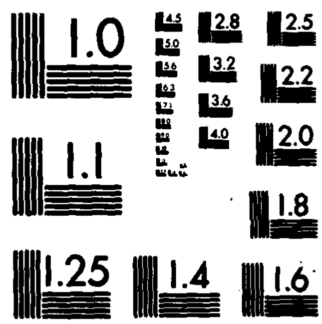
AD-A091 041 FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/G 6/5
THE CONVERGENCE OF THE STATE PROBABILITIES IN A CLASS OF M-DIME--ETC(U)
APR 80 H LACAYO, N A LANGBERG AFOSR-74-2581
UNCLASSIFIED FSU STATISTICS--M464R AFOSR-TR-80-0996 NL

END
DATE
FILMED
12 80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

The Convergence Of The State Probabilities In A Class Of m-Dimensional Simple Epidemic Models

by

Herbert Lacayo and Naftali A. Langberg

FSU Statistics Report M464R
AFOSR 84-R
AFOSR 9-R

April, 1980

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

Research sponsored by the Air Force of Scientific Research, AFSC, USAF, under Grants AFOSR 74-2581D and AFOSR [redacted].
78-3678

The Convergence Of The
State Probabilities In A Class Of
m-Dimensional Simple Epidemic Models

by

Herbert Lacayo and Naftali A. Langberg

ABSTRACT

A population of susceptible individuals exposed to m contagious diseases is considered. The progress of this epidemic among individuals is modeled by an m-dimensional stochastic progress. The components of this process represent the number of infective individuals with the respective diseases at time t.

A class of m-dimensional stochastic processes is constructed. These processes describe the progress of the epidemic models considered in the sequel. Exact and approximate formulas for the joint and marginal state probabilities of these models are obtained. It is shown that the approximate formulas are very simple functions of time while, the derivation of the exact formulas involve tedious computations.

Key-words: m-dimensional simple epidemics and stochastic processes, exponential and negative binomial random variables, convergence in distribution.

1. Introduction and Summary.

In a simple epidemic situation we assume that a population of susceptible individuals (susceptibles) is exposed only to one contagious disease (disease) [Bailey (1975)]. However, frequently susceptibles are exposed simultaneously to more than one disease, as is the case with different types of flu. In this paper a population of susceptibles exposed to m diseases, m = 1, 2, ..., is considered. We say that this population undergoes an m-dimensional simple epidemic if the following six assumptions hold.

(1.1) At each point in time at most one susceptible contracts a disease.

(1.2) Each susceptible contracts at most one disease.

(1.3) Once a susceptible contracts disease r, r = 1, ..., m, he remains contagious for the duration of the epidemic.

(1.4) An infective individual (infective) with disease r r = 1, ..., m, can transmit only that disease.

(1.5) All interactions between a susceptible and an infective with a specified disease are equally likely to result in an infection.

(1.6) Individuals neither join nor do they depart from the population.

For m = 1 the m-dimensional simple epidemic models reduce to the traditional simple epidemics.

Let T_0 denote the first time we have at least one infective with each of the various diseases, and n the number of susseptibles at time T_0. The progress of a m-dimensional simple epidemic among susceptibles is described by an m-dimensional stochastic process $\underline{X}_n(t) = [X_{n,1}(t),\ldots,X_{n,m}(t)]$, $t\epsilon[0,\infty)$. The components of $\underline{X}_n(t)$ represent the number of infectives with the respective diseases at time t measured from T_0. In Section 2 a class of m-dimensional stochastic processes is constructed. These processes describe the progress of the epidemic models considered in the sequel.

The computation of state probabilities in epidemic models is of major interest to researchers. In Section 3 we derive formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t) = P\{\underline{X}_n(t)-\underline{X}_n(0)=\underline{k}\}$, and $P_{n,k,r}(t) = P\{X_{n,r}(t)-X_{n,r}(0)=k\}$, where $\underline{k} = [k_1,\ldots,k_m]$, $k, k_1, \ldots, k_m \in \{0,1,\ldots\}$, $r = 1, \ldots, m$, and $t \in [0,\infty)$. These formulas are calculated without the traditional use of the differential equations associated with the state probabilities. This is done by utilizing a formula for the distribution function (df) of a sum of independent exponential random variables (rva's) given by Billard, Lacayo and Langberg (BLL) (1980).

