AD-A091 298 CALIFORNIA INST OF TECH PASADENA DIV OF ENGINEERING--ETC F/G 20/11
ON SINGULAR PROBLEMS IN LINEARIZED AND FINITE ELASTOSTATICS.(U)
AUG 80 E STERNBERG N00014-75-C-0196
UNCLASSIFIED TR-45 NL

END
DATE
FILMED
12 80
DTIC

LEVEL II

(12)

AD A091298

DTIC ELECTE NOV 3 1980 C

CALIFORNIA INSTITUTE OF TECHNOLOGY

DDC FILE COPY

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

80 10 14 216

12

Office of Naval Research

Contract N00014-75-C-0196

15

6

On singular problems in linearized and

finite elastostatics.

9

Technical Report No.45

by

10

Eli Sternberg



14 TR-45

12 34

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91125

11

August 1980



071585

# On singular problems in linearized and finite elastostatics*

by

Eli Sternberg

California Institute of Technology

Summary

The predictions of linear elasticity theory for various basic types of singular equilibrium problems are illustrated and issues associated with such solutions are discussed. Attention is then turned to recent studies concerning the implications of finite elastostatics for certain singular problems, including some that have no counterpart in the linearized theory.

## Introduction

This expository paper is devoted to a subject with which its author has been preoccupied on occasions for a good many years. I should make plain from the outset that the present article is not intended as a comprehensive survey of the literature on singular problems in elastostatics: its purpose is illustrative rather than compilatory. Further, the selection of material included here is heavily biased in favor of issues that have been close to my own interests.

Some of the investigations discussed in the section concerning the linearized theory originated long ago. In contrast, the second section, which deals with

---

*This paper is the written version of a General Lecture presented at the Fifteenth International Congress of Theoretical and Applied Mechanics in Toronto, August 1980.

pertinent implications of the finite theory, is essentially confined to recent studies carried out in collaboration with J.K.Knowles and to closely related work. In order to make the paper accessible to readers who are not specialists in elasticity theory, I have tried to keep the presentation primarily descriptive and to avoid an excessive encumbrance with technical detail.

## 1. Singular problems in linear elastostatics

In attempting to categorize singular problems within the linearized equilibrium theory of elastic bodies according to the principal source of the singularities in the ensuing elastostatic field, one is led to distinguish among the following circumstances:

(i) load-induced singularities, such as those arising in the presence of concentrated or discontinuous loadings;

(ii) shape-induced singularities, which are typically due to body geometries involving sharp notches or cracks;

(iii) singularities induced by mixed boundary conditions, as exemplified by those encountered in various indentation and contact problems;

(iv) singularities attributable to material discontinuities, like those emerging in bonded assemblies of distinct homogeneous elastic materials, which are characteristic of inclusion and load-transfer problems.

The preceding classification is at once incomplete and somewhat misleading. In particular, it leaves out of account dislocation problems, which owe their singular nature to discontinuities in the displacement field. At the same time, the appearance of singularities within either of the first two categories is ordinarily contingent upon some collusion between the load and body geometries. Thus uniformly distributed normal tractions confined to two parallel faces of a rectangular elastic slab[1] produce a uniform uni-axial field of stress despite the prevailing traction discontinuities at the corners, and this regular stress field remains undisturbed by a traction-free plane crack that is parallel to the load direction.

Among problems in category (i), those involving concentrated loads merit special attention because of their pivotal role in the relevant theory of Green's functions and in view of the conceptual issues attending the admission of such loads into linear elastostatics. Indeed, while the notion of a "concentrated load" is a natural ingredient of the mechanics of particle systems and rigid bodies, it is inherently alien to the mechanics of deformable continua, unless properly clarified.

A conceptual guide for a physically natural and mathmatically sound approach to concentrated-load problems in linear elasticity theory is supplied by Kelvin's [1] original treatment of the problem corresponding to a concentrated load applied at a point of an elastic body occupying the entire space. Kelvin deals with this basic singular problem by starting with the regular problem appropriate to a suitably smooth uni-directional distributed body-force loading that vanishes outside a sphere centered at the intended point of application of the concentrated load (load-point) and that is otherwise arbitrary. If the displacements are required to vanish at infinity, the uniqueness of the solution to the latter problem is assured. This solution, furthermore, admits an explicit integral representation. Kelvin then proceeds to the limit as the region of load application is contracted to its center, while the resultant body force is made

---

[1] In the absence of explicit exemptions, it is to be taken for granted that the elastic solids considered in this paper are both homogeneous and isotropic.

to tend to the given concentrated load.[1] The limit process discribed above serves a dual purpose: it attaches an unambiguous meaning to Kelvin's problem and at the same time leads to its familiar solution in closed elementary form.

Kelvin's solution, which is the elastostatic analogue of the fundamental singular solution of Laplace's equation supplied by the Newtonian potential of a mass point, has the following properties:

(a) It is a regular solution of the governing field equations, in the absence of body forces, throughout the complement of the load-point with respect to the entire space;

(b) its displacement field vanishes at infinity;

(c) the resultant force of the tractions on any spherical surface centered at and facing the load-point equals the prescribed concentrated load;

(d) if $r$ denotes the distance from the load-point, the displacements and stresses are, respectively, $O(r^{-1})$ and $O(r^{-2})$ as $r \to 0$, both being unbounded at the load-point.

Kelvin's limit treatment of his problem is abandoned in most of the subsequent treatise literature in favor of a direct formulation of this singular problem on the basis of properties (a), (b), (c), no limitation being placed on the orders of the displacement or stress singularities admitted at the load-point. These three requirements alone, however, fail to characterize the solution uniquely, as is seen by adding to Kelvin's solution an arbitrary multiple of the solution corresponding to a center of dilatation, situated at the load-point, which has a "self-equilibrated singularity" of higher order at that point.

[1]If the "replacement loading" fails to be uni-directional, the existence of this limit requires a supplemental restriction of the distributed body forces. See [2], where Kelvin's limit process is spelled out in detail.

Nor is this lack of uniqueness in conflict with the classical uniqueness theorem of elastostatics, which does not encompass such a formulation of the problem.

On the other hand, one can show that properties (a), (b), (c), and (d) are in fact sufficient to determine Kelvin's solution uniquely.[1] Accordingly, (a), (b), (c), together with (d), constitute a complete direct formulation of Kelvin's problem. Further, it is clear from the foregoing observations that there exists an infinity of "pseudo-solutions" to Kelvin's problem, each of which conforms to (a), (b), (c), but violates (d) because it is more severely singular.

