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USER'S MANUAL FOR THE BRL SUBROUTINE TO EVALUATE SINE, COSINE, AND EXPONENTIAL INTEGRALS FOR COMPLEX ARGUMENT

James N. Walbert
Emma M. Wineholt

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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The formulas used in the BRL Sine, Cosine and Exponential Integral subroutine are derived in sufficient detail to enable the user to correctly implement the computer code. Estimates of precision, sample output, and an annotated listing of the computer code are included in the appendices.

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TABLE OF CONTENTS

NOTATION

This user's manual is designed to assist the mathematician or programmer using the BRL Sine, Cosine, and Exponential Integral Subroutine. FORTRAN symbols for variables and arithmetic operations are used in the body of the report for consistency with excerpts from the coding.

As an aid to the reader unfamiliar with standard FORTRAN, the following symbols are defined:

	Symbol	Operation	Algebraic Notation		FORTRAN Notation
1.	+	add	$a + b$	=	A + B
2.	-	subtract	$a - b$	=	A - B
3.	*	multiply	$a \times b$	=	A * B
4.	/	divide	$a \div b$	=	A / B

Numbers are written in specific ways to define their type:

1. Integer: 2
2. Real: 2. or 2.0
3. Standard notation 2.78×10^5: 2.78 E+05

   (double precision) 2.78 D+05

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I. INTRODUCTION

Sine and cosine integrals occur in applications of Tranter's method[1] to the evaluation of stresses in thick-walled cylinders. The need for highly precise values of the integrals for complex argument led to the development of this BRL subroutine. Although tables of sine, cosine, and exponential integrals have been published, these tables are of necessity limited in scope. Moreover, interpolation between given values in such tables results in loss of accuracy.

While computer subroutines exist for the computation of certain of these integrals for restricted values of the argument, the authors know of no subroutine which is valid for the wide range of complex argument and order of the exponential integral or which has the degree of precision of the subroutine presented in this report.

The three methods used to compute the values of the integrals are

1. Series,
2. Gauss continued fractions, and
3. Asymptotic series.

Each of these methods will be discussed in sufficient detail to enable the user to understand the subroutine. Recourse to special multiple precision codes or integer arithmetic has been deliberately avoided. The goal was to provide a high degree of precision through careful attention to analytic detail.

The subroutine has been written in double-precision FORTRAN IV and has been code-checked on the CDC 7600. Examples run on the CDC 7600 have agreed to 25 significant digits with tables of the sine integral generated by C-B Ling[2]. (See Appendix C). Complex arithmetic has not been used in the computer code; the annotated listing in Appendix B, together with the analysis presented in section III, will serve to illustrate the manner in which real and imaginary parts of the integrals are computed. As a general rule, whenever the real and imaginary parts of a number are stored in an array of length 2, the real part is in the first location and the imaginary part is in the second location.

---

[1] *C. J. Tranter, Integral Transforms in Mathematical Physics, Methuen and Co., Ltd., London, England, 1966.*

[2] *Chih-Bing Ling, Collected Papers, Vol. II, Virginia Polytechnic Institute and State University, Blacksburg, VA, 1979.*

II. INPUT AND OUTPUT VARIABLES

The subroutine statement is

```
SUBROUTINE SCINT (X, Y, SI, CI, EX, NORDER, ICODE, IERR).
```

The input variables are X, Y, NORDER, and ICODE. X and Y are double-precision real variables, while NORDER and ICODE are integer variables. ICODE describes to the subroutine the manner in which X and Y are to be interpreted. If

ICODE = 1, the complex argument z is x+iy

= 2, the complex argument z is x*EXP(iy)

NORDER is the order n of the exponential integral $E_n(z)$ to be computed.

The output variables are SI, CI, EX, and IERR. SI, CI, and EX are double-precision real arrays of length 2, and IERR is an integer variable used as an error code.

SI(1) = Re Si(z)

SI(2) = Im Si(z)

CI(1) = Re Ci(z)

CI(2) = Im Ci(z)

EX(1) = Re $E_n(z)$, n = NORDER

EX(2) = Im $E_n(z)$, n = NORDER

IERR = 0, no errors occurred;

= 1, input value for ICODE was not 1 or 2;

= 2, the magnitude of z was less than 1.D-48, and interpreted to be 0.;

= 3, the argument of z was 180° degrees;

= 4, the magnitude of z was negative;

= 5, negative order was specified for $E_n(z)$.

It should be noted that not all of the nonzero values for IERR indicate fatal errors. If IERR - 1 or 4, no computations are performed. If IERR = 2 (that is, if z = 0), then the sine integral Si(0) = 0, while the cosine integral Ci(0) is undefined. If n = NORDER = 0 or 1, $E_n(0)$ is also undefined. If n = NORDER >1, $E_n(0) = 1/(n-1)$. If IERR = 3, Ci(z) and $E_n(z)$ are not defined for z on the negative real axis; the values of

Si(z) are computed. Finally, if IERR = 5, $E_n(z)$ is not computed, but Si(z) and Ci(z) are.

Appropriate error messages are printed to accompany the nonzero values of IERR. It is necessary to declare the double-precision arrays SI(2), CI(2), and EX(2) in a DIMENSION statement in the calling program.

III. METHODS OF COMPUTATION

A. Definitions

The sine integral is defined by

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\,dt \tag{1}$$

for all complex z.

The cosine integral is defined by

$$\mathrm{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t}\,dt \tag{2}$$

for $|\arg z|<\pi$, where $\gamma = .57721 \ldots$ is Euler's constant and ln z is the complex logarithm of z,

$$\ln z = \log_e |z| + i \arg z. \tag{3}$$

The exponential integral is defined by

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}\,dt \tag{4}$$

for Re z > 0 and order n = 0,1,2,...

B. Series: $|z| \leq 10$

Series expansions for Si(z) and Ci(z) are given by

$$Si(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)(2k+1)!} \tag{5}$$

and

$$Ci(z) = \gamma + \ln z + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)(2k)!}, \quad |\arg z| < \pi \tag{6}$$

For the exponential integral, one has

$$E_n(z) = \frac{(-1)^{n-1} z^{n-1}}{(n-1)!} \left(-\gamma - \ln z + \sum_{k=1}^{n-1} \frac{1}{k} \right) - \sum_{\substack{k=0 \\ k \neq n-1}}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!}, \tag{7}$$

$|\arg z| < \pi$

Since the infinite series in Eqs. (5), (6), and (7) are majorized by an infinite series for e^z, they are absolutely convergent throughout the finite complex plane. The presence of the term ln z in Eqs. (6) and (7) invalidates these equations at the origin and along the negative real axis. As $|z|$ increases, the rate of convergence decreases. Numerical experimentation showed agreement between the series and the continued fractions (to be discussed next) in the region $8 \; |z| \leq 12$ to in excess of 28 significant digits. $|z| = 10$ was therefore chosen as the cutoff point for utilization of the series expansion. See Appendix D for the derivation of these expansions.

C. <u>Gauss Continued Fraction</u>: $10 < |z| \leq 75$

For the exponential integral, the continued fraction is given by

$$E_n(z) = e^{-z} \cfrac{1}{z + \cfrac{n}{1 + \cfrac{1}{z + \cfrac{n+1}{1 + \cfrac{2}{z + \cdots}}}}}\;, \tag{8}$$

valid for $|\arg z| < \pi$. (See Appendix D)

The continued fraction expansion in Eq. (8) can also be used to evaluate the sine and cosine integrals. In particular, for $|\arg z| < \frac{\pi}{2}$,

$$Si(z) = \frac{1}{2i}\left[E_1(iz) - E_1(-iz)\right] + \frac{\pi}{2} \quad , \tag{9}$$

and

$$Ci(z) = -\frac{1}{2}\left[E_1(iz) + E_1(-iz)\right] \quad . \tag{10}$$

(See Appendix D for the derivations.) Then using the continued fraction in Eq. (8) to evaluate $E_1(\pm\, iz)$, one obtains the values of Si(z) and Ci(z) from Eqs. (9) and (10), respectively, for $|\arg z| < \frac{\pi}{2}$ and $10<|z|\leq 75$. For z such that $\frac{\pi}{2} <|\arg z|< \pi$, Si(-z) and Ci(-z) are computed, and then use is made of the fact that

$$Si(z) = -\,Si(-z) \tag{11}$$

and

$$Ci(z) = Ci(-z) + i\pi \tag{12}$$

For $|\arg z| = \frac{\pi}{2}$, a problem arises in Eqs. (9) and (10), since either iz or -iz may lie on the negative real axis, the branch cut for $E_1(z)$. It will be shown in Appendix D that the continued fraction expansion in Eq. (8) is still valid when $z < 0$, if used properly. That is, if z is real and negative, say $z = -x$, $x > 0$, then define

$$E_1(-x) = -\,Ei(x) \mp i\,\pi \ , \tag{13}$$

where

$$Ei(x) = -\,P.V. \int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt \quad , \tag{14}$$

P.V. denotes the Cauchy principal value of the integral, and the sign of the $i\pi$ term is chosen according to the following convention:

$$\lim_{y \to 0^+} E_1(-x + iy) = -\,Ei(x) - i\pi$$
$$\tag{15}$$
$$\lim_{y \to 0^-} E_1(-x + iy) = -\,Ei(x) + i\pi$$

The continued fraction expansion in Eq. (8) converges to $-Ei(x)$ when $z = -x$, $x < 0$. Suppose, then, that $z = ix$, $x > 0$. In this case, $iz = -x$, and the continued fraction converges to $-Ei(x)$. Since z has been rotated in the positive (anti-clockwise) sense to the negative real axis, π is subtracted from the imaginary part of $-Ei(x)$ (which is 0) to provide the value of $E_1(iz)$. Similarly, if $z = -ix$, then $-iz = -x$ by rotation in the negative sense, and π is added to the imaginary part of $-Ei(x)$ to obtain the value of $E_1(-iz)$. With these conventions, Eqs. (9) and (10) may be used to compute Si and Ci on the imaginary axis.

The value $|z| = 75$ was chosen as the cutoff point for use of the continued fraction, since numerical experimentation showed agreement between the continued fraction and the asymptotic series (to be discussed next) to in excess of 26 significant digits in the range $|z| = 73$ to $|z| = 78$.

D. Asymptotic Series: $|z| > 75$

For the exponential integral, the asymptotic series

$$E_n(z) \sim \frac{e^{-z}}{z} \quad 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \ldots \tag{16}$$

is valid for $|\arg z| < \frac{3}{2}\pi$, provided that $i\pi$ is added or subtracted from the value of the series, as appropriate, when the branch cut is crossed (see Appendix D). In particular, when z is on the negative real axis, the series in expression (16) is asymptotic to $-Ei(x)$, so the computation of Si and Ci from this asymptotic series is accomplished using Eq. (9) and (10) in exactly the same manner as was described in section C for the continued fraction.

E. <u>Recurrence Relation for the Exponential Integral</u>

For $n \geq 1$, it is shown in Appendix D that

$$E_{n+1}(z) = \frac{1}{n} \quad e^{-z} - z\,E_n(z) \quad , \tag{17}$$

for all z. This recurrence relation is stable for increasing n whenever $n < |z|$, and is stable for decreasing n whenever $n > |z|$.

F. Programming Methods

Throughout this section, reference is made to program line numbers which can be found in Appendix B. All computations within the subroutine are done using the rectangular coordinate form of the complex quantities. Since input to the subroutine may be in rectangular or polar form, lines 180 through 440 check the form of the input, and convert polar input to rectangular form. Polar input angles are modified, if necessary, to values greater than - 180° and less than or equal to 180° by adding or subtracting an appropriate multiple of 360°. It should be noted that input variables are not modified by the subroutine; in lines 160 and 170, auxiliary variables are assigned.

