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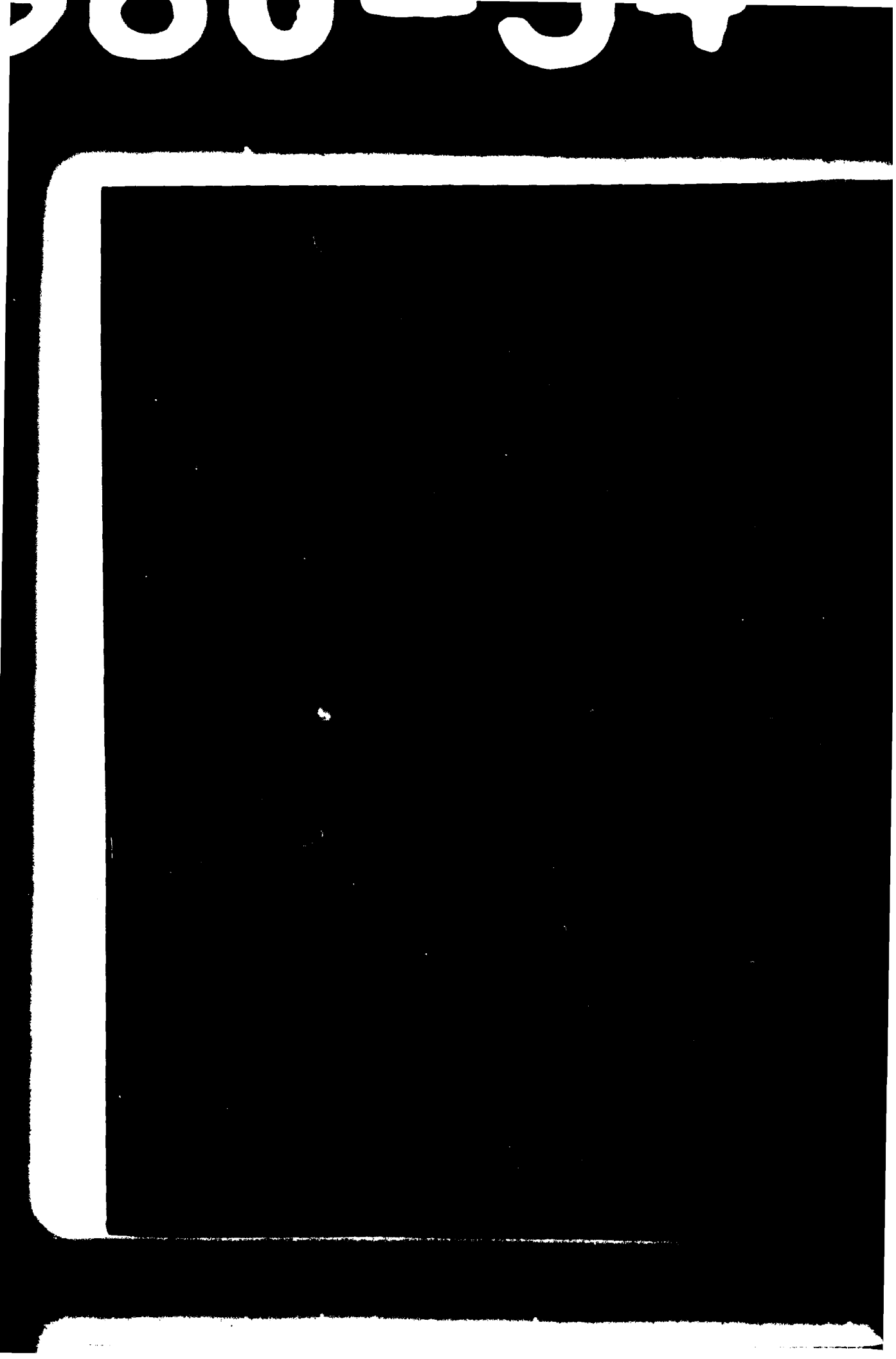
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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ITERATIVE TECHNIQUES FOR MINIMUM PHASE SIGNAL RECONSTRUCTION FROM PHASE OR MAGNITUDE

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ABSTRACT

In this report, we develop iterative algorithms for reconstructing a minimum phase sequence from the phase or magnitude of its Fourier transform. The iterative techniques result in two potentially important computational algorithms. The first is a means of implementing the Hilbert transform of the log-magnitude or phase of the Fourier transform of a minimum phase sequence. This procedure avoids problems of phase unwrapping and aliasing inherent in the direct discrete Fourier transform implementation of the Hilbert transform. The second algorithm is a new method of computing samples of the unwrapped phase. As compared with other available phase unwrapping algorithms, this approach does not rely on adding 2π multiples to samples of the principal value of the phase.

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CONTENTS

1. INTRODUCTION

Under certain conditions a signal can be reconstructed from a partial specification in the time domain, in the frequency domain, or in both domains. A minimum or maximum phase signal, in particular, can be recovered from the phase or magnitude of its Fourier transform [1]. The conventional reconstruction algorithm involves applying the Hilbert transform to the log-magnitude or phase of the Fourier transform to obtain the unknown component.

In this report, we take an alternative approach by developing iterative algorithms for reconstructing a minimum (or maximum) phase signal from the phase or magnitude of its Fourier transform. Specifically, we develop algorithms which impose causality in the time domain and the given phase or magnitude in the frequency domain, in an iterative fashion.

Iterative algorithms similar to those we discuss here have been useful in a number of areas where partial information in the two domains is available. In particular, the algorithms presented in this paper are similar in style to the Gerchberg-Saxton algorithm [2], and an iterative algorithm by Fienup [3] in alternately incorporating partial information in the time and frequency domains. The Gerchberg-Saxton algorithm recovers a two-dimensional complex signal by iteratively imposing the finite extent of the signal in the space-domain and its magnitude in both the space and frequency domains. Similarly, Fienup's algorithm recovers a real two-dimensional signal by iteratively imposing the finite extent and positivity of the signal in the space-domain and its magnitude in the frequency domain. Another iteration in this same style recovers a finite length mixed phase signal from the phase of its Fourier transform by imposing a finite length constraint in the time domain and the known phase in the frequency domain [4,5].

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In this report, we begin in section 2 with a discussion of a number of equivalent conditions for a sequence to be minimum phase. In sections III and IV, we use these conditions in developing two iterative reconstruction algorithms, one for reconstruction when the phase is known and the other for reconstruction when the magnitude is known.

In section V, we discuss the discrete Fourier transform (DFT) realizations of the algorithms and illustrate the reconstruction process with examples.

In sections VI and VII, we describe two computational algorithms which rely on the iterative procedures of sections III and IV. We first investigate the use of the algorithms in implementing the Hilbert transform. Of particular importance is reconstruction of the log-magnitude from phase since the iterative approach requires only the principal value of the phase, while the direct DFT implementation of the Hilbert transform requires the unwrapped phase [6]. The former technique, therefore, avoids the complications and problems of phase unwrapping which is often computationally difficult [1,7]. In addition, the accuracy of the iterative process is not limited by the finite length of the DFT as in the direct approach.

In the second computational procedure, the iterative technique of section III is used as the basis for a new phase unwrapping algorithm. This algorithm does not rely on adding 2π multiples to the principal value of the phase as required by available unwrapping algorithm.

In section VIII, the main results of the report are summarized and some future research is discussed.