Although the analysis of Kelvin's problem outlined above suggest a parallel program for coping with problems involving concentrated surface loads, the execution of such a program presents considerably greater technical difficulties. As shown in [3] with the aid of the theory of elastostatic Green's functions, a limit process strictly analogous to Kelvin's confirms that the singularities at the point of application of a concentrated surface load are of the same order as those prevailing at the load-point in Kelvin's problem. Further, [3] contains a uniquess theorem that accommodates both concentrated surface — and internal concentrated loads, in addition to distributed surface tractions and body forces. Roughly speaking, the theorem asserts that the specification of conditions analogous to (a), (c), (d), accompanied by the boundary conditions for the given regular surface loads and — in the case of an unbounded region — supplemented by appropriate prescriptions at infinity, uniquely characterizes the solution of the singular problem at hand.[2] This result supplies a complete direct formulation of concentrated-load problems, thereby obviating a limit process that

---

[1] Actually, it suffices to adjoin to (a),(b),(c) the prescription of the order of either the displacement or the stress singularity at the load-point.

[2] An earlier attempt in [2] to establish such a generalization of the conventional uniqueness theorem fell short of its aim; see the Introduction to [3].

may be quite cumbersome to carry out explicitly and that is apt to lead to inconvenient representations of the solution in specific applications.

The foregoing uniqueness issue is not just an idle concern as is brought out in [4], which provided the initial impetus for the subsequent studies [2], [3]. Here a solution in series form is deduced for the particular problem of an elastic sphere in equilibrium under two equal and diametrically opposite concentrated surface load — a problem which is of some relevance to the stress analysis of ball bearings. This problem admits also a pseudo-solution in closed elementary form, exhibited in [4], which differs from the unique physically acceptable solution by an elastostatic field that keeps the boundary free of tractions — except at the load-points, where it has self-equilibrated singularities of an inadmissible higher order. Further, with the aid of a single pseudo-solution one can evidently construct infinitely many such solutions that exhibit arbitrarily large departures — throughout the entire body — from the correct solution. The latter, which is validated in [4] on the basis of a limit process starting with distributed surface loads, is found to be in satisfactory agreement with the results of a photoelastic investigation by Frocht and Guernsey [5], who determined the normal stresses on the equatorial plane of symmetry. This experimental verification reflects the fact that the fiction of "concentrated loads" has immediate practical value, apart from its mathematical significance in the linearized theory.

The direct formulation of a concentrated-load problem opens the possibility of removing in advance the singular parts of the desired solution with a view toward a reduction of the original problem to one that is governed by a least finite and continuous surface tractions and body forces. The analysis carried out in [4] reveals that such a reduction of the problem of the sphere considered there necessitates the introduction of additional singularities of a lower order, beyond the dominant singularities furnished by Boussinesq's solution for a half-space under a concentrated load applied normal to its plane boundary. The question as to the influence of the local curvature at the point of application of a concentrated surface load upon the detailed structure of the singularity arising at such a point is dealt with in [6] on the assumption that the boundary is locally a smooth surface of revolution, whose axis coincides with the load-axis.[1]

An example revealing that the admission of concentrated couples into linear elastostatics gives rise to difficulties more subtle than those accompanying the introduction of concentrated forces, is analyzed in [8]. The plane problem treated here is that of a wedge loaded exclusively by a "concentrated couple" applied to its vertex. The well-known corresponding plane-strain solution, apparently due to Carothers [9], rests on a direct formulation of this singular problem on the basis of the *two-dimensional field equations*, the boundary conditions for the traction-free faces of the wedge, together with the requirements that the stress field vanish at infinity and possess a singularity at the vertex having the prescribed couple as its stress resultant.

The stress distribution associated with this elementary closed solution meets the conditions listed above rigorously for all but a single opening angle of the wedge. Moreover, this stress field is antisymmetric about the wedge axis and becomes unbounded at the vertex like $r^{-2}$, if $r$ is the distance from this point. On the other hand, the stresses become unbounded at all field points (rather than merely at the vertex) as the opening angle $\alpha$ approaches $\alpha_*$, where $\alpha_*$ is the unique real root of $\tan\alpha = \alpha$ $(0 < \alpha < 2\pi)$ —

[1]In this connection see also [7].

or, approximately, 257 degrees.[1]

The curious breakdown of the solution under discussion at $\alpha = \alpha_*$ leads one to question the soundness of the underlying direct formulation of the problem. Indeed, this formulation is shown in [8] to remain incomplete even if it is augmented by the stipulation that the stresses and displacements be $O(r^{-2})$ and $O(r^{-1})$, respectively, as $r \to 0$. For there exist plane-strain elastostatic fields with self-equilibrated singularities of a lower order at the vertex that meet all conditions imposed in the traditional direct characterization of the problem except for that concerning the stress resultant of the singularity at the tip of the wedge.[2] This lack of uniqueness, in turn, motivates an approach to the problem by means of the limit process pursued in [8]. Here the original singular loading is initially replaced by regular distributed tractions applied to two finite segments of the boundary, each issuing from the vertex and of the same length, the entire loading being statically equivalent to the given couple. If the replacement loading is assumed to be normal to the faces of the wedge and antisymmetric about its axis, the classical solution is recovered by proceeding to the limit in the solution to the modified problem as the two load-segments are contracted to the vertex, provided $0 < \alpha < \alpha_*$. In contrast, this limit fails to exist when $\alpha_* \le \alpha < 2\pi$, so that the idealization of a concentrated couple at the vetex of the wedge is deficient in meaning for this range of the opening angle.[3] The physical role of the critical angle $\alpha_*$

[1] For $\pi < \alpha < 2\pi$, the plane region at hand is actually the complement of a "wedge" with respect to the entire plane.

[2] At the same time the order of the stress singularity inherent in these fields exceeds $O(r^{-1})$, the latter order being characteristic of concentrated-force problems in two dimensions.

[3] If the replacement loading is no longer restricted to be antisymmetric, one is similarly led to exclude $\pi < \alpha < 2\pi$ from the range of validity of the classical solution.

is, however, in need of further clarification.[1]

We turn now to illustrative examples involving discontinuous distributed surface loadings and recall first the nature of the singularities encountered in the plane-strain solution for a half-plane subjected to normal or tangential edge tractions that exhibit a finite jump discontinuity. In either case the displacement field remains continuous at the point of load-discontinuity. Further, a jump in the normal tractions induces merely a finite discontinuity in the stresses, although the scalar rotation is logarithmically unbounded. On the other hand, at finite jump in the tangential tractions the normal stress on planes perpendicular to the edge becomes logarithmically infinite, whereas the corresponding shear stress, the resultant stress on planes parallel to the edge, as well as the scalar rotation, stay bounded.