Lines 500 through 630 comprise the series computation section. The array TSAVE, of length 2, contains the real and imaginary parts of the term

$$\frac{(-z)^{\mathrm{NORDER}-1}}{(\mathrm{NORDER}-1)!} ,$$

for use in the exponential integral in lines 620 and 630. In lines 580 through 610, the terms of the summation in the quantity

$$- \gamma - \ln |z| + \sum_{I=1}^{\mathrm{NORDER}-1} 1/I$$

are computed.

Lines 660 through 880 make up the continued fraction section. The first call to subroutine CONTFR (line 670) is to compute the value of $E_n(z)$, n = NORDER, z input. Lines 680 through 760 compute $E_1(iz)$ for use in Eqs. (9) and (10), as described in section III C. If z lies in the right half-plane (line 690), z is rotated 90° in the counterclockwise sense by setting XT = - Y and YT = X. That is, if z = X + iY, then iz = - Y + iX (lines 730 and 740). If z is in the left half-plane or on the imaginary axis, z is replaced by - z before rotation by ± i (lines 700 through 720). Subsequent to the computation, Si and Ci are modified in accordance with Eqs. (11) and (12) of section III C (lines 810 through 880). Having computed $E_1(iz)$ in line 750, iz is converted to -iz in lines 770 and 780, and $E_1(-iz)$ is computed in line 790. If the input value of Z was imaginary (X = 0), then either YT (line 740) or YT (line 780) will be 0 with the corresponding XT being negative. As described in section III C, the imaginary part of the computed E_1 is modified by $\pm \pi$, as appropriate, in line 760 or in line 800. Lines 810 through 840 apply Eqs. (9) and (10).

The asymptotic series computation section is in lines 900 through 1130. The first call to subroutine ASYMP (line 920) computes $E_n(z)$, n = NORDER, z input. Lines 950 through 1130 implement Eq. (9), (10), (11), and (12) exactly as described in the continued fraction section above, except of course that subroutine ASYMP is called instead of subroutine CONTFR.

Lines 1160 through 1510 make up the error handling section of the subroutine. For details, see the description of IERR, section II.

Subroutine SERIES is comprised of lines 1640 through 2380. The arrays PSS(100,2), PSC(100,2), and PSE(200,2) are used to store the computed terms of the series for SI, CI, and E_n, respectively. The first index is in each case the term number; the second index corresponds to the real (1) and imaginary (2) part of the term. The terms are computed in descending order and summed in ascending order, to minimize round-off error. For the range of application of the series representations, the terms are monotone decreasing except for perhaps the first two or three terms when $|z| > 6$, and hence it is unnecessary to sum the series from two directions.

For SI and CI, one requires terms of the form

$$\frac{(-1)^n z^{2n}}{(2n)(2n)!}$$

and

$$\frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!} ,$$

respectively. Letting z=X + iY, these terms are computed as follows:

$$\frac{z^m}{m!} = \text{FACTR} + i\ \text{FACTI}$$

$$(-1)^n = \text{SW}$$

$$m \quad = \text{EM}$$

In line 1740, EM is initialized as 1.D0, and in lines 1760-1770, FACTR and FACTI are initialized as X/EM and Y/EM, respectively. SW is initially given the value + 1.D0 (line 1800). Lines 1820 through 2220 make up the computational loop. The terms for SI are computed first (lines 1830-1840) as

SW*FACTR/EM

and

SW*FACTI/EM

EM is incremented by 1.D0 in line 1940, The sign of SW is changed in line 1950, and FACTR+iFACTI is multiplied by X/EM + i Y/EM (lines 1970-2010). The terms for CI are then computed in lines 2020-2030 as

SW*FACTR/EM

and

SW*FACTI/EM

EM is again incremented by 1.D0 (line 2150), and FACTR+iFACTI is multiplied by X/EM+iY/EM (lines 2170-2210). Note that the sign of SW is not changed this time, since the next term in the series representation for SI has the same sign as the term just computed for CI. The loop index increments, and the next term in the series representation for SI is computed.

Computation of the series representation for E_n requires terms of the form

$$\frac{(-1)^m z^m}{(m-n+1)(m!)}$$

Again letting z=X + iY, these terms are computed as

$$\frac{z^m}{m!} = \text{FACTR+iFACTI}$$

$(-1)^m$ = ESW

m = EM

n-1 = EXFACT

$\frac{1}{m-n+1}$ = EXCOEF

Since the series representation for E_n contains terms both even and odd powers of z, two terms of the series for E_n are computed each time through the loop. (Hence the dimension PSE (200,z)).

As indicated elsewhere, the array TSAVE, dimensioned 2, holds the real and imaginary parts of the factor

$$\frac{(-1)^{n-1}z^{n-1}}{(n-1)!}$$

used in the series for $E_n(z)$. If n=1, TSAVE(1)=1.00 and TSAVE(2)=0.D0, the values given these variables initially in lines 1780-1790. Once into the computational loop, so long as $m \neq n-1$, that is, EM $\neq$ EXFACT (lines 1850 and 2040), the terms PSE are computed as

ESW*EXCOEF*FACTR

and

ESW*EXCOEF*FACTI

The sign of ESW is changed after each such computation, and FACTR and FACTI are modified as described above.

If m=n-1, that is, EM = EXFACT, the term in z^{n-1} is omitted from the summation, so that PSE=0.D0 (lines 1880-1890 or lines 2070-2080). In this case,

TSAVE(1)=-ESW*FACTR

and

TSAVE(2)=-ESW*FACTI

(lines 1860-1870 or lines 2050-2060). The minus sign occurs because the loop computes terms of the series

$$-\sum_{\substack{k=0\\k\neq n-1}}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!} = \sum_{\substack{k=0\\k\neq n-1}}^{\infty} \frac{(-1)^{k+1} z^k}{(k-n+1)k!} ,$$

and TSAVE is required to have the value

$$\frac{(-1)^k z^k}{k!} ,$$

where k=n-1.

The variable TERM is

$$\frac{z^m}{m!}$$

and is computed in line 2130. When TERM < EPS, the computation loop is exited (line 2140). Note that if N $\leq$ NORDER, the loop is not exited, since in this case TSAVE has not yet been computed. In any event, no more than NORDER+1 terms (NMAX) are computed. If, after NMAX terms, TERM is not less than EPS, a message is printed (line 2230). In lines 2250 through 2340, the terms are summed. If NORDER>1, the first term in the series representation is EXFACT, which is added to EX(1) in line 2350.

Subroutine CONTFR is in lines 2390 through 2680. This subroutine computes the 2*NMAXth convergent of the continued fraction in Eq. (8), where NMAX=100. The variable ADD, initialized to NMAX in line 2420, will take on the values 1 to NMAX, in descending order. The variable ADDZ, initialized to NORDER+NMAX in line 2430, will take on the values NORDER to NORDER+NMAX, in descending order. W(1) and W(2) are the real and imaginary parts, respectively, of the value of the convergent at each stage of computation. W(1)+iW(2) is initialized to the value (NORDER+NMAX)+i0.D0 in lines 2440-2450.

Lines 2460 through 2560 comprise the main computational loop. z=X+iY is added to W(1)+iW(2) in lines 2470 and 2480. W(1)+iW(2) is inverted by writing

1.DO/(W(1)+iW(2)) = W(1)/R - iW(2)/R,

where R=W(1)*W(1)+W(2)*W(2)

R is computed in line 2490, and in lines 2500 and 2510, the new values of W(1) and W(2) are computed by inverting, multiplying by the current value of ADD, and adding 1.DO.

Next, ADDZ is decreased by 1.DO, and new values of W(1) and W(2) are computed by inverting and multiplying by the current value of ADDZ. Finally, ADD is decreased by 1.DO and the loop begins again. After the final pass through the loop, W(1) and W(2) contain the real and imaginary parts, respectively, of

$$W = \cfrac{NORDER}{1 + \cfrac{1}{z + \cfrac{NORDER+1}{1 + \cfrac{2}{z + \ddots \cfrac{1}{z+NORDER+NMAX}}}}}$$

The desired convergent is

$$\frac{1}{z+W},$$

so, in lines 2570 through 2610, z is added to W and the sum is inverted. In lines 2620 through 2660, this result is multiplied by

$$\begin{aligned} e^{-z} &= e^{-X-iY} \\ &= e^{-X}\cos Y - ie^{-X}\sin Y. \end{aligned}$$

Subroutine ASYMP is in lines 2690 through 3120. As in subroutine SERIES, the values of the terms are saved in the PSE array, and are summed in descending order of the index. As in subroutine CONTFR, only the function E_n is computed. The asymptotic expansion requires terms of the form

$$\frac{(-1)^{k-1}(n)(n+1)(\ldots)(n+k-2)}{z^k}$$

for k=2,3,..., NMAX.

These terms are computed in the DO-LOOP in lines 2840 through 2950. The Nth term is obtained from the previous one by multiplying by -(NORDER+N-1) and dividing by z (lines 2850 through 2870). TERM is the magnitude squared of each term; the computation loop is exited prior to the computation of all NMAX terms if TERM is larger than $|z|^{-2}$, or if TERM is less than 1.D-44.

The terms are summed in reverse order in lines 3000 through 3050. To these sums is added z^{-1} (lines 3060 and 3070), after which they are multiplied by e^{-z} (lines 3080 and 3090).

IV. CONCLUSIONS

The accuracy of the derivation and coding of the computational formulas used in this subroutine has been carefully checked. Because sufficiently precise tables are not available, the only means of checking the subroutine for complex z is through alternate computational methods. As has been noted, agreement between series, continued fraction, and asymptotic expansion in their regions of overlap provides some degree of verification. In certain parts of the complex plane, the functions have known values with which the subroutine may be compared. For example, on the imaginary axis, Re Si(z) = 0 and Im Ci(z) = $\pm \frac{\pi}{2}$, precisely the values given by the subroutine in verification runs. Some precision estimates are given in Appendix A.

V. ACKNOWLEDGEMENTS

The authors wish to express their appreciation to Mr. A. S. Elder, who provided much insight to some of the problem areas and who suggested the use of the Gauss continued fraction.

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APPENDIX A

PRECISION ESTIMATES

APPENDIX A

PRECISION ESTIMATES

A full discussion of error analysis and alternate means of computation of Si, Ci, and E_n will be the subject of a future report by the principal investigator. Preliminary results show that evaluation by Gaussian quadratures agrees with the values obtained by the continued fraction to 27 significant digits in double precision CDC FORTRAN. Applications of the recurrence relation (equation 17) for $E_n(z)$ in a Miller-type algorithm have revealed similar agreement with the series computations and the asymptotic expansions.

Internally, the subroutine determines the limits of computation as follows: the series computations are stopped when the magnitude of the last computed term is less than 1.D-144, or when 100 terms have been computed, whichever comes first. The continued fraction computes the 100th convergent. This value was found to provide the most stable results in the range of application. The asymptotic expansion is terminated in one of three ways: when 200 terms have been computed, when the magnitude of the last computed term was less than 1.D-144, or when the magnitudes of the computed terms reach a minimum.

Internal computations are performed using the rectangular coordinate form of the complex quantities. For this reason, the output of the subroutine will be inherently more precise when the input is in rectangular form. For example, an input of X=0, Y=A, ICODE=1, will produce better results than an input of X=A, Y=90°, ICODE=2, because of both the truncation error involved in converting 90° to $\frac{\pi}{2}$ radians, and the subsequent buildup of errors due to the fact that the computed value of X may not be identically 0.