2. THE MINIMUM PHASE CONDITION

In general, a signal cannot be uniquely specified by only the phase or magnitude of its Fourier transform. However, one condition under which the magnitude and phase are related is the minimum phase condition and under this condition a signal can be uniquely recovered from the magnitude of its Fourier transform or, to within a scale factor, the phase of its Fourier transform. In this section, we discuss a number of equivalent conditions for a signal to be minimum phase. These conditions will be of particular importance in section III in developing the iterative algorithms.

In the following discussion, we restrict the z-transform of the sequence $h(n)$ to be a rational function which we express in the form

$$H(z) = Az^{n_0} \frac{\prod_{k=1}^{M_1} (1-a_k z^{-1}) \prod_{k=1}^{M_0} (1-b_k z)}{\prod_{k=1}^{P_1} (1-c_k z^{-1}) \prod_{k=1}^{P_0} (1-d_k z)} \qquad (1)$$

where $|a_k|$, $|b_k|$, $|c_k|$ and $|d_k|$ are less than or equal to unity, z^{n_0} is a linear phase factor, and A is a scale factor. When, in addition, $h(n)$ is stable, i.e., $\sum_n |h(n)| < \infty$, $|c_k|$ and $|d_k|$ are strictly less than one.

A complex function $H(z)$ of a complex variable z is defined to be minimum phase if it and its reciprocal $H^{-1}(z)$ are both analytic for $|z| \geq 1$. A minimum phase sequence is then defined as a sequence whose z-transform is minimum phase. For $H(z)$ rational, as in (2), the minimum phase condition excludes poles or zeros on or outside the unit circle in the z-plane or at infinity. As a consequence, the factors of the form $(1-b_k z)$ corresponding to zeros on or outside the unit circle and the factors of the form $(1-d_k z)$ corresponding to poles on or outside the unit circle will not be present. Furthermore, in (2), $n_0 = 0$ to exclude poles

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or zeros at infinity. Thus for H(z) minimum phase, (1) reduces to

$$H(z) = A \frac{\prod\limits_{k=1}^{M_1} (1-a_k z^{-1})}{\prod\limits_{k=1}^{P_1} (1-c_k z^{-1})} \tag{2}$$

where $|a_k|$ and $|c_k|$ are both strictly less than unity.

From (2) other conditions can be formulated for a signal to be minimum phase. Two in particular which we discuss below are particularly useful in the context of the iterative algorithms to be discussed in section III and IV.

<u>Minimum Phase Condition A</u>: Consider h(n) stable and H(z) rational in the form of (1) with no zeros on the unit circle. A necessary and sufficient condition for h(n) to be minimum phase is that h(n) be causal, i.e., h(n)=0   n < 0, and n_o in (1) be zero.

From (2), it follows that these conditions are necessary. To show that they are sufficient, we want to show that they force (1) to reduce to (2). Clearly, factors of the form $(1-d_k z)$ $|d_k|$ < 1 in the denominator introduce poles outside the unit circle which would violate the causality condition. With n_o=0 in (1), factors of the form $(1-b_k z)$ would introduce positive powers of z in the Laurent expansion of H(z), requiring h(n) to have some non-zero values for negative values of n, thereby again violating the causality condition. Therefore, these factors cannot be present and with n_o=0, (1) reduces to (2). Finally, because our condition assumes h(n) is stable and that H(z) has no zeros on the unit circle, h(n) is minimum phase.

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The above minimum phase conditions require that $h(n)$ be causal and that the unwrapped phase function have no linear phase component. Another slightly different set of necessary and sufficient conditions for a signal to be minimum phase can be stated as follows:

<u>Minimum Phase Condition B</u>: Consider $h(n)$ stable and $H(z)$ rational in the form of (1) with no zeros on the unit circle. A necessary and sufficient condition for $h(n)$ to be minimum phase is that $h(n)=0$ $n < 0$ and $h(0)=A$ where A is the scale factor in (1).

Again, from (2) it follows that these conditions are necessary since (2) has no poles or zeros outside the unit circle or at infinity, guaranteeing causality, and from the Initial Value Theorem $h(0) = \lim_{z \to \infty} H(z) = A$. To demonstrate that these conditions are sufficient we note that again causality of $h(n)$ will eliminate factors of the form $(1-d_k z)$ in the denominator of (1). Furthermore, since the conditions require that $h(n)$ be causal, the Initial Value Theorem can be applied with the result that

$$h(0) = \lim_{z \to \infty} H(z) = \lim_{z \to \infty} A \, z^{n_o} \prod_{k=1}^{M_o} (1-b_k z) \qquad (3)$$

Since $h(0)=A$,

$$\lim_{z \to \infty} z^{n_o} \prod_{k=1}^{M_o} (1-b_k z) = 1 \qquad (4)$$

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and since $|b_k| < 1$ this requires that $n_o = 0$ and the b_k's be equal to zero. Thus, again (1) reduces to (2).

Another condition which can be shown to be equivalent to minimum phase condition A or B or our original definition of a minimum phase sequence is that the log-magnitude and unwrapped phase of $H(\omega)$ are related through the Hilbert transform [1]. The Hilbert transform relation guarantees that a minimum phase sequence can be uniquely specified from the Fourier transform phase and, to within a scale factor, from the Fourier transform magnitude. Consequently, this unique characterization can be made when minimum phase condition A or B holds, and the Fourier transform phase or magnitude is given.

One technique for minimum phase signal reconstruction from phase or magnitude relies on a DFT implementation of the Hilbert transform [6]. Two drawbacks to this algorithm are the requirement of samples of the un-wrapped phase and inaccuracies due to aliasing. In the next two sections, we develop iterative algorithms for reconstructing a minimum phase sequence $h(n)$ from the phase or magnitude of its Fourier transform which bypass these problems. Motivated by the minimum phase condition A, when the phase is given we impose, in an iterative fashion, causality in the time domain and the known phase (which has no linear phase component) in the frequency domain. When the resulting sequence satisfies minimum phase condition A and has the given phase, it must equal $h(n)$ to within a scale factor. Likewise, motivated by the minimum phase condition B, when the magnitude is given we impose, in an iterative fashion, causality and the initial value $h(0)$ in the time domain, and the known magnitude in the frequency domain. When the algorithm results in a sequence which satisfies minimum phase condition B and has the given magnitude, it must equal $h(n)$.

3. AN ITERATIVE ALGORITHM FOR SIGNAL RECONSTRUCTION FROM PHASE

The iterative algorithm for reconstructing a minimum phase signal from its phase function is shown in Figure 1. The functions $h_k(n)$, $\theta_k(\omega)$ and $M_k(\omega)$ represent the signal and its phase and magnitude estimates on the kth iteration. The function $\theta_k(\omega)$ is the known phase and $\tilde{h}_{k+1}(n) = h_k(n)u(n)$, where $u(n)$ is the unit step function.

The algorithm begins with an initial guess $M_0(\omega)$ of the desired Fourier transform magnitude and the inverse Fourier transform of $M_0(\omega)\exp[j\theta_h(\omega)]$ is taken. This step yields $h_0(n)$, the initial estimate of $h(n)$. Next, causality is imposed so that $\tilde{h}_0(n)$ is set to zero for $n < 0$ to obtain $\tilde{h}_1(n)$. The phase of the Fourier transform of $\tilde{h}_1(n)$ is then replaced by the given phase and the procedure is repeated.