The formulas for the state probabilities obtained in Section 3 are rather complicated. To overcome this deficiency we derive very simple approximations to the joint and marginal state probabilities when the initial number of susceptibles: n, is sufficiently large. Let us denote $X_{n,r}(0)$ by b_r, $r = 1, \ldots, m$. In Section 4 it is proven that for all $t \in (0,\infty)$, and $r = 1, \ldots, m$,

$$(1.7) \qquad \lim_{n\to\infty} P_{n,\underline{k}}(t) = \prod_{r=1}^{m} \binom{b_r+k_r-1}{b_r-1} e^{-\alpha_r b_r t}(1-e^{-\alpha_r t})^{k_r},$$

$$(1.8) \qquad \lim_{n\to\infty} P_{n,k,r}(t) = \binom{b_r+k-1}{b_r-1} e^{-\alpha_r b_r t}(1-e^{-\alpha_r t})^{k},$$

$$(1.9) \qquad \lim_{n\to\infty} EX_{n,r}(t) = b_r(e^{\alpha_r t}-1), \text{ and that}$$

$$(1.10) \qquad \lim_{n\to\infty} \operatorname{Var}\{X_{n,r}(t)\} = b_r(e^{2\alpha_r t}-e^{\alpha_r t}).$$

BLL (1979) consider a special class of m-dimensional simple epidemic models and name them the symmetric m-dimensional simple epidemics. These models are a subclass of the ones considered by us, and are presented in Section 2. BLL (1979) prove Statements (1.7) through (1.10) for the symmetric models. The results proven in Section 4 generalize those obtained by BLL (1979). Further, since the methods used by BLL do not apply to the more general case considered by us a new way of attacking the problem is presented.

Finally we mention that the process $\underline{X}_n(t)$ can be used to describe various competative situations other than the m-dimensional simple epidemic. Two such competative situations are presented.

(I) In marketing, let $X_{n,r}(0)$ represent the number of customers who initially own brand r of a product. Let n be the number of customers who at the time we start to observe do not possess the product. Then $X_{n,r}(t)$ can represent the random number of customers who own brand r at time t.

(II) Consider an election campaign. Let $X_{n,r}(0)$ denote the number of electors who support party (or candidate) r at the beginning of the campaign; and let n be the number of uncommitted voters at that time. Then $X_{n,r}(t)$ can describe the random number of voters that support party (or candidate) r at time t.

The various brand manufacturers (political parties or candidates) are competing for customers (voters) as diseases are "competing" for susceptibles. Thus, the results obtained in this paper are of interest not only to researchers in Epidemiology but as well to those who deal with various compotitive real life situations.

2. Model Construction.

In this section we construct a class of m-dimensional stochastic processes. These processes describe the progress of m-dimensional simple epidemic models considered in the sequel.

Some notation is needed. Let $X_n(t) = \sum_{r=1}^{m} X_{n,r}(t)$, be the total number of infectives at time t measured from T_0, $t\in[0,\infty)$, and $T_{n,k}$ the k^{th} inter-infection time defined as the time that elapses between the $X_n(0)+k-1$ and the $X_n(0)+k$ infection, $k = 1, \ldots, n$. Further, let $\xi_{n,k}$ be a rva assuming values in the set$\{1,\ldots,m\}$ designating the disease responsible for the $X_n(0)+k$ infection, $k = 1, \ldots, n$, and let $\xi_{n,0} = 0$. Finally, let $S_{n,k} = \sum_{q=1}^{k} T_{n,q}$ be the time measured from T_0 until the $X_n(0)+k$ infection, $k = 1, \ldots, n$, $S_{n,0} = 0$, $S_{n,n+1} = \infty$, and let I be the indicator function. In particular $S_{n,n}$ is the duration time of the m-dimensional simple epidemic.

For $k = 0, \ldots, n$, $r = 1, \ldots, m$, and $t\in(0,\infty)$, the following event equality holds.

$$(2.1) \qquad \{X_{n,r}(t)-X_{n,r}(0)=k\} = \bigcup_{q=k}^{n} \{S_{n,q}\le t<S_{n,q+1}, \sum_{j=1}^{q} I(\xi_{n,j}=r)=k\}.$$

Thus, to construct the process $X_n(t)$ it suffices to determine the df of the random vector (rve) $[T_{n,q},\xi_{n,q},q=1,\ldots,n]$. Before determining the df of this rve we introduce some more notation. Let $J_{n,k,r}$ be the index set of all infectives with disease r at time $T_0+S_{n,k-1}$, and let $C_{n,k,r} = X_n(0)+\sum_{q=0}^{k-1} I(\xi_{n,q}=r)$ be the number of infectives with disease r at time $T_0+S_{n,k-1}$, $k = 1,\ldots,n$, $r = 1, \ldots, m$. Further, let $\tau_{n,i,k}$ be a rva that describes the time measured from the $X_n(0)+k-1$ infection until the i^{th} contagious individual causes the

$X_n(0)+k$ infection, $i = 1, \ldots, X_n(0)+k-1$, $k = 1, \ldots, n$, and let $\underline{\alpha_1,\ldots,\alpha_m}\in(0,\infty)$.
Throughout we assume that

(2.2) The conditional rva's $\{\tau_{n,i,k}|\xi_{n,0},\ldots,\xi_{n,k-1}\}$, $i = 1, \ldots, X_n(0)+k-1$, $k = 1, \ldots, n$, are independent exponentially distributed, and that

(2.3) $E\{\tau_{n,i,k}|\xi_{n,0},\ldots,\xi_{n,k-1}\} = n[\alpha_r(n-k+1)]^{-1}$ for $i\in J_{n,k,r}$, $k = 1, \ldots, n$, $r = 1, \ldots, m$.