APPENDIX B

LISTING OF SUBROUTINE SCINT

```
      SUBROUTINE SCINT(U,V,SI,CI,EX,NORDER,ICODE,IERR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION SI(2),CI(2),EX(2),TSAVE(2)
      DATA PI/3.141592653589793238462643383279502884197D0/,
     *  GAMMA/.5772156649015328606065120900824024310421D0/
      IERR=0
      X=U
      Y=V
      IF(ICODE.EQ.1) GOTO 5
      IF(ICODE.EQ.2) GOTO 10
      GOTO 905
5     RHO=DSQRT(X*X+Y*Y)
      IF(RHO.LT.1.D-48) GOTO 910
      THET=DATAN2(Y,X)
      THETA=THET*180.D0/PI
      IF(X.GE.0.D0.OR.DABS(Y).GE.1.D-48) GOTO 50
      Y=0.D0
      IERR=3
      GOTO 50
10    RHO=X
      IF(RHO.LT.0.D0) GOTO 930
      IF(RHO.LT.1.D-48) GOTO 910
      THETA=Y
15    IF(THETA.LE.180.D0) GOTO 20
      THETA=THETA-360.D0
      GOTO 15
20    IF(THETA.GT.-180.D0) GOTO 30
      THETA=THETA+360.D0
      GOTO 20
30    IF(THETA.NE.180.D0) GOTO 40
      THETA=0.D0
      IERR=3
40    THET=THETA*PI/180.D0
      X=RHO*DCOS(THET)
      Y=RHO*DSIN(THET)
```

```
50    IF(NORDER.GE.1) GOTO 100
      NSRDER=NORDER
      NORDER=1
      IERR=5
100   IF(RHO.GT.10.D0) GOTO 200
      NMAX=MAX0(100,NORDER+1)
      EPS=1.D-144
      EXSUM=-GAMMA-DLOG(RHO)
      CALL SERIES(X,Y,SI,CI,EX,NORDER,TSAVE,NMAX,EPS)
      CI(1)=CI(1)-EXSUM
      CI(2)=CI(2)+THET
      IF(NORDER.LE.1) GOTO 120
      NEND=NORDER-1
      TERM=DBLE(FLOAT(NEND))
      DO 110 I=1,NEND
      EXSUM=EXSUM+1.D0/TERM
110   TERM=TERM-1.D0
120   EX(1)=EX(1)+TSAVE(1)*EXSUM+TSAVE(2)*THET
      EX(2)=EX(2)+TSAVE(2)*EXSUM-TSAVE(1)*THET
      GOTO 900
200   IF(RHO.GT.75.D0) GOTO 300
      NMAX=100
      CALL CONTFR(X,Y,EX(1),EX(2),NORDER,NMAX)
      SW=1.D0
      IF(X.GE.0.D0) GOTO 210
      SW=-1.D0
      X=-X
      Y=-Y
210   XT=-Y
      YT=X
      CALL CONTFR(XT,YT,E1,E2,1,NMAX)
      IF(XT.LT.0.D0.AND.YT.EQ.0.D0) E2=E2-PI
      XT=-XT
      YT=-YT
      CALL CONTFR(XT,YT,E3,E4,1,NMAX)
```

```
      IF(XT.LT.0.D0.AND.YT.EQ.0.D0) E4=E4+PI
      SI(1)=(E2-E4+PI)*SW*.5D0
      SI(2)=.5D0*SW*(E3-E1)
      CI(1)=-.5D0*(E1+E3)
      CI(2)=-.5D0*(E2+E4)
      IF(SW.GT.0.D0) GOTO 900
      CI(2)=CI(2)-PI
      X=-X
      Y=-Y
      GOTO 900
300   NMAX=200
      EPS=1.D-144
      CALL ASYMP(X,Y,EX(1),EX(2),NORDER,NMAX,EPS)
      SW=1.D0
      IF(X.GE.0.D0) GOTO 310
      SW=-1.D0
      X=-X
      Y=-Y
310   XT=-Y
      YT=X
      CALL ASYMP(XT,YT,E1,E2,1,NMAX,EPS)
      IF(XT.LT.0.D0.AND.YT.EQ.0.D0) E2=E2-PI
      XT=-XT
      YT=-YT
      CALL ASYMP(XT,YT,E3,E4,1,NMAX,EPS)
      IF(XT.LT.0.D0.AND.YT.EQ.0.D0) E4=E4+PI
      SI(1)=.5D0*SW*(E2-E4+PI)
      SI(2)=.5D0*SW*(E3-E1)
      CI(1)=-.5D0*(E1+E3)
      CI(2)=-.5D0*(E2+E4)
      IF(SW.GT.0.D0) GOTO 900
      CI(2)=CI(2)-PI
      X=-X
      Y=-Y
      GOTO 900
```

```
900   IF(IERR.EQ.0) GOTO 999
      GOTO (905,910,920,930,940),IERR
905   IERR=1
      WRITE(6,1000)
      GOTO 999
910   IERR=2
      SI(1)=0.D0
      SI(2)=0.D0
      IF(NORDER.GT.1) GOTO 915
      WRITE(6,1010)
      GOTO 999
915   EX(1)=1.D0/DBLE(FLOAT(NORDER-1))
      EX(2)=0.D0
      WRITE(6,1015)
      GOTO 999
920   IERR=3
      SI(1)=-SI(1)
      SI(2)=-SI(2)
      CI(1)=0.D0
      CI(2)=0.D0
      EX(1)=0.D0
      EX(2)=0.D0
      WRITE(6,1020)
      GOTO 999
930   IERR=4
      WRITE(6,1030)
      GOTO 999
940   IF(NSRDER.LT.0) GOTO 945
      IERR=0
      NORDER=0
      ERCT=DEXP(-X)/(RHO*RHO)
      EX(1)=ERCT*(X*DCOS(Y)-Y*DSIN(Y))
      EX(2)=-ERCT*(Y*DCOS(Y)+X*DSIN(Y))
      GOTO 999
945   WRITE(6,1040)
```

```
      EX(1)=0.D0
      EX(2)=0.D0
999   RETURN
1000  FORMAT(1H1,5X,36HINCORRECT VALUE SPECIFIED FOR ICODE.//
     *  5X,26HNO COMPUTATIONS PERFORMED.)
1010  FORMAT(1H1,5X,38HCI(0), EX(0) NOT DEFINED. SI SET TO 0.)
1015  FORMAT(1H1,5X,48HCI(0) NOT DEFINED. SI SET TO 0, EX=1/(NORDER-1).)
1020  FORMAT(1H1,5X,41HCI, EX NOT DEFINED ON NEGATIVE REAL AXIS.//
     *  5X,24HSI VALUES WERE COMPUTED.)
1030  FORMAT(1H1,5X,41HINPUT ERROR - RHO NEGATIVE IN POLAR FORM.//
     *  5X,26HNO COMPUTATIONS PERFORMED.)
1040  FORMAT(1H1,5X,27HNEGATIVE ORDER SPECIFIED - /
     *  5X,36HSI AND CI WERE COMPUTED, EX WAS NOT.)
      END
```

```
      SUBROUTINE SERIES(X,Y,SI,CI,EX,NORDER,TSAVE,NMAX,EPS)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION SI(2),CI(2),EX(2),TSAVE(2),PSS(100,2),PSC(100,2)
     *          ,PSE(200,2)
      SI(1)=0.D0
      SI(2)=0.D0
      CI(1)=0.D0
      CI(2)=0.D0
      EX(1)=0.D0
      EX(2)=0.D0
      EM=1.D0
      EXFACT=DBLE(FLOAT(NORDER-1))
      FACTR=X/EM
      FACTI=Y/EM
      TSAVE(1)=1.D0
      TSAVE(2)=0.D0
      SW=1.D0
      ESW=1.D0
      DO 30 N=1,NMAX
      PSS(N,1)=SW*FACTR/EM
      PSS(N,2)=SW*FACTI/EM
      IF(EM.NE.EXFACT) GOTO 5
      TSAVE(1)=-ESW*FACTR
      TSAVE(2)=-ESW*FACTI
      PSE(2*N-1,1)=0.D0
      PSE(2*N-1,2)=0.D0
      GOTO 10
    5 EXCOEF=1.D0/(EM-EXFACT)
      PSE(2*N-1,1)=ESW*EXCOEF*FACTR
      PSE(2*N-1,2)=ESW*EXCOEF*FACTI
   10 EM=EM+1.D0
      SW=-SW
      ESW=-ESW
      TEMPR=X/EM
      TEMPI=Y/EM
```

```
      TFACT=FACTR*TEMPR-FACTI*TEMPI
      FACTI=FACTR*TEMPI+FACTI*TEMPR
      FACTR=TFACT
      PSC(N,1)=SW*FACTR/EM
      PSC(N,2)=SW*FACTI/EM
      IF(EM.NE.EXFACT) GOTO 15
      TSAVE(1)=-ESW*FACTR
      TSAVE(2)=-ESW*FACTI
      PSE(2*N,1)=0.D0
      PSE(2*N,2)=0.D0
      GOTO 20
15    EXCOEF=1.D0/(EM-EXFACT)
      PSE(2*N,1)=ESW*EXCOEF*FACTR
      PSE(2*N,2)=ESW*EXCOEF*FACTI
20    TERM=FACTR*FACTR+FACTI*FACTI
      IF(TERM.LT.EPS.AND.N.GT.NORDER) GOTO 40
      EM=EM+1.D0
      ESW=-ESW
      TEMPR=X/EM
      TEMPI=Y/EM
      TFACT=FACTR*TEMPR-FACTI*TEMPI
      FACTI=FACTR*TEMPI+FACTI*TEMPR
      FACTR=TFACT
30    CONTINUE
      WRITE(6,100) NMAX,TERM
      N=NMAX
40    L=N+1
      DO 50 I=1,N
      K=L-I
      CI(1)=CI(1)+PSC(K,1)
      CI(2)=CI(2)+PSC(K,2)
      SI(1)=SI(1)+PSS(K,1)
      SI(2)=SI(2)+PSS(K,2)
      EX(1)=EX(1)+PSE(2*K,1)+PSE(2*K-1,1)
      EX(2)=EX(2)+PSE(2*K,2)+PSE(2*K-1,2)
```

```
50    CONTINUE
      IF(NORDER.GT.1) EX(1)=EX(1)+1.D0/EXFACT
      RETURN
100   FORMAT(1X,I5,34H TERMS INSUFFICIENT. LAST TERM WAS,D37.30)
      END
```

```
      SUBROUTINE CONTFR(X,Y,EA,EB,NORDER,NMAX)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION W(2)
      ADD=DBLE(FLOAT(NMAX))
      ADDZ=DBLE(FLOAT(NORDER+NMAX))
      W(1)=ADDZ
      W(2)=0.D0
      DO 10 N=1,NMAX
      W(1)=W(1)+X
      W(2)=W(2)+Y
      R=W(1)*W(1)+W(2)*W(2)
      W(1)=ADD*W(1)/R+1.D0
      W(2)=-ADD*W(2)/R
      R=W(1)*W(1)+W(2)*W(2)
      ADDZ=ADDZ-1.D0
      W(1)=ADDZ*W(1)/R
      W(2)=-ADDZ*W(2)/R
   10 ADD=ADD-1.D0
      W(1)=W(1)+X
      W(2)=W(2)+Y
      R=W(1)*W(1)+W(2)*W(2)
      W(1)=W(1)/R
      W(2)=-W(2)/R
      EMRCT=DEXP(-X)
      CRSTH=DCOS(Y)
      SRSTH=DSIN(Y)
      EA=EMRCT*(CRSTH*W(1)+SRSTH*W(2))
      EB=EMRCT*(CRSTH*W(2)-SRSTH*W(1))
      RETURN
      END
```

```
      SUBROUTINE ASYMP(X,Y,ER,EI,NORDER,NMAX,EPS)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION PSE(200,2)
      CRST=DCOS(Y)
      SRST=DSIN(Y)
      EMRCOS=DEXP(-X)
      RHOSQ=X*X+Y*Y
      ZINVR=X/RHOSQ
      ZINVI=-Y/RHOSQ
      EFACT=DBLE(FLOAT(NORDER))
      EFACTR=ZINVR
      EFACTI=ZINVI
      SAVET=EFACTR*EFACTR+EFACTI*EFACTI
      ER=0.D0
      EI=0.D0
      DO 10 N=1,NMAX
      TEFACT=-EFACT*(EFACTR*ZINVR-EFACTI*ZINVI)
      EFACTI=-EFACT*(EFACTR*ZINVI+EFACTI*ZINVR)
      EFACTR=TEFACT
      TERM=EFACTR*EFACTR+EFACTI*EFACTI
      IF(TERM.GT.SAVET) GOTO 15
      SAVET=TERM
      PSE(N,1)=EFACTR
      PSE(N,2)=EFACTI
      EFACT=EFACT+1.D0
      IF(DABS(TERM).LT.EPS) GOTO 20
10    CONTINUE
      WRITE(6,100) NMAX,TERM
      N=NMAX
      GOTO 20
15    N=N-1
20    L=N+1
      DO 30 I=1,N
      K=L-I
      ER=ER+PSE(K,1)
```

```
      EI=EI+PSE(K,2)
30    CONTINUE
      ETEMP=ER+ZINVR
      EI=EI+ZINVI
      ER=EMRCOS*(CRST*ETEMP+SRST*EI)
      EI=EMRCOS*(CRST*EI-SRST*ETEMP)
      RETURN
100   FORMAT(1X,I5,34H TERMS INSUFFICIENT. LAST TERM WAS,D37.30)
      END
```