The results referred to above are consistent with the appropriate limit of the solution to the corresponding regularized problem, in which the given discontinuous loading is replaced by one suitably smooth to justify an appeal to the conventional uniqueness theorem of elastostatics. Moreover, the mathematical idealization of a load-discontinuity evidently owes its physical relevance to such a limit-definition. Yet no limit process is needed in order to motivate a direct formulation of singular plane problems of this kind, in which the obvious field requirements and regular boundary conditions are accompanied by the mere stipulation that the displacements remain bounded at points of surface-traction discontinuity (without any restriction upon the admitted order of the stress singularities). The completeness of this direct formulation emerges from a generalization of the classical uniqueness theorem in two dimensions due

[1] See also Budiansky and Carrier [10], as well as Barenblatt [11], for additional discussions of the wedge problem treated in [8].

to Knowles and Pucik [12]. The theorem established in [12], which is not confined to isotropic elastic solids, aims specifically at an economical direct approach to plane crack problems. As pointed out in [12], however, the proof devised there is readily modified to accommodate two-dimensional problems involving discontinuous surface loads, as well as all singular plane problems belonging to categories (ii) and (iii). In all such problems the boundedness of the displacements by itself is a supplementary condition sufficient to insure the completeness of the appropriate direct formulation.[1] Although one would expect an analogous uniqueness theorem to hold true for the corresponding three-dimensional problems, which involve line-singularities, the argument employed in [12] does not appear to tolerate such an extension.

In view of their significance in fracture mechanics, crack problems furnish especially important examples of singular problems belonging to class (ii). For illustrative purposes we mention here merely the three fundamental two-dimensional solutions pertaining to a traction-free plane crack of constant width and doubly-infinite extent in an all-around infinite elastic body under various homogeneous loadings at infinity.[2]

The first two of these solutions are associated with plane deformations at right angels to the edges of the crack, appropriate to an in-plane loading of either tension perpendicular to the crack-faces (Mode I) or pure shear on planes parallel and perpendicular to the faces of the crack (Mode II). Since the case of uni-axial tension parallel to the crack-faces is trivial, the preceding two loading modes suffice to accommodate — within the linearized theory of plane strain (or generalized plane stress) — any uniform in-plane loading at

[1] The hypotheses underlying [12] presuppose a finite domain and require suitable augmentation in case the domain is unbounded.

[2] See, for instance, the expository article by Rice [13].

infinity that is consistent with a traction-free crack. The third fundamental solution refers to anti-plane shear deformations produced by longitudinal shearing tractions at infinity that are confined to planes parallel to the crack-faces (Mode III) and act parallel to the edges of the crack.

In all three of the preceding loading modes, the displacements remain finite and continuous at the crack-tips, whereas the stresses are unbounded and $O(r^{-1/2})$, if $r$ is the distance from a tip. Further, the ensuing elastostatic field is symmetric about the plane of the crack in Mode I, but anti-symmetric in Modes II and III. Also, the last two loading cases are "gliding modes", in which the two crack-faces slide upon each other and the crack fails to open, according to the linear theory. Each of the three solutions under consideration coincides with the limit of the appropriate regular two-dimensional solution for an infinite body with an elliptic cylindrical hole, as the latter degenerates into a plane crack. This observation is commonly invoked to justify the physical relevance of the singular solutions to which we have alluded. In view of the generalized uniqueness theorem [12], referred to earlier, no such limit process is needed, however, in order to motivate a physically plausible and complete direct formulation of plane crack-problems in linearized elastostatics.

We proceed now to <u>singular problems of type</u> (iii) and in this connection use the mixed plane problem of a half-plane that is indented by means of a rigid, flat-ended (axially loaded) punch as an example. The nature of the boundary conditions arising at the edge of the half-plane depends on the traditional distinction between the case of a perfectly "smooth punch" and that of an ideally "rough punch". In either instance the normal displacement is to be constant along the contact segment, while the traction vector must vanish on the remainder of the boundary; in the first case, however, the contact condition demands vanishing shearing tractions, whereas the tangential contact displacement must be zero in the second case.

As is clear from our previous remarks concerning uniqueness, the conventional direct formulation of either of these punch problems is rendered complete by the additional requirement that the displacements remain bounded at the punch-corners. Nor is such a formulation in need of physical motivation on the basis of a limit process that takes a suitably related regular problem as its point of departure. In fact, it is not at all obvious how mixed singular problems of this kind can be "regularized" in a natural and manageable manner.

The well-known plane-strain solutions of the above two punch problems[1] predict that all stresses become unbounded at the corners of the punch like $r^{-1/2}$, where $r$ is the distance from such a corner. With the exception of the special case of an incompressible (linear elastic) material, in which both solutions happen to be identical, the elastostatic field pertaining to the rough punch exhibits an additional pathological feature: it is oscillatory in the immediate vicinity of the punch-corners; in particular, the normal contact tractions here alternate between compression and tension infinitely often near the ends of the contact-segment. This anomaly cannot be safely dismissed as insignificant on the grounds that the oscillations in the contact pressure are confined to extremely narrow zones adjacent to the punch corners: the presence of any such oscillations is incompatible with the unilateral contact-constraint implied by the absence of an enforced bond between the indenter and the semi-infinite body to which it is applied.

A related, even more disturbing pathology afflicts the plane problem

---

[1] See, for example, Muskhelishvili [14].

corresponding to two semi-infinite bodies of distinct elastic properties that are bonded along their plane interface, except for a crack of finite width, and are subjected to a Mode I loading. The alleged solution to this problem exhibits oscillatory displacements that would require the two deformed crack-faces to overlap in the vicinity of the edges of the crack.