We are ready to determine the df of the rve $[T_{n,q},\xi_{n,q},q=1,\ldots,n]$. The following two lemmas are needed.

<u>Lemma 2.1</u>. Assume Conditions (2.2), (2.3) hold. Then for $r = 1, \ldots, m$, and $k = 1, \ldots, n$

(2.4)
$$P\{\xi_{n,k}=r|\xi_{n,0},\ldots,\xi_{n,k-1}\} = \alpha_r C_{n,k,r}[\textstyle\sum_{\ell=1}^{m}\alpha_\ell C_{n,k,\ell}]^{-1}.$$

<u>Proof</u>. Let k, r, be fixed. Then the event $\{\xi_{n,k}=r\}$ is equal to
$$\{\min[\tau_{n,i,k}:i\in J_{n,k,r}]<\min[\tau_{n,i,k}:i\in\bigcup_{\substack{\ell=1\\ \ell\neq r}}^{m} J_{n,k,\ell}]\}.$$

Consequently Statement (2.4) follows by Conditions (2.2), (2.3) and some simple integral evaluations. ||

Let $\underline{\ell_1,\ldots,\ell_n}\in\{1,\ldots,m\}$, and $\underline{\ell_0} = 0$. Then

(2.5)
$$P\{\xi_{n,q}=\ell_q,q=1,\ldots,n\} = \prod_{q=1}^{n} P\{\xi_{n,q}=\ell_q|\xi_{n,j}=\ell_j,j=0,\ldots,q-1\}.$$

Thus, Statement (2.4) determines the df of the rve $[\xi_{n,1},\ldots,\xi_{n,n}]$.

<u>Lemma 2.2</u>. Assume Conditions (2.2), (2.3) hold. Then the conditional

rva's $\{T_{n,q}|\xi_{n,0},\ldots,\xi_{n,q-1}\}$, $q = 1, \ldots, n$, are independent exponentially distributed with means respectively equal to $[(n-q+1)(\sum_{r=1}^{m}\alpha_r C_{n,q,r})n^{-1}]^{-1}$.

Proof. Note that $T_{n,q} = \min[\tau_{n,i,q}: i=1,\ldots,X_n(0)+k]$, $q = 1, \ldots, n$. Consequently the result of the lemma follows by Conditions (2.2), (2.3), and some simple properties of exponential rva's. ||

Clearly Lemmas 2.1, 2.2 together determine the df of the rve $[T_{n,q},\xi_{n,q},q=1,\ldots,n]$.

Although the transition rates of the various diseases are not used explicitly they are presented for the sake of completeness.

Definition 2.3. The transition rate of disease r at time t, $r = 1, \ldots, m$, $t\in[0,\infty)$, is given by

$$\lim_{h\to 0^+} h^{-1}P\{X_{n,r}(t+h)-X_{n,r}(t)=1|\underline{X}_n(t)\},$$

and is denoted by $R(\underline{X}_n(t),r)$.

Finally it is shown that the transition rates of an m-dimensional simple epidemic that satisfies Condition (2.2), as expected, determine uniquely the epidemic. We need the following lemma.

Lemma 2.4. Assume Conditions (2.2), (2.3) hold. Then for $r = 1, \ldots, m$, and $t\in[0,\infty)$

$$R(\underline{X}_n(t),r) = n^{-1}\alpha_r X_{n,r}(t)(n-X_n(t)+X_n(0)). \tag{2.6}$$

Proof. The results of the lemma clearly follow by the memoryless property of exponential rva's [Barlow, Proschan (1975), p. 56], Equation (2.1) and Conditions (2.2), (2.3). ||

Consequently we obtain for $i\in J_{n,k,r}$, $k = 1, \ldots, n$, and $r = 1, \ldots, m$, that

(2.7) $$E\{\tau_{n,i,k} | \xi_{n,0}, \ldots, \xi_{n,k-1}\} =$$
$$= C_{n,k,r}\{R([C_{n,k,1}, \ldots, C_{n,k,m}],r)\}^{-1}.$$

Thus, m-dimensional simple epidemic models that satisfy Condition (2.2) are uniquely determined by their transition rates.

