{m-2}(t)$, $v = -(1-t^2)^{N-2}/[2(N-2)]$, to obtain the final result

$$\mu_m(N,0) = \frac{2N-4}{2N-3}\mu_m(N-1,0) - \frac{m(m-1)}{(2N-3)(2N-4)}\mu_{m-2}(N-1,0) \text{ for } N \geq 3 \text{ and } m \geq 3. \qquad \text{(A-35)}$$

(Equation (A-35) actually holds for $m = 2$ also; in fact, it reduces to (A-33).)

In order to start this recursion, we need to know

$$\mu_m(2,0) = m\int_0^1 dt\,A^{m-1}(t) = \frac{m}{2^{m-1}}\int_0^1 dt\left(\ln\frac{1+t}{1-t}\right)^{m-1}. \qquad \text{(A-36)}$$

We already know $\mu_2(2,0)$ in (A-34), so keep $m \geq 3$ in the following. Let $x = (1-t)/(1+t)$ and obtain

$$\mu_m(2,0) = \frac{m}{2^{m-2}}\int_0^1 \frac{dx}{(1+x)^2}\left(-\ln x\right)^{m-1}. \qquad \text{(A-37)}$$

Integrate by parts with $u = (-\ln x)^{m-1}$, $v = x/(1+x)$, to obtain

$$\mu_m(2,0) = \frac{m(m-1)}{2^{m-2}} \int_0^1 \frac{dx}{1+x} \left(-\ln x\right)^{m-2}. \tag{A-38}$$

Now let $x = e^{-t}$ and get

$$\mu_m(2,0) = \frac{m(m-1)}{2^{m-2}} \int_0^\infty \frac{dt\, t^{m-2}}{e^t + 1}. \tag{A-39}$$

Use reference 6, 23.2.8 and obtain

$$\mu_m(2,0) = m!\, \frac{2^{m-2} - 1}{2^{2(m-2)}} \zeta(m-1) \quad \text{for} \quad m \geq 3, \tag{A-40}$$

where ζ is the Riemann zeta function. Thus, from reference 6, 23.2.24 and 23.2.25,

$$\mu_3(2,0) = \frac{\pi^2}{4}, \quad \mu_5(2,0) = \frac{7}{48}\pi^4. \tag{A-41}$$

The value for $\mu_4(2,0)$ requires knowledge of $\zeta(3)$; values of $\zeta(n)$ for $n \geqslant 2$ are given in reference 6, page 811. Thus, we now have all the starting values necessary to use *recursion (A-35) for* $m \geqslant 3$.

Appendix B
Numerical Evaluation of Moments of D

Our starting point is the top line of (A-4), modified to the case for $C \neq 0$. We then express the hyperbolic functions in terms of exponentials and obtain

$$\mu_m(N, C) = 8 \int_0^\infty dt \, E \, \frac{1-E}{(1+E)^3} \, p_1\left(\left(\frac{1-E}{1+E}\right)^2\right) t^m , \tag{B-1}$$

where

$$E = \exp(-2t), \tag{B-2}$$

and p_1 is given by (1). A program for evaluation of (B-1) is given below. Inputs are N, C, and M(=m) in the first three lines. This program was used to deduce the behaviors listed in (7), (8), and (9).

```
10    N=128
20    C=.5
30    M=1
40    S1=0
50    S2=8/SQR(N)+16/N+1
60    COM N,C,M
70    F=(N-1)*(1-C)^N
80    PRINT "N =";N,"C =";C,"M =";M,S1;S2
90    S3=(FNS(S1)+FNS(S2))*.5
100   S4=0
110   S5=2
120   S6=(S2-S1)*.5
130   S7=2/3
140   FOR S8=1 TO S5-1 STEP 2
150   S4=S4+FNS(S1+S6*S8)
160   NEXT S8
170   Q=(S3+2*S4)*S6*S7*F
180   PRINT USING "M.12DE,7D";Q,S5
190   S3=S3+S4
200   S4=0
210   S5=S5*2
220   S6=S6*.5
230   GOTO 140
```

```
240   DEF FNS(T)
250   COM N,C,M
260   E=EXP(-2*T)
270   D=1-E
280   S=1+E
290   A=(D/S)^2
300   RETURN 8*E*D/S^3*FNP(A)*T^M
310   FNEND
320   DEF FNF21(A,B,C,X)
330   F=F4=1
340   FOR F5=1 TO 100
350   F6=F5-1
360   F4=F4*(A+F6)*(B+F6)*X/((C+F6)*F5)
370   F=F+F4
380   IF ABS(F4)<1E-12*ABS(F) THEN 420
390   NEXT F5
400   DISP "100 TERMS AT ";A;B;C;X
410   PAUSE
420   RETURN F
430   FNEND
440   DEF FNP(X)
450   COM N,C,M
460   B=C*X
470   P=(1-X)^(N-2)/(1-B)^(2*N-1)*FNF21(1-N,1-N,1,B)
480   RETURN P
490   FNEND
```