We now show that the mean squared error between $h(n)$ and $h_k(n)$ or equivalently between $H(\omega)$ and $H_k(\omega)$ is non-increasing on successive iterations. The mean squared error on the kth iteration from Parseval's Theorem can be written as

$$E_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega) - H_k(\omega)|^2 \, d\omega$$

$$= \sum_n |h(n) - h_k(n)|^2$$

$$= \sum_{n < 0} |h(n) - h_k(n)|^2 + \sum_{n \geq 0} |h(n) - h_k(n)|^2 \tag{5}$$

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$M_0(\omega)\exp[j\theta_h(\omega)]$

F^{-1}

$h_k(n)$

Impose Causality

$\tilde{h}_{k+1}(n) = h_k[n]u[n]$

F

$M_{k+1}(\omega)\exp[j\theta_{k+1}(\omega)]$

$\theta_h(\omega) \rightarrow \theta_{k+1}(\omega)$
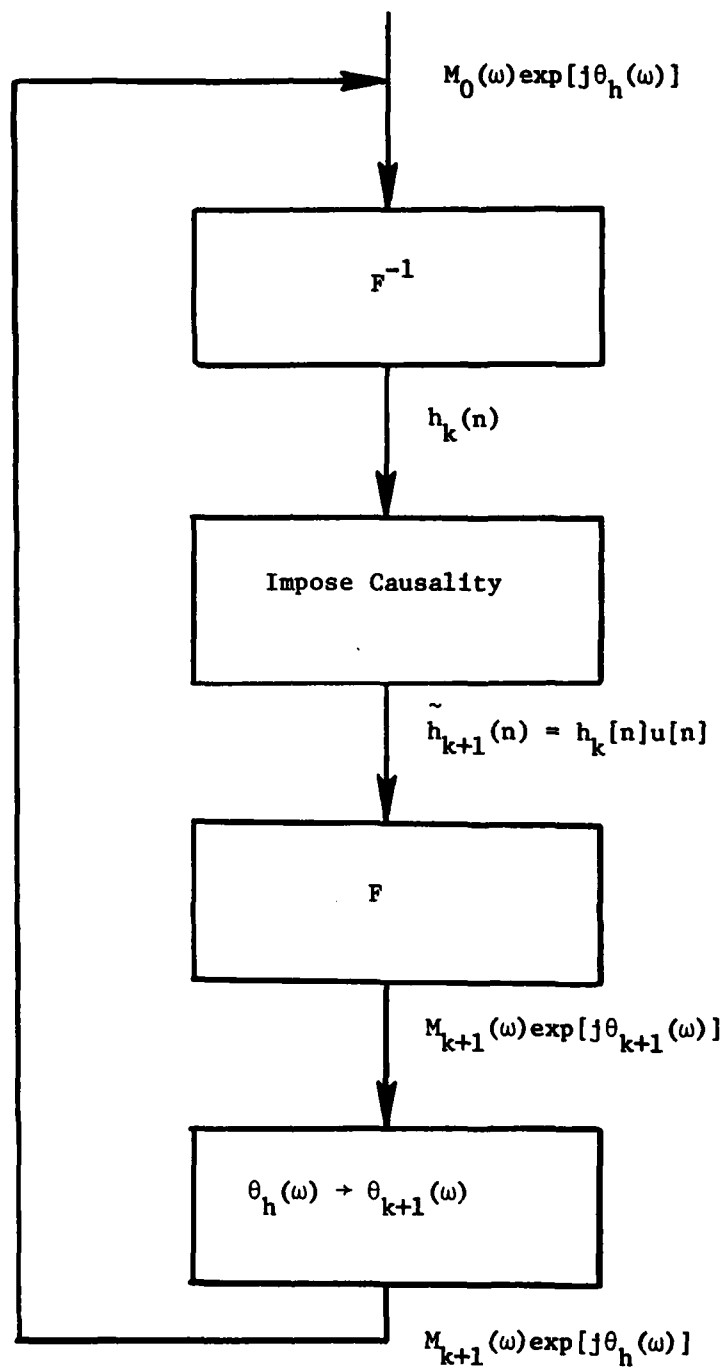
$M_{k+1}(\omega)\exp[j\theta_h(\omega)]$

Fig. 1. Iterative algorithm to recover h(n) from its phase.

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Since $\tilde{h}_{k+1}(n) = h_k(n)u(n)$,

$$|h(n) - h_k(n)| = |h(n) - \tilde{h}_{k+1}(n)| \ , \ n \geq 0 \tag{6}$$

and

$$|h(n) - h_k(n)| \geq |h(n) - \tilde{h}_{k+1}(n)| = 0 \ , \ n < 0 \tag{7}$$

Summing (6) and (7) over all n, we obtain

$$E_k = \sum_n |h(n) - h_k(n)|^2$$

$$\geq \sum_n |h(n) - \tilde{h}_{k+1}(n)|^2 \tag{8}$$

Next, from Parseval's Theorem, we write (8) as

$$E_k \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega) - \tilde{H}_{k+1}(\omega)|^2 \, d\omega \tag{9}$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| M(\omega)\exp[j\Theta(\omega)] - M_{k+1}(\omega)\exp[j\Theta_{k+1}(\omega)] \right|^2 d\omega$$

With the triangle inequality for vector differences, we have at each frequency ω:

$$\left| M(\omega)\exp[j\Theta(\omega)] - M_{k+1}(\omega)\exp[j\Theta_{k+1}(\omega)] \right|$$

$$\geq \left| M(\omega)\exp[j\Theta(\omega)] \right| - \left| M_{k+1}(\omega)\exp[j\Theta_{k+1}(\omega)] \right|$$

$$\geq |M(\omega) - M_{k+1}(\omega)| \tag{10}$$

Therefore, from (9) and (10), and the identity $\left|\exp[j\theta(\omega)]\right|^2 = 1$:

$$E_k \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|M(\omega) - M_{k+1}(\omega)\right|^2 d\omega$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|M(\omega)\exp[j\theta(\omega)] - M_{k+1}(\omega)\exp[j\theta(\omega)]\right|^2 d\omega$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|H(\omega) - H_{k+1}(\omega)\right|^2 d\omega$$

$$\geq E_{k+1} \tag{11}$$

Since E_k is, therefore, non-increasing and has a lower bound of zero, E_k must converge to a unique limit [8]. The non-increasing nature of E_k, ho·   r, is not sufficient to guarantee that the iterates $h_k(n)$ converge. Neve. ieless, if a converging solution with a rational z-transform exists, we can show that

$$\lim_{k \to \infty} h_k(n) = \alpha h(n) \tag{12}$$

where α is a positive constant.

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To see this, note from (6), (7) and (10) that the equality in (11) holds if and only if $h_k(n) = \tilde{h}_{k+1}(n) = 0$ for $n < 0$, and $\theta_h(\omega) = \theta_{k+1}(\omega)$. Therefore, since $\theta_h(\omega)$ contains no linear phase component (i.e., $n_o = 0$), if $h_k(n)$ converges to a sequence whose z-transform is of the form in (1), the converging solution must satisfy the minimum phase condition A. Consequently, the converging solution is minimum phase with phase $\theta_h(\omega)$, and (12) must hold.*

When h(n) is of finite duration (i.e., H(z) has no poles), we can impose not only causality, but also a finite duration constraint within the iteration. Under these particular constraints, the DFT realization of our iterative procedure (see section V) converges to a limit of the form in (12) [9].

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* The constant α in (12) is constrained to be positive since a negative value introduces an additive factor of π into the phase function.