In connection with plane problems of type (ii) or (iii) we refer at this place to an asymptotic scheme apparently first employed bv Knein [15] and later on systematically exploited by Williams [16], which aims at the local structure of the two-dimensional elastostatic field singularities arising in these circumstances. For this purpose Airy's stress function is used in [16] to construct global solutions of the homogeneous field equations of generalized plane stress in a wedge-shaped domain of arbitrary opening angle, for vanishing body forces and various homogeneous boundary conditions.[1] As far as the latter are concerned, three distinct cases are treated: (1) both legs of the boundary are free of tractions; (2) both legs are fixed; (3) one leg is traction-free and the other fixed. Further, the only solutions admitted are those in which each cartesian component of displacement — and hence each of the components of stress — is the product of power of the radial distance $r$ from the vertex and a function of the polar angel $\theta$ around this point. Such fields are necessarily generated by a biharmonic Airy function of the form $\phi(r,\theta) = r^{\lambda+1}f(\theta)$, and this Ansatz leads to a linear eigenvalue problem for the ordinary fourth-order differential equation to be satisfied by $f$, with $\lambda$ as the (possibly complex) eigenvalue parameter. In this manner one is led to an infinite sequence of (real-valued) solutions of the homogeneous boundary-value problem at hand. Moreover, each

[1]The analogous solutions appropriate to plane strain are obtained by replacing Poisson's ratio $\nu$ with $\nu/(1-\nu)$.

member of this sequence is fully determinate but for an arbitrary amplitude parameter; its displacements are $O(r^{\beta})$ and its stresses $O(r^{\beta-1})$ as $r \to 0$, if $\beta$ denotes the real part of the appropriate eigenvalue $\lambda$.

An elastostatic field belonging to the foregoing sequence thus has bounded displacements but unbounded stresses if and only if $0 < \beta = \mathrm{Re}\{\lambda\} < 1$. In Case 1 and Case 2 there is at least one such eigenvalue for each opening angle $\alpha$ in the range[1] $\pi < \alpha \leq 2\pi$, the same being true in Case 3 for $\pi \leq \alpha \leq 2\pi$. Suppose now $\lambda$ denotes the unique eigenvalue with the smallest real part $\beta$ in (0,1). In the first two cases $\lambda$ is real and $1/2 \leq \lambda = \beta < 1$. While $\lambda$ is independent of the elastic constants in Case 1, it varies with Poisson's ratio in Case 2 and Case 3. Further, $\lambda$ is no longer real in Case 3 so that the stresses in this instance are oscillatory in their dependence upon $r$, which is of the form

$$r^{\beta-1}[\cos(\kappa \log r) \quad \text{or} \quad \sin(\kappa \log r)] \quad ,$$

where $\kappa$ is a function of Poisson's ratio.[2]

The elastostatic fields associated with the particular eigenvalue singled out above is evidently a candidate for the description of the <u>dominant</u> asymptotic field behavior near a singular point in pertinent plane boundary-value problems belonging to categories (ii) or (iii). Indeed, the predictions thus arrived at are found to be consistent with available global solutions to such singular problems: for example the plane crack problems and the rough-punch problem discussed earlier. The appropriate value of the amplitude parameter, which remains indeterminate in the direct asymptotic analysis just outlined, in each

[1] Note the $\alpha = \pi$ corresponds to a half-plane, $\pi < \alpha < 2\pi$ to a re-entrant corner, and $\alpha = 2\pi$ to a crack.

[2] The discussion in [16] is confined to $\nu = 0.3$. I am indebted to R.Muki for an analysis supporting the more general conclusions stated above.

instance is bound to depend on the global body geometry, the complete boundary conditions, and — in the event of an unbounded domain — also upon the conditions imposed at infinity. An a priori determination of this parameter would greatly facilitate an efficient treatment of such singular problems, and in crack problems, where a complete knowledge of the crack-tip singularities is the information of primary physical concern, would actually eliminate altogether the need for dealing with the global problem.

Rice [17], supplies a special example of a crack problem in which the foregoing objective is attainable with the aid of a conservation law originally due to Eshelby [18], which is also valid in the finite equilibrium theory. More recently Freund [19], successfully applied an additonal conservation law deduced in [20][1] to the direct determination of the stress-intensity factors in several physically interesting problems pertaining to special crack and load configurations. There seems to be no generally applicable scheme, however, for accomplishing this purpose and the available conservation laws appear to be inadequate to cope with this issue even for the basic Mode I and Mode II problems.

We conclude this discussion of singular problems in linear elastostatics with some cursory remarks on problems in category (iv). The singularity arising at a regular interface between an elastic body and a fully bonded inclusion of different elastic properties involves merely finite jump discontinuities in the displacement gradient and the stresses; the displacements themselves are continuous by virtue of the prevailing bond, while the continuity of the tractions across the interface is an equilibrium requirement.

Among the most important problems in class (iv) are "load-transfer problems", which aim at the mechanical interaction between two bodies of different elastic

[1]See [20] and [21] for references to related work on conservation laws in elastostatics.

properties, bonded along a common portion of their boundaries, under circumstances in which one of the two contiguous bodies may be regarded as an essentially one-dimensional elastic continuum. Plane load-transfer problems concerning plate-stringer assemblies are of primary interest in connection with aircraft structures. On the other hand, problems pertaining to the diffusion of load from — or the absorption of load by — an elastic rod that is bonded to a three-dimensional elastic body are relevant to certain civil engineering structures and play a significant role in the mechanics of fiber-reinforced materials. A selective survey of analytical work (up to 1970) on plane and spatial load-transfer problems may be found in [22]. We mention here merely that the character of the singularity arising at the endpoints of the attachement depends on the particular manner in which the stringer or rod is modelled.

## 2. Some recent applications of nonlinear elastostatics to singular problems

In almost all of the singular problems discussed in the preceding section the linearized equilibrium theory — oblivious to the approximative assumption upon which it rests — gives rise to locally unbounded displacement gradients and stresses, regardless of the magnitude of the loads. The predictions of linear elastostatics for such problems may therefore be presumed at best to be realistic at finite distances from the singular points in the presence of sufficiently small loads, but cannot possibly be valid uniformly in the vicinity of these points, no matter how small the loads.

Misgivings about this state of affairs, in particular with regard to the appearance of infinite crack-tip stresses, have prompted ad hoc modifications of the implications of linear elasticity theory in fracture mechanics, such as that proposed by Barenblatt [23], and have motivated various studies of crack problems within plasticity theory[1], in most of which the hypothesis of infinitesimal deformations is retained. The same motivation underlies a series of investigations summarized in [25], aiming at the effect of couple-stresses upon singular stress concentrations in elastic solids and carried out within a linearized version of couple-stress theory due to Mindlin and Tiersten [26].