BLL(1979) define a class of m-dimensional simple epidemic models and name them the symmetric m-dimensional simple epidemics. Specifically a population of susceptibles exposed to m diseases undergoes a symmetric m-dimensional simple epidemic if Assumptions (1.1) through (1.6) hold, Condition (2.2) is satisfied, and the transition rate of disease r at time t, $r = 1, \ldots, m$, $t \in [0,\infty)$, is given by

(2.8) $$R(\underline{X}_n(t),r) = \alpha n^{-1} X_{n,r}(t)(n - X_n(t) + X_n(0)), \text{ where } \alpha \in (0,\infty).$$

The transition rates given by Equation (2.6) reduce to those of Equation (2.8) by selecting $\alpha_1 = \alpha_2 \ldots = \alpha_m = \alpha$. Thus, the symmetric m-dimensional simple epidemic models are a particular case of the ones constructed in this section. For the symmetric m-dimensional simple epidemic we obtain from Lemma 2.2 that the interinfection times: $T_{n,1}, \ldots, T_{n,n}$, are independent exponential rva's. independent of the infection causes: $\xi_{n,1}, \ldots, \xi_{n,n}$. This special structure does not hold in general for m-dimensional simple epidemic models. BLL (1979) use this special structure to prove Statements (1.7) through (1.10) for the case of symmetric m-dimensional simple epidemic models. Their method of proof does not apply to the models considered by us. To prove Statements (1.7) through (1.10), for the models constructed in this section, a new way of attacking the problem is needed. One such way is presented in Section 4.

3. Formulas for the State Probabilities.

This section contains formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t)$ and $P_{n,k,r}(t)$, $r = 1, \ldots, m$, $t\epsilon(0,\infty)$. These formulas are calculated without the traditional use of the differential equations associated with the state probabilities. Rather, we utilize an available formula for the distribution function of a sum of independent exponential rva's. For the sake of completeness this formula is presented. We need some notation. Let $\underline{\theta_1, \theta_2, \ldots,}$ be positive real numbers, and $\underline{U_1, U_2, \ldots,}$ be iid exponential rva's with mean equal to one.

Theorem 3.1. [BLL(1980), Theorem 2]. Let M be a positive integer, $A_M(M) = (-1)^{M-1}\prod_{q=1}^{m}\theta_q$, and $A_M(j) = (-1)^M \sum_{j_1+\ldots+j_m=j}\prod_{q=1}^{m}\theta_q^{j_q+1}$, $j = M+1$, $M+2, \ldots$. Then

$$(3.1)\qquad P\{\textstyle\sum_{q=1}^{M}\theta_q^{-1}U_q \le t\} = \\ = \textstyle\sum_{j=M}^{\infty}(j!)^{-1}A_M(j)(-t)^j,\ t\epsilon(0,\infty).$$

To aid in computing the joint and marginal state probabilities some more notation is introduced. Let $\underline{\ell_1, \ell_2, \ldots,}$ be a sequence of positive integers assuming values in the set $\{1,\ldots,m\}$, $\underline{\ell_0} = 0$, and let $\underline{s(\underline{k})} = \sum_{r=1}^{m}k_r$. Further, let $\underline{B(\underline{k})} = \{[\ell_1,\ldots,\ell_{s(\underline{k})}]: \sum_{q=0}^{s(\underline{k})}I(\ell_q=r)=k_r, r=1,\ldots,m\}$, and $\underline{A(r,j,k)} = \{[\ell_1,\ldots,\ell_j]: \sum_{q=0}^{j}I(\ell_q=r)=k\}$ for $j = k, k+1, \ldots$, $r = 1, \ldots, m$, and $k = 0, 1, \ldots$. By Equation (2.1) we obtain for $t\epsilon(0,\infty)$ and $r = 1, \ldots, m$, that

$$(3.2)\qquad P_{n,\underline{k}}(t) = \\ = \sum_{[\ell_1,\ldots,\ell_{s(\underline{k})}]}[P\{S_{n,s(\underline{k})}\le t<S_{n,s(\underline{k})+1}\mid \xi_{n,q}=\ell_q, q=0,\ldots,s(\underline{k})\}\cdot \\ \cdot P\{\xi_{n,q}=\ell_q, q=0,\ldots,s(\underline{k})\}\, I\,([\ell_1,\ldots,\ell_{s(\underline{k})}]\epsilon B(\underline{k})],$$

(3.3) $P_{n,k,r}(t) =$

$$= \sum_{j=k}^{n} \sum_{[\ell_1,\ldots,\ell_j]} [P\{S_{n,j} \le t < S_{n,j+1} | \xi_{n,q} = \ell_q, q=0,\ldots,j\} \cdot$$

$$\cdot P\{\xi_{n,q} = \ell_q, q=0,\ldots,j\} I([\ell_1,\ldots,\ell_j] \epsilon A(r,j,k))].$$