APPENDIX C

SAMPLE OUTPUT FROM SUBROUTINE SCINT

TABLE C-1 THE SINE INTEGRAL

This table contains the values of

$$\frac{2}{\pi} * SI(N*\pi/2),$$

for integral N from 1 to 200, inclusive. These values may be compared directly with those obtained in reference 2.

N	(2/PI)*SI(N*PI/2)	N	(2/PI)*SI(N*PI/2)
1	0.87265 42994 60602 71576 59444	2	1.17897 97444 72167 27023 20288
3	1.02392 18965 04422 23019 58924	4	0.90282 33335 80280 62679 57003
5	0.99047 27689 20713 65943 47007	6	1.06618 64752 36543 96509 75798
7	1.00503 75068 72085 67801 11276	8	0.94993 93397 67310 15465 36691
9	0.99690 22442 90777 44985 53440	10	1.04021 43283 82617 03471 96493
11	1.00209 20197 11019 90508 47287	12	0.96641 04349 95494 26493 25285
13	0.99849 43053 14195 12479 07743	14	1.02883 19942 46610 50207 26712
15	1.00113 47443 16457 77656 98086	16	0.97474 84501 15639 94344 74694
17	0.99911 45365 95291 30065 71405	18	1.02246 03052 73994 91365 66249
19	1.00071 00044 66726 52463 71334	20	0.97977 63423 07757 31916 68549
21	0.99941 81070 82184 04864 70160	22	1.01839 14840 16433 23254 46128
23	1.00048 55271 49853 04663 16720	24	0.98313 67033 37201 75822 05588
25	0.99958 87662 42385 00253 69317	26	1.01556 93156 25132 14680 36262
27	1.00035 27585 51912 42962 72297	28	0.98554 04187 49606 26750 80883
29	0.99969 40864 49691 43415 47590	30	1.01349 73889 44023 15655 25645
31	1.00026 78093 80702 67594 45612	32	0.98734 48303 56441 91493 03565
33	0.99976 35992 98588 37331 29185	34	1.01191 18158 34353 66238 08553
35	1.00021 02068 95110 09048 84742	36	0.98874 91056 67325 04003 38538
37	0.99981 18651 15064 97551 67193	38	1.01065 94208 80317 72301 85080
39	1.00016 93635 44275 09910 36288	40	0.98987 29991 41390 94210 96987
41	0.99984 67335 98608 32833 16659	42	1.00964 52146 08343 75151 79828
43	1.00013 93588 02178 01884 71784	44	0.99079 28301 69714 31425 26948
45	0.99987 27389 26921 55349 88709	46	1.00880 71703 96100 86688 04080
47	1.00011 66723 78909 85603 48290	48	0.99155 95322 73286 32998 77242
49	0.99989 26481 30255 80902 15142	50	1.00810 30716 87889 28938 63943
51	1.00009 91048 84017 39563 46595	52	0.99220 83949 77031 69168 32104
53	0.99990 82272 84145 22256 41294	54	1.00750 31900 63818 32163 26775
55	1.00008 52249 32120 31058 01877	56	0.99276 46403 30704 12630 23448
57	0.99992 06464 61712 01691 17974	58	1.00698 59865 55880 83463 10099
59	1.00007 40685 01726 61833 23759	60	0.99324 67732 62360 12817 66438
61	0.99993 07057 16834 87030 41640	62	1.00653 54738 95543 64174 01925
63	1.00006 49671 25010 86008 77414	64	0.99366 86777 10445 51666 48310
65	0.99993 89670 86469 05293 80332	66	1.00613 95364 04374 93280 86601
67	1.00005 74454 80208 26109 85510	68	0.99404 09740 55334 08400 84418

N	(2/PI)*SI(N*PI/2)	N	(2/PI)*SI(N*PI/2)
69	0.99994 58347 57874 40733 67344	70	1.00578 88251 11402 65118 65970
71	1.00005 11581 06205 67804 17552	72	0.99437 19246 71929 49823 56069
73	0.99995 16054 19623 34501 91136	74	1.00547 60107 64876 58576 90627
75	1.00004 58490 48185 96751 74038	76	0.99466 80538 53813 26108 04821
77	0.99995 65008 26602 30242 78821	78	1.00519 52664 31855 29868 91313
79	1.00004 13254 08622 85673 64161	80	0.99493 45819 55121 35507 47711
81	0.99996 06894 12282 20903 77628	82	1.00494 19013 81642 95992 60668
83	1.00003 74395 86020 47543 54041	84	0.99517 57356 08786 21337 02601
85	0.99996 43009 64177 02092 79362	86	1.00471 20970 49802 66863 00000
87	1.00003 40770 87051 78185 42379	88	0.99539 49734 11874 66900 50593
89	0.99996 74367 94869 84115 88774	90	1.00450 27133 54636 18457 45969
91	1.00003 11479 99564 76517 26042	92	0.99559 51527 82044 71306 21265
93	0.99997 01769 21966 59407 04500	94	1.00431 11444 28245 93586 04851
95	1.00002 85809 21728 95430 47878	96	0.99577 86551 29336 79143 34011
97	0.99997 25852 21506 93655 62952	98	1.00413 52096 54903 89741 37840
99	1.00002 63185 70896 47967 84044	100	0.99594 74810 06163 65303 08501
101	0.99997 47131 83076 09817 61104	102	1.00397 30703 31559 02687 06298
103	1.00002 43145 61326 21908 71289	104	0.99610 33233 12667 58732 90783
105	0.99997 66026 83968 45155 89339	106	1.00382 31651 86890 17222 74400
107	1.00002 25310 08274 30280 05381	108	0.99624 76242 43145 85320 18419
109	0.99997 82880 64679 30532 85575	110	1.00368 41599 61506 31079 50077
111	1.00002 09367 24849 15809 39900	112	0.99638 16200 35867 12562 63389
113	0.99997 97976 99807 30936 43160	114	1.00355 49075 97650 30782 04760
115	1.00001 95058 49773 05553 74984	116	0.99650 63764 68669 16193 53727
117	0.99998 11551 99841 09402 78961	118	1.00343 44165 22231 41656 41019
119	1.00001 82167 92265 01109 96340	120	0.99662 28172 58551 19802 22667
121	0.99998 23803 39719 05957 65456	122	1.00332 18251 66618 11430 66137
123	1.00001 70514 12977 26982 65139	124	0.99673 17469 66820 90525 64853
125	0.99998 34897 82912 38959 62430	126	1.00321 63813 37883 14436 84709
127	1.00001 59943 82507 15581 88055	128	0.99683 38696 11170 30592 18023
129	0.99998 44976 50915 67847 34767	130	1.00311 74253 97023 08291 26155
131	1.00001 50326 74812 49505 24469	132	0.99692 98038 94952 01803 49445
133	0.99998 54159 74745 17835 33792	134	1.00302 43764 48986 78193 09382

N	(2/PI)*SI(N*PI/2)	N	(2/PI)*SI(N*PI/2)
135	1.00001 41551 64057 04797 00129	136	0.99702 00957 49841 75992 69305
137	0.99998 62550 55577 65556 69163	138	1.00293 67209 33653 12480 79424
139	1.00001 33523 01438 22690 38087	140	0.99710 52287 29011 91014 62013
141	0.99998 70237 54840 38150 51455	142	1.00285 40031 44500 72739 40536
143	1.00001 26158 54364 07989 78798	144	0.99718 56326 68627 23177 29005
145	0.99998 77297 29094 41639 19897	146	1.00277 58173 05399 87832 75491
147	1.00001 19386 94602 15665 70651	148	0.99726 16909 45170 15938 01786
149	0.99998 83796 21399 52662 02387	150	1.00270 18009 14769 85874 98251
151	1.00001 13146 25167 21652 66686	152	0.99733 37465 87178 51569 50978
153	0.99998 89792 18137 11256 89254	154	1.00263 16291 26741 00025 27407
155	1.00001 07382 38058 85715 42346	156	0.99740 21074 46953 64632 74454
157	0.99998 95335 78237 08489 11061	158	1.00256 50099 85600 95366 82694
159	1.00001 02047 96722 47845 65935	160	0.99746 70505 96698 41028 11427
161	0.99999 00471 40221 94970 44150	162	1.00250 16803 66083 89756 26074
163	1.00000 97101 38442 52193 63130	164	0.99752 88260 81464 48332 00768
165	0.99999 05238 11315 44030 55109	166	1.00244 14025 00475 48428 30251
167	1.00000 92505 92896 64701 71018	168	0.99758 76601 36080 51487 64667
169	0.99999 09670 41969 59089 42621	170	1.00238 39609 95905 18604 24111
171	1.00000 88229 13883 66454 80878	172	0.99764 37579 53299 21950 92765
173	0.99999 13798 88474 74298 33799	174	1.00232 91602 62963 70774 14196
175	1.00000 84242 21845 14593 05725	176	0.99769 73060 74544 13134 54206
177	0.99999 17650 65781 49932 44991	178	1.00227 68222 90957 03813 60572
179	1.00000 80519 55273 66809 73386	180	0.99774 84744 61949 98419 02999
181	0.99999 21249 92245 11547 95617	182	1.00222 67847 16479 74946 57545
183	1.00000 77038 29471 49625 20694	184	0.99779 74183 00185 44733 18019
185	0.99999 24618 27673 75070 49156	186	1.00217 88991 41160 92016 37601
187	1.00000 73778 01415 92101 98946	188	0.99784 42795 68295 41119 29655
189	0.99999 27775 05801 79634 65639	190	1.00213 30296 61869 63313 67326
191	1.00000 70720 39719 32020 79236	192	0.99788 91884 15095 26793 58783
193	0.99999 30737 62102 63458 37074	194	1.00208 90515 82721 48127 80903
195	1.00000 67848 98856 82396 41275	196	0.99793 22643 66176 08090 57106
197	0.99999 33521 57689 86839 01807	198	1.00204 68502 83181 42750 51309
199	1.00000 65148 96982 51677 00953	200	0.99797 36173 86091 00499 79706

TABLE C-2 THE SINE, COSINE, AND EXPONENTIAL INTEGRALS

This table contains values of SI, CI, EX_1, EX_5, and EX_{10} for $|z| = 1$, 40, and 80, and arg z ranging from -90 degrees to + 90 degrees in 15 degree increments. For this table, the notation

$$E(n,z) = E_n(z)$$

has been used. The exponent of each value, and the sign of this exponent, appear in parentheses in front of each value.