Returning to classical elastostatics we note that the field behavior near a point that is a source of infinite stress concentration according to the linear theory, is bound to involve the material's response to severe deformations. It is therefore natural to inquire into the corresponding implications of the finite theory, which allows for arbitrarily large deformations and takes account of constitutive nonlinearities as well. On the other hand, the common notion that local infinities in the stress field reflect merely the rebellion of a singular problem against its linearization and will automatically give way to finite stress concentrations in the absence of such a linearization, is quite unfounded. <u>Geometrically</u> induced infinities in the displacement gradients must be expected to be accompanied by locally unbounded stresses whenever the nonlinear mechanical response of the elastic material to the relevant homogeneous deformation leads to stresses that become infinite as the deformation grows beyond bounds. Furthermore, the <u>statically</u> induced infinity in the stresses at a point of application of a "concentrated load" cannot disappear in the nonlinear theory, although such loads may be precluded altogether by the particular constitution of the material.

We proceed now to some recent work on singular problems within the framework of finite elastostatics, which comprises two interrelated sequences of studies: one of these deals with locally unbounded deformation gradients and aims primarily at the asymptotic character of the elastostatic field near the tip of a crack; the other is concerned with discontinuous deformation gradients,

[1] See Rice [13] and Knowles [24] for pertinent references.

which have no counterpart in the linear theory of homogeneous elastic solids. Earlier stages of this work were summarized in [21] and in previous surveys by Knowles [24], [27], [28]. Although the present survey is overlapping with those cited, it describes also some results not previously available.

The analysis of crack problems on the basis of the nonlinear equilibrium theory of elastic solids appears to have its origins in a paper by Wong and Shield [29], who deduced an approximate global solution to the problem of a finite crack in an all-around infinite incompressible elastic sheet of a Neo-Hookean material, subjected to bi-axial tension at infinity. The approximative scheme employed in [29] requires the deformations to be large throughout the sheet.

An investigation of the elastostatic field near the tip of a crack for the nonlinear analogue of the Mode I problem and a class of compressible elastic materials, is contained in [30], [31]. Later on Knowles [32], confining his attention to certain incompressible elastic solids, carried out a related local analysis pertaining to the Mode III problem in the finite theory. Further, Stephenson [33] — in a doctoral dissertation just completed — deals asymptotically with the finite plane-strain crack problem for a class of incompressible materials in the absence of any symmetry restrictions upon the loading at infinity; his results thus encompass in particular Mode II as well as Mode I loading conditions.[1]

An asymptotic approach common to these local studies is best illustrated by means of the Mode III problem treated in [32], which is free of extraneous mathematical complications encountered in the other investigations mentioned

[1]The special case of the Mode I loading for a Mooney-Rivlin material had been treated earlier in an unpublished study, the results of which are described briefly in [24].

above.

The class of incompressible materials considered in [32] is defined in terms of an elastic potential that is completely determined by the response to a homogeneous deformation of simple shear. Materials of this kind are capable of sustaining nonhomogeneous anti-plane shear deformations and permit one to reduce the nonlinear Mode III problem to a boundary-value problem governed by a single quasi-linear partial differential equation of the second order for the scalar displacement $u$ parallel to the edges of the crack; the two in-plane components of displacement vanish identically. The constitutive law adopted in [32] is then specialized to a sub-class of materials whose simple-shear response obeys a power-law that involves a "hardening-parameter" $n$. For $n \geq 1/2$ the induced shear stress $\tau(k)$ increases steadily with the amount of shear $k$; $\tau(k) \rightarrow \infty$ as $k \rightarrow \infty$ when $n > 1/2$, but approaches a finite ultimate value when $n = 1/2$. If $0 < n < 1/2$, the stress $\tau(k)$ grows monotonically to a maximum with increasing $k$ and thereafter declines steadily toward zero. Moreover, the slope of the response curve is steadily increasing for $n > 1$ (hardening material) and steadily diminishing for $1/2 < n < 1$ (softening material); $n = 1$ corresponds to the special case of a Neo-Hookean material.

If one restricts the range of $n$ to $(1/2, \infty)$, the nonlinear crack problem at hand yields to a local treatment that is a counterpart of the scheme applied by Williams [16] to singular plane-strain problems in the linear theory. The starting point in this local analysis is the assumption that $u$ in the vicinity of — say — the right-hand crack-tip is asymptotic to $r^m v(\theta)$, where $(r,\theta)$ are polar coordinates based at this point, $m$ is an as yet unknown exponent and $v(\theta)$ an initially unknown function of the polar angle. Upon invoking the displacement equation of equilibrium and the boundary conditions for a traction-free crack to leading order, one arrives at an eigenvalue problem for an ordinary nonlinear second-order differential equation governing $v(\theta)$ and involving $n$, with $m$ as the eigenvalue parameter.[1] The solution to this subsidiary eigenvalue problem is deducible in closed elementary form and — but for an arbitrary amplitude parameter — is unique, provided $0 < m < 1$. The ensuing asymptotic respresentation for $u$, together with the corresponding representation for the stresses derivable from it, thus furnishes the dominant near-field behavior consistent with a finite crack-tip displacement. Furthermore, for <u>small</u> load intensities one can obtain a precise estimate for the amplitude parameter with the aid of the available global solution to the analogous linearized crack problem, on the basis of the conservation law supplied in [18].[2]

The asymptotic results appropriate to $n > 1/2$ thus emerging reveal that the order and specific structure of the stress singularities depends crucially on the value of the hardening parameter. The two nonvanishing shear stresses become unbounded at the crack-tip, as is to be anticipated in the present instance.[3] Hardening $(n > 1)$ aggravates, while softening $(1/2 < n < 1)$ mitigates the shear-stress singularities predicted by the linear theory.[4] In contrast the axial normal stress, which vanishes identically according to linearized elastostatics, becomes infinite like $1/r$ as $r \to 0$ (regardless of the

---

[1]For reasons as yet obscure precisely the same eigenvalue problem arises also in the asymptotic studies [30] and [33] under quite different circumstances.

[2]In this connection it is taken for granted that the solution of the linearized problem, for small enough loads, uniformly approximates its counterpart in the nonlinear theory at all material points a <u>finite</u> distance away from the crack-tips.

[3]Recall that $\tau(k) \to \infty$ as $k \to \infty$ when $n > 1/2$.

[4]Nor $n = 1$ (Neo-Hookean material) the global solutions to the exact and the linearized crack problem coincide, as far as $u$ and the shear stresses are concerned.

particular value of $n$), the same being true of the strain-energy density.