Thus, to compute $P_{n,\underline{k}}(t)$ and $P_{n,k,r}(t)$ it suffices to evaluate $P\{\xi_{n,q}=\ell_q, q=0,\ldots,k\}$ and $P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q}=\ell_q, q=0,\ldots,k\}$ for k = 0, ..., n. Let $\underline{\mu_0} = 1$, and $\underline{\mu_q} = \sum_{r=1}^{m} \alpha_r [b_r + \sum_{j=0}^{q-1} I(\ell_j = r)]$, q = 1, 2, ... For simplicity in notation the dependence of μ_q on $\ell_0, \ldots, \ell_{q-1}$, q = 1, 2, ..., is suppressed. First $P\{\xi_{n,q}=\ell_q, q=0,\ldots,k\}$ is evaluated.

Lemma 3.2. Let $k \epsilon \{0,1,\ldots\}$, and $d_r = \sum_{q=0}^{k} I(\ell_q = r)$.

Then

(3.4) $$P\{\xi_{n,q}=\ell_q, q=0,\ldots,k\} =$$

$$= \left[\prod_{r=1}^{m} d_r! \alpha_r^{d_r} \binom{b_r+d_r-1}{b_r-1} \right] [\prod_{q=0}^{k} \mu_q^{-1}].$$

Proof. Note that $P\{\xi_{n,q}=\ell_q, q=0,\ldots,k\} = \prod_{q=1}^{k} P\{\xi_{n,q}=\ell_q | \xi_{n,j}=\ell_j, j=0,\ldots,q-1\}$. Consequently Statement (3.4) follows by Equation (2.4). ||

Next $P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q}=\ell_q, q=0,\ldots,k\}$ is computed. Let $\underline{f_q(\theta_1,\ldots,\theta_q,t)}$ be the density function of the rva $\sum_{j=1}^{q} \theta_j^{-1} U_j$, q = 1, 2, ... Then for k = 0, 1, ..., and $t \epsilon (0,\infty)$.

$$(3.5) \qquad P\{\textstyle\sum_{q=1}^{k}\theta_q^{-1}U_q \le t < \sum_{q=1}^{k+1}\theta_q^{-1}U_q\} =$$

$$= \theta_{k+1}^{-1} f_{k+1}(\theta_1,\ldots,\theta_{k+1},t).$$

Thus, by Lemma 2.2 and Theorem 3.1 for $k = 0, 1, \ldots,$ and $t \in (0,\infty)$

$$(3.6) \qquad P\{S_{n,k} \le t < S_{n,k+1} \mid \xi_{n,q} = \ell_q, q=0,\ldots,k\} =$$

$$= \begin{cases} e^{-\mu_1 t} & k = 0 \\ \sum_{j=n+1}^{\infty} (j!)^{-1} t^j (-1)^{n+j} \sum_{j_1+\ldots+j_n=j} \prod_{q=1}^{n} \mu_q^{j_q+1} + (-1)^{n-1}(n!)^{-1} t^n \prod_{q=1}^{n} \mu_q & k = n \\ \mu_{k+1}^{-1} \sum_{j=k+2}^{\infty} (-1)^{k+1+j} t^{j-1} [(j-1)!]^{-1} \sum_{j_1+\ldots+j_{k+1}=j} \prod_{q=1}^{k+1} \mu_q^{j_q+1} & 1 \le k < n \\ + \mu_{k+1}^{-1} t^k (k!)^{-1} \prod_{q=1}^{k+1} \mu_q & \end{cases}$$

The formulas for the joint and marginal state probabilities are obtained from Equations (3.2) and (3.3) by subtitution. Note that, as expected, the exact formulas for the joint and marginal state probabilities are rather complicated.

4. Asymptotic Approximations.

In this section we present the asymptotic approximations of the joint and marginal state probabilities and some related moments. All limits are calculated as $n \to \infty$.

First it is shown that the joint and marginal state probabilities satisfy Statements (1.7) and (1.8). Three lemmas are needed. Throughout these lemmas let $\nu_q = \sum_{r=1}^{m} \alpha_r \sum_{j=0}^{q} I(\ell_j = r)$, and as in Section 3 let $\mu_0 = 1$, and $\mu_q = \nu_q + \sum_{r=1}^{m} \alpha_r b_r$, $q = 1, 2, \ldots$. For simplicity we suppress the dependence of ν_q and μ_q on the sequence $\ell_1, \ell_2, \ldots$.