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -90

	REAL PART		IMAGINARY PART
SI(Z)	(-28) .29666 24887 79092 09678		(+ 1) -.10572 50875 37572 85145
CI(Z)	(+ 0) .83786 69409 80208 240[illegible]9		(+ 1) -.15707 96326 79489 66192
EX(1,Z)	(+ 0) -.33740 39229 00968 13466		(+ 0) .62471 32564 27713 60428
EX(5,Z)	(- 1) .63443 19256 79930 81070		(+ 0) .22384 87702 63972 32728
EX(10,Z)	(- 1) .47573 65782 62179 83944		(- 1) .99212 56691 88739 26430

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -90

	REAL PART		IMAGINARY PART
SI(Z)	(+ 1) .15707 96326 79786 75969		(+16) -.30198 59131 80562 41906
CI(Z)	(+16) .30198 59131 80562 41906		(-11) .29709 79754 76384 22880
EX(1,Z)	(- 1) -.19020 00789 62087 66961		(- 1) -.16188 79255 98878 87544
EX(5,Z)	(- 1) -.20321 92419 84824 36218		(- 1) -.14101 26268 73744 56188
EX(10,Z)	(- 1) -.21316 58872 67299 75968		(- 1) -.11315 38889 06891 02018

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -90

	REAL PART		IMAGINARY PART
SI(Z)	(+ 6) .69932 64331 38801 22249		(+33) -.35073 00002 45239 99848
CI(Z)	(+33) .35073 00002 45239 99848		(+ 6) .69932 48623 42474 42759
EX(1,Z)	(- 1) .12402 50115 50709 58192		(- 2) -.15345 60117 59069 61199
EX(5,Z)	(- 1) .12280 19470 66883 74446		(- 2) -.21448 64614 58175 61117
EX(10,Z)	(- 1) .12046 09812 36901 69426		(- 2) -.28784 84048 93317 65838

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -75

	REAL PART		IMAGINARY PART
SI(Z)	(+ 0) .29974 02274 92505 46979		(+ 1) -.10056 33346 56684 65838
CI(Z)	(+ 0) .79892 87750 41746 96141		(+ 1) -.11747 41660 92934 82827
EX(1,Z)	(+ 0) -.14479 80723 99690 16638		(+ 0) .49812 40483 32539 28630
EX(5,Z)	(- 1) .57203 80457 33726 32389		(+ 0) .15925 77958 24796 77768
EX(10,Z)	(- 1) .38777 29620 71565 23877		(- 1) .72718 05184 20066 25334

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -75

	REAL PART		IMAGINARY PART
SI(Z)	(+15) -.47283 80894 65682 38819		(+15) .61026 45952 99039 55217
CI(Z)	(+15) -.61026 45952 99039 55217		(+15) -.47283 80894 65683 95899
EX(1,Z)	(- 6) -.48079 13205 39153 35524		(- 6) .62919 95805 72658 54553
EX(5,Z)	(- 6) -.40792 77989 04173 45642		(- 6) .64819 06966 55219 15754
EX(10,Z)	(- 6) -.32173 01129 25119 28353		(- 6) .65368 44780 03363 63999

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -75

	REAL PART		IMAGINARY PART
SI(Z)	(+32) .22952 39934 18520 48663		(+30) .45941 13087 62031 92634
CI(Z)	(+30) -.45941 13087 62031 92634		(+32) .22952 39934 18520 48663
EX(1,Z)	(-10) -.12673 95439 94881 78993		(-12) -.39676 62813 98469 74101
EX(5,Z)	(-10) -.12490 29897 44391 50572		(-12) .19527 32198 41604 55629
EX(10,Z)	(-10) -.12197 24444 82197 02556		(-12) .88193 09022 08863 76213

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -60

		REAL PART		IMAGINARY PART
SI(Z)	(+ 0)	.55637 44092 28347 57934	(+ 0)	-.86455 74828 89181 88368
CI(Z)	(+ 0)	.69677 43139 40552 45068	(+ 0)	-.82167 28110 09608 70376
EX(1,Z)	(- 2)	-.19456 04279 57614 88333	(+ 0)	.39007 88918 17426 93769
EX(5,Z)	(- 1)	.59342 92637 91145 10476	(+ 0)	.11186 44426 43028 86784
EX(10,Z)	(- 1)	.35873 57743 72981 86837	(- 1)	.51873 97120 49347 20471

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -60

		REAL PART		IMAGINARY PART
SI(Z)	(+13)	.81491 50990 12170 02984	(+14)	-.11574 17629 71922 48409
CI(Z)	(+14)	.11574 17629 71922 48409	(+13)	.81491 50990 12012 95021
EX(1,Z)	(-10)	-.22625 91485 73859 75628	(-10)	-.45559 14643 49524 70431
EX(5,Z)	(-10)	-.24749 38599 98904 12704	(-10)	-.41373 72807 90046 48447
EX(10,Z)	(-10)	-.26324 90739 36884 82847	(-10)	-.36438 00981 39203 94363

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -60

		REAL PART		IMAGINARY PART
SI(Z)	(+28)	.75978 25855 48224 35041	(+28)	.15404 02557 38707 04288
CI(Z)	(+28)	-.15404 02557 38707 04288	(+28)	.75978 25855 48224 35041
EX(1,Z)	(-19)	.18945 18554 48414 96710	(-19)	.49249 58708 02778 50194
EX(5,Z)	(-19)	.20415 97334 52290 23065	(-19)	.47182 29981 26363 14790
EX(10,Z)	(-19)	.21925 28816 31394 73650	(-19)	.44605 78368 49913 46583

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -45

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 0)	.74519 21553 53659 28422	(+ 0)	-.66666 48174 19506 33905
CI(Z)	(+ 0)	.56680 20982 59308 90460	(+ 0)	-.53562 96173 22429 89740
EX(1,Z)	(- 1)	.99862 71916 01974 34444	(+ 0)	.28997 45541 18807 43759
EX(5,Z)	(- 1)	.63542 36380 27992 15763	(- 1)	.76072 99919 57240 76634
EX(10,Z)	(- 1)	.35453 15002 32377 86525	(- 1)	.35524 42968 14939 40464

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -45

	REAL PART		IMAGINARY PART	
SI(Z)	(+11)	.17438 43424 32340 68560	(+11)	.17136 76188 22945 12960
CI(Z)	(+11)	-.17136 76188 22945 12960	(+11)	.17438 43424 16632 72233
EX(1,Z)	(-14)	-.90994 99898 28422 07254	(-14)	-.89756 79949 34719 79228
EX(5,Z)	(-14)	-.89856 48783 09158 07708	(-14)	-.78302 23432 94590 22757
EX(10,Z)	(-14)	-.87121 62564 30965 41059	(-14)	-.66272 38913 29436 99466

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -45

	REAL PART		IMAGINARY PART	
SI(Z)	(+23)	-.16287 98278 48661 77274	(+23)	-.16643 51044 63585 26635
CI(Z)	(+23)	.16643 51044 63585 26635	(+23)	-.16287 98278 48661 77274
EX(1,Z)	(-26)	.23453 37641 82110 25263	(-26)	.23987 79073 88748 00420
EX(5,Z)	(-26)	.23392 78595 98617 08913	(-26)	.22397 86382 00258 40002
EX(10,Z)	(-26)	.23194 47875 04430 64082	(-26)	.20570 33464 64836 98670

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -30

	REAL PART		IMAGINARY PART
SI(Z)	(+ 0) .86460 65733 55143 20983	(+ 0)	-.44529 16450 65800 71921
CI(Z)	(+ 0) .44723 72493 45376 86804	(+ 0)	-.31611 08616 22326 47169
EX(1,Z)	(+ 0) .16778 31669 48629 43299	(+ 0)	.19274 91065 56940 33612
EX(5,Z)	(- 1) .67264 52671 14151 54400	(- 1)	.47432 88004 84870 58924
EX(10,Z)	(- 1) .35821 64103 57870 27453	(- 1)	.22209 04666 13963 62340

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -30

	REAL PART		IMAGINARY PART
SI(Z)	(+ 7) .51181 60029 36333 75742	(+ 7)	.33873 01491 69562 96080
CI(Z)	(+ 7) -.33873 01491 69562 95654	(+ 7)	.51181 56458 56701 07337
EX(1,Z)	(-17) -.20189 92805 59119 20847	(-16)	.22004 78076 79644 64041
EX(5,Z)	(-18) -.99454 79182 05069 35292	(-16)	.20339 23641 44354 52169
EX(10,Z)	(-19) -.71379 18629 49964 90217	(-16)	.18506 47537 64260 78647

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -30

	REAL PART		IMAGINARY PART
SI(Z)	(+16) -.11513 64309 88702 34453	(+15)	-.93023 54076 64162 40543
CI(Z)	(+15) .93023 54076 64162 40543	(+16)	-.11513 64309 88702 50161
EX(1,Z)	(-32) -.95581 03663 08972 87012	(-32)	.32010 19095 37987 54961
EX(5,Z)	(-32) -.90913 94977 30063 88053	(-32)	.32792 23219 01497 75947
EX(10,Z)	(-32) -.85557 27025 42686 96878	(-32)	.33425 37625 71799 63580

MAGNITUDE OF Z = 1, ARGUMENT OF Z = -15

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 0)	.92708 06031 16271 82709	(+ 0)	-.22111 80484 81933 62471
CI(Z)	(+ 0)	.36591 61208 81623 79109	(+ 0)	-.14559 16753 46085 50583
EX(1,Z)	(+ 0)	.20670 45715 25099 62243	(- 1)	.96314 43837 30428 66655
EX(5,Z)	(- 1)	.69646 72318 95616 56534	(- 1)	.22817 13966 26851 63533
EX(10,Z)	(- 1)	.36232 32605 62719 76442	(- 1)	.10691 83372 33688 28971

MAGNITUDE OF Z = 40, ARGUMENT OF Z = -15

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 3)	-.15027 59111 85608 31562	(+ 3)	-.36356 34070 80294 15565
CI(Z)	(+ 3)	.36356 34075 61085 47619	(+ 3)	-.15184 67068 83203 63166
EX(1,Z)	(-18)	-.15304 26437 56411 27990	(-18)	-.37545 48397 35917 57442
EX(5,Z)	(-18)	-.14746 67070 56811 62937	(-18)	-.34039 84094 53101 45892
EX(10,Z)	(-18)	-.14013 30034 92025 42706	(-18)	-.30434 69589 03545 83384

MAGNITUDE OF Z = 80, ARGUMENT OF Z = -15

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 7)	.32435 10537 63041 89058	(+ 7)	-.52353 83308 61328 19961
CI(Z)	(+ 7)	.52353 83308 61328 20088	(+ 7)	.32435 08966 83409 21105
EX(1,Z)	(-35)	-.17615 89030 60227 01677	(-35)	.29133 01137 22908 64752
EX(5,Z)	(-35)	-.16489 88483 46799 12302	(-35)	.28012 81437 64506 12496
EX(10,Z)	(-35)	-.15248 53889 79257 69950	(-35)	.26715 33819 26509 06426

MAGNITUDE OF Z = 1. ARGUMENT OF Z = 0

		REAL PART		IMAGINARY PART
SI(Z)	(+ 0)	.94608 30703 67183 01494	(0)	0.00000 00000 00000 00000
CI(Z)	(+ 0)	.33740 39229 00968 13466	(0)	0.00000 00000 00000 00000
EX(1,Z)	(+ 0)	.21938 39343 95520 27367	(0)	0.00000 00000 00000 00000
EX(5,Z)	(- 1)	.70454 23746 17203 98335	(0)	0.00000 00000 00000 00000
EX(10,Z)	(- 1)	.36393 99403 14164 01634	(0)	0.00000 00000 00000 00000