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4. AN ITERATIVE ALGORITHM FOR SIGNAL RECONSTRUCTION FROM MAGNITUDE

In this section we present an iterative algorithm for reconstruction of a minimum phase signal from the magnitude of its Fourier transform. The algorithm is shown in Figure 2. The functions $h_k(n)$, $\theta_k(\omega)$ and $M_k(\omega)$ represent the signal, phase, and magnitude estimates, respectively on the kth iteration and $\tilde{h}_{k+1}(n)$ is defined by

$$\tilde{h}_{k+1}(n) = \begin{cases} h_k(n), & n > 0 \\ h(0), & n = 0 \\ 0, & n < 0 \end{cases} \tag{13}$$

The algorithm begins with an initial guess $\theta_0(\omega)$ of the desired phase, and the inverse transform of $M(\omega)\exp[j\theta_0(\omega)]$ is taken where $M(\omega) = |H(\omega)|$ is the given magnitude. This step yields $h_0(n)$, the initial estimate of $h(n)$. Next, on the basis of the minimum phase condition B, causality and the known value of $h(0)$ are imposed so that $h_0(n)$ is set to zero for $n < 0$ and set to $h(0)$ for $n=0$, to obtain $h_1(n)$. The magnitude of the Fourier transform of $\tilde{h}_1(n)$ is then replaced by the given magnitude and the procedure is repeated.

It has not been possible to show that the mean squared error, as considered in section III, is non-increasing for this algorithm. However, an error function that is non-increasing is the mean squared difference between the known magnitude and the estimate $M_k(\omega)$ on each iteration; i.e.,

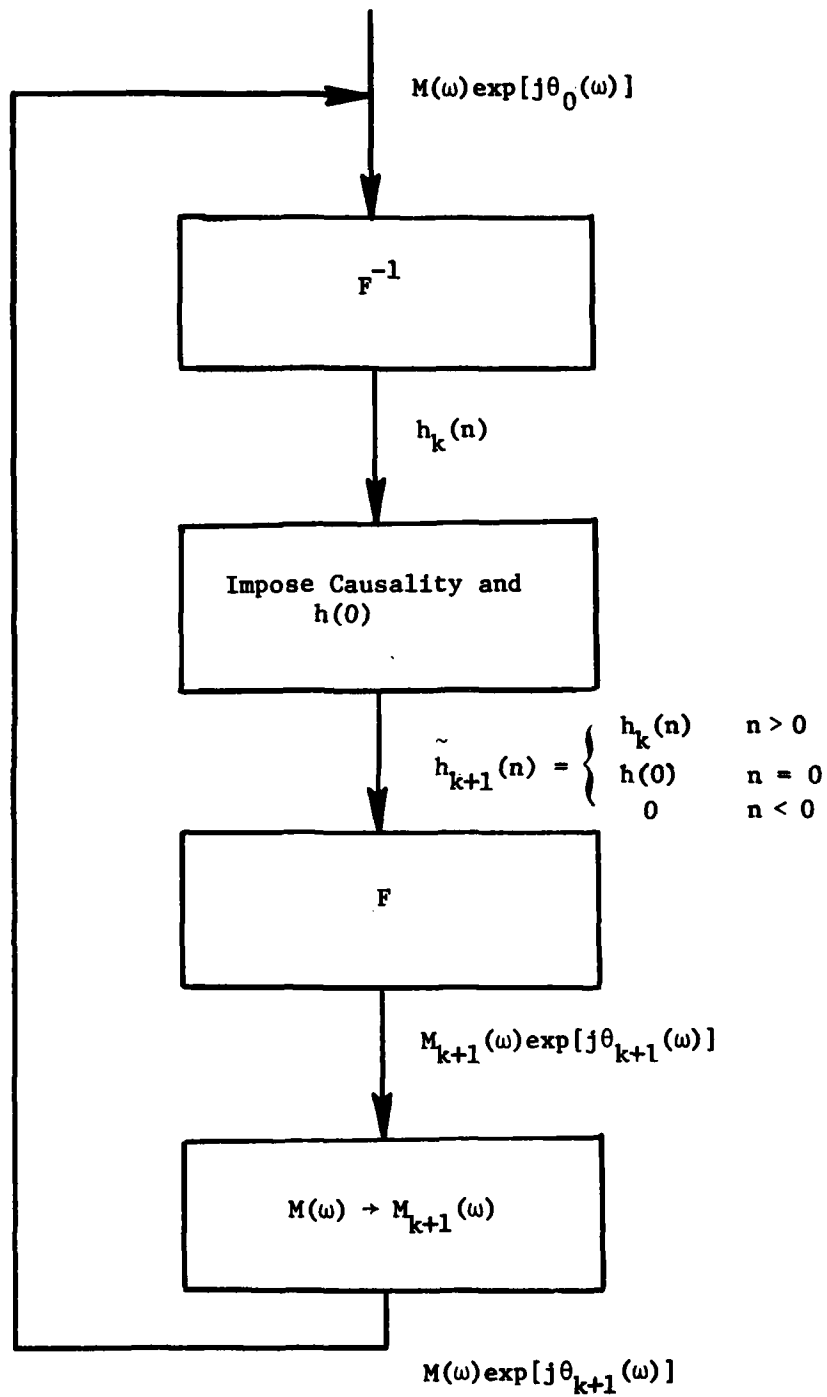$$E_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |M(\omega) - M_k(\omega)|^2 d\omega \tag{14}$$

12

$$M(\omega)\exp[j\theta_0(\omega)]$$

$$\boxed{F^{-1}}$$

$$h_k(n)$$

$$\boxed{\text{Impose Causality and } h(0)}$$

$$\tilde{h}_{k+1}(n) = \begin{cases} h_k(n) & n > 0 \\ h(0) & n = 0 \\ 0 & n < 0 \end{cases}$$

$$\boxed{F}$$

$$M_{k+1}(\omega)\exp[j\theta_{k+1}(\omega)]$$

$$\boxed{M(\omega) \rightarrow M_{k+1}(\omega)}$$

$$M(\omega)\exp[j\theta_{k+1}(\omega)]$$

Fig. 2.   Iterative algorithm to recover h(n) from its magnitude.

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To show that E_k is non-increasing, we first use the identity $\left|\exp[j\theta_k(\omega)]\right|^2 = 1$ to express E_k as

$$E_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |M(\omega) - M_k(\omega)|^2 \left|\exp[j\theta_k(\omega)]\right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |M(\omega)\exp[j\theta_k(\omega)] - M_k(\omega)\exp[j\theta_k(\omega)]|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_k(\omega) - \tilde{H}_k(\omega)|^2 d\omega \tag{15}$$

where $H_k(\omega)$ and $\tilde{H}_k(\omega)$ are the Fourier transforms of $h_k(n)$ and $\tilde{h}_k(n)$, respectively. From Parseval's Theorem equation (15) is given in the time domain by

$$E_k = \sum_n |h_k(n) - \tilde{h}_k(n)|^2 \tag{16}$$

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From (13), it follows that

$$|h_k(n) - \tilde{h}_k(n)| \geq |h_k(n) - \tilde{h}_{k+1}(n)| = 0 \ , \ n > 0 \qquad (17)$$

and

$$|h_k(n) - \tilde{h}_k(n)| = |h_k(n) - \tilde{h}_{k+1}(n)| \ , \quad n \leq 0 \qquad (18)$$

Summing (17) and (18) over all n, we obtain

$$E_k = \sum_n |h_k(n) - \tilde{h}_k(n)|^2 \geq \sum_n |h_k(n) - \tilde{h}_{k+1}(n)|^2 \qquad (19)$$

Next, we apply the triangle inequality for vector differences, to yield

$$|H_k(\omega) - H_{k+1}(\omega)| \geq |\tilde{H}_k(\omega)| - |\tilde{H}_{k+1}(\omega)| \qquad (20)$$

Therefore, we have from Parseval's Theorem, and (19) and (20)

$$E_k \geq \sum_n |h_k(n) - \tilde{h}_{k+1}(n)|^2$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_k(\omega) - \tilde{H}_{k+1}(\omega)|^2 d\omega$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |M(\omega) - M_{k+1}(\omega)|^2 d\omega$$

$$\geq E_{k+1} \qquad (21)$$

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Since E_k is non-increasing and has a lower bound of zero, it must converge to a limit point [8].