Lemma 4.1. Let $k \in \{0,1,\ldots\}$, and $d_r = \sum_{q=0}^{k} I(\ell_q = r)$, $r = 1, \ldots, m$. Then

$$\lim P\{\xi_{n,q} = \ell_q, q=0,\ldots,k\} = \tag{4.1}$$

$$= \left[\prod_{r=1}^{m} d_r! \alpha_r^{d_r} \binom{b_r + d_r - 1}{b_r - 1}\right] \left[\prod_{q=0}^{k} \mu_q\right]^{-1}.$$

Proof. The desired result follows clearly from Lemma 3.2. ||

Lemma 4.2. Let $k \in \{1,2,\ldots\}$. Then the conditional rva's $\{S_{n,k} | \xi_{n,q} = \ell_q, q=0,\ldots,k-1\}$ converge in distribution to the rva $\sum_{q=1}^{k} \mu_q^{-1} U_q$.

Proof. To prove the desired result it suffices, by the Cramer-Wold device [Billingsley (1968), p. 49], to show that the conditional rve's $\{[T_{n,1},\ldots,T_{n,k}] | \xi_{n,q} = \ell_q, q=0,\ldots,k-1\}$ converge in distribution to the rve $[\mu_1^{-1} U_1, \ldots, \mu_k^{-1} U_k]$. The preceding statement follows by Lemma 2.2. ||

To prove the result of the next lemma we need [Rényi (1970), p. 203], the following

$$\frac{dP\{\sum_{q=1}^{N} \lambda_q^{-1} U_q \le t\}}{dt} = f_N(\lambda_1,\ldots,\lambda_N,t) = \sum_{q=1}^{N} C_N(q) e^{-\lambda_q t}, \tag{4.2}$$

where $\lambda_1,\ldots,\lambda_N$, are distinct positive real numbers and $C_N(q) = [\prod_{j=1}^{N} \lambda_j][\prod_{\substack{j=1 \\ j\neq q}}^{N} (\lambda_j-\lambda_q)]^{-1}$, $q = 1, \ldots, N$.

Lemma 4.3. Let $k_1, \ldots, k_m \in \{0,1,\ldots\}$, $\underline{k} = [k_1,\ldots,k_m]$, and $H(t) = (\prod_{r=1}^{m} k_r!\alpha_r^{k_r}) \sum_{B(\underline{k})} \sum_{j=1}^{s(\underline{k})+1} e^{-\nu_j t} [\prod_{\substack{i=1 \\ i\neq j}}^{s(\underline{k})+1} (\nu_i-\nu_j)]^{-1}$, $t\in[0,\infty)$. Then

$$H(t) = \prod_{r=1}^{m} (1-e^{-\alpha_r t})^{k_r}, \quad t\in[0,\infty). \tag{4.3}$$

Proof. The proof is by an induction argument on k.

First note that Statement (4.3) holds for $k = 1$. Assume Statement (4.3) holds for $k - 1$ $(k>1)$. Let $\gamma_q = \sum_{j=2}^{q} \sum_{r=1}^{m} \alpha_r I(\ell_q=r)$, $q = 2, 3, \ldots$, and $B(r,\underline{k}) = \{[r,\ell_2,\ldots,\ell_k]: [\ell_1,\ldots,\ell_k]\in B(\underline{k})\}$. Now by the induction assumption

$$\frac{dH(t)}{dt} = \tag{4.4}$$

$$= \sum_{r=1}^{m} e^{-\alpha_r t} k_r\alpha_r[(k_r-1)!\alpha_r^{k_r-1} \prod_{\substack{i=1 \\ i\neq r}}^{m} k_i\alpha_i^{k_i} \sum_{B(r,\underline{k})} \sum_{j=2}^{k+1} e^{-\gamma_j t} [\prod_{\substack{i=2 \\ i\neq j}}^{s(\underline{k})+1} (\gamma_i-\gamma_j)]^{-1} =$$

$$= \sum_{r=1}^{m} k_r\alpha_r(1-e^{-\alpha_r t})^{-1} e^{\alpha_r t} \prod_{r=1}^{m} (1-e^{-\alpha_r t})^{k_r} = \frac{d}{dt}\{\prod_{r=1}^{m} (1-e^{-\alpha_r t})^{k_r}\}.$$

By Equation (4.2) $P\{\sum_{q=1}^{N} \lambda_q^{-1} U_q \le t\} = 1-\sum_{q=1}^{N} \lambda_q^{-1} C_N(q) e^{-\lambda_q t}$. Since $\mu_1, \mu_2, \ldots$, are distinct, we conclude by (4.2) that $H(0) = 0$. Consequently Statement (4.3) for k follows by integrating both sides of Equation (4.4). ||

We are now ready to prove Statement (1.7).