MAGNITUDE OF Z = 40. ARGUMENT OF Z = 0

		REAL PART		IMAGINARY PART
SI(Z)	(+ 1)	.15869 85119 35478 45067	(0)	0.00000 00000 00000 00000
CI(Z)	(- 1)	.19020 00789 62087 66961	(0)	0.00000 00000 00000 00000
EX(1,Z)	(-18)	.10367 73261 45165 69721	(0)	0.00000 00000 00000 00000
EX(5,Z)	(-19)	.94632 77239 39156 81904	(0)	0.00000 00000 00000 00000
EX(10,Z)	(-19)	.85297 77609 98886 39960	(0)	0.00000 00000 00000 00000

MAGNITUDE OF Z = 80. ARGUMENT OF Z = 0

		REAL PART		IMAGINARY PART
SI(Z)	(+ 1)	.15723 30886 91248 73153	(0)	0.00000 00000 00000 00000
CI(Z)	(- 1)	-.12402 50115 50709 58192	(0)	0.00000 00000 00000 00000
EX(1,Z)	(-36)	.22285 43258 68847 29112	(0)	0.00000 00000 00000 00000
EX(5,Z)	(-36)	.21247 93451 66407 25163	(0)	0.00000 00000 00000 00000
EX(10,Z)	(-36)	.20078 21545 71055 66614	(0)	0.00000 00000 00000 00000

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 15

		REAL PART		IMAGINARY PART
SI(Z)	(+ 0)	.92708 06031 16271 82709	(+ 0)	.22111 80484 81933 62471
CI(Z)	(+ 0)	.36591 61208 81623 79109	(+ 0)	.14559 16753 46085 50583
EX(1,Z)	(+ 0)	.20670 45715 25099 62243	(- 1)	-.96314 43837 30428 66655
EX(5,Z)	(- 1)	.69646 72318 95616 56534	(- 1)	-.22817 13966 26851 63533
EX(10,Z)	(- 1)	.36232 32605 62719 76442	(- 1)	-.10691 83372 33688 28971

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 15

		REAL PART		IMAGINARY PART
SI(Z)	(+ 3)	-.15027 59111 85608 31562	(+ 3)	.36356 34070 80294 15565
CI(Z)	(+ 3)	.36356 34075 61085 47619	(+ 3)	.15184 67068 83203 63166
EX(1,Z)	(-18)	-.15304 26437 56411 27990	(-18)	.37545 48397 35917 57442
EX(5,Z)	(-18)	-.14746 67070 56811 62937	(-18)	.34039 84094 53101 45892
EX(10,Z)	(-18)	-.14013 30034 92025 42706	(-18)	.30434 69589 03545 83384

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 15

		REAL PART		IMAGINARY PART
SI(Z)	(+ 7)	.32435 10537 63041 89058	(+ 7)	.52353 83308 61328 19961
CI(Z)	(+ 7)	.52353 83308 61328 20088	(+ 7)	-.32435 08966 83409 21105
EX(1,Z)	(-35)	-.17615 89030 60227 01877	(-35)	-.29133 01137 22908 64752
EX(5,Z)	(-35)	-.16489 88483 46799 12302	(-35)	-.28012 81437 64506 12496
EX(10,Z)	(-35)	-.15248 53889 79257 69450	(-35)	-.26715 33819 26509 06426

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 30

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 0)	.86460 65733 55143 20983	(+ 0)	.44529 16450 65800 71921
CI(Z)	(+ 0)	.44723 72493 45376 86804	(+ 0)	.31611 08616 22326 47169
EX(1,Z)	(+ 0)	.16778 31689 48629 43299	(+ 0)	-.19274 91065 56940 33612
EX(5,Z)	(- 1)	.67264 52671 14151 54400	(- 1)	-.47432 88004 84870 58924
EX(10,Z)	(- 1)	.35821 64103 57870 27453	(- 1)	-.22209 04666 13963 62340

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 30

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 7)	.51181 60029 36333 75742	(+ 7)	-.33873 01491 69562 96880
CI(Z)	(+ 7)	-.33873 01491 69562 95854	(+ 7)	-.51181 58458 56701 07337
EX(1,Z)	(-17)	-.20189 92805 59119 20847	(-16)	-.22004 78076 79644 64041
EX(5,Z)	(-18)	-.99454 79182 05069 35292	(-16)	-.20339 23641 44354 52169
EX(10,Z)	(-19)	-.71379 18629 49964 90217	(-16)	-.18506 47537 64260 78647

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 30

	REAL PART		IMAGINARY PART	
SI(Z)	(+16)	-.11513 64309 88702 34453	(+15)	.93023 54076 64162 40543
CI(Z)	(+15)	.93023 54076 64162 40543	(+16)	.11513 64309 88702 50161
EX(1,Z)	(-32)	-.95581 03663 08972 87012	(-32)	-.32010 19095 37987 54961
EX(5,Z)	(-32)	-.90913 94977 30063 88053	(-32)	-.32792 23219 01497 75947
EX(10,Z)	(-32)	-.85557 27025 42686 96878	(-32)	-.33425 37625 71799 63580

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 45

		REAL PART		IMAGINARY PART
SI(Z)	(+ 0)	.74519 21553 53659 28422	(+ 0)	.66666 48174 19506 32005
CI(Z)	(+ 0)	.56680 20982 59308 90460	(+ 0)	.53562 96173 22429 89740
EX(1,Z)	(- 1)	.99862 71916 01974 34444	(+ 0)	-.28997 45541 18807 43759
EX(5,Z)	(- 1)	.63542 36380 27992 15763	(- 1)	-.76072 99919 57240 76634
EX(10,Z)	(- 1)	.35453 15002 32377 86525	(- 1)	-.35524 42968 14939 40464

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 45

		REAL PART		IMAGINARY PART
SI(Z)	(+11)	.17438 43424 32340 68560	(+11)	-.17136 76188 22945 12960
CI(Z)	(+11)	-.17136 76188 22945 12960	(+11)	-.17438 43424 16632 72233
EX(1,Z)	(-14)	-.90994 99898 28422 07254	(-14)	.89756 79949 34719 79228
EX(5,Z)	(-14)	-.89856 48783 09158 07708	(-14)	.78302 23432 94590 22757
EX(10,Z)	(-14)	-.87121 62564 30965 41059	(-14)	.66272 38913 29436 99466

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 45

		REAL PART		IMAGINARY PART
SI(Z)	(+23)	-.16287 98278 48661 77274	(+23)	.16643 51044 63585 26635
CI(Z)	(+23)	.16643 51044 63585 26635	(+23)	.16287 98278 48661 77274
EX(1,Z)	(-26)	.23453 37641 82110 25263	(-26)	-.23987 79073 88748 00420
EX(5,Z)	(-26)	.23392 78595 98617 08913	(-26)	-.22397 86382 00258 40002
EX(10,Z)	(-26)	.23194 47875 04430 64082	(-26)	-.20570 33464 64836 98670

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 60

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 0)	.55637 44092 28347 57934	(+ 0)	.86455 74828 89181 88368
CI(Z)	(+ 0)	.69677 43139 40552 45068	(+ 0)	.82167 28110 09608 70376
EX(1,Z)	(- 2)	-.19456 04279 57614 88333	(+ 0)	-.39007 88918 17426 93769
EX(5,Z)	(- 1)	.59342 92637 91145 10476	(+ 0)	-.11186 44426 43028 86784
EX(10,Z)	(- 1)	.35873 57743 72981 86837	(- 1)	-.51873 97120 49347 20471

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 60

	REAL PART		IMAGINARY PART	
SI(Z)	(+13)	.81491 50990 12170 02984	(+14)	.11574 17629 71922 48409
CI(Z)	(+14)	.11574 17629 71922 48409	(+13)	-.81491 50990 12012 95021
EX(1,Z)	(-10)	-.22625 91485 73859 75628	(-10)	.45559 14643 49524 70431
EX(5,Z)	(-10)	-.24749 38599 98904 12704	(-10)	.41373 72807 90046 48447
EX(10,Z)	(-10)	-.26324 90739 36884 82847	(-10)	.36438 00981 39203 94363

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 60

	REAL PART		IMAGINARY PART	
SI(Z)	(+28)	.75978 25855 48224 35041	(+28)	-.15404 02557 38707 04288
CI(Z)	(+28)	-.15404 02557 38707 04288	(+28)	-.75978 25855 48224 35041
EX(1,Z)	(-19)	.18945 18554 48414 96710	(-19)	-.49249 58708 02778 50194
EX(5,Z)	(-19)	.20415 97334 52290 23085	(-19)	-.47182 29981 26363 14790
EX(10,Z)	(-19)	.21925 28816 31394 73650	(-19)	-.44605 78368 49913 46583

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 75

		REAL PART		IMAGINARY PART
SI(Z)	(+ 0)	.29974 02274 92505 46979	(+ 1)	.10056 33346 56684 65838
CI(Z)	(+ 0)	.79892 87750 41746 96141	(+ 1)	.11747 41660 92934 82827
EX(1,Z)	(+ 0)	-.14479 80723 99690 16638	(+ 0)	-.49812 40483 32539 28630
EX(5,Z)	(- 1)	.57203 80457 33726 32389	(+ 0)	-.15925 77958 24796 77768
EX(10,Z)	(- 1)	.38777 29620 71565 23877	(- 1)	-.72718 05184 20066 25334

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 75

		REAL PART		IMAGINARY PART
SI(Z)	(+15)	-.47283 80894 65682 38819	(+15)	-.61026 45952 99039 55217
CI(Z)	(+15)	-.61026 45952 99039 55217	(+15)	.47283 80894 65683 95899
EX(1,Z)	(- 6)	-.48079 13205 39153 35524	(- 6)	-.62919 95805 72658 54553
EX(5,Z)	(- 6)	-.40792 77989 04173 45642	(- 6)	-.64819 06966 55219 15754
EX(10,Z)	(- 6)	-.32173 01129 25119 28353	(- 6)	-.65368 44780 03363 63999

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 75

		REAL PART		IMAGINARY PART
SI(Z)	(+32)	.22952 39934 18520 48663	(+30)	-.45941 13087 62031 92634
CI(Z)	(+30)	-.45941 13087 62031 92634	(+32)	-.22952 39934 18520 48663
EX(1,Z)	(-10)	-.12673 95439 94881 78993	(-12)	.39676 62813 98469 74101
EX(5,Z)	(-10)	-.12490 29897 44391 50572	(-12)	-.19527 32198 41604 55629
EX(10,Z)	(-10)	-.12197 24444 82197 02556	(-12)	-.88193 09022 08863 76213

MAGNITUDE OF Z = 1, ARGUMENT OF Z = 90

	REAL PART		IMAGINARY PART	
SI(Z)	(-28)	.29666 24887 79092 09678	(+ 1)	.10572 50875 37572 85145
CI(Z)	(+ 0)	.83786 69409 80208 24089	(+ 1)	.15707 96326 79489 66192
EX(1,Z)	(+ 0)	-.33740 39229 00968 13466	(+ 0)	-.62471 32564 27713 60428
EX(5,Z)	(- 1)	.63443 19256 79930 81070	(+ 0)	-.22384 87702 63972 32728
EX(10,Z)	(- 1)	.47573 65782 62179 83944	(- 1)	-.99212 56691 88739 26430

MAGNITUDE OF Z = 40, ARGUMENT OF Z = 90

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 1)	.15707 96326 79786 75989	(+16)	.30198 59131 80562 41906
CI(Z)	(+16)	.30198 59131 80562 41906	(-11)	-.29709 79754 76384 22880
EX(1,Z)	(- 1)	-.19020 00789 62087 66961	(- 1)	.16188 79255 98878 87544
EX(5,Z)	(- 1)	-.20321 92419 84824 36218	(- 1)	.14101 26268 73744 56188
EX(10,Z)	(- 1)	-.21316 58872 67299 75968	(- 1)	.11315 38889 06891 02018

MAGNITUDE OF Z = 80, ARGUMENT OF Z = 90

	REAL PART		IMAGINARY PART	
SI(Z)	(+ 6)	.69932 64331 38801 22249	(+33)	.35073 00002 45239 99848
CI(Z)	(+33)	.35073 00002 45239 99848	(+ 6)	-.69932 48623 42474 42759
EX(1,Z)	(- 1)	.12402 50115 50709 58192	(- 2)	.15345 60117 59069 61199
EX(5,Z)	(- 1)	.12280 19470 66883 74446	(- 2)	.21448 64614 58175 61117
EX(10,Z)	(- 1)	.12046 09812 36901 69426	(- 2)	.28784 84048 93317 65838

APPENDIX D

DERIVATIONS OF THE COMPUTATIONAL FORMULAS

APPENDIX D

DERIVATIONS OF THE COMPUTATIONAL FORMULAS

The series expansions in Eqs. (5) and (6) are derived from term-by-term integration of the series expansions of the integrands in the integrals in Eqs. (1) and (2), respectively.