As with the algorithm in section III, although we have shown that the error E_k is non-increasing, we have not shown that the iterates $h_k(n)$ converge. However, if the iterates converge to a sequence whose z-transform is rational with no zeros on the unit circle and which is causal with initial value $h(0)$, from the minimum phase condition B, the converging solution must be minimum phase. Consequently, if in addition the magnitude of the Fourier transform of the converging solution equals $|H(\omega)|$, the solution is the unique minimum phase sequence associated with $|H(\omega)|$, i.e., $h(n)$.

The convergence of $h_k(n)$ has yet to be rigorously proven even when a finite length constraint is imposed within the iteration [9]. Empirically, however, we have found the DFT realization of the algorithm to always converge. In the next section, we shall illustrate the convergence of $h_k(n)$ to $h(n)$ with an example.

5. REALIZATIONS OF THE ITERATIVE ALGORITHMS USING THE DFT

Since the iterative algorithms will be implemented on a digital computer, we can compute a Fourier transform at only a finite number of points. In particular, we shall use the DFT.

One consequence of the DFT realization is that our desired sequence $h(n)$ must be of finite duration. Imposing a finite duration constraint within the iterations, however, does not change the non-increasing nature of the error functions, as can be seen from (8) and (19).

A second consequence of the DFT realization is that only uniformly spaced samples of the phase and magnitude functions are available. Nevertheless, it is again possible to show that the non-increasing nature of E_k is not altered when we use samples of the magnitudes and Fourier transforms in the expressions for E_k in (5) and (14) [4,10].

Finally, questions of convergence need to be addressed. Consider first minimum phase reconstruction from phase samples. When $H(z)$ is constrained to have no conjugate reciprocal zero pairs and no zeros on the unit circle, a unique sequence $h(n)$ of length M (to within a scale factor) is guaranteed when given M-1 or more phase samples of $\theta_h(\omega)$ in the open frequency interval $(0,\pi)$ [4]. A minimum phase sequence, in particular, satisfies these constraints. Therefore, the DFT realization of the iterative algorithm to reconstruct a minimum phase sequence from its phase samples can be implemented with a DFT of length $N \geq 2M$. Furthermore, this iteration will converge to $\alpha h(n)$ for $0 \leq n \leq N-1$, where α is positive [9].

Consider next the dual problem of developing a DFT realization of the iterative algorithm to recover a minimum phase sequence of length M from a magnitude function. In this case, there exists only one M-point sequence, i.e., the minimum phase sequence $h(n)$, when $h(0)$ is specified along with M or more uniformly spaced samples of the magnitude in the half open frequency interval $[0,\pi)$ [10]. Therefore, a DFT realization of the iteration can be implemented with DFT length $N \geq 2M-1$. If the algorithm converges to a causal

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sequence of length M with initial value h(0) and the known magnitude samples, the converging solution must equal h(n) for $0 \leq n \leq N-1$.

We now consider two examples where the DFT length is 512 points which is twice the length of h(n). In the first example, the initial magnitude guess is unity, and in the second example the initial phase guess is zero.


Example 5.1:   Signal Reconstruction From Phase

Consider a 256-point minimum phase signal h(n) illustrated in Figure 3. The phase is known and we wish to reconstruct h(n). The functions $h_k(n)$ and $\log[M_k(\omega)]$ are depicted in Figures 3 and 4 along with the originals for k equal to 1, 5 and 45. The signal $h_k(n)$ (to within a multiplicative constant) and the spectrum $\log[M_k(\omega)]$ (to within an additive constant) are indistintuishable from the originals after 45 iterations.

Example 5.2:   Signal Reconstruction From Magnitude

In this example, we consider the sequence of example 5.1, but where the Fourier transform magnitude is given. The functions $h_k(n)$ and $\theta_k(\omega)$ are depicted in Figure 5 and 6 with the originals for k equal to 1, 5 and 25. The functions $h_k(n)$ and $\theta_k(\omega)$ are indistinguishable from the originals after 25 iterations.
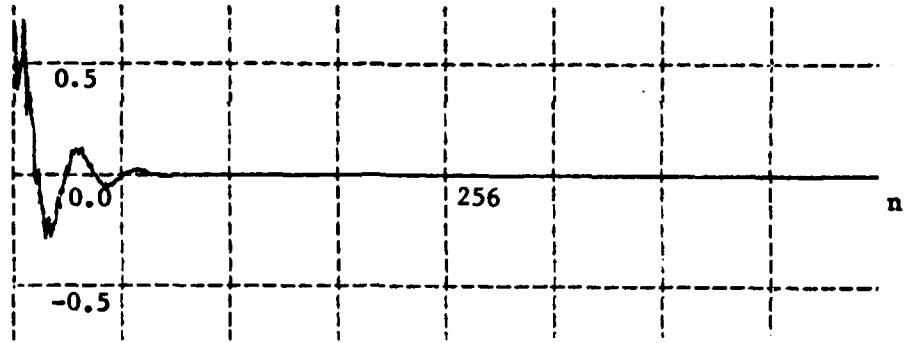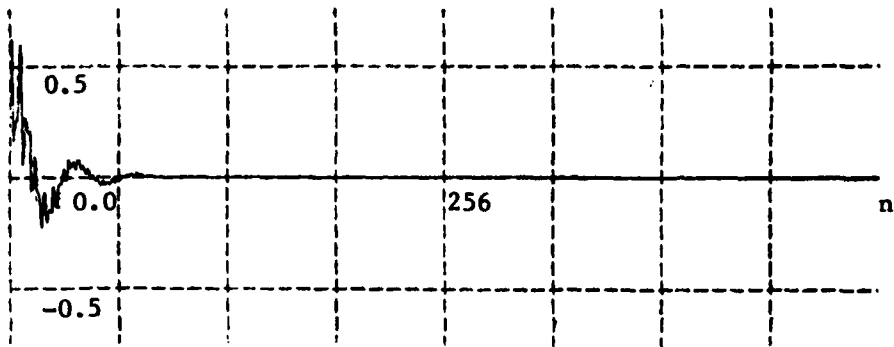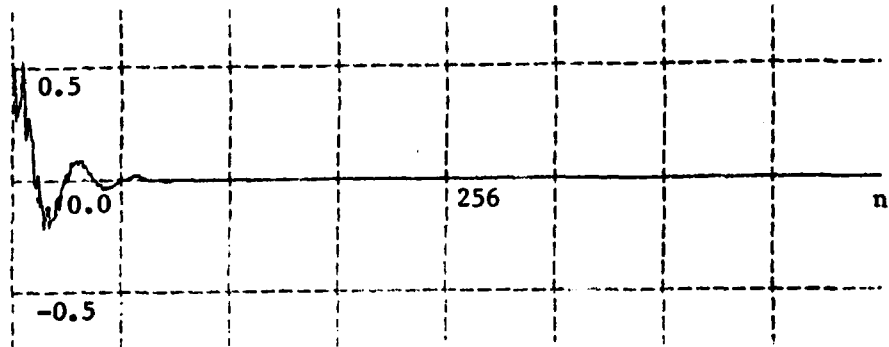
(a) original
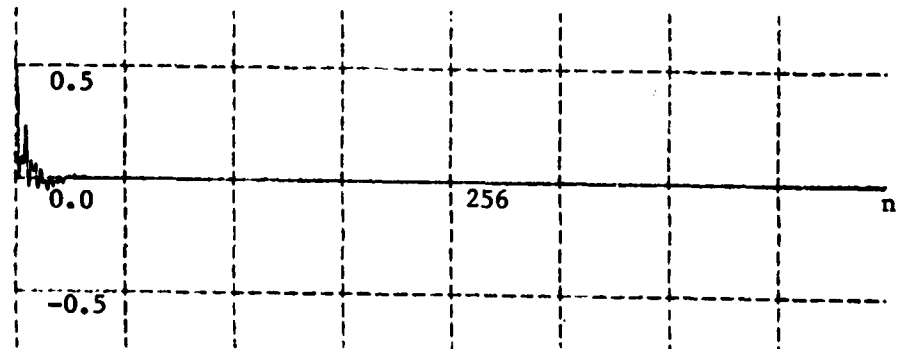
(b) 1 iteration