Theorem 4.4. Let $t\in(0,\infty)$. Then

$$\lim P_{n,k}(t) = \prod_{r=1}^{m} \begin{pmatrix} b_r+k_r-1 \\ b_r-1 \end{pmatrix} e^{-\alpha_r b_r t} (1-e^{-\alpha_r t})^{k_r}.$$

<u>Proof.</u> By Equation (3.2) and Lemmas 4.1, 4.2 we obtain that

$$(4.5)\qquad \ell im\ P_{n,\underline{k}}(t) = \sum_{[\ell_1,\ldots,\ell_{s(\underline{k})}]} [P\{\textstyle\sum_{q=1}^{s(\underline{k})} \mu_q^{-1}U_q \le t < \sum_{q=1}^{s(\underline{k})+1} \mu_q^{-1}U_q\} \cdot$$

$$\cdot \left\{\prod_{r=1}^{m} k_r!\alpha_r^{k_r}\binom{b_r+k_r-1}{b_r-1}\right\}\left\{\prod_{q=1}^{s(\underline{k})} \mu_q^{-1}\right\} I([\ell_1,\ldots,\ell_k]\in B(\underline{k}))]$$

By Equations (3.5) and (4.2)

$$(4.6)\qquad \ell im\ P_{n,\underline{k}}(t) = \left\{\prod_{r=1}^{m}\binom{b_r+k_r-1}{b_r-i} e^{-\alpha_r b_r t}\right\} H(t).$$

Consequently the desired result follows by Lemma 4.3. ||

For the sake of completeness we present the following

<u>Definition 4.5.</u> We say that W is a <u>negative binomial</u> rva with parameters $a\in\{1,2,\ldots\}$ and $p\in(0,1)$, and write $W\sim NB(a,p)$ if $P\{W=k\} = \binom{a+k-1}{a-1}p^a(1-p)^k$, $k=0,1,\ldots$. Next Statement (1.8) is proven. Let $\underline{X}(t) = [X_1(t),\ldots,X_m(t)]$, $t\in[0,\infty)$ be a collection of rve's with independent components such that $X_r(0) = b_r$, and $X_r(t)\sim NB(b_r, e^{-\alpha_r t})$, $r = 1, \ldots, m$, $t\in(0,\infty)$. Since $\underline{X}_n(t)$ and $\underline{X}(t)$ are discrete rve's we obtain by Theorem 4.4 and a well known result [Billingsley (1968), p. 16], that

<u>Corollary 4.6.</u> Let $t\in(0,\infty)$. Then the rve's $\underline{X}_n(t)$ converge in distribution to the rve $\underline{X}(t)$.

By the Cramer-Wold device and Corollary 4.6 we conclude

<u>Corollary 4.7.</u> Let $t\in(0,\infty)$ and $r = 1, \ldots, m$. Then $X_n(t)$ and $X_{n,r}(t)$ converge in distribution to $\sum_{r=1}^{m} X_r(t)$ and $X_r(t)$, respectively.

By Corollary 4.7 we conclude that Statement (1.8) holds.

<u>Theorem 4.8.</u> Let $t\in(0,\infty)$ and $r = 1, \ldots, m$. Then

$$\ell im\ P_{n,k,r}(t) = \binom{b_r+k-1}{b_r-1} e^{-\alpha_r b_r t}(1-e^{-\alpha_r t})^k.$$

To obtain Statements (1.9) and (1.10) the following two lemmas are needed.

Lemma 4.9. Let k, $a\epsilon\{1,2,\ldots\}$ and $[\sigma_{a+k-1,1},\ldots,\sigma_{a+k-1,a+k-1}]$ be the order statistic of a sample of size $a+k-1$ taken from the population U_1. Then $\sum_{q=1}^{k}(a+q-1)^{-1}U_q$ and $\sigma_{a+k-1,k}$ are identically distributed.

Proof. The spacings $\sigma_{a+k-1,1}$, $\sigma_{a+k-1,2}-\sigma_{a+k-1,1}$, ..., $\sigma_{a+k-1,k}-\sigma_{a+k-1,k-1}$, are independent exponential rva's with means respectively equal to $(a+k-1)^{-1}$, $(a+k-2)^{-1}$, ..., b^{-1}, [Barlow-Proschan (1975), p. 59]. Hence $\sum_{q=1}^{k}(a+q-1)^{-1}U_q$ and $\sum_{q=1}^{k}(\sigma_{a+k-1,k-q+1}-\sigma_{a+k-1,k-q})$ are equal in distribution, $(\sigma_{a+k-1,0}=0)$. Consequently the desired result follows. ||

Lemma 4.10. Let t, $\beta\epsilon(0,\infty)$, and $r = 1, \ldots, m$. Then $\ell im\ E\{X_n(t)\}^\beta = E\{X(t)\}^\beta$ and $\ell im\ E\{X_{n,r}(t)\}^\beta = E\{X(t)\}^\beta$.