From Eq. (1),

$$\mathrm{Si}(z) = \int_0^z \frac{1}{t} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\, dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^z t^{2k} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)(2k+1)!} .$$

Similarly, from Eq. (2),

$$\mathrm{Ci}(z) - \gamma - \ln z = \int_0^z \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} - 1\right) dt$$

$$= \int_0^z \frac{1}{t} \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}\, dt$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^z t^{2k-1} dt$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)(2k)!} .$$

To derive the series expansion in Eq. (7), one proceeds as follows: from Eq. (4),

$$E_1(z) = \int_1^\infty \frac{e^{-z\tau}}{\tau}\, d\tau$$

$$= \int_z^\infty \frac{e^{-t}}{t}\, dt \ .$$

It is known[3] that Euler's constant γ is given by

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}\, dt - \int_1^\infty \frac{e^{-t}}{t}\, dt \ .$$

Thus

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}\, dt - \int_1^z \frac{e^{-t}}{t}\, dt - \int_z^\infty \frac{e^{-t}}{t}\, dt$$

$$= \int_0^1 \frac{1-e^{-t}}{t}\, dt - \int_1^z \frac{e^{-t}}{t}\, dt - E_1(z) \quad ,$$

provided z is not on the negative real axis or at the origin. Then

$$E_1(z) = -\gamma + \int_0^1 \frac{1-e^{-t}}{t}\, dt - \int_1^z \frac{e^{-t}}{t}\, dt$$

[3] *E.J. Whittaker and G.N. Watson,* *A Course of Modern Analysis,* *Cambridge University Press, London, 1927, p. 246.*

$$E_1(z) = -\gamma + \int_0^1 \frac{1}{t}\left\{1 - \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!}\right\} dt - \int_1^z \frac{1}{t} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!}\, dt$$

$$= -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k-1} dt - \int_1^z \frac{1}{t}\, dt - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_1^z t^{k-1} dt$$

$$= -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^z t^{k-1} dt \ ,$$

or,

$$E_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{(k)\ k!}$$

Starting with the integral in Eq. (4), an integration by parts yields the recurrence relation of Eq. (17):

$$\int_1^{\infty} \frac{e^{-zt}}{t^n}\, dt = \left. \frac{e^{-zt} t^{1-n}}{1-n} \right|_{t=1}^{t=\infty} + \frac{z}{1-n} \int_1^{\infty} \frac{e^{-zt}}{t^{n-1}}\, dt$$

$$= \frac{1}{n-1} e^{-z} - \frac{z}{n-1} \int_1^{\infty} \frac{e^{-zt}}{t^{n-1}}\, dt \ ,$$

provided $n > 1$. That is,

$$E_n(z) = \frac{1}{n-1} \left[e^{-z} - z\, E_{n-1}(z)\right].$$

From the series expansion for $E_1(z)$ and the recurrence relation in Eq. (17), one can derive the series expansion for $E_n(z)$. To this end, apply the recurrence relation (n-1) times:

$$E_n(z) = \frac{1}{n-1}\left[e^{-z} - z\,E_{n-1}(z)\right]$$

$$= \frac{e^{-z}}{n-1} - \frac{z}{n-1}\left[\frac{e^{-z}}{n-2} - \frac{z}{n-2}E_{n-2}(z)\right]$$

$$= \frac{e^{-z}}{n-1} - \frac{ze^{-z}}{(n-1)(n-2)} + \frac{z^2}{(n-1)(n-2)}\left[\frac{e^{-z}}{n-3} - \frac{z}{n-3}E_{n-3}(z)\right]$$

$$= \frac{(-1)^0(z)^0e^{-z}}{(n-1)} + \frac{(-1)^1z^1e^{-z}}{(n-1)(n-2)} + \frac{(-1)^2z^2e^{-z}}{(n-1)(n-2)(n-3)} + \ldots$$

$$+ \frac{(-1)^{n-2}z^{n-2}e^{-z}}{(n-1)!} + \frac{(-1)^{n-1}z^{n-1}}{(n-1)!}E_1(z)$$

Expanding each of the e^{-z} terms in a power series, and using the series expansion for $E_1(z)$,

$$E_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^kz^k}{(n-1)k!} + \sum_{k=0}^{\infty}\frac{(-1)^{k+1}z^{k+1}}{(n-1)(n-2)k!} + \sum_{k=0}^{\infty}\frac{(-1)^{k+2}z^{k+2}}{(n-1)(n-2)(n-3)k!}$$

$$+\ldots+\sum_{k=0}^{\infty}\frac{(-1)^{k+n-2}z^{k+n-2}}{(n-1)!\;k!}$$

$$+ \frac{(-1)^{n-1}z^{n-1}}{(n-1)!}\left(-\gamma - \ln z - \sum_{k=1}^{\infty}\frac{(-1)^kz^k}{k\;k!}\right).$$

Redefining the indices of summation, and collecting all terms with z^{n-1},

$$E_n(z) = \sum_{\substack{k=0\\k\neq n-1}}^{\infty}\frac{(-1)^kz^k}{(n-1)k!} + \sum_{\substack{k=1\\k\neq n-1}}^{\infty}\frac{(-1)^kz^k}{(n-1)(n-2)(k-1)!}$$

(Equation continued on next page)

$$+\sum_{\substack{k=2\\k\neq n-1}}^{\infty} \frac{(-1)^k z^k}{(n-1)(n-2)(n-3)(k-2)!} + \ldots + \sum_{\substack{k=n-2\\k\neq n-1}}^{\infty} \frac{(-1)^k z^k}{(n-1)!(k-n+2)}$$

$$+ \frac{(-1)^{n-1} z^{n-1}}{(n-1)!} \left(-\gamma - \ln z + \sum_{k=1}^{n-1} \frac{1}{k} \right)$$

$$- \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(n-1)!(k-n+1)(k-n+1)!}$$

From this last equation, one sees that the only contributions to powers of z less than (n-1) come from the first (n-1) series. In particular, for any integer ℓ such that $0 \leq \ell \leq (n-2)$, there are $(\ell+1)$ terms with z^ℓ given by

$$(-1)^\ell z^\ell \left\{ \frac{1}{(n-1)\ell!} + \frac{1}{(n-1)(n-2)(\ell-1)!} + \ldots + \frac{1}{(n-1)(n-2)\ldots(n-\ell-1)(0)!} \right\}$$

Repeated factoring of the expression above in braces yields

$$\frac{(-1)^\ell z^\ell}{(\ell-n+1)\ell!} \left\{ \frac{\ell-n+1}{n-1} + \frac{\ell(\ell-n+1)}{(n-1)(n-2)} + \ldots + \frac{\ell!(\ell-n+1)}{(n-1)(n-2)\ldots(n-\ell)(n-\ell-1)} \right\}$$

$$= \frac{(-1)^\ell z^\ell}{(\ell-n+1)\ell!} \left\{ \frac{\ell-n+1}{n-1} \left(1 + \frac{\ell}{n-2} \left(1 + \frac{\ell-1}{n-3} \left(\ldots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{3}{n-\ell+1} \left(1 + \frac{2}{n-\ell} \left(1 + \frac{1}{n-\ell-1} \right) \right) \ldots \right) \right\}$$

Since

$$1 + \frac{2}{n-\ell} \left(1 + \frac{1}{n-\ell-1} \right) = 1 + \frac{2}{n-\ell} \left(\frac{n-\ell}{n-\ell-1} \right)$$

(Equation continued on next page)

$$= 1 + \frac{2}{n-\ell-1}$$

$$= \frac{n-\ell+1}{n-\ell-1} \, ,$$

one sees that the factored expression compresses to

$$\frac{(-1)^{\ell} z^{\ell}}{(\ell-n+1)\ell!} \left\{ \frac{\ell-n+1}{n-1} \left(1 + \frac{\ell}{n-2} \left(\frac{n-2}{n-\ell-1} \right) \right) \right\}$$

$$= \frac{(-1)^{\ell} z^{\ell}}{(\ell-n+1)\ell!} \left\{ \frac{\ell-n+1}{n-1} \left(1 + \frac{\ell}{n-\ell-1} \right) \right\}$$

$$= \frac{(-1)^{\ell} z^{\ell}}{(\ell-n+1)\ell!} \left\{ \frac{\ell-n+1}{n-1} \left(\frac{n-1}{n-\ell-1} \right) \right\}$$

$$= \frac{-(-1)^{\ell} z^{\ell}}{(\ell-n+1)\ell!} \; .$$

Using this result in the previous equation,

$$E_n(z) = \frac{(-1)^{n-1} z^{n-1}}{(n-1)!} \left(- \gamma - \ln z + \sum_{k=1}^{n-1} \frac{1}{k} \right) - \sum_{k=0}^{n-2} \frac{(-1)^k z^k}{(k-n+1)k!}$$

$$+ \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(n-1)k!} + \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(n-1)(n-2)(k-1)!} + \cdots$$

$$+ \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(n-1)!(k-n+2)!} - \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(n-1)!(k-n+1)(k-n+1)!}$$

In the series in this equation, the terms involving z^{ℓ}, where $\ell \geq n$, may be treated in a manner analogous to the terms for which $\ell \leq (n-2)$. Collecting these last n series, one has

$$\sum_{k=n}^{\infty} (-1)^k z^k \left\{ \frac{1}{(n-1)k!} + \frac{1}{(n-1)(n-2)(k-1)!} +\cdots \right.$$

$$\left. + \frac{1}{(n-1)!(k-n+2)!} - \frac{1}{(n-1)!(k-n+1)(k-n+1)!} \right\}$$

$$= \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!} \left\{ \frac{k-n+1}{n-1} \left(1+ \frac{k}{n-2} \left(1+ \frac{k-1}{n-3} \left(1+\cdots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{k-n+4}{2} \left(1+ \frac{k-n+3}{1} \left(1- \frac{k-n+2}{k-n+1} \right)\right)\cdots\right)\right\}$$

$$= \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!} \left\{ \frac{k-n+1}{n-1} \left(1+ \frac{k}{n-2} \left(1+ \frac{k-1}{n-3} \left(1+\cdots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{k-n+4}{2} \left(1- \frac{k-n+3}{k-n+1} \right)\right) \cdots\right)\right\}$$

$$\vdots$$

$$= \sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!} \left\{ \frac{k-n+1}{n-1} \left(\frac{1-n}{k-n+1} \right)\right\}$$

$$= -\sum_{k=n}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!}$$

Thus finally, the series representation for

$E_n(z)$ is given by

$$E_n(z) = \frac{(-1)^{n-1} z^{n-1}}{(n-1)!}\left(-\gamma - \ln z + \sum_{k=1}^{n-1} \frac{1}{k}\right) - \sum_{\substack{k=0 \\ k\neq n-1}}^{\infty} \frac{(-1)^k z^k}{(k-n+1)k!}$$

which is Eq. (7).