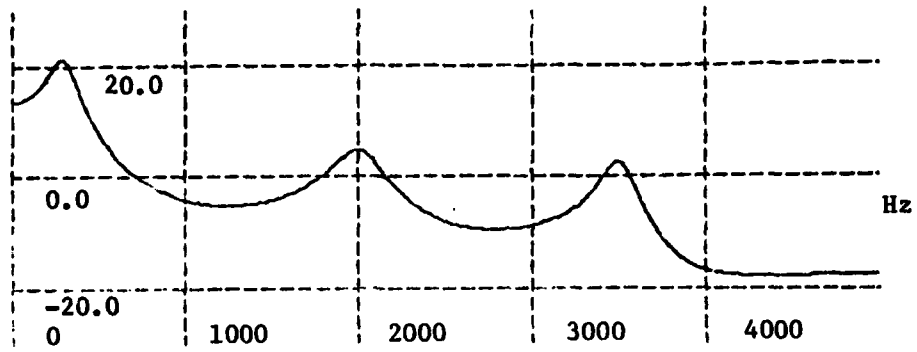Fig. 3. Convergence of $h_k(n)$ in example 5.1.
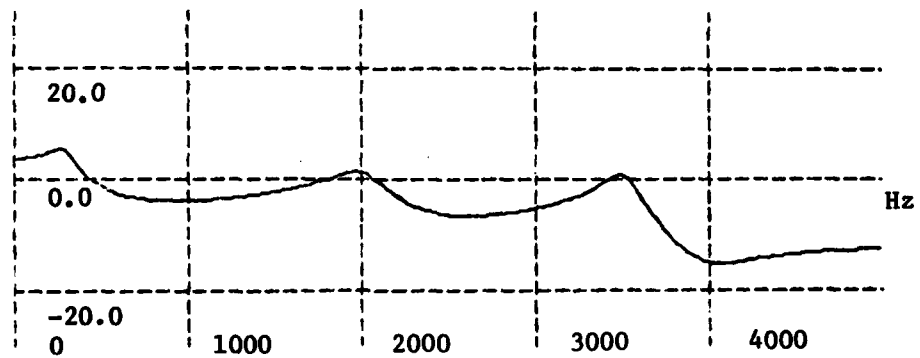
19

(c) 5 iterations

(d) 45 iterations

Fig. 3.* Continued.

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* The sequences in Figures 3c and 3d have been scaled by a factor of two.
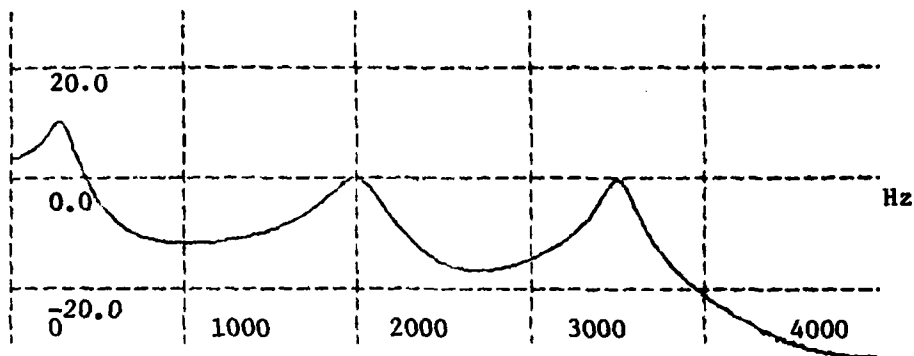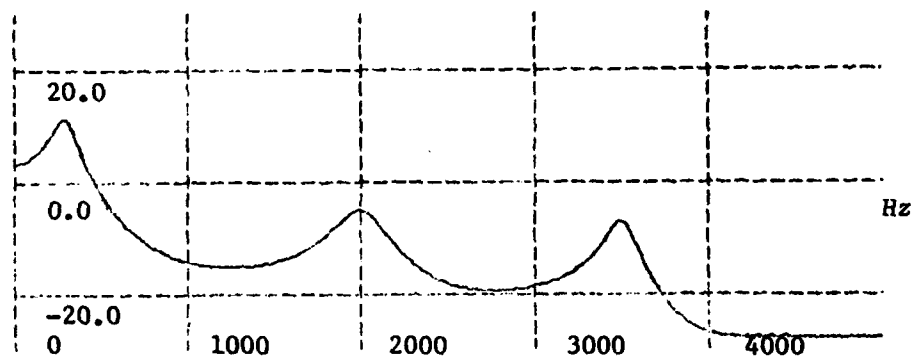
(a)   original

(b)   1 iteration

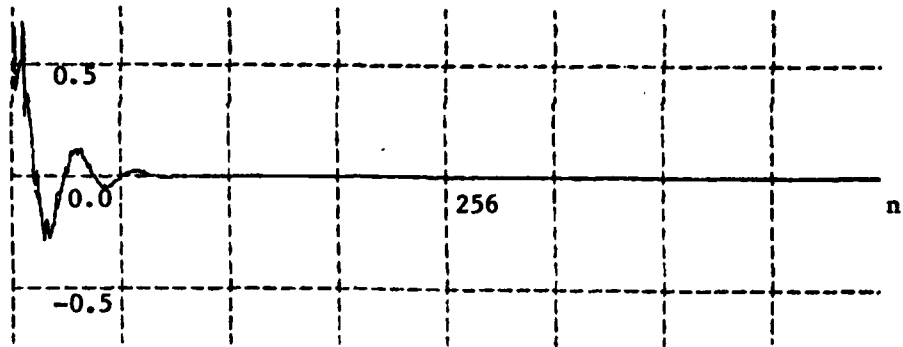Fig. 4.   Convergence of $\log|H_k(\omega)|$
(in decibels) in example 5.2.
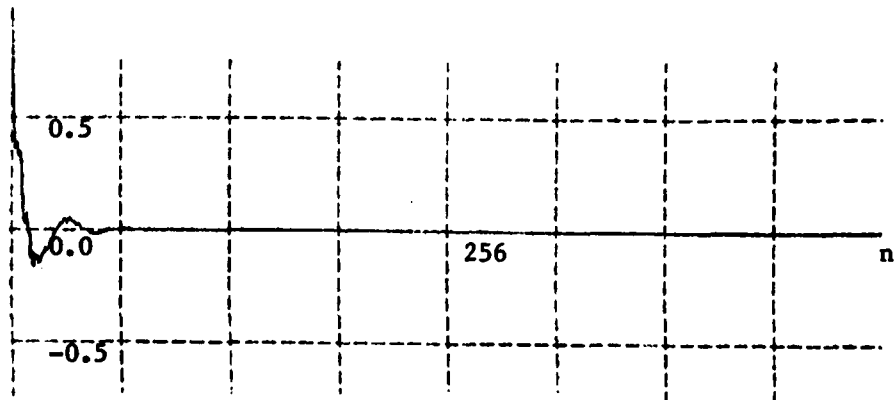
21