Appendix C
Program for the Evaluation of (13)

```
10    N=16
20    OUTPUT 0;"N =";N
30    PLOTTER IS "GRAPHICS"
40    GRAPHICS
50    DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60    DIM G(1:15)              ! GRID LINES
70    READ G(*)
80    Xm=1
90    Ym=FNInvphi(G(15))
100   SCALE 0,Xm,-Ym,Ym
110   FOR X=0 TO Xm STEP Xm/10
120   MOVE X,Ym
130   DRAW X,-Ym
140   NEXT X
150   FOR I=1 TO 15
160   Y=FNInvphi(G(I))
170   MOVE 0,Y
180   DRAW Xm,Y
190   NEXT I
200   PENUP
210   DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220   DIM C(1:13)            ! TRUE COHERENCE VALUES
230   READ C(*)
240   FOR I=3 TO 11
250   C=C(I)
260   T=(1-C)^2
270   Bias=T*(1+2*C/N)/N
280   Sigma=(N-1)*T*(2*C+(1-6*C+13*C*C)/N)/(N*(N+1))
290   Sigma=SQR(Sigma)
300   LINE TYPE 3
310   FOR Td=0 TO Xm STEP Xm/100
320   Y=(Td-C-Bias)/Sigma
330   PLOT Td,Y
340   NEXT Td
350   PENUP
360   LINE TYPE 1
370   FOR Td=0 TO Xm STEP Xm/100
380   Tc=Td
390   Y=FNCdfmsc(N,C,Tc)
400   PLOT Td,FNInvphi(Y)
410   IF Y>.99 THEN 430
420   NEXT Td
430   PENUP
440   NEXT I
450   NEXT K
460   END
```

```
470   DEF FNCdfmsc(N,C,Ct)
480   O=1-Ct
490   P=C*Ct
500   H=1-P
510   R=O/H
520   Tnew=0
530   S=T=1
540   FOR K=1 TO N-2
550   Told=Tnew
560   Tnew=T
570   T=((2*K-1+(N-K)*P)*Tnew-(K-1)*O*Told)*R/K
580   S=S+T
590   NEXT K
600   P=Ct*((1-C)/H)^N*S
610   RETURN P
620   FNEND
630   DEF FNInvphi(X)
640   IF (X>=0) AND (X<=1) THEN 670
650   PRINT "ARGUMENT ";X;"IS DISALLOWED FOR INVERSE PHI FUNCTION"
660   STOP
670   IF (X>0) AND (X<1) THEN 700
680   P=9.99999999999E99*(2*X-1)
690   GOTO 750
700   P=X
710   IF X>.5 THEN P=.5-(X-.5)
720   P=SQR(-2*LOG(P))
730   P=P-(2.515517+P*(.802853+P*.010328))/(1+P*(1.432788+P*(.189269+P*.001308))
)
740   IF X<.5 THEN P=-P
750   RETURN P
760   FNEND
770   DEF FNTanh(X)
780   S=EXP(2*X)
790   RETURN (S-1)/(S+1)
800   FNEND
810   DEF FNArctanh(X)
820   RETURN .5*LOG((1+X)/(1-X))
830   FNEND
```

Appendix D
Program for the Evaluation of (12)

The functions Cdfmsc, Invphi, Tanh, and Arctanh have already been listed in appendix C.

```
10    N=8
20    OUTPUT 0;"N =";N
30    PLOTTER IS "GRAPHICS"
40    GRAPHICS
50    DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60    DIM G(1:15)                ! GRID LINES
70    READ G(*)
80    Xm=4
90    Ym=FNInvphi(G(15))
100   SCALE 0,Xm,-Ym,Ym
110   FOR X=0 TO Xm STEP Xm/8
120   MOVE X,Ym
130   DRAW X,-Ym
140   NEXT X
150   FOR I=1 TO 15
160   Y=FNInvphi(G(I))
170   MOVE 0,Y
180   DRAW Xm,Y
190   NEXT I
200   PENUP
210   DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220   DIM C(1:13)            ! TRUE COHERENCE VALUES
230   READ C(*)
240   FOR I=1 TO 13
250   C=C(I)
260   B=(1-C^2)/(2*(N-1))
270   Mean=FNArctanh(SQR(C+B))
280   Sigma=1/SQR(2*(N-1))
290   LINE TYPE 3
300   FOR Td=0 TO Xm STEP Xm/100
310   Y=(Td-Mean)/Sigma
320   PLOT Td,Y
330   NEXT Td
340   PENUP
350   LINE TYPE 1
360   FOR Td=0 TO Xm STEP Xm/100
370   Tc=FNTanh(Td)^2
380   Y=FNCdfmsc(N,C,Tc)
390   PLOT Td,FNInvphi(Y)
400   IF Y>.99 THEN 420
410   NEXT Td
420   PENUP
430   NEXT I
440   NEXT K
450   END
```