In the limiting case $n = 1/2$ of a monotone increasing response function $\tau(k)$, the asymptotic Ansatz $u \sim r^m v(\theta)$ as $r \to 0$ leads to an eigenvalue problem that has no solution and is therefore inadmissible. This transition case is treated in [32] by a different asymptotic approach — limited _a priori_ to small amounts of shear at infinity — in which the crack of finite width is replaced by a semi-infinite one, while the _far_ field is required to match the elastostatic field _near_ the crack-tip predicted by the linearized theory. This "small-scale nonlinear crack problem", which is an analogue of what is known as the small-scale yielding problem in the related plasticity literature, is then solved by means of the hodograph method. The results thus obtained furnish _finite_ shear stresses at the crack-tip, as is to be expected since $n = 1/2$ gives rise to a finite ultimate stress in simple shear, but the induced axial normal stress and the strain-energy density remain unbounded as $r \to 0$. Although the solution to the small-scale nonlinear Mode III crack problem has unlimited smoothness, it exhibits rapid changes in the angular distribution of the elastostatic field around the crack-tip that signal an impending breakdown of smoothness once the response curve for simple shear undergoes a reversal of slope. Examples of this kind will be mentioned later on.

We now refer briefly to some conclusions reached in [33] concerning the plane-strain crack problem in the finite theory for a class of incompressible materials. Here again the asymptotic method employed in [32] for $n > 1/2$ leads to a consistent near-field approximation if the relevant hardening parameter is suitably restricted. The asymptotic results arrived at in this manner are compatible with the existence of a global solution to the appropriate Mode I problem that is _symmetric_ about the plane of the crack; on the other hand, they suggest the nonexistence of an antisymmetric global solution to the corresponding Mode II problem. Indeed, as is proved in [33] by recourse to the Mooney-Rivlin material, the latter problem in the finite theory — unlike its counterpart in infinitesimal elastostatics — in general does not admit a solution antisymmetric about the crack-plane. This, at first sight surprising inference is rendered plausible by the observation that the governing nonlinear field equations, in contrast to those of the linearized theory, are not invariant under the corresponding parity transformation.

The asymptotic results in [33] lead to another somewhat startling departure from the predictions of the linear theory for the Mode II problem: at least for a certain range of the pertinent hardening parameter, the crack is found to open in the vicinity of its tips, while the global solution based on the linear theory implies that the crack-faces fail to separate in this instance.

The preceding result gives rise to the intriguing question as to the transition from the finite to the infinitesimal theory in the Mode II crack problem. Still more perplexing is the related question concerning the precise approximative status within the nonlinear theory of the solution to the rough-punch problem discussed in the previous section. For according to an asymptotic study [34], which is confined to compressible materials of "harmonic type" introduced by John [35], nonoscillatory contact tractions are found to be consistent with the finite theory.

Difficulties encountered in an unsuccessful attempt to adapt the asymptotic treatment [30], [31] of the nonlinear Mode I crack problem to a particular constitutive law proposed by Blatz and Ko [36] suggested that the global Mode I problem may not admit a solution of unlimited smoothness in this instance. Such an eventuality, in turn, pointed to a breakdown in the ellipticity of the elastostatic field equations for the Blatz-Ko material in the presence of sufficiently severe deformations — a conjecture confirmed in [37]. This special investigation gave impetus to several studies concerning a potential loss of ellipticity in finite elastostatics and the concomitant possibility of equilibrium solution fields that possess continuous displacements but exhibit finite jump discontinuities in the first displacement gradients. Singular solutions of this kind are associated with localized shear failures and had been considered earlier by Rudnicki and Rice [38] in a different constitutive setting. Mathematically, such singular solutions bear a more than casual resemblance to gas-dynamical shocks in steady transonic flows; it is therefore natural to speak of "elastostatic shocks" or "equilibrium shocks" in the present context.

Explicit necessary and sufficient conditions for ordinary and strong ellipticity of the displacement equations of equilibrium governing plane deformations of compressible isotropic elastic solids are deduced in [39]. The resulting inequalities involve the local principal stretches directly as well as through the first and second gradients of the plane-strain elastic potential with respect to the principal stretches. Unfortunately these inequalities do not appear to admit a convenient physical interpretation. Abeyaratne [40] deals with the analogous issue for plane deformations of incompressible isotropic elastic materials and arrives at ellipticity conditions that have a rather simple physical meaning. Even more transparent is the ellipticity condition for anti-plane shear deformations of incompressible elastic solids of the kind considered in [32]. Here the governing displacement equation of equilibrium is found to be locally elliptic at a solution if and only if the response curve for simple shear has a positive slope at an amount of shear equal to the magnitude of the corresponding local displacement gradient. The analysis in [32] of the small-scale nonlinear crack problem for a hardening parameter $n = 1/2$ thus pertains to the limiting elliptic case within the class of power-law materials introduced there.

As shown in [41], the emergence of an elastostatic shock in a plane deformation of a homogeneous but possibly anisotropic, compressible elastic medium necessitates a breakdown of strong ellipticity in the displacement equations of equilibrium at some homogeneous deformation. The local structure of such singular solution fields near a surface of discontinuity in the displacement gradient is explored extensively in [41], particular attention being given to "weak" equilibrium shocks. For this purpose it is sufficient to consider a plane shock-surface that separates two distinct homogeneous deformations. Further, an example of such a "piecewise homogeneous shock" of finite strength, based on the Blatz-Ko material, is analyzed in detail. Finally, in the same paper, a dissipation inequality — akin to the entropy inequality of gas dynamics — is proposed on energetic grounds. Results parallel to those in [41], for elastostatic shocks associated with plane deformations of incompressible materials are deduced in [40]. Knowles [42] deals with the energetics of elastostatic shocks in greater generality and detail. Here nonhomogeneous shocks with curved shock surfaces are admitted and compressible as well as incompressible materials are considered. It is found that the dissipation inequality arrived at in [41] remains applicable in these broader circumstances.

Abeyaratne [43] presents a comprehensive analysis of a one-dimensional boundary-value problem in which elastostatic shocks arise. The example studied here concerns the finite twisting of a hollow cylinder for a class of incompressible elastic solids whose constitutive behavior admits a loss of ellipticity in the presence of sufficiently severe simple shearing deformations.