(c) 5 iterations

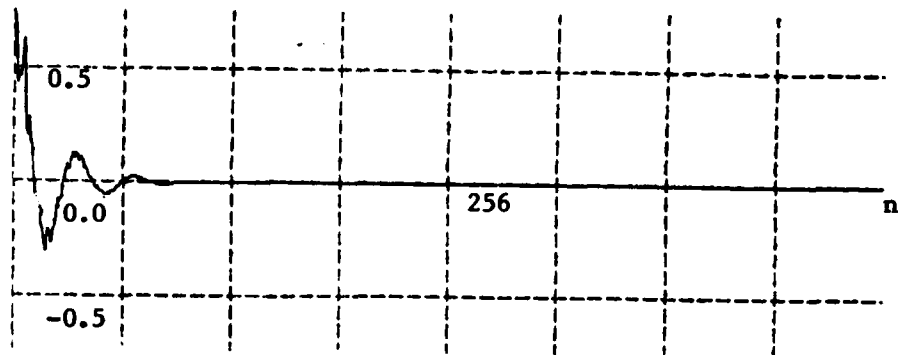(d) 45 iterations

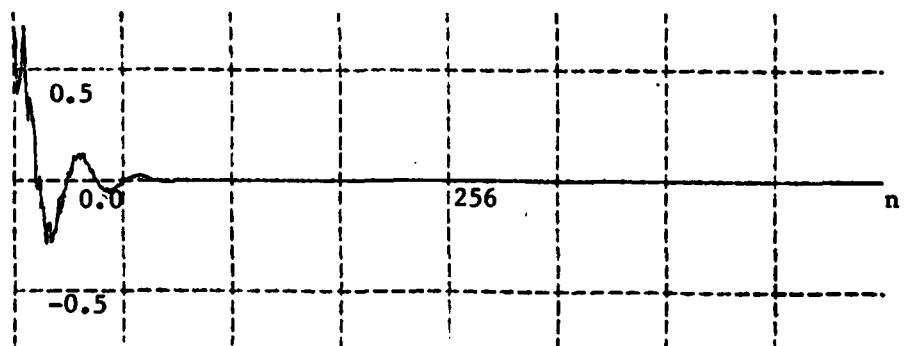Fig. 4. Continued.

22

(a) original

(b) 1 iteration
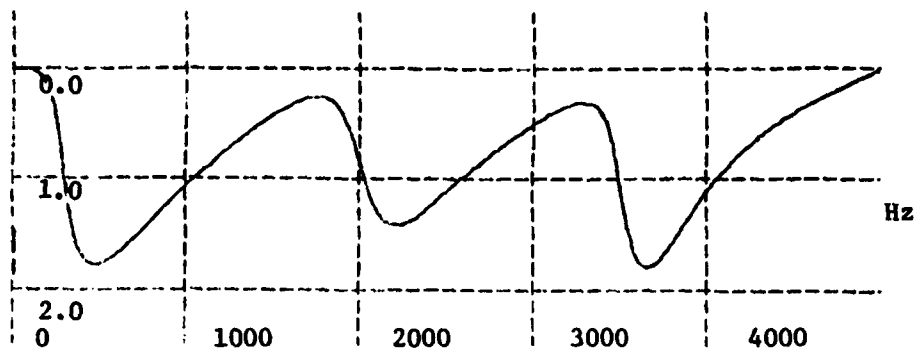
Fig. 5. Convergence of $h_k(n)$ in example 5.2.
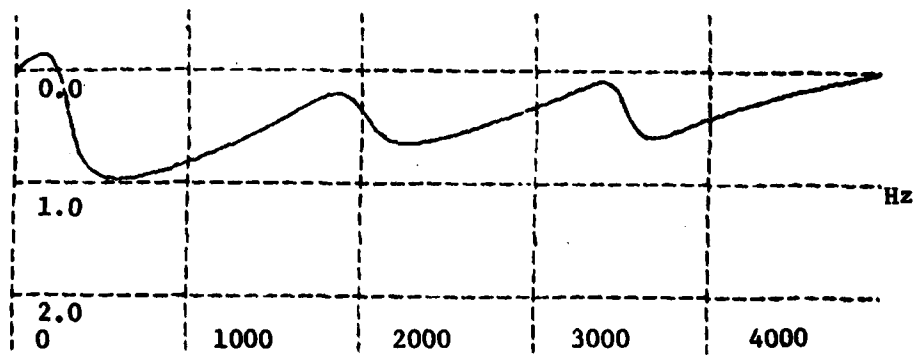
23

(c) 5 iterations

(d) 25 iterations
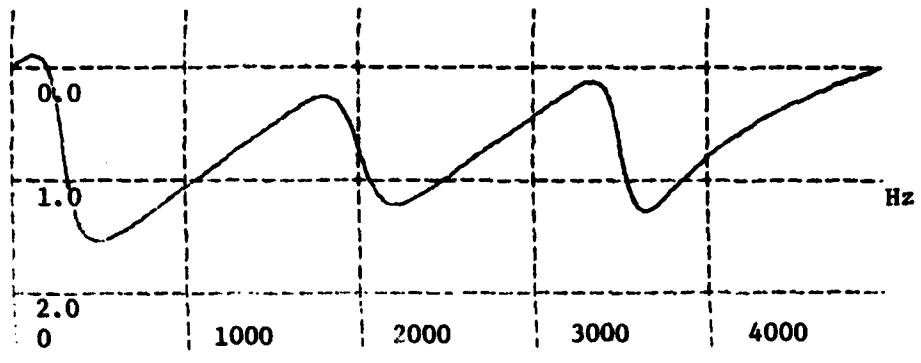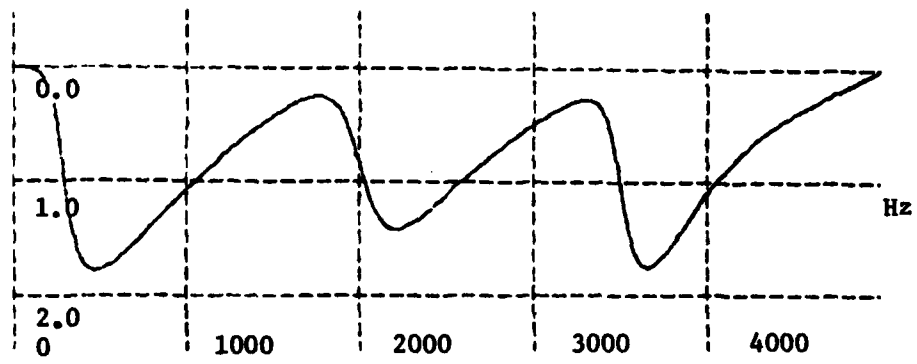
Fig. 5. Continued.