Proof. By Corollary 4.7 to prove the results of the theorem it suffices, [Chung(1974), p. 95], to show that $\sup_n E\{X_{n,r}(t)\}^{\beta+1}$, and $\sup_n E\{X_n(t)\}^{\beta+1}$ are finite. Since, $0\le X_{n,r}(t)\le X_n(t)$, it is enough to show that $\sup_n E\{X_n(t)\}^{\beta+1}<\infty$.

Next we prove the preceding statement. Let $b = \sum_{r=1}^{m} b_r$, and $\bar{\alpha} = \sum_{r=1}^{m}\alpha_r$. Then by Lemmas 2.2 and 4.9 for all $n\epsilon\{1,2,\ldots\}$, $P\{X_n(t)-b\ge k\} \le P\{\sum_{q=1}^{k}q^{-1}U_q\le\bar{\alpha}bt\} =$ $= (1-e^{-\bar{\alpha}bt})^k$, $k = 0, 1, \ldots$. Further for all $n\epsilon\{1,2,\ldots\}$, $E\{X_n(t)\}^{\beta+1} =$ $= (\beta+1)\int_0^\infty (z+b)^\beta\ P\{X_n(t)-b>z\}dz \le (\beta+1)\sum_{q=0}^{\infty}(q+b+1)^\beta(1-e^{-\bar{\alpha}bt})^q < \infty$. Consequently the desired results follow. ||

Clearly from Lemma 4.10 we obtain Statements (1.9) and (1.10).

Theorem 4.11. Let $t\epsilon(0,\infty)$, and $r = 1, \ldots, m$. Then (i) $\ell im\ EX_{n,r}(t) = b_r(e^{\alpha_r t}-1)$, and (ii) $\ell im\ Var\{X_{n,r}(t)\} = b_r(e^{2\alpha_r t}-e^{\alpha_r t})$.

Finally from Corollary 4.7 and Lemma 4.10 we obtain for $t\epsilon(0,\infty)$ that

(4.7) $$\ell im\ EX_n(t) = \sum_{r=1}^{m} b_r(e^{\alpha_r t}-1),$$ and that

(4.8) $$\ell im\ Var\{X_n(t)\} = \sum_{r=1}^{m} b_r(e^{2\alpha_r t}-e^{\alpha_r t}).$$

References

(1) Bailey, N.T.J. (1975). The mathematical theory of infectious disease and its Applications. Charles Griffin and Co., London.

(2) Billard, L., Lacayo, H., and Langberg, N.A. (1979). The symmetric m-dimensional simple epidemic model J.R.S.S. Vol. 41 196-202.

(3) ______________________________ (1980). Generalizations of the simple epidemic process. J. App. Prob. Dec.

(4) Billingsley, P. (1968). Convergence of probability measures. John Wiley and Sons, New York.

(5) Barlow, R. and Proschan, F. (1975). Statistical theory of reliability and life testing probability models. Holt, Rinehart and Winston.

(6) Chung, K.L. (1974). A course in probability theory. Academic Press, New York.

(7) Rényi, A. (1970). Probability theory. North Holland Publishing Co., Amsterdam.

UNCLASSIFIED
Security Classification of this Page

REPORT DOCUMENTATION PAGE

1. REPORT NUMBERS AFOSR-TR-80-0996	2. GOVT. ACCESSION NO. AD-A091047	3. RECIPIENT'S CATALOG NUMBER
4. TITLE The Convergence of the State Probabilities in a Class of m-Dimensional Simple Epidemic Models		5. TYPE OF REPORT & PERIOD COVERED INTERIM 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Herbert Lacayo and Naftali A. Langberg		8. CONTRACT OR GRANT NUMBER(s) AFOSR 78-3678
9. PERFORMING ORGANIZATION NAME & ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306		10. PROGRAM ELEMENT, PROJECT, TASK AREA AND WORK UNIT NOS. 61102F 2304/A5
11. CONTROLLING OFFICE NAME & ADDRESS Air Force Office of Scientific Research/NM Bolling Air Force Base, D.C. 20332		12. REPORT DATE April, 1980 13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

m-dimensional simple epidemics and stochastic processes, exponential and negative binomial random variables, convergence in distribution.

20. ABSTRACT

A population of susceptible individuals exposed to m contagious diseases is considered. The progress of this epidemic among individuals is modeled by an m-dimensional stochastic progress. The components of this process represent the number of infective individuals with the respective diseases at time t.

A class of m-dimensional stochastic processes is constructed. These processes describe the progress of the epidemic models considered in the sequel. Exact and approximate formulas for the joint and marginal state probabilities of these models are obtained. It is shown that the approximate formulas are very simple functions of time while, the derivation of the exact formulas involve tedious computations.

UNCLASSIFIED