A pilot study illustrating the emergence of equilibrium shocks in a two-dimensional boundary-value problem is analyzed in [44], which aims at the small-scale nonlinear Mode III crack problem for a particular incompressible elastic solid within the constitutive category underlying [32]. In [44] the shear stress $\tau(k)$, induced by a simple shear of amount $k$, is taken to rise linearly to a finite peak-value at a critical amount of shear $k_o$ and is assumed to decline steadily to zero in a definite manner as $k$ is further increased. Accordingly the governing scalar displacement equation of equilibrium suffers a loss of ellipticity at any solution whose local gradient exceeds $k_o$.

A formal application of the hodograph transformation to this nonlinear second-order partial differential equation enables one to generate, in closed elementary form, exact elliptic and hyperbolic solutions on certain subdomains of the exterior of the semi-infinite crack. These solutions, in turn, may be pieced together so as to produce a global solution of restricted smoothness to the particular small-scale Mode III problem at hand. The explicit solution thus arrived at displays finite jump discontinuities in the displacement gradient across two curved shock lines that are symmetrically situated with respect to the crack, issue from its tip, and terminate in the interior of the body — ahead of the crack. This solution is hyperbolic in a bulb-shaped domain whose boundary consists of the two shock lines and a circular arc joining their endpoints; it is elliptic outside the closure of this region. While the displacements and tractions remain continuous as the shock lines are traversed, the stresses suffere jump discontinuities. All stresses remain finite at the tip of the crack except for the axial normal stress, which becomes unbounded if the crack-tip is approached from within the hyperbolic domain.

If the nonlinear portion of the response curve for simple shear is generalized to allow for different rates of decay past critical shear, the results described above undergo certain modifications that are discussed at the end of [44]. Of particular interest is the limiting non-elliptic case in which $\tau(k)$

is constant for $k_o < k < \infty$. In this instance one is led to a solution free of shocks, whose asymptotic character near the crack-tip is the same as that established in [32] for the limiting elliptic case of a power-law material ($n = 1/2$). As is to be expected, the solution based on a flat shear-response past critical shear is also closely related to results obtained by Hult and McClintock [45] in dealing with the Mode III crack problem for an elastic-perfectly plastic body.

A considerbaly broader generalization of the specific example treated in [44] is carried out in [46]. Here the analysis of the small-scale nonlinear Mode III problem in the presence of a potential loss of ellipticity is extended to a class of incompressible elastic solids that includes power-law materials with a hardening parameter $n < 1/2$. In addition, the constitutive assumptions underlying [46] also admit a response to simple shear in which $\tau(k)$ increases (nonlinearly) to a peak value and thereafter declines steadily to a positive ultimate shear stress as $k \to \infty$.

The analysis in [46] relies once again on the hodograph method and is guided by the pilot study [44]. The generalized problem is, however, no longer amenable to an explicit solution. Instead, the existence of a solution involving elastostatic shocks and its relevant features are established in [46] on the basis of qualitative arguments. The conclusions thus reached are in some respect quite similar to those arrived at in [44].

The studies [44],[46] of equilibrium shocks near the tip of a crack in the small-scale nonlinear Mode III problem leave two significant questions unanswered. First, the nonexistence of a smooth (shock-free) solution, though plausible, remains to be demonstrated. Next, the stability of the solution found has yet to be examined. In this connection it should be remarked that a homogeneous deformation of simple shear is readily seen to be dynamically unstable at amounts of shear beyond the critical amount. This, however, does not preclude the stability of the solutions under consideration: nor would their possible instability render them devoid of physical interest.

Finally, the work reported in [44],[46], which depends crucially on the hodograph method, supplies no clue as to the treatment of the more important small-scale nonlinear plane-strain crack problem under constitutive assumptions that entail a potential loss of ellipticity, since the latter problem is governed by a system of partial differential equations of the fourth order.

Acknowledgment

This paper was prepared — and a major part of the work discussed in it was carried out — under the sponsorship of the Office of Naval Research in Washington, D.C. I am grateful to J.K.Knowles for most helpful discussions and criticisms.

## REFERENCES

[1] W.Thompson (Lord Kelvin) and P.G.Tait, Treatise on Natural Philosophy (reprinted as Principles of mechanics and dynamics), Part 2, Dover, New York, 1962.

[2] E.Sternberg and R.A.Eubanks, On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity, J.Rat.Mech.Anal., 4 (1955), 1, p.135.

[3] M.J.Turteltaub and E.Sternberg, On concentrated loads and Green's functions in elastostatics, Arch.Rat.Mech.Anal., 29 (1968), 3, p.193.

[4] E.Sternberg and F.Rosenthal, The elastic sphere under concentrated loads, J.Appl.Mech., 19 (1952), 4, p.413.

[5] M.M.Frocht and R.Guernsey, Jr., A special investigation to develop a general method for three-dimensional photoelastic stress analysis, NACA Report under Contract NAw-5959, September 1951.

[6] E.Sternberg and R.A.Eubanks, On the singularity at a concentrated load applied to a curved surface, Proc., Second U.S.Nat.Congress of Appl.Mech., Ann Arbor, Mich., ASME, New York, 1954.

[7] G.L.Neidhardt and E.Sternberg, On the transmission of a concentrated load into the interior of an elastic body, J.Appl.Mech., 23 (1956), 4, p.541.

[8] E.Sternberg and W.T.Koiter, The wedge under a concentrated couple, J.Appl. Mech., 25 (1958), 4, p.575.

[9] S.D.Carothers, Plane strain in a wedge, Proc., Royal Soc.Edinburgh, 23 (1912), p.292.

[10] B.Budiansky and G.F.Carrier, The pointless wedge, SIAM J.Appl.Math., 25 (1973), 3, p.378.

[11] G.I.Barenblatt, Similarity, self-similarity, and intermediate asymptotics, English translation, Consultants Bureau, New York, 1979.

[12] J.K.Knowles and T.A.Pucik, Uniqueness for plane crack problems in linear elastostatics, J.Elasticity, 3 (1973), 3, p.155.

[13] J.R.Rice, Mathematical analysis in the mechanics of fracture, in Fracture, Vol.II, Chapter 3, Academic Press, New York, 1968.

[14] N.I.Muskhelishvili, Some basis problems of the mathematical theory of elasticity, English translation, Noordhoff, Groningen, 1953.

[15] M.Knein, Zur Theorie des Druckversuchs, Zeit.Angew.Math.Mech., 6 (1952), p.526.

[16] M.L.Williams, Stress singularities resulting from various boundary conditions in angular corners of plates in extension, J.Appl.Mech., 19 (1952), 4, p.526.