(a) original

(b) 1 iteration

Fig. 6. Convergence of $\Theta_k(\omega)$ (in radians) in example 5.2.

(c) 5 iterations

(d) 25 iterations

Fig. 6. Continued.

6. IMPLEMENTATION OF THE HILBERT TRANSFORM

In this section and the next, we investigate two computational algorithms based on the iterative procedures of sections III and IV: (i) a new means of implementing the Hilbert transform, and (ii) the use of this implementation as the basis for a new phase unwrapping algorithm.

For a minimum phase signal, the log-magnitude and phase of the Fourier transform are related through the Hilbert transform and the direct implementation of the Hilbert transform using the DFT has been extensively investigated [6]. One disadvantage of this implementation is that in computing the log-magnitude from the phase, samples of the unwrapped phase are required and are often difficult to compute. A second drawback is that aliasing occurs in the inverse discrete Fourier transform of samples of the log-magnitude and unwrapped phase due to a finite DFT length, limiting the accuracy of the computed samples of the unknown component.

An alternative to the direct implementation of the Hilbert transform exploits the iterative algorithms of sections III and IV. When the phase is given, through the use of the algorithm in section III, $\alpha h(n)$ is first obtained from the phase and, in particular, does not require samples of the unwrapped phase. From $\alpha h(n)$ the log-magnitude of $\alpha h(\omega)$, representing the Hilbert transform of the phase to within an additive factor is then computed. Furthermore, by increasing the number of iterations we can come arbitrarily close to samples of the log-magnitude so that accuracy is not limited by a fixed DFT length as in the direct approach.

A similar procedure can, of course, be applied through the use of the iterative algorithm in section IV to implement the Hilbert transform of a given logmagnitude function. As before, for a fixed DFT length, we can come arbitrarily close to samples of the phase by increasing the number of iterations. If $h(0)$ is not known a priori (recall (13)), it can be obtained (at least in theory) from the magnitude, although in practice $h(0)$ can be computed only approximately [1]. However, it was found empirically that the iterates always converge to $h(n)$ when only causality is imposed in the time domain (i.e., $h(0)$ is assumed unknown) and the initial phase $\theta_0(\omega)$ is set to zero.

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7. A NEW PHASE UNWRAPPING ALGORITHM

There are a variety of applications in which it is desired to obtain an unwrapped phase. Current algorithms rely on adding multiples of 2π to samples of the principal value of the phase [1,7]. In this section, we present a phase unwrapping algorithm which avoids such considerations and which relies primarily on the iterative algorithm of section III.

We assume either that there is no linear phase component in $H(z)$ as given in (2), or that we can separately determine z^{n_o}. Let $\theta(\omega)$ denote the desired unwrapped phase function and $\theta_p(\omega)$ its value modulo 2π. The proposed phase unwrapping algorithm proceeds as follows:

(i) Remove the linear phase component to obtain the principal value of the phase of $H(\omega)\exp[-jn_o\omega]$, denoted by $\hat{\theta}_p(\omega)$.

(ii) Apply the iterative algorithm of section III with a causality constraint and with phase $\hat{\theta}_p(\omega)$ to obtain a minimum phase sequence $h_{mp}(n)$.

(iii) Compute $\log|H_{mp}(\omega)|$ where $H_{mp}(\omega)$ is the Fourier transform of $h_{mp}(n)$.

(iv) Apply the Hilbert transform to $\log|H_{mp}(\omega)|$ to obtain the unwrapped phase function $\theta(\omega)-n_o\omega$.

(v) Add back the linear phase component to obtain the desired unwrapped phase.

Of particular interest is step (ii) which yields the same minimum phase sequence $h_{mp}(n)$ that would be obtained by a Hilbert transform of the unwrapped phase but bypasses the need for phase unwrapping.

There are two major considerations in the use of this algorithm. First, the minimum phase sequence $h_{mp}(n)$ derived from the iteration is of infinite extent regardless of whether the original sequence $h(n)$ is of finite duration [10]. Therefore, a possible problem with aliasing arises. The DFT length must be sufficiently large so that the minimum phase sequence $h_{mp}(n)$ decays effectively to zero. In particular, when $h_{mp}(n)=0$ for $n > M$ the DFT length, from our discussion in section V, should be at least 2M.

The second consideration is the linear phase factor of $H(z)$. The presence of this term represents a potential drawback to the algorithm since a priori knowledge of such a factor is sometimes difficult to obtain.

8. SUMMARY AND CONCLUSIONS

In this report, we have developed iterative algorithms for recon-
structing a minimum phase sequence from either the phase or magnitude of
its Fourier transform. When the phase is known, the mean squared error
between the desired Fourier transform and its estimate was shown to be non-
increasing on successive iterations. Likewise, when the magnitude is
given, on successive iterations, the mean squared error between the known
magnitude and its estimate is non-increasing. In addition, we noted that
convergence of the iteration with known phase samples (i.e., the DFT
realization) has been demonstrated, but convergence of the iteration with
magnitude samples has been shown only empirically.

Finally, we developed two computational algorithms based on the
iterative procedures: (i) a new means of implementing the Hilbert transform
which doesn't require an unwrapped phase and potentially provides greater
accuracy than the direct approach, and (ii) a new phase unwrapping
algorithm which doesn't require adding multiples of 2π to the principal
value of the phase.

The iterative algorithms, as presented, rely on exact knowledge of
the magnitude, phase, and the initial value of the desired signal. Sensi-
tivity to the inexactness of these quantities, to quantization noise, and
other forms of degradation is not understood and is a significant area
of future research.

In practice, we have found that the iterative algorithm converge
sometimes slowly (e.g., after several hundred iterations) and sometimes
quickly (e.g., after a few iterations). Consequently, determining rates of
convergence in terms of characteristics of the minimum phase signal and
initial magnitude or phase estimates, and methods of speeding up convergence
should be explored.

Another area being considered is the interchange of the signal re-
construction problems. In particular, we have found empirically that

when $|H(\omega)|$ and $\theta_h(\omega)$ are effectively interchanged through $j\log H(\omega)$, a slightly modified version of the iterative algorithm of section III, requiring a phase function, will recover $h(n)$ from its magnitude. Likewise, when the phase is known, $h(n)$ is recovered by a procedure similar to the iteration in section IV which requires a magnitude function. These results have led to some interesting theoretical speculations about the duality of the reconstruction problems and their iterative solutions.

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In this report, we develop iterative algorithms for reconstructing a minimum phase sequence from the phase or magnitude of its Fourier transform. The iterative techniques result in two potentially important computational algorithms. The first is a means of implementing the Hilbert transform of the log-magnitude or phase of the Fourier transform of a minimum phase sequence. This procedure avoids problems of phase unwrapping and aliasing inherent in the direct discrete Fourier transform implementation of the Hilbert transform. The second algorithm is a new method of computing samples of the unwrapped phase. As compared with other available phase unwrapping algorithms, this approach does not rely on adding 2π multiples to samples of the principal value of the phase.

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