The continued fraction of Eq. (8) is derived from the Gauss continued fraction by a method found in Wall[4]. The Gauss continued fraction is given by

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \cfrac{1}{1 - \cfrac{\frac{a(c-b)}{c(c+1)}z}{1 - \cfrac{\frac{(b+1)(c-a+1)}{(c+1)(c+2)}z}{1 - \cfrac{\frac{(a+1)(c-b+1)}{(c+2)(c+3)}z}{1 - \ddots}}}}$$

where $F(a,b,c;z)$ *is the hypergeometric function,*

$$F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad ,$$

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

the quotient of two gamma functions.

If, in the series for the hypergeometric function, one replaces z by $-cz$ and takes the limit as $c \to \infty$, the divergent series

[4]*H. S. Wall,* *Continued Fractions,* *D. Van Nostrand Co., Inc., New York, 1948, pages 336-352.*

$$\Omega(a,b;-z) = 1 - abz + a(a+1)(b)(b+1)\,\frac{z^2}{2!}$$

$$- a(a+1)(a+2)(b)(b+1)(b+2)\,\frac{z^3}{3!} + \ldots$$

is obtained. Using the same transformation and limit in the continued fraction of Gauss,

$$\frac{\Omega(a,b+1;-z)}{\Omega(a,b;-z)} = \cfrac{1}{1 + \cfrac{az}{1 + \cfrac{(b+1)z}{1 + \cfrac{(a+1)z}{1 + \cfrac{(b+2)z}{1 + \ddots}}}}}$$

Using the divergent series for Ω,

$$\Omega(a,b;-z) = 1 - abz + a(a+1)(b)(b+1)\,\frac{z^2}{2!} - \ldots$$

$$= \frac{\Gamma(a)}{\Gamma(a)} + \frac{\Gamma(a+1)}{\Gamma(a)}\binom{-b}{1}z + \frac{\Gamma(a+2)}{\Gamma(a)}\binom{-b}{2}z^2 + \ldots$$

$$= \frac{1}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}du + \frac{1}{\Gamma(a)}\binom{-b}{1}z\int_0^\infty e^{-u}u^{a}du$$

$$+ \frac{1}{\Gamma(a)}\binom{-b}{2}z^2\int_0^\infty e^{-u}u^{a+1}du + \ldots$$

$$= \frac{1}{\Gamma(a)}\int_0^\infty\left(1 + \binom{-b}{1}zu + \binom{-b}{2}(zu)^2 + \ldots\right)e^{-u}u^{a-1}du$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty (1+zu)^{-b} e^{-u} u^{a-1} du,$$

where use has been made of the binomial coefficient

$$\binom{j}{k} = \frac{j!}{(j-k)!(k!)} = (-1)^k \binom{k-j-1}{k}$$

and of the integral representation

$$\Gamma(p) = \int_0^\infty e^{-u} u^{p-1} du.$$

It follows that

$$\frac{\Omega(a,b+1;-z)}{\Omega(a,b;-z)} = \frac{\displaystyle\int_0^\infty \frac{e^{-u}u^{a-1}}{(1+zu)^{b+1}}\,du}{\displaystyle\int_0^\infty \frac{e^{-u}u^{a-1}}{(1+zu)^{b}}\,du} .$$

Choosing b=0 and using the continued fraction expansion for the quotient on the left,

$$\frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u}u^{a-1}}{(1+zu)}\,du = \cfrac{1}{1 + \cfrac{az}{1 + \cfrac{1z}{1 + \cfrac{(a+1)z}{1 + \cfrac{2z}{1 + \cdots}}}}}$$

Since $\Omega(a,b;-z) = \Omega(b,a;-z)$,

$$\frac{1}{\Gamma(a)} \int_0^\infty (1+zu)^{-b} e^{-u} u^{a-1} du$$

$$= \frac{1}{\Gamma(b)} \int_0^\infty (1+zu)^{-a} e^{-u} u^{b-1} du.$$

Setting b=1,

$$\int_0^\infty \frac{e^{-u}}{(1+zu)^a} du = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{1+zu} du$$

$$= 1 + \cfrac{1}{\cfrac{az}{1 + \cfrac{1z}{1 + \cfrac{(a+1)z}{1 + \ddots}}}}$$

It can be shown that the integrals in this last equation converge for all values of z not on the negative real axis. Replace z by $\frac{1}{z}$ in the first of these integrals and let $t = 1 + \frac{u}{z}$.

$$\int_0^\infty \frac{e^{-u}}{(1 + \frac{1}{z} u)^a} du = ze^z \int_1^\infty \frac{e^{-zt}}{t^a} dt \quad ,$$

or, on letting a = n,

$$E_n(z) = \frac{e^{-z}}{z} \quad \cfrac{1}{1 + \cfrac{n/z}{1 + \cfrac{1/z}{1 + \cfrac{(n+1)/z}{1 + \ddots}}}}$$

This continued fraction may be simplified to the form

$$E_n(z) = e^{-z} \cfrac{1}{z + \cfrac{n}{1 + \cfrac{1}{z + \cfrac{n+1}{1 + \cfrac{2}{z + \ddots}}}}}$$

for $|\arg z| < \pi$, which is Eq. (8).

To derive Eqs. (9) and (10), note that for $|\arg z| < \frac{\pi}{2}$,

$$E_1(iz) = \int_{iz}^{\infty} \frac{e^{-t}}{t}\, dt$$

$$= \int_{z}^{\infty} \frac{e^{-iu}}{iu}\, d(iu)$$

$$= \int_{z}^{\infty} \frac{1}{u} (\cos u - i \sin u)\, du$$

$$= \int_{z}^{\infty} \frac{\cos u}{u}\, du - i \int_{z}^{\infty} \frac{\sin u}{u}\, du .$$

Now

$$\int_{z}^{\infty} \frac{\sin u}{u}\, du = \int_{0}^{\infty} \frac{\sin u}{u}\, du - \int_{0}^{z} \frac{\sin u}{u}\, du$$

$$= \frac{\pi}{2} - \mathrm{Si}(z) .$$

Moreover,

$$
\begin{aligned}
Ci(z) &= \int_0^z \frac{\cos t - 1}{t}\, dt + \gamma + \ln z \\
&= \int_0^s \frac{\cos t - 1}{t}\, dt + \int_s^z \frac{\cos t - 1}{t}\, dt + \gamma + \ln z \\
&= \int_0^s \left\{ \frac{\cos t}{t} - \frac{1}{t} + \frac{1}{t(1+t)} - \frac{1}{t(1+t)} \right\} dt + \int_s^z \frac{\cos t}{t}\, dt \\
&\quad - \int_s^z \frac{dt}{t} + \gamma + \ln z = \left[-\int_0^s \left(\frac{1}{1+t} - \cos t \right) \frac{dt}{t} + \gamma \right] \\
&\quad + \left[\int_0^s \left\{ -\frac{1}{t} + \frac{1}{t(1+t)} \right\} dt - \int_s^z \frac{dt}{t} + \ln z \right] - \int_z^s \frac{\cos t}{t}\, dt
\end{aligned}
$$

The first term in square brackets above tends to 0

as $s \to \infty$, since[5]

$$
\gamma = \int_0^\infty \left\{ \frac{1}{1+t} - \cos t \right\} \frac{dt}{t} .
$$

The second term in square brackets is

$$
\int_0^s \left\{ -\frac{1}{t} + \frac{1}{t(1+t)} \right\} dt + \int_z^s \frac{dt}{t} + \ln z = -\int_0^s \frac{dt}{1+t} + \ln s - \ln z + \ln z
$$

---

[5] *W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer Verlag, New York, 1966, page 35.*

$$= - \ln(1+s) + \ln 1 + \ln s = \ln \frac{s}{1+s}$$

and

$$\lim_{s\to\infty} \ln \frac{s}{1+s} = \lim_{s\to\infty} \frac{1}{1+\frac{1}{s}} = \ln 1 = 0.$$

Therefore,

$$Ci(z) = - \int_z^\infty \frac{\cos t}{t} \, dt,$$

so that

$$E_1(iz) = - Ci(z) + i \, Si(z) - i \frac{\pi}{2} .$$

Using Eqs. (11) and (12), it follows that

$$E_1(-iz) = - Ci(-z) + i \, Si(-z) - i \frac{\pi}{2}$$

$$= - Ci(z) + i\pi - i \, Si(z) - i \frac{\pi}{2}$$

$$= - Ci(z) - i \, Si(z) + i \frac{\pi}{2}$$

Hence,

$$Si(z) = \frac{1}{2i} \left[E_1(iz) - E_1(-iz) \right] + \frac{\pi}{2}$$

and

$$Ci(z) = - \frac{1}{2} \left[E_1(iz) + E_1(-iz) \right] ,$$

proving Eqs. (9) and (10).

It remains to derive the asymptotic expansion for $E_n(z)$. To this end, rearrange the recurrence relation in Eq. (17) in the form

$$E_n(z) = \frac{1}{z}\left(e^{-z} - nE_{n+1}(z)\right),$$

and use it repeatedly:

$$\begin{aligned}
E_n(z) &= \frac{1}{z}\left(e^{-z}-n\left(\frac{1}{z}\left(e^{-z}-(n+1)E_{n+2}(z)\right)\right)\right)\\
&= \frac{e^{-z}}{z}\left(1-\frac{n}{z}+\frac{n(n+1)}{ze^{-z}}E_{n+2}(z)\right)\\
&= \frac{e^{-z}}{z}\left(1-\frac{n}{z}+\frac{n(n+1)}{ze^{-z}}\left[\frac{1}{z}\left(e^{-z}-(n+2)E_{n+3}(z)\right)\right]\right)\\
&= \frac{e^{-z}}{z}\left(1-\frac{n}{z}+\frac{n(n+1)}{z^2}-\frac{n(n+1)(n+2)}{z^2e^{-z}}E_{n+3}(z)\right)\\
&\quad\vdots\\
&= \frac{e^{-z}}{z}\left(1-\frac{n}{z}+\frac{n(n+1)}{z^2}-\frac{n(n+1)(n+2)}{z^3}+\ldots\right.\\
&\qquad\left.+\frac{n(n+1)(\ldots)(n+N)}{z^N e^{-z}}E_{n+N+1}(z)\right)
\end{aligned}$$

Therefore,

$$E_n(z) \sim \frac{e^{-z}}{z}\left(1-\frac{n}{z}+\frac{n(n+1)}{z^2}-\frac{n(n+1)(n+2)}{z^3}+\ldots\right).$$

This derivation is equivalent to repeated integration by parts starting with the integral in Eq. (4). (see, e.g., Olver[6], page 67).

[6] *F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.*

All of the computational formulas used in this report can be found in reference 7. It should be noted that this reference contains several other computational formulas which would, at first glance, seem to provide a more simple means of evaluating Si and Ci for large $|z|$ than the method used in this report. In particular, the sine and cosine integrals may be written in terms of the auxiliary functions

$$f(z) = \int_0^\infty \frac{e^{-zt}}{t^2+1}\, dt \ ,$$

and

$$g(z) = \int_0^\infty \frac{te^{-zt}}{t^2+1}\, dt.$$

The functions f and g have asymptotic expansions which are easily derived, but which fail to represent Si and Ci correctly on the imaginary axis. This problem will be examined in more detail in a subsequent report on verification of the present subroutine.

[7] *M. Abramowitz and I. Stegun, editors, Handbook of Mathematical Functions, National Bureau of Standards, U.S. Dept. of Commerce, 1965.*

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