[17] J.R.Rice, A path-independent integral and the approximate analysis of strain concentrations by notches and cracks, J.Appl.Mech., 35 (1968), 2, p.379.

[18] J.D.Eshelby, The continuum theory of lattice defects, in Progress in solid state physics, Vol.3, Academic Press, New York, 1956.

[19] L.B.Freund, Stress intensity factor calculations based on a conservation integral, Int.J.Solids and Structures, 14 (1978), 3, p.241.

[20] J.K.Knowles and E.Sternberg, On a class of conservation laws in linearized and finite elastostatics, Arch.Rat.Mech.Anal., 44 (1972), 3, p.187.

[21] E.Sternberg, Some recent advances in the application of nonlinear elastostatics to singular problems, in Trends in solid mèchanics 1979, Proc. Symp. dedicated to the 65th Birthday of W.T.Koiter, Delft University Press, Sijthoff and Noordhoff, 1979.

[22] E.Sternberg, Load-transfer and load-diffusion, Proc.Sixth U.S.Nat.Congress Appl.Mech., Cambridge, MA., ASME, New York, 1970.

[23] G.I.Barenblatt, The mathematical theory of equilibrium cracks in brittle fractures, in Advances in applied mechanics, Vol.VII, Academic Press, New York, 1962.

[24] J.K.Knowles, Crack problems in finite elastostatics, Proc., Symposium on Mathematical problems in fracture mechanics, AMS-SIAM, New York, 1978.

[25] E.Sternberg, Couple-stresses and singular stress concentrations in elastic solids, Proc., IUTAM Symposium on Mechanics of generalized continua, Freudenstadt, Springer, Berlin, 1968.

[26] R.D.Mindlin and H.F.Tiersten, Effects of couple stresses in linear elasticity, Arch.Rat.Mech.Anal., 11 (1962), 5, p.415.

[27] J.K.Knowles, Finite elastostatic fields with unbounded deformation gradients, Proc., ASME Symposium on Finite elasticity, Atlanta, GA., AMD Vol.27, ASME, New York, 1977.

[28] J.K.Knowles, On some inherently nonlinear singular problems in finite elastostatics, Proc., Eighth U.S.Nat.Congress Appl.Mech., Los Angeles, CA., Western Periodicals Co., North Hollywood, 1978.

[29] F.S.Wong and R.T.Shield, Large plane deformations of thin elastic sheets of Neo-Hookean material, Zeit.angew.Math.Physik, 20 (1969), p.176.

[30] J.K.Knowles and E.Sternberg, An asymptotic finite-deformation analysis of the elastostatic field near the tip of a crack, J.Elasticity, 3 (1973), 2, p.67.

[31] J.K.Knowles and E.Sternberg, Finite-deformation analysis of the elastostatic field near the tip of a crack: Reconsideration and higher-order results, J.Elasticity, 4 (1974), 3, p.201.

[32] J.K.Knowles, The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids, Int.J.Fracture, 13 (1977), 5, p.611.

[33] R.A.Stephenson, The equilibrium field near the tip of a crack for finite plane strain of incompressible elastic materials, Doctoral Dissertation, Div.Eng.Appl.Sci., Calif.Inst.Technology, June 1980.

[34] J.K.Knowles and E.Sternberg, On the singularity induced by certain mixed boundary conditions in linearized and nonlinear elastostatics, Int.J.Solids and Structures, 11 (1975), 11, p.1173.

[35] F.John, Plane strain problems for a perfectly elastic material of harmonic type, Comm.Pure and Appl.Math., XIII (1966), p.309.

[36] P.J.Blatz and W.L.Ko, Application of finite elasticity to the deformation of rubbery materials, Trans., Soc.Rheology, 6 (1962), p.223.

[37] J.K.Knowles and E.Sternberg, On the ellipticity of the equations of nonlinear elastostatics for a special material, J.Elasticity, 5 (1975), 3-4, p.341.

[38] J.W.Rudnicki and J.R.Rice, Conditions for the localization of deformation in pressure-sensitive dilatant materials, J.Mech.Phys.of Solids, 23 (1975), p.371.

[39] J.K.Knowles and E.Sternberg, On the failure of ellipticity of the equations for finite elastostatic plane strain, Arch.Rat.Mech.Anal., 63 (1977), 4, p.321.

[40] R.C.Abeyaratne, Discontinuous deformation gradients in plane finite elastostatics of incompressible materials, TR No.41, ONR Contract N00014-75-C-0196, Calif.Inst.Tech., May 1979 (to appear in J.Elasticity).

[41] J.K.Knowles and E.Sternberg, On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics, J.Elasticity, 8 (1978), 4, p.329.

[42] J.K.Knowles, On the dissipation associated with equilibrium shocks in finite elasticity, J.Elasticity, 9 (1979), 2, p.131.

[43] R.C.Abeyaratne, Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube, TR No.42, ONR Contract N00014-75-C-0196, Calif.Inst.Tech., June 1979 (to appear in J.Elasticity).

[44] J.K.Knowles and E.Sternberg, Discontinuous deformation gradients near the tip of a crack: an example, J.Elasticity, 10 (1980), 1, p.81.

[45] J.A.H.Hult and F.A.McClintock, Elastic-plastic stress and strain distribution around sharp notches under repeated shear, Proc., Ninth IUTAM Congress of Applied Mechanics, Brussels, 1956.

[46] J.K.Knowles and E.Sternberg, Anti-plane shear fields with discontinuous deformation gradients near the tip of a crack in finite elastostatics, TR No.43, ONR Contract No.N00014-75-C-0196, Calif.Inst.Tech., September 1979 (to appear in J.Elasticity).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | |
|---|---|
| 1. REPORT NUMBER: 45 | 2. GOVT ACCESSION NO.: AD-A092298 |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (and Subtitle): On singular problems in linearized and finite elastostatics | 5. TYPE OF REPORT & PERIOD COVERED: Research Report |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): Eli Sternberg | 8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0196 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: California Institute of Technology, Pasadena, California 91125 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Washington D.C. | 12. REPORT DATE: August 1980 |
| | 13. NUMBER OF PAGES: 31 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report): Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear elastostatics, nonlinear elastostatics, singular problems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The predictions of linear elasticity theory for various basic types of singular equilibrium problems are illustrated and issues associated with such solutions are discussed. Attention is then turned to recent studies concerning the implications of finite elastostatics for certain singular problems, including some that have no counterpart in the linearized theory.



Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)