Appendix E
Program for the Evaluation of (16)

The functions Invphi and Tanh have already been listed in appendix C.

```
10    INPUT P,N,C           ! PROBABILITY,PIECES,COHERENCE
20    Alpha=SQR(C+(1-C*C)/(2*(N-1)))
30    T=FNInvphi(P)/SQR(2*(N-1))
40    Beta=FNTanh(T)
50    A=((Alpha+Beta)/(1+Alpha*Beta))^2
60    PRINT A,P;N;C
70    END
```

Appendix F
Program for the Evaluation of (23)

The functions Cdfmsc, Invphi, and Arctanh have already been listed in appendix C.

```
10    C=.1
20    N=32
30    PRINT "C =";C,"N =";N
40    B=(1-C*C)/(2*(N-1))
50    Mean=FNArctanh(SQR(C+B))
60    Sigma=1/SQR(2*(N-1))
70    PLOTTER IS "GRAPHICS"
80    GRAPHICS
90    SCALE 0,1,-.5,3.5
100   GRID .1,.5
110   PENUP
120   LINE TYPE 3
130   FOR X=0 TO .995 STEP .005
140   Y=FNArctanh(SQR(X))                   ! ORIGINAL NON-LINEARITY
150   PLOT X,Y
160   NEXT X
170   PENUP
180   LINE TYPE 1
190   FOR X=.002 TO .998 STEP .002
200   P1=FNCdfmsc(N,C,X)
210   IF P1<.01 THEN 250
220   IF P1>1-.01 THEN 260
230   Y=Mean+Sigma*FNInvphi(P1)             ! DESIRED NON-LINEARITY
240   PLOT X,Y
250   NEXT X
260   END
```

Appendix G
Program for Evaluation of (26)

The functions Cdfmsc, Invphi, Tanh, and Arctanh have already been listed in appendix C.

```
10    N=32
20    OUTPUT 0;"N =";N
30    PLOTTER IS "GRAPHICS"
40    GRAPHICS
50    DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60    DIM G(1:15)                    ! GRID LINES
70    READ G(*)
80    Xm=4
90    Ym=FNInvphi(G(15))
100   SCALE 0,Xm,-Ym,Ym
110   FOR X=0 TO Xm STEP Xm/8
120   MOVE X,Ym
130   DRAW X,-Ym
140   NEXT X
150   FOR I=1 TO 15
160   Y=FNInvphi(G(I))
170   MOVE 0,Y
180   DRAW Xm,Y
190   NEXT I
200   PENUP
210   DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220   DIM C(1:13)                 ! TRUE COHERENCE VALUES
230   READ C(*)
240   FOR I=1 TO 13
250   C=C(I)
260   B=(1-C^2)/(2*(N-1))
270   Mean=FNArctanh(SQR(C+B))
280   Sigma=1/SQR(2*(N-1))
290   LINE TYPE 3
300   FOR Td=0 TO Xm STEP Xm/100
310   Y=(Td-Mean)/Sigma
320   PLOT Td,Y
330   NEXT Td
340   PENUP
350   LINE TYPE 1
360   FOR Td=0 TO Xm STEP Xm/100
370   Tc=FNTanh(Td)
380   Tc=FNInvg(Tc,N)                                   ! DISTORTION IS g(N)
390   Y=FNCdfmsc(N,C,Tc)
400   PLOT Td,FNInvphi(Y)
410   IF Y>.99 THEN 430
420   NEXT Td
430   PENUP
440   NEXT I
450   NEXT K
460   END
```

```
470   DEF FNInvg(Y,N)
480   X1=(Y+1.5/N)^2
490   X2=X1+.01
500   F1=SQR(X1)+.5*(1-X1)^(1*N)*LOG(.5*(N-1)*X1)/(N+2)-Y
510   F2=SQR(X2)+.5*(1-X2)^(1*N)*LOG(.5*(N-1)*X2)/(N+2)-Y
520   IF ABS(F1-F2)<1E-10 THEN RETURN X2
530   T=(X1*F2-X2*F1)/(F2-F1)
540   X1=X2
550   X2=T
560   F1=F2
570   GOTO 510
580   FNEND
```

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